A1. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Answer. The six ordered pairs are $(1009, 2018), (2018, 1009), (1009 \cdot 337, 674) = (350143, 674), (1009 \cdot 1346, 673) = (1358114, 673), (674, 1009 \cdot 337) = (674, 350143)$, and $(673, 1009 \cdot 1346) = (673, 1358114)$.

Solution. First rewrite the equation as $2 \cdot 1009(a + b) = 3ab$, and note that 1009 is prime, so at least one of a and b must be divisible by 1009. If both a and b are divisible by 1009, say with $a = 1009q, b = 1009r$, then we have $2(q + r) = 3qr$. But $qr \geq q + r$ for integers $q, r \geq 2$, so at least one of q, r is 1. This leads to the solutions $q = 1, r = 2$ and $r = 1, q = 2$, corresponding to the ordered pairs $(a, b) = (1009, 2018)$ and $(a, b) = (2018, 1009)$.

In the remaining case, just one of a and b is divisible by 1009, say $a = 1009q$. This yields $2 \cdot 1009(1009q + b) = 3 \cdot 1009qb$, which can be rewritten as $2 \cdot 1009q = (3q - 2)b$. Because the prime 1009 does not divide b, it must divide $3q - 2$; say $3q - 2 = 1009k$. Then $1009k + 2 = 3 \cdot 336k + k + 2$ is divisible by 3, so $k \equiv 1 \pmod 3$. For $k = 1$, we get $q = 337, a = 1009 \cdot 337, b = 2q = 674$. For $k = 4$, we get $q = 1346, a = 1009 \cdot 1346, b = q/2 = 673$. We now show there is no solution with $k > 4$. Assuming there is one, the corresponding value of q is greater than 1346, and so the corresponding

$$b = \frac{2q}{3q - 2} \cdot 1009$$

is less than 673. Because b is an integer, it follows that $b \leq 672$, which implies

$$\frac{1}{b} \geq \frac{1}{672} > \frac{3}{2018}, \quad \text{contradicting} \quad \frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Finally, along with the two ordered pairs (a, b) for which a is divisible by 1009 and b is not, we get two more ordered pairs by interchanging a and b.

A2. Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

Answer. The determinant is 1 if $n = 1$ and -1 if $n > 1$.

Solution. Note that if we take the nonempty subsets of $\{1, 2, \ldots, n\}$ in a different order, this will replace the matrix M by PMP^{-1} for some permutation matrix P, and the determinant will stay unchanged. Denote the matrix M by M_n to indicate its dependence on n. Then we will show that $\det(M_{n+1}) = -(\det(M_n))^2$, from which the given answer follows by induction on n because M_1 has the single entry 1.

Using the order $S_1, S_2, \ldots, S_{2^n-1}$ for the nonempty subsets of $\{1, 2, \ldots, n\}$, we arrange the nonempty subsets of $\{1, 2, \ldots, n + 1\}$ in the order

$$S_1, S_2, \ldots, S_{2^n-1}, \{n + 1\}, S_1 \cup \{n + 1\}, S_2 \cup \{n + 1\}, \ldots, S_{2^n-1} \cup \{n + 1\}.$$

Then any of the first $2^n - 1$ subsets has empty intersection with $\{n + 1\}$, any two of the last 2^n subsets have nonempty intersection because they both contain $n + 1$, and for any other choice of two nonempty subsets of $\{1, 2, \ldots, n + 1\}$, whether they have nonempty intersection is completely determined by the relevant entry in M_n. Thus the matrix M_{n+1} has the following block form:

$$M_{n+1} = \begin{pmatrix} M_n & \mathbf{0_n} & M_n \\ \mathbf{0_n}^T & 1 & \mathbf{1_n}^T \\ M_n & \mathbf{1_n} & E_n \end{pmatrix},$$

where $\mathbf{0_n}, \mathbf{1_n}$ denote column vectors of length $2^n - 1$, all of whose entries are $0, 1$ respectively, and E_n is the $(2^n - 1) \times (2^n - 1)$ matrix, all of whose entries are 1. To find $\det(M_{n+1})$, we subtract the middle row from each of the rows below it. This will not affect the lower left block, it will change the lower part of the middle column from $\mathbf{1_n}$ to $\mathbf{0_n}$, and it will replace the lower right block E_n by a block of zeros. Then we can expand the determinant using the middle column (whose only nonzero entry is now the "central" 1) to get

$$\det(M_{n+1}) = \det \begin{pmatrix} M_n & M_n \\ M_n & O \end{pmatrix} = -\det \begin{pmatrix} M_n & O \\ M_n & M_n \end{pmatrix},$$

where the last step is carried out by switching the ith row with the $(2^n - 1 + i)$th row for all $i = 1, 2, \ldots, 2^n - 1$. (Because this is an odd number of row swaps, the sign of the determinant is reversed.) Finally, the determinant of the block triangular matrix $\begin{pmatrix} M_n & O \\ M_n & M_n \end{pmatrix}$ is equal to the product of the determinants of the diagonal blocks, so it equals $(\det(M_n))^2$, and the result follows.

A3. Determine the greatest possible value of $\sum\limits_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum\limits_{i=1}^{10} \cos(x_i) = 0$.

Answer. The maximum value is $\dfrac{480}{49}$.

Solution. Let $z_i = \cos(x_i)$. Then the real numbers $z_1, z_2, \ldots, z_{10}$ must satisfy $-1 \le z_i \le 1$, and the given restriction on the x_i is equivalent to the additional restriction $z_1 + z_2 + \cdots + z_{10} = 0$ on the z_i. Also, note that

$$\begin{aligned}\cos(3x_i) &= \cos(2x_i)\cos(x_i) - \sin(2x_i)\sin(x_i) = (2\cos^2(x_i) - 1)\cos(x_i) - 2\sin^2(x_i)\cos(x_i) \\ &= 2\cos^3(x_i) - \cos(x_i) - 2(1 - \cos^2(x_i))\cos(x_i) = 4z_i^3 - 3z_i.\end{aligned}$$

Thus we can rephrase the problem as follows: Find the maximum value of the function f given by

$$f(z_1, \ldots, z_{10}) = \sum_{i=1}^{10} (4z_i^3 - 3z_i) = 4\sum_{i=1}^{10} z_i^3$$

on the set

$$S = \{(z_1, \ldots, z_{10}) \in [-1, 1]^{10} \mid z_1 + \cdots + z_{10} = 0\}.$$

Because f is continuous and S is closed and bounded (compact), f does take on a maximum value on this set; suppose this occurs at $(m_1, \ldots, m_{10}) \in S$. Let

$a = m_1 + m_2$. If we fix $z_3 = m_3, \ldots, z_{10} = m_{10}$ and $z_1 + z_2 = a$, then the function

$$g(z) = \frac{1}{4} f(z, a - z, m_3, \ldots, m_{10}) - (m_3^3 + \cdots + m_{10}^3) = z^3 + (a - z)^3 = a^3 - 3a^2 z + 3az^2$$

has a global maximum at $z = m_1$ on the interval for z that corresponds to $(z, a - z, m_3, \ldots, m_{10}) \in S$, which is the interval $[a - 1, 1]$ if $a \geq 0$ and the interval $[-1, a + 1]$ if $a < 0$. If $a \geq 0$ that maximum occurs at both endpoints of the interval (that is, when either $z = 1$ or $a - z = 1$), while if $a < 0$ the maximum occurs when $z = a - z = a/2$. We can conclude that either (in the first case) one of m_1, m_2 equals 1 or (in the second case) $m_1 = m_2$. Repeating this argument for other pairs of subscripts besides $(1, 2)$, we see that whenever two of the m_i are distinct, one of them equals 1. So there are at most two distinct values among the m_i, namely 1 and one other value v. Suppose that 1 occurs d times among the m_i, so that v occurs $10 - d$ times. Then because $m_1 + m_2 + \cdots + m_{10} = 0$, we have $d + (10 - d)v = 0$, so $v = d/(d - 10)$ and the maximum value of f on S is

$$f(m_1, \ldots, m_{10}) = 4 \cdot d \cdot 1^3 + 4 \cdot (10 - d) \cdot \left(\frac{d}{d - 10}\right)^3 = 4d - \frac{4d^3}{(10 - d)^2} = h(d), \text{ say.}$$

In order for $v = d/(d - 10)$ to be in the interval $[-1, 1]$, we must have $0 \leq d \leq 5$. So to finish, we can make a table of values

d	$h(d)$
0	0
1	320/81
2	15/2
3	480/49
4	80/9
5	0

and read off that the maximum value is $\frac{480}{49} = 9\frac{39}{49}$, for $d = 3$. (This value occurs for $x_1 = x_2 = x_3 = 0,\ x_4 = x_5 = \cdots = x_{10} = \arccos(-3/7)$, which corresponds to $z_1 = z_2 = z_3 = 1,\ z_4 = z_5 = \cdots = z_{10} = -3/7$.)

A4. Let m and n be positive integers with $\gcd(m, n) = 1$, and define

$$a_k = \left\lfloor \frac{mk}{n} \right\rfloor - \left\lfloor \frac{m(k-1)}{n} \right\rfloor$$

for $k = 1, 2, \ldots, n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1} gh^{a_2} \cdots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Solution. The proof is by induction on n. If $n = 1$, we have $a_1 = m$ and $gh^m = e$, so $g = h^{-m}$ and g, h commute. For the induction step, first consider the case that $m < n$. Then each a_k is 0 or 1, and $a_1 + \cdots + a_n = m$. The values $k_1, \ldots k_m$ of k for which $a_k = 1$ are the smallest values for which

$$\frac{mk}{n} \geq 1, \frac{mk}{n} \geq 2, \ldots, \frac{mk}{n} \geq m,$$

so they are

$$k_1 = \left\lceil \frac{n}{m} \right\rceil, k_2 = \left\lceil \frac{2n}{m} \right\rceil, \ldots, k_m = \left\lceil \frac{mn}{m} \right\rceil = n\,.$$

Thus the given relation $gh^{a_1}gh^{a_2}\cdots gh^{a_n} = e$ can be rewritten, by omitting all factors h^0, as

$$g^{k_1}hg^{k_2-k_1}h\cdots g^{k_m-k_{m-1}}h = e.$$

Taking the inverse of both sides, we get

$$h^{-1}(g^{-1})^{k_m-k_{m-1}}\cdots h^{-1}(g^{-1})^{k_2-k_1}h^{-1}(g^{-1})^{k_1} = e.$$

We claim that this new relation is one of the original form, but with g and h replaced by h^{-1} and g^{-1} respectively, and with the roles of m and n interchanged. Because $m < n$, it will follow by the induction hypothesis that h^{-1} and g^{-1} commute, and thus g and h commute. To prove the claim, note that the exponents of g^{-1} in the new relation are

$$b_1 = k_m - k_{m-1}, \ldots, b_{m-1} = k_2 - k_1, b_m = k_1,$$

and so, with the convention $k_0 = 0$, we have

$$\begin{aligned} b_i = k_{m-i+1} - k_{m-i} &= \left\lceil \frac{(m-i+1)n}{m} \right\rceil - \left\lceil \frac{(m-i)n}{m} \right\rceil \\ &= -\left\lfloor -\frac{(m-i+1)n}{m} \right\rfloor + \left\lfloor -\frac{(m-i)n}{m} \right\rfloor \\ &= \left\lfloor \frac{in}{m} - n \right\rfloor - \left\lfloor \frac{(i-1)n}{m} - n \right\rfloor \\ &= \left\lfloor \frac{in}{m} \right\rfloor - \left\lfloor \frac{(i-1)n}{m} \right\rfloor, \end{aligned}$$

as desired.

Now consider the remaining case, in which $m > n$. Write $m = qn + r$ with $0 \le r < n$. We have

$$a_k = \lfloor \tfrac{mk}{n} \rfloor - \lfloor \tfrac{m(k-1)}{n} \rfloor = qk + \lfloor \tfrac{rk}{n} \rfloor - q(k-1) - \lfloor \tfrac{r(k-1)}{n} \rfloor = q + a'_k$$

where $a'_k = \lfloor \frac{rk}{n} \rfloor - \lfloor \frac{r(k-1)}{n} \rfloor$. If we set $g' = gh^q$ then we have

$$g'h^{a'_1}g'h^{a'_2}\cdots g'h^{a'_n} = gh^{a_1}gh^{a_2}\cdots gh^{a_n} = e.$$

This is again a relation of the original form, but with m replaced by r (and g replaced by g'). So by the case considered above, g' and h commute. As a result, $(gh)h^q = g'h = hg' = (hg)h^q$ and it follows that $gh = hg$, completing the proof.

A5. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \ge 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

Solution. Suppose, to the contrary, that $f^{(n)}(x) \ge 0$ for all integers $n \ge 0$ and all $x \in \mathbb{R}$. To begin, note that f has a minimum at $x = 0$, so $f'(0) = 0$. Now we show by induction on n that for *every* function f satisfying the given conditions and such that $f^{(n)}(x) \ge 0$ for all $n \ge 0$ and all x, we have $f(2x) \ge 2^n f(x)$ for all $n \ge 0$ and all $x \ge 0$. Since $f' \ge 0$, we have $f(2x) \ge f(x)$ for $x \ge 0$, showing the base case $n = 0$. Suppose we have shown for some particular n that $f(2x) \geq 2^n f(x)$ for all relevant functions f and all $x \geq 0$. Note that because $f(0) = 0$ and $f(1) = 1$, $f'(x) > 0$ for some $x \in [0,1]$; because f' is nondecreasing, it follows that $f'(1) > 0$. Therefore, we can define a function g by $g(x) = f'(x)/f'(1)$, and for this infinitely differentiable function we have that $g(0) = 0$, $g(1) = 1$, and all derivatives of g are nonnegative. By the induction hypothesis, we then have $g(2x) \geq 2^n g(x)$, and therefore $f'(2x) \geq 2^n f'(x)$, for all $x \geq 0$. Integrating with respect to x from 0 to y gives $\frac{1}{2} f(2y) \geq 2^n f(y)$ for all $y \geq 0$, so this shows that $f(2x) \geq 2^{n+1} f(x)$ for all $x \geq 0$, completing the induction proof. In particular, we now have $f(2) \geq 2^n f(1) = 2^n$ for all $n \geq 0$, which is obviously false. So it must be the case that $f^{(n)}(x) < 0$ for some integer $n \geq 0$ and some $x \in \mathbb{R}$.

A6. Suppose that A, B, C, and D are distinct points in the Euclidean plane no three of which lie on a line. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$

is a rational number.

NOTE: I don't believe this is quite the final wording of this problem.

Solution. Let $\mathbf{v}, \mathbf{w}, \mathbf{z}$ be the displacement vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ from A to the points B, C, D respectively. Then the dot products $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$, $\mathbf{w} \cdot \mathbf{w}$, $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w})$ are all rational, so

$$\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}(\mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}))$$

is also rational; similarly, $\mathbf{v} \cdot \mathbf{z}$ and $\mathbf{w} \cdot \mathbf{z}$ are rational.

Because A, B, and C are not collinear, $\mathbf{v}$ and $\mathbf{w}$ form a basis for $\mathbb{R}^2$, so there are constants $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{z} = \lambda \mathbf{v} + \mu \mathbf{w}$. Then we have the system of linear equations

$$\begin{aligned} \mathbf{v} \cdot \mathbf{z} &= (\mathbf{v} \cdot \mathbf{v})\, \lambda + (\mathbf{v} \cdot \mathbf{w})\, \mu \\ \mathbf{w} \cdot \mathbf{z} &= (\mathbf{w} \cdot \mathbf{v})\, \lambda + (\mathbf{w} \cdot \mathbf{w})\, \mu \end{aligned}$$

for λ and μ. The determinant $(\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - (\mathbf{v} \cdot \mathbf{w})^2$ of the matrix of this system is positive (because $\mathbf{v}$ and $\mathbf{w}$ are linearly independent) by the Cauchy-Schwarz inequality. Thus, because all coefficients of the system are rational, λ and μ are also rational. Now we can rewrite the desired quotient as

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)} = \frac{|\det(\mathbf{v}\ \mathbf{w})|/2}{|\det(\mathbf{v}\ \mathbf{z})|/2} = \left|\frac{\det(\mathbf{v}\ \mathbf{w})}{\lambda \det(\mathbf{v}\ \mathbf{v}) + \mu \det(\mathbf{v}\ \mathbf{w})}\right| = \left|\frac{1}{\mu}\right|,$$

a rational number. (Note that $\mu \neq 0$ because A, B, D are not collinear.)

(The B section starts on the next page.)

B1. Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \ 0 \leq a \leq 2,\ 0 \leq b \leq 100,\ \text{and } a, b \in \mathbb{Z} \right\}.$$

Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

Answer. The vectors $\mathbf{v}$ of the form $\begin{pmatrix} 1 \\ b \end{pmatrix}$ with b even, $0 \leq b \leq 100$.

Solution. First note that if we add all the vectors in $\mathcal{P}$ by first summing over a for fixed b, we get the sum of $\begin{pmatrix} 3 \\ 3b \end{pmatrix}$ for $0 \leq b \leq 100$, which is $\begin{pmatrix} 303 \\ 3 \cdot (1 + \cdots + 100) \end{pmatrix} = \begin{pmatrix} 303 \\ 3 \cdot 50 \cdot 101 \end{pmatrix}$. Thus if the set $\mathcal{P} \setminus \{\mathbf{v}\}$ is to be partitioned into two sets of equal sum, the vector $\begin{pmatrix} 303 \\ 3 \cdot 50 \cdot 101 \end{pmatrix} - \mathbf{v}$ must have both coordinates even. For $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, this implies that a is odd and b is even, so because $\mathbf{v} \in \mathcal{P}$, we have $a = 1, b$ even, $0 \leq b \leq 100$. It remains to show that this necessary condition on $\mathbf{v}$ is also sufficient.

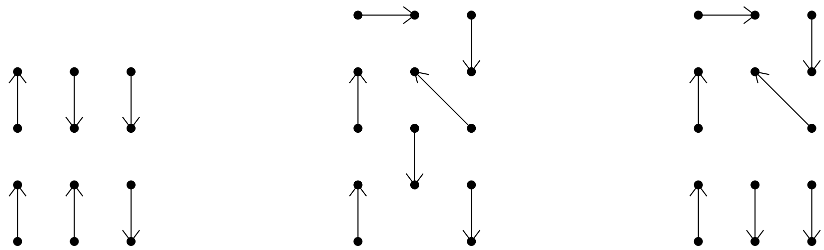
Identify each of the vectors $\mathbf{w} = \begin{pmatrix} c \\ d \end{pmatrix}$ in $\mathcal{P}$ with the lattice point (c, d). Given a particular vector $\mathbf{v} = \begin{pmatrix} 1 \\ b \end{pmatrix}$ in $\mathcal{P}$ with b even, there are 302 lattice points in $\mathcal{P} \setminus \{\mathbf{v}\}$. If we can number these points $P_1, P_2, \ldots, P_{302}$ such that the sum of the displacement vectors $\overrightarrow{P_1P_2}, \overrightarrow{P_3P_4}, \ldots, \overrightarrow{P_{301}P_{302}}$ is zero, then we can partition $\mathcal{P} \setminus \{\mathbf{v}\}$ into the set of points P_i with i odd and the set of points P_i with i even, and those sets will have equal size and equal sum. To do so, we first partition the set $\mathcal{P} \setminus \{\mathbf{v}\}$ into 24 rectangular sets of 4×3 lattice points, along with a single 5×3 rectangular set from which one of the points in the middle column is missing. For the 4×3 rectangular sets, we can take three displacement vectors pointing up and three displacement vectors pointing down, as shown in the first diagram below. For the 5×3 rectangular set with a single point missing, there are (up to symmetry) two cases, depending on whether the missing point is on an edge or at the center (the parity condition on b guarantees that it will be either on an edge or at the center). These are shown in the second and third diagrams below. As an example, if the 5×3 rectangle is the set

$$\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \ 0 \leq a \leq 2,\ 0 \leq b \leq 4,\ \text{and } a, b \in \mathbb{Z} \right\}$$

and $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the missing point, then the second diagram corresponds to the partition of the rectangular set minus that point into the two subsets of equal size and equal sum

$$\{(0,0), (0,2), (0,4), (1,2), (2,4), (2,2), (2,1)\} \quad \text{and}$$
$$\{(0,1), (0,3), (1,4), (1,1), (2,3), (1,3), (2,0)\},$$

which contain the starting and end points, respectively, of the displacement vectors shown.



B2. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \le 1\}$.

Solution. If z is a root of f_n, then $0 = (1-z)f_n(z) = n - \sum_{j=1}^n z^j$, so $\sum_{j=1}^n z^j = n$. On the other hand, when $|z| \le 1$, we have $\left|\sum_{j=1}^n z^j\right| \le \sum_{j=1}^n |z|^j \le n$, with equality only for $z = 1$. Because $z = 1$ is not a root of f_n, we are done.

B3. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

Answer. $n = 2^2, 2^4, 2^{16}, 2^{256}$.

Solution. We first prove that for positive integers a and b, $2^a - 1$ divides $2^b - 1$ if and only if a divides b. If a divides b then we can write $b = aq$, and modulo $2^a - 1$ we then have $2^b - 1 = (2^a)^q - 1 \equiv 1^q - 1 = 0$. Conversely, suppose that $2^a - 1$ divides $2^b - 1$. Let $b = aq + r$ with $0 \le r < a$; we then have $2^b - 1 = 2^r(2^{aq} - 1) + (2^r - 1)$. Because $2^a - 1$ divides $2^b - 1$ and $2^{aq} - 1$, it must also divide $2^r - 1$. Because $0 \le r < a$, it follows that $r = 0$, so a divides b.

A positive integer n divides 2^n if and only if n is a power of 2. So we may assume that $n = 2^m$ for some nonnegative integer m. Now $n - 1 = 2^m - 1$ divides $2^n - 1 = 2^{2^m} - 1$ if and only if m divides 2^m, so if and only if m is a power of 2. So we may assume that $m = 2^l$, that is, $n = 2^{2^l}$, for some nonnegative integer l, and we have to find all $n < 10^{100}$ of this form for which $n - 2$ divides $2^n - 2$.

Note that $n-2 = 2\cdot(2^{2^l-1}-1)$, so $n-2$ divides $2^n - 2 = 2\cdot(2^{n-1}-1) = 2\cdot(2^{2^{2^l}-1}-1)$ if and only if $2^l - 1$ divides $2^{2^l} - 1$, which is if and only if l divides 2^l, that is, if and only if l is a power of 2. We can write $l = 2^k$, so that $n = 2^{2^{2^k}}$, and to finish we have to find the values of $k \ge 0$ for which $n < 10^{100}$.

Suppose that n is a solution with $n < 10^{100}$. Then we have $2^m = n < 10^{100} < (2^4)^{100} = 2^{400}$. It follows that $m = 2^l < 400 < 2^9$, so $l = 2^k < 9 < 2^4$ and thus $0 \le k \le 3$. Conversely, $2^{2^{2^3}} = 2^{2^8} = 2^{256} < 2^{300} = (2^3)^{100} < 10^{100}$. So the solutions are $n = 2^{2^{2^k}}$ with $k = 0, 1, 2, 3$; in other words, n is equal to 2^2, 2^4, 2^{16} or 2^{256}.

B4. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \ge 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Solution. Let z be a (complex) root of the polynomial $z^2 - 2az + 1$. Then $z \ne 0$, and z satisfies $2az = z^2 + 1$, so $a = \dfrac{z + z^{-1}}{2}$. If the Fibonacci numbers $F_0, F_1, F_2, \ldots$ are given, as usual, by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, then we see that

$$x_n = \frac{z^{F_n} + z^{-F_n}}{2}$$

for $n = 0, 1, 2$. We now show by induction on n that this equation holds for all n. For the induction step, assuming the equation is correct for $n, n-1$, and $n-2$, we have

$$\begin{aligned} x_{n+1} &= 2x_n x_{n-1} - x_{n-2} \\ &= \frac{(z^{F_n} + z^{-F_n})(z^{F_{n-1}} + z^{-F_{n-1}}) - (z^{F_{n-2}} + z^{-F_{n-2}})}{2} \\ &= \frac{z^{F_n + F_{n-1}} + z^{-(F_n + F_{n-1})} + z^{F_n - F_{n-1}} + z^{-(F_n - F_{n-1})} - (z^{F_{n-2}} + z^{-F_{n-2}})}{2} \\ &= \frac{z^{F_{n+1}} + z^{-F_{n+1}}}{2}, \end{aligned}$$

because $F_n - F_{n-1} = F_{n-2}$.

Suppose that $x_n = 0$ for some n, say $x_m = 0$. Then $z^{2F_m} = -1$, so z is a dth root of unity, where $d = 4F_m$. Now note that the Fibonacci sequence modulo d is periodic: There are only finitely many (specifically, d^2) possibilities for a pair ($F_i \bmod d, F_{i+1} \bmod d$) of successive Fibonacci numbers modulo d, and when a pair reoccurs, say $(F_i \bmod d, F_{i+1} \bmod d) = (F_j \bmod d, F_{j+1} \bmod d)$ with $i < j$, it is straightforward to show by induction that $F_{i+k} = F_{j+k} \bmod d$ for all k, including negative k for which $i + k \geq 0$. But then, because $x_n = \dfrac{z^{F_n} + z^{-F_n}}{2}$ is determined by the value of F_n modulo d, the sequence (x_n) is also periodic, and we are done.

B5. Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

Solution. Consider the Jacobian matrix

$$J = J(x_1, x_2) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}$$

and the related symmetric matrix

$$A = \frac{1}{2}(J + J^T) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f_1}{\partial x_2}\right) & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

From the givens, the entries of A are positive everywhere, and the determinant of A is also positive everywhere. So the eigenvalues of A are positive (because their sum $\mathrm{tr}(A)$ and their product are both positive), and so A is positive definite. That is, for any nonzero vector $\mathbf{v}$, we have $\mathbf{v}^T A\mathbf{v} = \mathbf{v} \cdot A\mathbf{v} > 0$, so $\mathbf{v}^T(J + J^T)\mathbf{v} = \mathbf{v}^T J\mathbf{v} + (\mathbf{v}^T J\mathbf{v})^T > 0$.

But the scalar $\mathbf{v}^T J\mathbf{v}$ is its own transpose, showing that $\mathbf{v}^T J\mathbf{v} > 0$ at all points in $\mathbb{R}^2$ and for all nonzero vectors $\mathbf{v}$.

We now show that if P and Q are distinct points in $\mathbb{R}^2$, then the dot product of the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{f(P)f(Q)}$ is positive; it then follows that $f(P) \neq f(Q)$, showing that f is one-to-one. In doing so, we will identify points in $\mathbb{R}^2$ with vectors. In particular, the displacement vector $\overrightarrow{f(P)f(Q)}$ can be written as

$$\begin{aligned}
\overrightarrow{f(P)f(Q)} = f(Q) - f(P) &= f(P + t\overrightarrow{PQ})\Big|_{t=0}^{1} \\
&= \int_0^1 \frac{d}{dt}\left[f(P + t\overrightarrow{PQ})\right] dt \\
&= \int_0^1 J(P + t\overrightarrow{PQ})\,\overrightarrow{PQ}\; dt,
\end{aligned}$$

using the multivariable chain rule. Thus the dot product of this vector and $\overrightarrow{PQ}$ is

$$\begin{aligned}
\overrightarrow{PQ} \cdot \overrightarrow{f(P)f(Q)} &= \overrightarrow{PQ} \cdot \left(\int_0^1 J(P + t\overrightarrow{PQ})\,\overrightarrow{PQ}\; dt\right) \\
&= \int_0^1 \mathbf{v}^T J(P + t\mathbf{v})\mathbf{v}\; dt,
\end{aligned}$$

where $\mathbf{v} = \overrightarrow{PQ}$. By our earlier observation, the integrand is positive for all t, and so the integral is also, completing the proof.

B6. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

Solution. Note that the binary expansion of 2018 is

$$2018 = 1024 + 512 + 256 + 128 + 64 + 32 + 2 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^1,$$

and consequently

$$\frac{2018}{2048} = \frac{2018}{2^{11}} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + 2^{-10}.$$

We can interpret this as the probability that a random variable X takes on a value in the set $A = \{1, 2, 3, 4, 5, 6, 10\}$, if the possible values of the variable are all the positive integers, and the probability of taking on the value k is 2^{-k}. (Note that $\sum_{k=1}^{\infty} 2^{-k} = 1$, so this is a consistent assignment of probabilities.)

Now consider a sequence $\mathbf{X} = (X_n)_{n=1}^{2018}$ of independent random variables that each take positive integer values k with probability 2^{-k}. For any given sequence $(s_n) \in S$, the probability that $X_n = s_n$ for all n (with $1 \leq n \leq 2018$) equals $2^{-s_1}2^{-s_2}\cdots 2^{-s_{2018}} = 2^{-\sum s_n} = 2^{-3860}$. Therefore, the probability that the sequence (X_n) will be in S is

$$\mathbb{P}(\mathbf{X} \in S) = |S| \cdot 2^{-3860}.$$

On the other hand, this probability is less than the probability that each X_n takes on a value in A; as we have seen, for any X_n individually that probability is 2018/2048, and so because the X_n are independent, the probability that they will all take on values in A is $(2018/2048)^{2018}$. So we have the inequality

$$|S| \cdot 2^{-3860} = \mathbb{P}(\mathbf{X} \in S) < \left(\frac{2018}{2048}\right)^{2018},$$

and the desired result follows.