A1 A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Answer: 578.

Solution: Each hop can be described by a displacement vector $\langle p, q\rangle$ with $p^2 + q^2 = 25$; the twelve possible vectors are

$$\langle 3,4\rangle; \langle -3,4\rangle; \langle 3,-4\rangle; \langle -3,-4\rangle; \langle 4,3\rangle; \langle -4,3\rangle; \langle 4,-3\rangle; \langle -4,-3\rangle; \langle 5,0\rangle; \langle -5,0\rangle; \langle 0,5\rangle; \langle 0,-5\rangle.$$

One way to write the total displacement as a sum of 578 of these vectors is

$$\langle 2021, 2021\rangle = 288 \cdot \langle 3,4\rangle + 288 \cdot \langle 4,3\rangle + \langle 0,5\rangle + \langle 5,0\rangle.$$

To show that it cannot be done with fewer, note that each hop can increase the sum of the grasshopper's coordinates by at most $3 + 4 = 7$. Because this sum has to reach

$$2021 + 2021 = 4042 = 7 \cdot (577) + 3\,,$$

at least 578 hops are needed.

A2 For every positive real number x, let

$$g(x) = \lim_{r \to 0} \left((x+1)^{r+1} - x^{r+1} \right)^{\frac{1}{r}}.$$

Find $\lim\limits_{x \to \infty} \dfrac{g(x)}{x}$.

Answer: e.

Solution: Note that for $r > -1$ and any positive x, we have $(x+1)^{r+1} - x^{r+1} > 0$. Thus, by the continuity of the logarithm,

$$\begin{aligned}
\log g(x) &= \lim_{r \to 0} \log \left(((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}} \right) \\
&= \lim_{r \to 0} \frac{1}{r} \log \left((x+1)^{r+1} - x^{r+1} \right).
\end{aligned}$$

Applying L'Hôpital's rule, we get

$$\begin{aligned}
\log g(x) &= \lim_{r \to 0} \frac{(x+1)^{r+1} \log(x+1) - x^{r+1} \log x}{(x+1)^{r+1} - x^{r+1}} \\
&= \frac{(x+1)\log(x+1) - x \log x}{(x+1) - x} = \log \left((x+1)^{x+1} x^{-x} \right), \quad \text{so} \\
g(x) &= (x+1)^{x+1} x^{-x} = (x+1) \left(1 + \frac{1}{x} \right)^{x}.
\end{aligned}$$

Finally,

$$\lim_{x \to \infty} \frac{g(x)}{x} = \lim_{x \to \infty} \frac{x+1}{x} \left(1 + \frac{1}{x} \right)^{x} = 1 \cdot e = e.$$

A3 Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

Answer: A necessary and sufficient condition is that N be of the form $N = 3m^2$, where m is a positive integer.

Solution 1: To see that the condition is sufficient, note that the four points

$$(-m, -m, -m),\ (m, m, -m),\ (m, -m, m),\ (-m, m, m)$$

are the vertices of a regular tetrahedron inscribed in the sphere $x^2 + y^2 + z^2 = 3m^2$.

To show that the condition is necessary, we will use two lemmas:

Lemma 1. *If T is a tetrahedron whose vertices have integer coordinates, then its volume is of the form $V(T) = D/6$ for some integer D.*

Lemma 2. *The volume of a regular tetrahedron T inscribed in a sphere of radius R is given by $V(T) = \dfrac{8\sqrt{3}R^3}{27}$.*

Assuming that the sphere $x^2 + y^2 + z^2 = N$ has an inscribed regular tetrahedron T whose vertices have integer coordinates, we can combine the results of these lemmas (for $R = \sqrt{N}$) to get

$$D = 6\,V(T) = \frac{16\sqrt{3}\,N^{3/2}}{9} = \frac{16N}{9}\sqrt{3N}\,.$$

Because D is an integer, it follows that $\sqrt{3N}$ is a rational number. Thus the prime factorization of N must contain an odd number of factors 3 and an even number of factors p for any other prime p; therefore, $N = 3m^2$ for some positive integer m.

Proof of Lemma 1: Let P, Q, R, and S be the vertices of the tetrahedron. As these are all lattice points, the three vectors

$$\vec{PQ} = \langle q_1, q_2, q_3\rangle, \quad \vec{PR} = \langle r_1, r_2, r_3\rangle, \quad \vec{PS} = \langle s_1, s_2, s_3\rangle$$

have integer components. We can use a triple product to express the volume as

$$V = \frac{1}{6}|\vec{PQ}\cdot(\vec{PR}\times\vec{PS})| = \frac{1}{6}\left|\det\begin{pmatrix} q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3\end{pmatrix}\right|,$$

and as the determinant is an integer, we are done.

Proof of Lemma 2: If a regular tetrahedron T is inscribed in a sphere of radius R, we can choose a coordinate system in which the vertices of T are given by

$$P = (-d, -d, -d), \quad Q = (d, d, -d), \quad R = (d, -d, d), \quad S = (-d, d, d)$$

with $3d^2 = R^2$. We then have

$$\vec{PQ} = \langle 0, 2d, 2d\rangle, \quad \vec{PR} = \langle 2d, 0, 2d\rangle, \quad \vec{PS} = \langle 2d, 2d, 0\rangle,$$

and the volume is

$$V = \frac{1}{6}|\vec{PQ} \cdot (\vec{PR} \times \vec{PS})| = \frac{1}{6}\left| \det \begin{pmatrix} 0 & 2d & 2d \\ 2d & 0 & 2d \\ 2d & 2d & 0 \end{pmatrix} \right| = \frac{8}{3}d^3 = \frac{8\sqrt{3}R^3}{27}.$$

Solution 2: As in Solution 1, the condition on N is sufficient. To show that it is necessary, note that if

$$v_i = (x_i, y_i, z_i), \quad i = 1, 2, 3, 4$$

are the vertices of a regular tetrahedron inscribed in the sphere $x^2 + y^2 + z^2 = N$, we can add the four antipodal points

$$w_i = (-x_i, -y_i, -z_i), \quad i = 1, 2, 3, 4$$

to get the eight vertices of a cube. (This is easily seen by choosing an alternate coordinate system as in the proof of Lemma 2 above.) Because this cube is inscribed in the sphere, its space diagonals have length $2\sqrt{N}$; therefore, each edge of the cube has length $2\sqrt{N/3}$ and its volume is $8\left(\sqrt{N/3}\right)^3$. But the volume of the cube is the determinant of three vectors with integer coordinates, so it is an integer, and as in Solution 1 it follows that $N = 3m^2$ for some positive integer m.

A4 Let

$$I(R) = \iint\limits_{x^2+y^2\le R^2} \left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4} \right) dx\,dy\,.$$

Find

$$\lim_{R\to\infty} I(R)\,,$$

or show that this limit does not exist.

Answer: The limit exists and equals $\dfrac{\pi\sqrt{2}\log 2}{2}$.

Solution: First we symmetrize the integrand. Let

$$f(x,y) = \frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4}\,, \quad \text{so that}$$

$$f(x,y)+f(y,x) = \frac{2+2(x^2+y^2)}{1+x^4+6x^2y^2+y^4} - \frac{2+x^2+y^2}{2+x^4+y^4} \quad \text{and thus}$$

$$2\,I(R) = \iint\limits_{x^2+y^2\le R^2} \frac{2+2(x^2+y^2)}{1+x^4+6x^2y^2+y^4} - \frac{2+x^2+y^2}{2+x^4+y^4}\,dx\,dy.$$

Now consider the "first part" of this double integral, say

$$J(R) = \iint\limits_{x^2+y^2\le R^2} \frac{2+2(x^2+y^2)}{1+x^4+6x^2y^2+y^4}\,dx\,dy.$$

Let $u = x-y$ and $v = x+y$. Then

$$u^2+v^2 = (x+y)^2+(x-y)^2 = 2(x^2+y^2), \qquad u^4+v^4 = (x+y)^4+(x-y)^4 = 2x^4+2y^4+12x^2y^2$$

and $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$, so

$$\begin{aligned} J(R) &= \iint\limits_{x^2+y^2\le R^2} \frac{2+2(x^2+y^2)}{2+2x^4+12x^2y^2+2y^4}\,2\,dx\,dy \\ &= \iint\limits_{u^2+v^2\le 2R^2} \frac{2+u^2+v^2}{2+u^4+v^4}\,du\,dv. \end{aligned}$$

Note that if we rename the variables in this last integral x, y instead of u, v, the integrand will be the same as the integrand of the "second part" of the double integral for $2\,I(R)$ above. Thus we can recombine the parts to get

$$2\,I(R) = \iint\limits_{R^2<x^2+y^2\le 2R^2} \frac{2+x^2+y^2}{2+x^4+y^4}\,dx\,dy.$$

Converting to polar coordinates, we get

$$2\,I(R) = \int_{t=0}^{2\pi}\int_{r=R}^{R\sqrt{2}} \frac{2+r^2}{2+r^4(\cos^4 t+\sin^4 t)}\,r\,dr\,dt.$$

As $R \to \infty$, throughout the range of integration r also goes to infinity and

$$\frac{2r + r^3}{2 + r^4(\cos^4 t + \sin^4 t)} = \frac{1}{r(\cos^4 t + \sin^4 t)} + \mathcal{O}(1/r^3)\,,$$

where the error term makes a vanishing contribution to the integral. So

$$2\,I(R) \sim \left[\int_{r=R}^{R\sqrt{2}} \frac{dr}{r}\right] \left[\int_{t=0}^{2\pi} \frac{dt}{\cos^4 t + \sin^4 t}\right].$$

Now

$$\int_{r=R}^{R\sqrt{2}} \frac{dr}{r} = \log(R\sqrt{2}) - \log(R) = \log(\sqrt{2}) = \tfrac{1}{2}\log 2$$

and

$$\begin{aligned}\int_{t=0}^{2\pi} \frac{dt}{\cos^4 t + \sin^4 t} &= \int_{t=0}^{2\pi} \frac{dt}{1 - 2\sin^2 t\cos^2 t} = \int_{t=0}^{2\pi} \frac{2\,dt}{2 - \sin^2(2t)} \\ &= \int_{t=0}^{2\pi} \frac{2\,dt}{2\cos^2(2t) + \sin^2(2t)}\,.\end{aligned}$$

The integrand is periodic with period $\frac{\pi}{2}$ and is also even, so we can proceed as follows:

$$\begin{aligned}\int_{t=0}^{2\pi} \frac{dt}{\cos^4 t + \sin^4 t} &= 4\int_{t=-\pi/4}^{\pi/4} \frac{2\,dt}{2\cos^2(2t) + \sin^2(2t)} = 8\int_{t=0}^{\pi/4} \frac{2\,dt}{2\cos^2(2t) + \sin^2(2t)} \\ &= \int_{t=0}^{\pi/4} \frac{16\sec^2(2t)\,dt}{2 + \tan^2(2t)} \\ &= \int_{w=0}^{\infty} \frac{8\,dw}{2 + w^2} = \int_{w=0}^{\infty} \frac{4\,dw}{1 + w^2/2} \\ &= 4\sqrt{2}\tan^{-1}(w/\sqrt{2})\Big|_{w=0}^{\infty} = 2\pi\sqrt{2}\,.\end{aligned}$$

So

$$\lim_{R\to\infty} 2\,I(R) = (\frac{1}{2}\log 2)(2\pi\sqrt{2}) = \pi\sqrt{2}\log 2\,, \quad \text{and} \quad \lim_{R\to\infty} I(R) = \frac{\pi\sqrt{2}\log 2}{2}\,.$$

A5 Let A be the set of all integers n such that $1 \le n \le 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let

$$S(j) = \sum_{n \in A} n^j .$$

Determine all values of j such that $S(j)$ is a multiple of 2021.

Answer: All j that are not multiples of 42 or 46.

Solution: Note that modulo 2021, the set A consists precisely of the elements of the multiplicative group. Multiplying by an element of that group permutes the elements, so if x is relatively prime to 2021, then

$$x^j \cdot S(j) = \sum_{n \in A} (xn)^j \equiv \sum_{m \in A} m^j \equiv S(j) \pmod{2021}.$$

Therefore,

$$(x^j - 1)\, S(j) \equiv 0 \pmod{2021}.$$

Also note that $2021 = 2025 - 4 = 45^2 - 2^2 = 43 \cdot 47$ gives the prime factorization of 2021. Let x be a primitive root modulo 43 (that is, an integer between 1 and 42 that is a generator of the cyclic group $(\mathbb{Z}/43\,\mathbb{Z})^*$, which is the multiplicative group of the field with 43 elements). Then $x^j - 1 \equiv 0 \pmod{43}$ if and only if j is a multiple of 42; also, x is relatively prime to 2021. In particular, if j is not a multiple of 42 we have

$$(x^j - 1)\, S(j) \equiv 0 \pmod{2021} \Rightarrow (x^j - 1)\, S(j) \equiv 0 \pmod{43} \Rightarrow S(j) \equiv 0 \pmod{43}.$$

Similarly, if y is a primitive root modulo 47 and $y \ne 43$, we have

$$(y^j - 1)\, S(j) \equiv 0 \pmod{2021} \Rightarrow (y^j - 1)\, S(j) \equiv 0 \pmod{47} \Rightarrow S(j) \equiv 0 \pmod{47}$$

whenever j is not a multiple of 46. So if j is not a multiple of 42 or 46, then $S(j)$ is a multiple of both 43 and 47, hence of 2021.

Conversely, suppose that j is a multiple of 42. Then $n^j \equiv 1 \pmod{43}$ for all n in the sum, and $S(j)$ is therefore not a multiple of 43 (or of 2021), as

$$S(j) \equiv \sum_{n \in A} 1 = 42 \cdot 46 \cdot 1 \equiv -3 \equiv 40 \pmod{43}.$$

Similarly, if j is a multiple of 46, then $S(j) \equiv 5 \pmod{47}$.

A6 Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as the product of two nonconstant polynomials with integer coefficients. Does it follow that $P(2)$ is a composite integer?

Solution: Yes, we will show that $P(2)$ must be composite. Let $P(x) = F(x)G(x)$ have degree N, where $F(x)$ and $G(x)$ are nonconstant polynomials with integer coefficients, and suppose that $P(2) = p$ were prime. Then either $F(2)$ or $G(2)$ would be a unit, so without loss of generality we may assume that $G(2) = 1$. We have $N \geq 2$ because $F(x)$ and $G(x)$ are nonconstant, and we can write

$$P(x) = \sum_{n=0}^{N} \sigma_n x^n \,,$$

where each σ_n is either 0 or 1 and $\sigma_N = 1$.

Because the coefficients of $P(x)$ are nonnegative integers, $P(x)$ cannot have a positive real root, so $G(x)$ cannot have a positive real root either. Thus, as $G(2)$ is positive, $G(x)$ must be positive for all $x > 0$. In particular, because $\sigma_N = 1$ is the product of the leading coefficients of $F(x)$ and $G(x)$, the polynomial $G(x)$ must be monic. Let $r_1, \ldots, r_k$ be the (complex, not necessarily distinct) roots of $G(x)$, so that

$$G(x) = \prod_{j=1}^{k} (x - r_j) \,.$$

Consider the integer $G(1)$. Because $G(x) > 0$ for $x > 0$, we have $G(1) \geq 1 = G(2)$. In particular, $|G(1)| \geq |G(2)|$, and using the factorization of $G(x)$ we get

$$\prod_{j=1}^{k} |1 - r_j| \geq \prod_{j=1}^{k} |2 - r_j| \,.$$

It follows that $G(x)$ must have at least one root ρ with $|\rho - 1| \geq |\rho - 2|$, which is equivalent to $\mathrm{Re}(\rho) \geq \frac{3}{2}$. This implies that $|\rho| \geq \frac{3}{2}$; note that ρ is also a root of $P(x)$.

First consider the case $N = 2$. Dividing $P(\rho) = 0$ by ρ yields

$$\rho + \sigma_1 + \frac{\sigma_0}{\rho} = 0.$$

We have $\mathrm{Re}(\rho) \geq \frac{3}{2} > 0$ and hence

$$\mathrm{Re}\left(\frac{1}{\rho}\right) = \mathrm{Re}\left(\frac{\bar{\rho}}{|\rho|^2}\right) = \frac{1}{|\rho|^2}\mathrm{Re}\,(\rho) > 0.$$

But then

$$\mathrm{Re}(\rho) \leq \mathrm{Re}\left(\rho + \sigma_1 + \frac{\sigma_0}{\rho}\right) = 0\,,$$

which is a contradiction. (Alternatively, one can check the four possible polynomials $P(x)$ of degree 2 with coefficients from $\{0, 1\}$.)

For $N > 2$, we again divide $P(\rho) = 0$ by ρ^{N-1}, which now yields

$$\rho + \sigma_{N-1} + \frac{\sigma_{N-2}}{\rho} = -\frac{\sigma_{N-3}}{\rho^2} - \cdots - \frac{\sigma_0}{\rho^{N-1}}.$$

Once again, the terms on the left have nonnegative real parts. The triangle inequality gives

$$\mathrm{Re}(\rho) \leq \mathrm{Re}\left(\rho + \sigma_{N-1} + \frac{\sigma_{N-2}}{\rho}\right) \leq \left|\rho + \sigma_{N-1} + \frac{\sigma_{N-2}}{\rho}\right| \leq \frac{\sigma_{N-3}}{|\rho|^2} + \cdots + \frac{\sigma_0}{|\rho|^{N-1}}\,,$$

and we can estimate the sum on the right using an infinite geometric series:

$$\begin{aligned}\frac{\sigma_{N-3}}{|\rho|^2} + \cdots + \frac{\sigma_0}{|\rho|^{N-1}} &\le \frac{1}{|\rho|^2} + \cdots + \frac{1}{|\rho|^{N-1}} \\ &\le \frac{1}{|\rho|^2}\left(1 + \frac{1}{|\rho|} + \frac{1}{|\rho|^2} + \cdots\right) \\ &= \frac{1}{|\rho|^2} \cdot \frac{1}{1 - 1/|\rho|} = \frac{1}{|\rho|(|\rho| - 1)}\,.\end{aligned}$$

But $\dfrac{1}{x(x-1)}$ is a decreasing function of x for $x > 1$, and $|\rho| \ge \mathrm{Re}(\rho) \ge \dfrac{3}{2}$, so we get

$$\frac{3}{2} \le \mathrm{Re}(\rho) \le \frac{1}{|\rho|(|\rho| - 1)} \le \frac{1}{\frac{3}{2}(\frac{3}{2} - 1)} = \frac{4}{3}\,,$$

a contradiction.

Remark. There are polynomials like $x^7 + x^2 + x + 1 = (x+1)(x^2+1)(x^4 - x^3 + 1)$ or $x^7 + x^3 + x^2 + x + 1$ (which is irreducible) which have roots with real part greater than 1. The polynomial $x^{11} + x^3 + x^2 + x + 1$ has a root r with $|r - 2| < 1$. Hence one needs to take some care with this argument.

B1 Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

Answer: $\dfrac{2(\pi-3)}{\pi}$.

Solution: For convenience, choose the center of one of the squares of the checkerboard to be the origin, choose axes parallel to the sides of the squares, and let the squares have side length 2, so that one of them, say S, will have its corners at $(\pm 1, \pm 1)$. Let S' be the additional square that is dropped at random; we may assume that the center of S' is at some uniformly distributed random position in $[-1,1] \times [-1,1]$, and that S' is rotated clockwise relative to S by some uniformly distributed angle θ.

We will determine the allowable positions for S' by conditioning on the angle; by the eight-fold dihedral symmetry, we need only consider $0 \leq \theta < \frac{\pi}{4}$. Given such a θ, first suppose that the center of S' is at the origin (so it coincides with the center of S). Then one of the perpendiculars from the center of S' to its edge is the unit vector $\vec{u} = \langle \sin\theta, \cos\theta \rangle$. The upper right corner of S at $(1,1)$ projects along $\vec{u}$ to a vector of length $\langle 1,1 \rangle \cdot \langle \sin\theta, \cos\theta \rangle = \sin\theta + \cos\theta$. This means that S' can be shifted a distance of $\sin\theta + \cos\theta - 1$ in the direction of $\vec{u}$ before it hits that corner. By symmetry, the allowable region for the center of S' is a square with side length $2(\sin\theta + \cos\theta - 1)$ (centered at the origin, and rotated by an angle of θ).

The total probability is therefore

$$\begin{aligned}
\frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{\text{Area of allowable region}}{\text{Total area of square}}\, d\theta &= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{[2(\sin\theta + \cos\theta - 1)]^2}{4}\, d\theta \\
&= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} 2 + \sin(2\theta) - 2\sin\theta - 2\cos\theta\, d\theta \\
&= \frac{4}{\pi} \left(\frac{\pi}{2} + \frac{1}{2} + 2\left(\frac{1}{\sqrt{2}} - 1 \right) - 2 \cdot \frac{1}{\sqrt{2}} \right) = \frac{2(\pi - 3)}{\pi}.
\end{aligned}$$

B2 Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} \left(a_1 a_2 \cdots a_n\right)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1 \,.$$

Answer: The maximum value is $S = \dfrac{2}{3}$; it is achieved by the sequence $a_k = \dfrac{3}{4^k}$.

Solution: First consider geometric sequences, which are given by $a_k = a_1 r^{k-1}$ for all k, with $0 < r < 1$. For such a sequence we have

$$\left(a_1 a_2 \cdots a_n\right)^{1/n} = a_1 \left(1 \cdot r \cdot \, \cdots \, \cdot r^{n-1}\right)^{1/n} = a_1 \left(r^{n(n-1)/2}\right)^{1/n} = a_1 \, r^{(n-1)/2} \,,$$

and the constraint $\sum_{k=1}^{\infty} a_k = 1$ yields $a_1 = 1 - r$. Thus we can calculate S as a function of r:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{n}{2^n} \left(a_1 a_2 \cdots a_n\right)^{1/n} = (1-r) \sum_{n=1}^{\infty} \frac{n r^{(n-1)/2}}{2^n} = \frac{1-r}{\sqrt{r}} \sum_{n=1}^{\infty} n \left(\frac{\sqrt{r}}{2}\right)^n \\ &= \frac{1-r}{\sqrt{r}} f\left(\frac{\sqrt{r}}{2}\right) = \frac{2(1-r)}{(2-\sqrt{r})^2} \,, \quad \text{where } \; f(x) = \sum_{n=1}^{\infty} n x^n = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2} \,. \end{aligned}$$

By taking the derivative of S with respect to r, which is zero only for $r = \dfrac{1}{4}$, and comparing the values of $\dfrac{2(1-r)}{(2-\sqrt{r})^2}$ for $r = 0$, $r = \dfrac{1}{4}$, and $r = 1$, we find that the maximum value of S that can be obtained for a geometric sequence is $\dfrac{2(3/4)}{(3/2)^2} = \dfrac{2}{3}$, for $r = \dfrac{1}{4}$. It remains to show that this is actually the maximum value for *any* sequence.

Given any sequence of nonnegative numbers that sum to 1, consider the geometric mean, say G_n , of the first n numbers. This can be written as

$$G_n = \left(a_1 a_2 \cdots a_n\right)^{1/n} = \left[\frac{(4a_1)\cdot(4^2 a_2)\,\cdots\,(4^n a_n)}{4^1 \cdot 4^2 \,\cdots\, 4^n}\right]^{1/n} = \frac{1}{2^{n+1}} \left[(4a_1)\cdot(4^2 a_2)\,\cdots\,(4^n a_n)\right]^{1/n} ,$$

and we can then apply the AM-GM inequality to obtain

$$G_n \le \frac{1}{2^{n+1}} \frac{\left[(4a_1) + (4^2 a_2) + \,\cdots\, + (4^n a_n)\right]}{n} = \frac{1}{n 2^{n+1}} \sum_{k=1}^{n} 4^k a_k \,.$$

We then have

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} G_n \le \sum_{n=1}^{\infty} \left(\frac{n}{2^n} \cdot \frac{1}{n 2^{n+1}} \sum_{k=1}^{n} 4^k a_k\right) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{a_k}{4^{n-k}} \,.$$

This series is absolutely convergent, so we can change the order of summation to get

$$S \le \frac{1}{2} \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{a_k}{4^{n-k}} = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{a_k}{4^j} = \frac{1}{2} \left[\sum_{j=0}^{\infty} \frac{1}{4^j}\right] \left[\sum_{k=1}^{\infty} a_k\right] .$$

The first bracketed factor is a geometric series with sum $\dfrac{1}{1-\frac{1}{4}} = \dfrac{4}{3}$ and the second factor is 1 by the given constraint, so $S \le \frac{2}{3}$, and we are done.

B3 Let $h(x,y)$ be a real-valued function that is twice continuously differentiable throughout $\mathbb{R}^2$, and define

$$\rho(x,y) = yh_x - xh_y\,.$$

Prove or disprove: For any positive constants d and r with $d > r$, there is a circle $\mathcal{S}$ of radius r whose center is a distance d away from the origin such that the integral of ρ over the interior of S is zero.

Solution: We will prove the statement above. First we introduce polar coordinates R, θ centered at the origin, so that $x = R\cos\theta$ and $y = R\sin\theta$. Then

$$\frac{\partial h}{\partial \theta} = h_x\,x_\theta + h_y\,y_\theta = -R\sin\theta\,h_x + R\cos\theta\,h_y = -yh_x + xh_y\,.$$

So if we define $P(R,\theta) = \rho(x,y)$, then $P(R,\theta) = -\dfrac{\partial h}{\partial \theta}$ and consequently the integral of P over any circle centered at the origin is zero; that is,

$$\int_{\theta=0}^{2\pi} P(R,\theta)\,d\theta = 0 \quad \text{for every } R\,.$$

Now let $\mathcal{S}(\alpha)$ be the disc of radius r centered at $(x,y) = (d\cos\alpha,\,d\sin\alpha)$ and let

$$I(\alpha) = \iint\limits_{\mathcal{S}(\alpha)} \rho(x,y)\,dA\,;$$

our goal is to show that $I(\alpha) = 0$ for some value of α. We will set up $I(\alpha)$ using polar coordinates R, φ centered at the origin, but with the polar angle φ measured from α, so $\varphi = \theta - \alpha$. Note that the disk subtends an angle 2β at the origin, where

$$\beta = \sin^{-1}\left(\frac{r}{d}\right)\,.$$

Thus φ ranges from $-\beta$ to β, and for any fixed φ, R ranges from $R_-(\varphi)$ to $R_+(\varphi)$, where $R_-(\varphi)$, $R_+(\varphi)$ are the distances from the origin to the closest and farthest points of the disk along the ray for φ. (A short calculation using the law of cosines shows that $R_\pm(\varphi) = d\cos\varphi \pm \sqrt{r^2 - d^2\,\sin^2\varphi}$, but we won't need this formula.) Thus we have

$$I(\alpha) = \int_{\varphi=-\beta}^{\beta} \int_{R=R_-(\varphi)}^{R_+(\varphi)} P(R,\alpha+\varphi)\,R\,dR\,d\varphi.$$

Note that $I(\alpha)$ is a continuous (and even differentiable) function of α, because $\rho(x,y)$ is continuously differentiable. Finally, consider the average value M of this function over one period:

$$\begin{aligned}
M &= \frac{1}{2\pi}\int_{\alpha=0}^{2\pi} I(\alpha)\,d\alpha \\
&= \frac{1}{2\pi}\int_{\alpha=0}^{2\pi}\int_{\varphi=-\beta}^{\beta}\int_{R=R_-(\varphi)}^{R_+(\varphi)} P(R,\alpha+\varphi)\,R\,dR\,d\varphi\,d\alpha \\
&= \frac{1}{2\pi}\int_{\varphi=-\beta}^{\beta}\int_{\mathbb{R}=R_-(\varphi)}^{R_+(\varphi)} \left[\int_{\alpha=0}^{2\pi} P(R,\alpha+\varphi)\,d\alpha\right] R\,dR\,d\varphi \\
&= 0\,,
\end{aligned}$$

where the last step uses the fact that the integral of P over any circle centered at the origin is zero. But by the mean value theorem for integrals, $I(\alpha)$ must assume this mean value 0 at least once (actually, at least twice) on the interval $[0,2\pi]$, so we are done.

B4 Let $F_0, F_1, \ldots$ be the sequence of Fibonacci numbers, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m - 1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.

Solution: The *hyperfactorial* of any positive integer n is defined as

$$H(n) \equiv \prod_{k=1}^{n} k^k.$$

Thus, R_m is the remainder when $H(F_m - 1)$ is divided by F_m.

If $m = 3$, then $F_m = 2$ and $H(F_m - 1) = 1$, so $R_3 = 1$, a Fibonacci number.

If $m = 4$, then $F_m = 3$ and $H(F_m - 1) = 4 \equiv 1 \pmod 3$, so $R_4 = 1$.

For $m > 4$, we will show that the remainder R_m is one of the three Fibonacci numbers $F_0 = 0$, F_{m-1}, F_{m-2}.

If F_m is composite, then $F_m = qr$ for some integers $1 < q \leq r < F_m$. If q and r are distinct, then q^q and r^r are among the factors in the product $H(F_m - 1)$, so $H(F_m - 1)$ is divisible by $qr = F_m$ and $R_m = 0$. If $q = r$, then $F_m = q^2$ divides q^q and again, $H(F_m - 1)$ is divisible by F_m and $R_m = 0$. So we are left with the case that $F_m \geq 5$ is prime.

Let $p = F_m$, $p \geq 5$, be prime. We use two standard Fibonacci identities, which can be proved together by induction on i:

$$F_{2i} = F_i(F_{i-1} + F_{i+1}), \quad F_{2i+1} = F_i^2 + F_{i+1}^2.$$

The first identity shows that if F_m is prime and $m > 4$, m cannot be even. The second identity then shows that F_m is the sum of two squares, so $p = F_m \equiv 1 \pmod 4$.

Now consider $H(p-1)$ modulo p. Note that for each k with $1 \leq k \leq p-1$ we have

$$\begin{aligned} k^k \cdot (p-k)^{p-k} &\equiv k^k \cdot (-1)^{p-k}\, k^{p-k} \\ &= (-1)^{k+1}\, k^p \equiv (-1)^{k+1}\, k \pmod p \quad \text{by Fermat's little theorem, so} \end{aligned}$$

$$\begin{aligned} H(p-1)^2 &= \left(\prod_{k=1}^{p-1} k^k\right)\left(\prod_{k=1}^{p-1} (p-k)^{p-k}\right) = \prod_{k=1}^{p-1} k^k (p-k)^{p-k} \\ &\equiv \prod_{k=1}^{p-1} (-1)^{k+1} k = \left(\prod_{k=1}^{p-1} (-1)^{k+1}\right)(p-1)! \pmod p. \end{aligned}$$

But $(p-1)! \equiv -1 \pmod p$ by Wilson's theorem, so

$$H(p-1)^2 \equiv \prod_{k=0}^{p-1} (-1)^{k+1} \equiv (-1)^{p(p+1)/2} = -1 \pmod p,$$

where the last step uses that $p \equiv 1 \pmod 4$. Finally, we use a third Fibonacci identity:

$$F_j^2 = F_{j-1} F_{j+1} + (-1)^{j-1},$$

which can be shown by interpreting $F_{j-1}F_{j+1} - F_j^2$ as the determinant of the matrix $\begin{pmatrix} F_{j-1} & F_j \\ F_j & F_{j+1} \end{pmatrix}$, which is the jth power of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. In particular, for $j = m-1$ we see that, because m is odd and $F_m = p$,

$$F_{m-1}^2 = F_{m-2}\, p + (-1)^{m-2} \equiv -1 \pmod p.$$

It follows that $H(p-1)^2 \equiv F_{m-1}^2 \pmod p$, so, as p is prime, either $H(p-1) \equiv F_{m-1}$ or $H(p-1) \equiv -F_{m-1} \pmod p$. Then the remainder R_m is F_{m-1} in the first case and $F_m - F_{m-1} = F_{m-2}$ in the second, so we are done.

B5 Say that an n-by-n matrix $A = (a_{ij})_{1 \le i,j \le n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1, 2, \ldots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \ge 1$.

Solution: First of all, because we are only interested in determinants modulo 2, we can reduce the entries of A modulo 2; that is, we may assume that all entries of A are in $\{0, 1\}$.

Claim: Under this assumption, a necessary and sufficient condition for A to be very odd is that there exists a permutation π of $\{1, \ldots, n\}$ such that, when both the rows and columns of A are permuted by π, A becomes upper triangular with all diagonal entries 1. In other words, A is very odd if and only if there exists an n-by-n permutation matrix P such that PAP^{-1} is upper triangular with 1's along the diagonal.

Note that if PAP^{-1} is upper triangular with 1's along the diagonal, then so is $PA^kP^{-1} = (PAP^{-1})^k$. Therefore, the problem statement follows immediately from the claim.

Proof of the claim: To show the condition is sufficient, note that if A is upper triangular with 1's on the diagonal, then any submatrix $(a_{ij})_{i,j \in S}$ has that same form, so such a submatrix has determinant 1. Also, permuting the rows and columns of A by a permutation π does not affect the set of determinants of the submatrices.

Now we show the condition is necessary. Suppose that A is very odd (and has entries from $\{0, 1\}$). By taking the subsets $S = \{i\}$ of $\{1, \ldots, n\}$, we see that $a_{ii} = 1$ for all i. Now consider a two-element subset $\{i, j\}$. Because the determinant $a_{ii}a_{jj} - a_{ij}a_{ji}$ must be odd, at least one of a_{ij} and a_{ji} must be zero. Define a relation $\lhd$ on $\{1, \ldots, n\}$ by

$$i \lhd j \quad \text{if and only if} \quad a_{ij} = 1\,.$$

Then we've seen that for $i \neq j$, we cannot have both $i \lhd j$ and $j \lhd i$. In fact, we'll show that the relation $\lhd$ is *acyclic*, meaning that there is no cycle $i_1 \lhd i_2 \lhd \cdots \lhd i_k \lhd i_1$ with $k > 1$ (and $i_1 \neq i_2$). Suppose we do have such a cycle, and take one for which k is as small as possible. Consider the submatrix $M = (a_{ij})_{i,j \in S}$ of A corresponding to the subset $S = \{i_1, i_2, \ldots, i_k\}$. Then in the expression of $\det(M)$ as a sum of signed products of entries of M, each corresponding to a permutation of S, there will be exactly two nonzero terms, namely the "diagonal" term $a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_k i_k} = 1$ and a term $\pm a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = \pm 1$ corresponding to the cycle. (Any nonzero term in the determinant has to be $\pm$ a product of 1's, and unless the corresponding permutation is the identity it has at least one nontrivial cycle in its cycle decomposition, which is then a cycle for $\lhd$; because k is as small as possible, this can only be a k-cycle, which means it must involve all the elements of S, and if it weren't the original cycle $(i_1\, i_2\, \cdots\, i_k)$, it could be used together with the original cycle to construct a shorter cycle for $\lhd$.) But then $\det(M)$ is even, which is a contradiction.

Because $\lhd$ is acyclic, we can find a permutation π of $\{1, \ldots, n\}$ such that $i \lhd j$ implies $\pi(i) \le \pi(j)$. If we then use π to rearrange the rows and columns of A, the new matrix will have the desired upper triangular form with 1's on the diagonal. (An explicit procedure for constructing π is as follows: List the elements of $\{1, \ldots, n\}$ in stages, starting with the elements - in any order - that have no "predecessors" under the relation $\lhd$. At each subsequent stage, list, in any order, the elements all of whose predecessors have already been listed. When the list is complete, let $\pi(i)$ be the ith number on the list.)

B6 Given an ordered list of $3N$ real numbers, we can *trim* it to form a list of N numbers as follows: We divide the list into N groups of 3 consecutive numbers, and within each group, discard the highest and lowest numbers, keeping only the median.

Consider generating a random number X by the following procedure: Start with a list of 3^{2021} numbers, drawn independently and uniformly at random between 0 and 1. Then trim this list as defined above, leaving a list of 3^{2020} numbers. Then trim again repeatedly until just one number remains; let X be this number. Let μ be the expected value of $|X - \frac{1}{2}|$. Show that

$$\mu \geq \frac{1}{4}\left(\frac{2}{3}\right)^{2021}.$$

Solution: First, replace each random number x by $z = x - 1/2$, which will lie in the interval $[-1/2, 1/2]$. Let $\rho_n(z)$ be the probability density function on that interval for each of the numbers that remain after n trims. We know that $\rho_0(z) = 1$ because the initial distribution is uniform. Furthermore, $\rho_n(-z) = \rho_n(z)$ for all n, as the process is now symmetric with respect to the origin. This implies that

$$\int_{-\frac{1}{2}}^{0} \rho_n(t)\,dt = \int_{0}^{\frac{1}{2}} \rho_n(t)\,dt = \frac{1}{2}.$$

We proceed to calculate ρ_n , the probability density after n trims, from ρ_{n-1} . When we carry out the nth trim, there are $3! = 6$ equivalent orderings of the three numbers in a group, so we may first assume a fixed ordering of these numbers (specifically, let the first be the median, the second be the smallest, and the third be the largest) and then multiply by 6 to take the possible orderings into account. This yields the recursive formula

$$\begin{aligned}
\rho_n(z) &= 6\,\rho_{n-1}(z)\left[\int_{-\frac{1}{2}}^{z} \rho_{n-1}(t)\,dt\right]\left[\int_{z}^{\frac{1}{2}} \rho_{n-1}(t)\,dt\right] \\
&= 6\,\rho_{n-1}(z)\left[\frac{1}{2} + \int_{0}^{z} \rho_{n-1}(t)\,dt\right]\left[\frac{1}{2} - \int_{0}^{z} \rho_{n-1}(t)\,dt\right] \\
&= \frac{3}{2}\,\rho_{n-1}(z)\left[1 - 4\left(\int_{0}^{z} \rho_{n-1}(t)\,dt\right)^{2}\right].
\end{aligned}$$

It follows that $\rho_n(0) = \frac{3}{2}\rho_{n-1}(0)$, so by induction on n we have $\rho_n(0) = \left(\frac{3}{2}\right)^n$. Also by induction, for $n \geq 1$ the function $\rho_n(z)$ is monotonically decreasing with respect to $|z|$, and in particular $\rho_n(z) \leq \rho_n(0) = \left(\frac{3}{2}\right)^n$.

Now let $n = 2021$, so the expected value μ in the problem is given by

$$\mu = \int_{-1/2}^{1/2} |z|\rho_n(z)\,dz = 2\int_{0}^{1/2} z\rho_n(z)\,dz.$$

Let $M = \rho_n(0) = \left(\frac{3}{2}\right)^{2021}$. For $0 \leq z \leq \frac{1}{2}$, define the antiderivative

$$S(z) = \int_{t=0}^{z} \rho_n(t)\,dt \quad \text{of} \quad \rho_n(z);$$

note that

$$S(0) = 0,\ S(\tfrac{1}{2}) = \int_{0}^{\frac{1}{2}} \rho_n(t)\,dt = \frac{1}{2},$$

and that, by the monotonicity of ρ_n, we have the estimate $S(z) \le Mz$, so in fact

$$S(z) \le \min(Mz, \frac{1}{2}).$$

Finally, we integrate by parts to get

$$\begin{aligned}
\mu = 2\int_0^{1/2} z\rho_n(z)\,dz = 2zS(z)\Big|_{z=0}^{1/2} - 2\int_{z=0}^{1/2} S(z)\,dz \\
&= \frac{1}{2} - 2\int_{z=0}^{1/2} S(z)\,dz \\
&\ge \frac{1}{2} - 2\int_{z=0}^{1/(2M)} Mz\,dz - 2\int_{z=1/(2M)}^{1/2} \frac{1}{2}\,dz \\
&= \frac{1}{2} - \frac{1}{4M} - \left(\frac{1}{2} - \frac{1}{2M}\right) \\
&= \frac{1}{4M} = \frac{1}{4}\left(\frac{2}{3}\right)^{2021},
\end{aligned}$$

as desired.

Comment: The intuition behind the lower bound on μ is that if we consider all the nonincreasing functions $\rho(z)$ on $[0, \frac{1}{2}]$ that have value $\left(\frac{3}{2}\right)^n$ at $z = 0$ and whose integral over that interval is $\frac{1}{2}$, the smallest possible integral $\int_0^{1/2} z\rho(z)\,dz$ will occur for the step function which stays constant until $z = \frac{1}{2}\left(\frac{2}{3}\right)^n$ and is zero thereafter.