

Paul R. Halmos - Lester R. Ford Awards

Ben Blum-Smith and Japheth Wood

“Chords of an Ellipse, Lucas Polynomials, and Cubic Equations,” *The American Mathematical Monthly*, 127:8, 688–705. 10.1080/00029890.2020.1785253

Mark off n equally spaced points on a unit circle and draw the chords connecting one of the points to the remaining $n - 1$ others. The product of the lengths of these chords is n , even though all or most of the lengths of the chords are irrational. To start this paper, the authors give an interesting history of the circle problem and a generalization of this problem to the ellipse by Thomas Price. The authors then reorganize Price’s proof of the generalization for the ellipse, with some new ideas, and connect the result to classical mathematics. Connections include Cardano’s solution of the cubic equation, Newton’s theorem on power sums, generalized Lucas polynomials, and generalized Fibonacci polynomials.

Response

We are thrilled to be recognized for this honor, and to now have our names associated with Paul Halmos and Lester Ford, as well as the long list of other excellent expositors who have been so lauded.

This article was 10 years in the making, and would never have come into being if not for a fortuitous confluence of events. In 2010–2011, we were working together on the faculty of Bard College’s Master of Arts in Teaching program. Several of our colleagues in high school classrooms had just come back from the Secondary School Teachers’ Program at the Park City Mathematics Institute, and were very energized by the Developing Mathematics course taught by Bowen Kerins and Darryl Yong. Among these teachers was math blogger Sam Shah. Bowen and Darryl had thrown the circle problem and the ellipse generalization into a problem set, as a super-extra-challenge, and Sam blogged about it. At the time, Ben was reading the chapter on cyclotomic equations in Gauss’ *Disquisitiones Arithmeticae*. This made the pairing of the circle and ellipse problems tantalizing: the circle problem was immediately amenable to the ideas in Gauss—surely it couldn’t be that hard to adapt them to the ellipse? The method seemed to carry over, except a cyclotomic polynomial was replaced with a polynomial we didn’t immediately recognize—what was it? Meanwhile, Japheth had been developing math circle lessons about solving polynomial equations of degrees 2, 3, and 4 with a nod to the historical development. And somehow, we were co-teaching a course for the Bard College preservice teachers that included a unit on Cardano’s solution to the cubic. The ellipse problem cracked open for us when we recognized that the roots of our mystery ellipse polynomial had the same shape as Cardano’s formula—something we had just taught our students.

We pursued all this purely recreationally. The problem had come to us through a problem set—surely we were just reinventing some wheel, for edification and fun. It wasn’t until we had a complete solution in hand that we noticed how many different parts of mathematics it seemed to touch, and Japheth suggested it might be interesting enough to write down. Thus commenced a whole new project: to track down the source of the ellipse problem—it turned out to be the beautiful theorem of Thomas Price—and to sort through how what we’d done related to everything that had come before. It is these relationships, more than the solution itself, that drive the story we tell in the final version of the article.

We would like to express gratitude to the Halmos-Ford Award Committee for choosing to recognize our work, and also to the several individuals who gave us encouragement, suggestions, and feedback. We would particularly like to thank Bowen Kerins who offered early encouragement; and Tom Edgar who encouraged us much later, and also made the suggestion of compiling a history of the circle problem. The anonymous referees at the *American Mathematical Monthly* gave us extremely valuable feedback that

helped us bring out the paper's central story more clearly. And of course the whole inquiry would never have taken place without the beautiful work of Thomas Price.

Biographical Sketches

Ben Blum-Smith received a BA in anthropology from Yale University in 2000, an MAT in mathematics teaching from Tufts University in 2001, and a PhD in mathematics from NYU in 2017, with a thesis in representation and invariant theory of finite groups. He worked as a middle and high school teacher in public schools in Cambridge, MA and New York City, and then as a mathematics professional development specialist for high schools and as a faculty member of Bard College's MAT program, before beginning his training as a research mathematician in 2011. He is currently a part-time faculty member of Eugene Lang College's Department of Natural Sciences and Mathematics, and a Visiting Academic at the NYU Center for Data Science. His research interests lie in invariant theory, algebraic combinatorics, their applications to data science, and connections between mathematics and democracy. He is a founding organizer of the Math and Democracy Seminar at the NYU Center for Data Science, and serves on the editorial board of the American Mathematical Society's Teaching & Learning Blog. He is also engaged in mathematical outreach through the Bridge to Enter Advanced Mathematics, an organization focused on creating a realistic pathway for underserved students to enter the mathematical sciences.

Japheth Wood received his PhD in mathematics from U.C. Berkeley, and his undergraduate math degree from Washington University in Saint Louis. He is a continuing associate professor of mathematics at Bard College, where he is the director of quantitative literacy and also directs the Bard Math Circle. Japheth has taught in Bard College's MAT program, the Bard Prison Initiative, as well as in Bard College's undergraduate college. Japheth enjoys developing math enrichment opportunities for all students, and is serving on the Anneli Lax New Mathematical Library Editorial Board.