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William Lowell **PUTNAM** Mathematical Competition

Problems for Session A

The 82nd William Lowell Putnam Mathematical Competition 2021

A1 A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

A2 For every positive real number x, let

$$g(x) = \lim_{r \to 0} \left((x+1)^{r+1} - x^{r+1} \right)^{\frac{1}{r}}.$$

Find $\displaystyle\lim_{x \to \infty} \frac{g(x)}{x}$.

A3 Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

A4 Let

$$I(R) = \iint_{x^2+y^2 \leq R^2} \left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4} \right) dx\, dy\,.$$

Find

$$\lim_{R \to \infty} I(R)\,,$$

or show that this limit does not exist.

A5 Let A be the set of all integers n such that $1 \leq n \leq 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let

$$S(j) = \sum_{n \in A} n^j\,.$$

Determine all values of j such that $S(j)$ is a multiple of 2021.

A6 Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as the product of two nonconstant polynomials with integer coefficients. Does it follow that $P(2)$ is a composite integer?