Paul R. Halmos - Lester R. Ford Awards

Dominic Klyve and Erik R. Tou

In 1759, Euler proved that any prime number of the form $4n + 1$ can be written as the sum of two squares thus settling the Fermat's 1640 conjecture. At the same time, Euler devised a primality test that may be the history's first non-trivial primality test. This paper explores this test. Authors show how Euler did it at that time and compare with the modern-day methods.

Euler proves "if a number of the form $4N + 1$ can be written as the sum of two squares (prime to each other) in only one way, then it is certain to be a prime number." In modern terms, he proved "Let $N \cong 1$ (mod 4). Then N is prime if and only if there is exactly one pair of nonnegative integers a, b for which $N = a^2 + b^2$, and that pair satisfies $\gcd(a, b) = 1$."

At first glance, this test appears to be massively inefficient; in particular, finding representations of a large number as a sum of two squares seems more difficult than checking for prime divisors. Euler was cleverer than this, though: he used a series of arithmetic simplifications and practical shortcuts to allow a person to check this in much less time. They explain this using the prime number 82421. A clearly described algorithm is part of the example. In the later part of the paper, the authors explain the complexity of Euler's algorithm and compare that with the other available test of the time.

Response

It is an honor and a pleasure to receive this award from the MAA. Leonhard Euler's number theory has been a long-standing interest to both of us, and we very much enjoyed the historical and mathematical paths this paper took us down. Of course, Euler's work is but one piece of a larger puzzle, which we look forward to exploring in the future. In the meantime, we have the engaging and accessible articles of the *Monthly* to keep us company.

Biographical Sketches

Dominic Klyve is a professor of mathematics at Central Washington University. He is the author of more than 60 papers in number theory, the history of mathematics and science, and applied statistics. His interdisciplinary works have appeared in journals ranging from *Gastrointestinal Endoscopy* to *Shakespeare Quarterly*. For the last six years, Klyve has served as a PI on $1.5 Million grant from the National Science Foundation to develop classroom materials to teach mathematics from primary historical sources. During 2021, he took a leave of absence from his university to work in the role of "Lead Polymath" at Know Labs, a Seattle-based tech start-up. He was a 2014 winner of the MAA's Alder Award, a national teaching award for young faculty who have a demonstrated impact within and beyond the classroom. He currently serves as editor of the *College Mathematics Journal*.

Erik Tou received his PhD from Dartmouth College in 2007, after earning a BA from Gustavus Adolphus College in 2002. Since 2015, he has lived and worked in Tacoma, Washington, and is currently an associate professor of mathematics at the University of Washington Tacoma. He also serves as a director of the Euler Archive. Erik's interests include (of course) the work of Leonhard Euler, but also the mathematics of juggling, frequent jigsaw puzzles, occasional kite-flying, and a well-tended campfire.