Putnam 2025 Solutions

A1 Let m_0 and n_0 be distinct positive integers. For every positive integer k, define m_k and n_k to be the relatively prime positive integers such that

$$\frac{m_k}{n_k} = \frac{2m_{k-1}+1}{2n_{k-1}+1}.$$

Prove that $2m_k+1$ and $2n_k+1$ are relatively prime for all but finitely many nonnegative integers k.

Solution: Note that $m_k \neq n_k$ for all k by induction: $m_0 \neq n_0$ is given, and if $m_{k-1} \neq n_{k-1}$, then the given recurrence gives $m_k \neq n_k$. Write $|m_k - n_k| = 2^{a_k} b_k$, where b_k is odd. We have

$$\begin{aligned} m_k &= \frac{2m_{k-1}+1}{\gcd(2m_{k-1}+1, 2n_{k-1}+1)}, \\ n_k &= \frac{2n_{k-1}+1}{\gcd(2m_{k-1}+1, 2n_{k-1}+1)}, \\ |m_k - n_k| &= \frac{2|m_{k-1}-n_{k-1}|}{\gcd(2m_{k-1}+1, 2n_{k-1}+1)}. \end{aligned}$$

As $\gcd(2m_{k-1}+1, 2n_{k-1}+1)$ is odd, we get $a_k = a_{k-1}+1$ and

$$b_k = \frac{b_{k-1}}{\gcd(2m_{k-1}+1, 2n_{k-1}+1)}.$$

In particular, whenever $2m_{k-1}+1$ and $2n_{k-1}+1$ are not relatively prime, we have $\gcd(2m_{k-1}+1, 2n_{k-1}+1) > 1$ and $b_k < b_{k-1}$, and otherwise $b_k = b_{k-1}$. Thus, the number of k such that $2m_{k-1}+1$ and $2n_{k-1}+1$ are not relatively prime is finite.

Comment: In fact, the number of terms with a common divisor $\gcd(2m_{k-1}+1, 2n_{k-1}+1) > 1$ is at most the number of prime divisors, counting multiplicity, of b_0.

A2 Find the largest real number a and the smallest real number b such that

$$ax(\pi - x) \le \sin x \le bx(\pi - x)$$

for all x in the interval $[0, \pi]$.

Answer: $a = 1/\pi$ and $b = 4/\pi^2$.

Solution #1: Note that both $y = \sin x$ and $y = x(\pi - x)$ are symmetric with respect to the line $x = \pi/2$, so it suffices to consider $x \in [0, \pi/2]$. At $x = \pi/2$, we have $\sin \pi/2 = 1$ and $(\pi/2)(\pi - \pi/2) = \pi^2/4$, so $b \ge 4/\pi^2$. At $x = 0$, by L'Hôpital's rule,

$$a \le \lim_{x \to 0^+} \frac{\sin x}{x(\pi - x)} = \frac{1}{\pi}.$$

Thus, the constants are best possible and it remains to prove

$$\frac{1}{\pi} x(\pi - x) \le \sin x \le \frac{4}{\pi^2} x(\pi - x).$$

The Taylor series expansion for $\sin x$ satisfies the alternating series test criteria on $[0, \pi/2]$ and so, on this interval,

$$\sin x - \frac{1}{\pi} x(\pi - x) \ge x - x^3/6 - \frac{1}{\pi} x(\pi - x) = x^2 \left(\frac{1}{\pi} - \frac{x}{6} \right) \ge x^2 \left(\frac{1}{\pi} - \frac{\pi}{12} \right) \ge 0.$$

To verify the upper bound, set

$$f(x) = \frac{4}{\pi^2} x(\pi - x) - \sin x.$$

We have

$$f'(x) = \frac{4}{\pi} - \frac{8}{\pi^2} x - \cos x.$$

Because $f'(\pi/2) = 0$ and $f''(x) = -8/\pi^2 + \sin x$ has one root in $[0, \pi/2]$, the mean value theorem implies f' has at most one root in $(0, \pi/2)$, which must be a relative maximum because $f'(0) = 4/\pi - 1 > 0$. Thus, the absolute minimum is $f(0) = f(\pi/2) = 0$ and the inequality follows.

Solution #2: We show that $f(x) = \dfrac{\sin x}{x(\pi - x)}$ is increasing on $(0, \pi/2]$. Differentiation yields

$$f'(x) = \frac{(\pi x - x^2)\cos x - (\pi - 2x)\sin x}{x^2(\pi - x)^2}.$$

Set $g(x) = (\pi x - x^2)\cos x - (\pi - 2x)\sin x$ and note $g(0) = g(\pi/2) = 0$. Differentiation yields $g'(x) = (x^2 - \pi x + 2)\sin x$, which has one root, $x = (\pi - \sqrt{\pi^2 - 8})/2$, in $(0, \pi/2)$. Because $g'(x) > 0$ for $0 \le x < (\pi - \sqrt{\pi^2 - 8})/2$ and $g'(x) < 0$ for $(\pi - \sqrt{\pi^2 - 8})/2 < x < \pi/2$, we conclude that the minimum value of g is at one or both endpoints, where it is 0. Therefore, on $(0, \pi/2]$,

$$\frac{1}{\pi} = \lim_{x \to 0^+} \frac{\sin x}{x(\pi - x)} < \frac{\sin x}{x(\pi - x)} \le \frac{\sin \pi/2}{(\pi/2)(\pi - \pi/2)} = \frac{4}{\pi^2},$$

from which the desired inequality follows immediately.

A3 Alice and Bob play a game with a string of n digits, each of which is restricted to be 0, 1, or 2. Initially all the digits are 0. A legal move is to add or subtract 1 from one digit to create a new string that has not appeared before. A player with no legal move loses, and the other player wins. Alice goes first, and the players alternate moves. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

Answer: Bob can always win.

Solution: Bob can always win by changing the first nonzero digit from 1 to 2 or vice-versa every time.

To see this, group all nonzero strings into pairs that differ only in the first nonzero digit. If Bob's response to Alice's move is to create the second string of the pair that includes Alice's string, then whenever Alice is about to move, every pair will either be both used or be both unused. Therefore, if Alice is able to move, so is Bob. Because there are only finitely many strings, one player must eventually lose, and that must be Alice.

A4 Find the minimal value of k such that there exist k-by-k real matrices $A_1, \ldots, A_{2025}$ with the property that $A_i A_j = A_j A_i$ if and only if $|i - j| \in \{0, 1, 2024\}$.

Answer: The minimal k is 3.

Solution #1: We first note that no matrix in the list can be a scalar multiple of the identity. This rules out $k = 1$; we next rule out $k = 2$. Observe that conjugating any such set of matrices produces other such sets. We will show that there are no *complex* matrices satisfying the conditions, and in particular, we can convert any particular A_i to Jordan normal form. Thus, if some A_i is not diagonalizable, there is no loss of generality in assuming that $i = 1$ and that $A_1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Simple computation yields that those matrices commuting with A_1 have the form $\begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}$, so A_2 must be of this form. But then A_3 must have this form since it commutes with A_2 and so it commutes with A_1, a contradiction. Now consider all A_i diagonalizable, each with two distinct eigenvalues. We may assume A_1 is a diagonal matrix. If A_i is a diagonal matrix, then $A_i A_{i+1} = A_{i+1} A_i$ forces A_{i+1} to be diagonal. But then A_1 and A_3 must commute, another contradiction.

To see that such matrices exist when $k = 3$, choose unit column vectors $\mathbf{u}_i$ as follows. Choose $\mathbf{u}_1$ arbitrarily. Having chosen $\mathbf{u}_1, \ldots, \mathbf{u}_i$ for $1 \le i \le 2023$, choose $\mathbf{u}_{i+1}$ to be any unit vector in the orthogonal complement of $\mathbf{u}_i$ other than the finite set of vectors that are either multiples of or orthogonal to $\mathbf{u}_1, \ldots, \mathbf{u}_{i-1}$. Choose $\mathbf{u}_{2025}$ to be one of the two unit vectors orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_{2024}$.

Let $A_i = \mathbf{u}_i \mathbf{u}_i^T$. Then

$$A_i A_j = (\mathbf{u}_i \cdot \mathbf{u}_j) \mathbf{u}_i \mathbf{u}_j^T.$$

If $|i - j| \in \{1, 2024\}$, then $\mathbf{u}_i \cdot \mathbf{u}_j = 0$, hence $A_i A_j = A_j A_i = \mathbf{0}$. Conversely, if $A_i A_j = A_j A_i$ for $|i - j| \notin \{0, 1, 2024\}$, we must have $\mathbf{u}_i \mathbf{u}_j^T = \mathbf{u}_j \mathbf{u}_i^T$, which is equivalent to $\mathbf{u}_i \times \mathbf{u}_j = \mathbf{0}$, a contradiction.

Solution #2: We show that $k \le 2$ is impossible as in Solution 1.

To see that such matrices exist when $k = 3$, let $\mathbf{v}_1, \ldots, \mathbf{v}_{2025}$ be vectors in $\mathbb{R}^3$ such that no three are linearly dependent. (For instance, one can take $\mathbf{v}_k = [1, k, k^2]^T$.) For notational convenience, extend $\mathbf{v}_i$ periodically by defining $\mathbf{v}_{i \pm 2025} = \mathbf{v}_i$. Let $\mathbf{w}_i = \mathbf{v}_{i-1} \times \mathbf{v}_{i+1}$. Because the cross product of two linearly independent vectors is a nonzero vector orthogonal to both, the linear independence of any three $\mathbf{v}_i$ implies that $\mathbf{w}_{i-1}$, $\mathbf{w}_{i+1}$, and $\mathbf{v}_i \times \mathbf{v}_{i+1}$ are in the orthogonal complement of $\mathbf{v}_i$, any two of which are linearly independent. Furthermore, the linear independence of $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, and $\mathbf{v}_{i+1}$ implies $\mathbf{w}_i$ is not in the orthogonal complement of $\mathbf{v}_i$. Therefore, there is a unique linear transformation that takes $\mathbf{w}_i$ to itself and the orthogonal complement of $\mathbf{v}_i$ to $\mathbf{0}$. Let the matrix A_i represent this linear transformation with respect to the standard basis. Because $\mathbf{w}_i$, $\mathbf{w}_{i+1}$, and $\mathbf{v}_i \times \mathbf{v}_{i+1}$ form a basis of eigenvectors for both

A_i and A_{i+1}, these matrices commute. On the other hand, if A_i and A_k commute for $2 \le k - 1 \le 2024$,

$$A_i A_k \mathbf{w}_i = A_k A_i \mathbf{w}_i = A_k \mathbf{w}_i.$$

Therefore, $A_k \mathbf{w}_i$ is in the 1-eigenspace of A_i and $A_k \mathbf{w}_i = c_k \mathbf{w}_i$. Because $\mathbf{w}_i$ is also an eigenvector of A_k, we must have $c_k \in \{0, 1\}$. If $c_k = 0$, then $\mathbf{w}_i$ is orthogonal to $\mathbf{v}_{i-1}$, $\mathbf{v}_{i+1}$, and $\mathbf{v}_k$, a contradiction. If $c_k = 1$, then $\mathbf{w}_i$ is orthogonal to $\mathbf{v}_{i-1}$, $\mathbf{v}_{i+1}$, $\mathbf{v}_{k-1}$, and $\mathbf{v}_{k+1}$, also a contradiction.

Comment: The first construction only works for $n \ge 5$ matrices, whereas the second construction works for $n \ge 4$ matrices.

A5 Let n be an integer with $n \geq 2$. For a sequence $s = (s_1, \ldots, s_{n-1})$, where each $s_i = \pm 1$, let $f(s)$ be the number of permutations $(a_1, \ldots, a_n)$ of $(1, 2, \ldots, n)$ such that $s_i(a_{i+1} - a_i) > 0$ for all i. For each n, determine the sequences s for which $f(s)$ is maximal.

Answer: The two sign strings s for which $f(s)$ is maximal are $s_i = (-1)^i$ and $s_i = (-1)^{i+1}$.

Solution #1: We say that a permutation a is ordered by a string of signs s if it obeys $s_i(a_{i+1} - a_i) > 0$ for each i, in other words, if $s_i = 1$ then $a_{i+1} > a_i$ while if $s_i = -1$ then $a_i > a_{i+1}$. Let $P(s)$ denote the set of permutations ordered by s. Consider s other than $s_i = (-1)^i$ or $s_i = (-1)^{i+1}$. Note that the inversion $a_i \mapsto n+1-a_i$ restricts to a bijection from $P(s)$ to $P(-s)$, so that $f(s) = f(-s)$; this completes the $n = 2$ case. We assume $n \geq 3$ and that there exists $k \in \{1, n-2\}$ such that $s_k = s_{k+1}$, which we take to be minimal. Via the inversion, we may assume $s_k = s_{k+1} = -1$. Let $s' = (s_1, \ldots, s_{k-1}, -1, 1, -s_{k+2}, \ldots, -s_{n-1})$ and note that s' has one more sign change than s. It suffices to show that there is an injective, non-surjective map $\beta \colon P(s) \to P(s')$.

For $a = (a_1, \ldots, a_n) \in P(s)$, define a permutation b by

- $b_i = a_i$ for $i \leq k$;
- if a_k and a_{k+1} are the jth and mth smallest of $\{a_k, \ldots, a_n\}$, respectively, then b_{k+1} is the $(j-m)$th smallest of $\{a_k, \ldots, a_n\}$. Note that since $a_k > a_{k+1}$ we have $j > m$ and $j - m < j$, so $b_{k+1} \neq a_k$;
- for $i > k+1$, if a_i is the ℓth smallest of $\{a_{k+2}, \ldots, a_n\}$, then b_i is the ℓth largest of $\{a_{k+1}, \ldots, a_n\} \setminus \{b_{k+1}\}$.

It follows immediately from the definition of b that $s_i(b_{i+1} - b_i) > 0$ for $i \in \{1, \ldots, k\}$ and that $-s_i(b_{i+1} - b_i) > 0$ for $i \in \{k+2, \ldots, n-1\}$. Thus, to show that $b \in P(s')$, it suffices to show that $b_{k+1} < b_{k+2}$. We have that a_{k+2} is at most the $(m-1)$st smallest of $\{a_{k+2}, \ldots, a_n\}$. Therefore, b_{k+2} is no smaller than the $(m-1)$st largest of $\{a_{k+1}, \ldots, a_n\} \setminus \{b_{k+1}\}$, hence no smaller than the mth largest or $(n-k+1-m)$th smallest of $\{a_{k+1}, \ldots, a_n\}$. Because $j - m < n - k + 1 - m$, we have $b_{k+1} < b_{k+2}$. This completes the construction of β, which is clearly injective.

To show that β is not surjective, note that if $b = \beta(a)$, we have $b_k = a_k \geq 3$ if $k = 1$ and $b_k = a_k \geq 4$ if $k > 1$. However, if $k = 1$, then $P(s')$ contains a permutation b with $b_2 = 1$ and $b_1 = 2$. If $k > 1$, then $P(s')$ contains a permutation b with $b_{k-1} = 1$, $b_{k+1} = 2$, and $b_k = 3$. Therefore, $f(s) < f(s')$.

Comments: The maximal value of $f(s)$ is called the nth *zigzag*, *up-down*, or *Euler* number.

A6 Let $b_0 = 0$ and, for $n \geq 0$, define $b_{n+1} = 2b_n^2 + b_n + 1$. For each $k \geq 1$, show that $b_{2^{k+1}} - 2b_{2^k}$ is divisible by 2^{2k+2} but not by 2^{2k+3}.

Solution #1: We prove by induction the following two lemmas, of which the second one immediately solves the problem:

Lemma 1. *For $k \geq 1$, $b_{n+2^k} - b_n \equiv 2^{k+1} \pmod{2^{k+2}}$.*

Lemma 2. *For $k \geq 1$, $b_{n+2^{k+1}} - 2b_{n+2^k} + b_n \equiv 2^{2k+2} \pmod{2^{2k+3}}$.*

Proof of Lemma 1. Let $c_{n,k} = b_{n+2^k} - b_n$, so that the lemma may be written as

$$c_{n,k} \equiv 2^{k+1} \pmod{2^{k+2}}. \tag{1}$$

For the base case $k = 1$, observe that the b_n are alternately even and odd so

$$c_{n,1} = b_{n+2} - b_n = 2(b_{n+1}^2 + b_n^2) + 2 \equiv (2 + 0) + 2 \equiv 4 \pmod{8}.$$

Now suppose that (1) holds for a given k. Observe that

$$\begin{aligned} c_{n+1,k} - c_{n,k} &= b_{n+2^k+1} - b_{n+2^k} - b_{n+1} + b_n \\ &= (2b_{n+2^k}^2 + 1) - (2b_n^2 + 1) = 2b_{n+2^k}^2 - 2b_n^2 \\ &= 2(b_n + c_{n,k})^2 - 2b_n^2 = 4b_n c_{n,k} + 2c_{n,k}^2. \end{aligned} \tag{2}$$

We save (2) for future reference. At the moment, the relevant observation is that both terms are divisible by 2^{k+3} since $2^{k+1} \mid c_{n,k}$. Hence the value $c_{n,k}$ modulo 2^{k+3} is independent of n, and

$$c_{n,k+1} = c_{n,k} + c_{n+2^k,k} \equiv 2c_{n,k} \equiv 2^{k+2} \pmod{2^{k+3}},$$

establishing the lemma. □

Proof of Lemma 2. Let $d_{n,k} = b_{n+2^{k+1}} - 2b_{n+2^k} + b_n = c_{n+2^k,k} - c_{n,k}$, so that the lemma may be written as

$$d_{n,k} \equiv 2^{2k+2} \pmod{2^{2k+3}}. \tag{3}$$

For the base case $k = 1$, using (2), we write

$$\begin{aligned} d_{n,1} = c_{n+2,1} - c_{n,1} &= (c_{n+1,1} - c_{n,1}) + (c_{n+2,1} - c_{n+1,1}) \\ &= 4b_n c_{n,1} + 2c_{n,1}^2 + 4b_{n+1}c_{n+1,1} + 2c_{n+1,1}^2 \\ &\equiv 16(b_n + b_{n+1}) \equiv 16 \pmod{32}. \end{aligned}$$

Now suppose that (3) holds for a given k. Observe that

$$\begin{aligned} d_{n+1,k} - d_{n,k} &= c_{n+2^k+1,k} - c_{n+2^k,k} - c_{n+1,k} + c_{n,k} \\ &= 4b_{n+2^k}c_{n+2^k,k} + 2c_{n+2^k}^2 - 4b_n c_{n,k} - 2c_{n,k}^2 \\ &= 4(b_{n+2^k} - b_n)c_{n+2^k,k} + 4b_n(c_{n+2^k,k} - c_{n,k}) + 2(c_{n+2^k} - c_{n,k})(c_{n+2^k} + c_{n,k}) \\ &= 4c_{n,k}c_{n+2^k} + 4b_n d_{n,k} + 2d_{n,k}(2c_{n,k} + d_{n,k}) \\ &= 4c_{n,k}c_{n+2^k} + 4b_n d_{n,k} + 4c_{n,k}d_{n,k} + 2d_{n,k}^2. \end{aligned}$$

Since $2^{k+1} \parallel c_{n,k}$ and $2^{2k+2} \parallel d_{n,k}$, the last two terms vanish modulo 2^{2k+5} and we get

$$d_{n+1,k} - d_{n,k} \equiv 2^{2k+4}(1+b_n) \pmod{2^{2k+5}}.$$

So

$$\begin{aligned} d_{n,k+1} &= d_{n+2^{k+1},k} + 2d_{n+2^k,k} + d_{n,k} \\ &= \sum_{\ell=n}^{n+2^{k+1}-1} (d_{\ell+1,k} - d_{\ell,k}) + 2 \sum_{\ell=n}^{n+2^k-1} (d_{\ell+1,k} - d_{\ell,k}) + 4d_{n,k} \\ &\equiv 2^{2k+4} \sum_{\ell=n}^{n+2^{k+1}-1} (1+b_\ell) + 0 + 4d_{n,k} \pmod{2^{2k+5}}. \end{aligned}$$

Since the b_ℓ are alternately even and odd and the number of terms in the sum is a multiple of 4, the sum vanishes modulo 2^{2k+5} and we get

$$d_{n,k+1} \equiv 4d_{n,k} \equiv 2^{2k+4} \pmod{2^{2k+5}},$$

as desired. □

Solution #2: We begin with a lemma.

Lemma 1. *For $k \geq 1$,*

$$b_{2^k+n} \equiv b_{2^k} \prod_{i=0}^{n-1}(4b_i+1) + b_n \pmod{2^{2k+3}}.$$

Consequently, $b_{2^k} \equiv 2^{k+1} \pmod{2^{k+2}}$ and $b_{2^k+n} \equiv b_n \pmod{2^{k+1}}$.

Proof. The third congruence follows immediately from the first two. We combine proofs of the first two congruences. We have equality, not just congruence, for $n = 0$ in the first congruence. Compute that $b_2 = 4$, verifying the second congruence for $k = 1$. Assume both congruences hold for the pair k and n. By the given recurrence,

$$\begin{aligned} b_{2^k+n+1} &\equiv 2\left(b_{2^k}\prod_{i=0}^{n-1}(4b_i+1) + b_n\right)^2 + \left(b_{2^k}\prod_{i=0}^{n-1}(4b_i+1) + b_n\right) + 1 \\ &\equiv 4b_n b_{2^k}\prod_{i=0}^{n-1}(4b_i+1) + 2b_n^2 + b_{2^k}\prod_{i=0}^{n-1}(4b_i+1) + b_n + 1 \\ &\equiv b_{2^k}\prod_{i=0}^{n}(4b_i+1) + b_{n+1} \pmod{2^{2k+3}}, \end{aligned}$$

establishing the first claim for k and $n+1$. Setting $n = 2^k$ in the first claim and working only modulo 2^{k+3},

$$b_{2^{k+1}} \equiv 2b_{2^k} \equiv 2^{k+2} \pmod{2^{k+3}},$$

establishing the second congruence for $k+1$. □

To complete the proof, we show that, for $k \geq 1$,

$$b_{2^{k+1}} - 2b_{2^k} \equiv 2^{2k+2} \pmod{2^{2k+3}}$$

by induction. We compute $b_4 = 2776$ (or just $b_4 \equiv 24 \pmod{32}$) to verify the claim for $k = 1$. We assume the result for $k-1$. Equating the results of our lemma and the inductive hypothesis,

$$b_{2^k} \equiv b_{2^{k-1}} \left(\prod_{i=0}^{2^{k-1}-1} (4b_i + 1) + 1 \right) \equiv 2b_{2^{k-1}} + 2^{2k} \pmod{2^{2k+1}}.$$

It follows that

$$\prod_{i=0}^{2^{k-1}-1} (4b_i + 1) \equiv 2^k + 1 \pmod{2^{k+1}}.$$

Furthermore, from our lemma, $4b_{2^{k-1}} + 1 \equiv 1 \pmod{2^{k+2}}$ and

$$\prod_{i=2^{k-1}+1}^{2^k-1} (4b_i + 1) \equiv \prod_{i=1}^{2^{k-1}-1} (4b_i + 1) \equiv \prod_{i=0}^{2^{k-1}-1} (4b_i + 1) \pmod{2^{k+2}}.$$

Therefore,

$$b_{2^{k+1}} \equiv b_{2^k} \left(\left(\prod_{i=0}^{2^{k-1}-1} (4b_i + 1) \right)^2 + 1 \right) \equiv b_{2^k}(2^{k+1} + 2) \equiv 2^{2k+2} + 2b_{2^k} \pmod{2^{2k+3}}.$$

B1 Suppose that each point in the plane is colored either red or green, subject to the following condition: For every three noncollinear points A, B, C of the same color, the center of the circle passing through A, B, and C is also this color. Prove that all points of the plane are the same color.

Solution #1: Note that if R is a red point, then every circle centered at R has at most two green points on it. Assume for the sake of contradiction that there is a red point R and a green point G, at some distance $d = RG > 0$. Consider three circles centered at R and three circles centered at G, all of radii strictly between $d/2$ and d. The circles intersect at $2 \cdot 3 \cdot 3 = 18$ points $X_1, \ldots, X_{18}$. Since each circle centered at R has at most two green points on it, at most $2 \cdot 3 = 6$ of the intersection points X_i are green. But by the same token, at most 6 of the X_i are red. But $6 + 6 < 18$ so we have a contradiction.

Solution #2: We repeatedly use the fact that a circle can have at most two points of the color different from that of its center. Suppose there are both red and green points. Assign a coordinate system so that a red point is at $(-1, 0)$ and a green point is at $(1, 0)$. Let C_R denote the circle of radius 2 centered at $(-1, 0)$ and C_G denote the circle of radius 2 centered at $(1, 0)$. Each circle contains the other center of the opposite color. Thus, their intersections $(0, \pm\sqrt{3})$ must be distinct colors and all remaining points on the these circles must have the same colors as their centers. The circle with center $(0, \sqrt{3})$ that passes through the origin intersects C_R in two red points and C_G in two green points, so the origin and $(0, \sqrt{3})$ must be the same color. Similarly, the origin and $(0, -\sqrt{3})$ must be the same color, a contradiction.

Comment: If one begins with any finite number of colors, similar arguments lead to the same conclusion.

B2 Let $f\colon [0,1] \to [0,\infty)$ be strictly increasing and continuous. Let R be the region in the xy-plane bounded by $x=0$, $x=1$, $y=0$, and $y=f(x)$. Let x_1 be the x-coordinate of the centroid of R. Let x_2 be the x-coordinate of the centroid of the solid generated by rotating R about the x-axis. Prove that $x_1 < x_2$.

Solution #1: We have

$$x_1 = \frac{\int_0^1 x f(x)\,dx}{\int_0^1 f(x)\,dx}$$

and

$$x_2 = \frac{\int_0^1 x\pi f(x)^2\,dx}{\int_0^1 \pi f(x)^2\,dx} = \frac{\int_0^1 x f(x)^2\,dx}{\int_0^1 f(x)^2\,dx}.$$

Thus, $x_1 < x_2$ is equivalent to

$$\int_0^1 x f(x)\,dx \cdot \int_0^1 f(x)^2\,dx < \int_0^1 x f(x)^2\,dx \cdot \int_0^1 f(x)\,dx.$$

We may rewrite this inequality as

$$\int_0^1 \int_0^1 x f(x) f(y)^2\,dy\,dx < \int_0^1 \int_0^1 x f(x)^2 f(y)\,dy\,dx,$$

or

$$0 < \int_0^1 \int_0^1 x f(x) f(y)\big(f(x) - f(y)\big)\,dy\,dx.$$

By symmetry,

$$\begin{aligned}
2&\int_0^1 \int_0^1 x f(x) f(y)\big(f(x) - f(y)\big)\,dy\,dx \\
&= \int_0^1 \int_0^1 x f(x) f(y)\big(f(x) - f(y)\big) + y f(x) f(y)\big(f(y) - f(x)\big)\,dy\,dx \\
&= \int_0^1 \int_0^1 (x - y) f(x) f(y)\big(f(x) - f(y)\big)\,dy\,dx.
\end{aligned}$$

Because f is strictly increasing, the integrand is always nonnegative and is positive for $y \neq x$, proving the claim.

Solution #2: Let S_1 be the solid in $\mathbb{R}^3$ defined by the inequalities

$$0 \le x \le 1, \quad 0 \le y \le f(x), \quad 0 \le z \le f(x_1).$$

Observe that the centroid of S_1 has x-coordinate x_1, because S_1 is a prism with base R and height $f(x_1)$. Now let S_2 be the solid defined by the inequalities

$$0 \le x \le 1, \quad 0 \le y \le f(x), \quad 0 \le z \le f(x).$$

Observe that the centroid of S_2 has x-coordinate x_2, because the cross-sections of S_2 parallel to the yz-plane are squares with $1/\pi$ times the area of the disks obtained by slicing the solid of revolution defining x_2. However, S_2 can be obtained from S_1 by removing the region defined by

$$0 \leq x < x_1, \quad 0 \leq y \leq f(x), \quad f(x) < z \leq f(x_1)$$

and adding the region defined by

$$x_1 < x \leq 1, \quad 0 \leq y \leq f(x), \quad f(x_1) < z \leq f(x).$$

Since we are removing a volume to the left of $x = x_1$ and adding a volume to the right of $x = x_1$ (considering the x-axis as left-right), the centroid necessarily shifts to the right, so $x_1 < x_2$.

B3 Suppose S is a nonempty set of positive integers with the property that if n is in S, then every positive divisor of $2025^n - 15^n$ is in S. Must S contain all positive integers?

Answer: Yes.

Solution: First note that $2025/15 = 135 = 3^3 \cdot 5$, so that $2025^n - 15^n = 15^n(135^n - 1)$. If n is in S, then S contains all divisors of $2025^n - 15^n$, which include 1, 2, and 3. We show by induction that S contains all positive integers. For the inductive step, suppose all integers less than n are in S, and let $n = 3^a 5^b c$, where c is relatively prime to 15. By Euler's theorem (or by the pigeonhole principle), there exists k with $1 \leq k \leq c - 1$ such that c divides $135^k - 1$. If $a = b = 0$, we are done. If $a + b > 0$, then by, say, an easy induction or the tangent line approximation to $f(x) = y^x$ at $x = 0$, one sees that $a < 3^a$ and $b < 5^b$, implying

$$a + b < 3^a + 5^b - 1 = 3^a 5^b + (3^a - 1)\left(5^b - 1\right) \leq 3^a 5^b.$$

Hence

$$(a + b)k \leq 3^a 5^b (c - 1) < n,$$

c divides $135^{(a+b)k} - 1$, and $3^a 5^b$ divides $15^{(a+b)k}$, so that n divides $2025^{(a+b)k} - 15^{(a+b)k}$. Therefore, n is in S.

B4 For $n \geq 2$, let $A = [a_{i,j}]_{i,j=1}^{n}$ be an n-by-n matrix of nonnegative integers such that

(a) $a_{i,j} = 0$ when $i + j \leq n$;

(b) $a_{i+1,j} \in \{a_{i,j}, a_{i,j} + 1\}$ when $1 \leq i \leq n-1$ and $1 \leq j \leq n$; and

(c) $a_{i,j+1} \in \{a_{i,j}, a_{i,j} + 1\}$ when $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

Let S be the sum of the entries of A, and let N be the number of nonzero entries of A. Prove that

$$S \leq \frac{(n+2)N}{3}.$$

Solution #1: We will use induction on the size n of the matrix A. The base case $n = 2$ is trivial, as the only possible matrices A are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Observe that the entries $a_{i,j}$ of A on the diagonal $i + j = n + 1$ are 0's or 1's. If they are all 0's, then the $(n-1) \times (n-1)$ matrix A_0 obtained from A by deleting the top row and leftmost column, which consist entirely of 0's, already satisfies the conditions (a)–(c). By applying the induction hypothesis to A_0, we get

$$S \leq \frac{(n+1)N}{3} \leq \frac{(n+2)N}{3},$$

as desired. So we assume that there is at least one 1 on this diagonal.

Let A' be the $(n-1) \times (n-1)$ matrix obtained from A by decreasing each nonzero entry by 1 and then deleting the top row and leftmost column, which now consist entirely of 0's. We observe that A' satisfies the conditions (a)–(c) (for (a), observe that $a'_{i,j} = \max\{a_{i+1,j+1} - 1, 0\} = 0$ for $i + j \leq n$). Let S' be the sum of the entries of A', and let N' be the number of nonzero entries of A'. We have $N = N' + r$, where r is the number of entries of A that equal 1. We claim that $r \geq n$. For this, we can find the following 1's:

- We have one $a_{i_0,j_0} = 1$ with $i_0 + j_0 = n + 1$ as mentioned above;
- For every $i > i_0$, row i must have an $a_{i,j} = 1$ with $j \leq j_0$;
- For every $j > j_0$, column j must have an $a_{i,j} = 1$ with $i \leq i_0$.

The spatial arrangement of these 1's ensures that they are non-coincident, and we get $r \geq 1 + (n - i_0) + (n - j_0) = n$. Now, utilizing the inductive hypothesis, we have

$$\begin{aligned} S = N + S' \leq N + \frac{(n+1)N'}{3} &= \frac{(n+4)N - (n+1)r}{3} \\ \leq \frac{(n+4)N - (n+1)n}{3} &\leq \frac{(n+2)N}{3}, \end{aligned}$$

where the last step uses that there are $N \leq (n+1)n/2$ spots $a_{i,j}$, $i + j \geq n + 1$, that can hold a nonzero entry.

Solution #2: Consider any nonzero A that maximizes the average S/N. As in Solution #1, by induction we need only consider the case of at least one nonzero entry on the long diagonal $i+j = n+1$ (hereafter "the diagonal"). If $a_{i-1,j} = a_{i,j-1} = a_{i,j} > 0$ for some i, j, choose such a pair with maximal $i + j$. Then either $i = n$ or $a_{i+1,j} = a_{i,j} + 1$, and either $j = n$ or $a_{i,j+1} = a_{i,j} + 1$. In any of the four cases, replacing $a_{i,j}$ with $a_{i,j} + 1$ satisfies the conditions and increases S, leaving N unchanged.

Therefore, if we have a run of $k \geq 1$ consecutive 1's on the diagonal, the $k \times k$ submatrix of A containing them is forced to have the shape

$$\begin{matrix} 0 & 0 & \cdots & 0 & \mathbf{1} \\ 0 & 0 & \cdots & \mathbf{1} & 2 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \mathbf{1} & \cdots & k-2 & k-1 \\ \mathbf{1} & 2 & \cdots & k-1 & k, \end{matrix} \tag{1}$$

which we call a *k-pyramid*. (Bold indicates the diagonal.)

If the diagonal is all 1's, then the entire matrix A is an n-pyramid. We have $N = (n+1)n/2$ and, by the hockey-stick identity or the sum of the first n squares,

$$S = \sum_{i=1}^{n} \sum_{k=1}^{i} k = \sum_{i=1}^{n} \frac{(n+1)n}{2} = \frac{(n+2)(n+1)n}{6} = \frac{(n+2)N}{3}.$$

Otherwise, consider a maximal run of $k \geq 1$ ones along the diagonal, terminated on at least one end (WLOG the bottom) by a 0. Then the entries bordering (1) are forced as well:

$$\begin{array}{c|cccccc|c} [0] & [0] & [0] & [\cdots] & [0] & [0] & [\mathbf{0}] \\ \hline 0 & 0 & 0 & \cdots & 0 & \mathbf{1} & [1] \\ 0 & 0 & 0 & \cdots & \mathbf{1} & 2 & [2] \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & [\vdots] \\ 0 & 0 & \mathbf{1} & \cdots & k-2 & k-1 & [k-1] \\ 0 & \mathbf{1} & 2 & \cdots & k-1 & k & [k] \\ \hline \mathbf{0} & 1 & 2 & \cdots & k-1 & k & [k+1] \end{array}$$

Here the square brackets mean that there is either an entry in A with the specified value, or else there is no entry because we have reached the edge of the matrix A.

Observe that if we decrease each nonzero entry in the k-pyramid by 1, then the conditions of the problem are still satisfied, and N decreases by k while S decreases by $k(k+1)/2$. Maximality of S/N implies that $S/N \leq (k+1)/2$.

On the other hand, let the "key zero" be the lowest 0 in the column to the left of the k-pyramid. The conditions force the entries $0, 1, \ldots, k$ to repeat in each row until we reach the row of the key zero:

$$\begin{array}{rc|ccccc|c}
 & [0] & [0] & [0] & [\cdots] & [0] & [0] & [\mathbf{0}] \\
\cline{2-8}
 & 0 & 0 & 0 & \cdots & 0 & \mathbf{1} & [1] \\
 & 0 & 0 & 0 & \cdots & \mathbf{1} & 2 & [2] \\
 & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & [\vdots] \\
 & 0 & 0 & \mathbf{1} & \cdots & k-2 & k-1 & [k-1] \\
 & 0 & \mathbf{1} & 2 & \cdots & k-1 & k & [k] \\
\cline{2-8}
 & \mathbf{0} & 1 & 2 & \cdots & k-1 & k & [k+1] \\
 & 0 & 1 & 2 & \cdots & k-1 & k & [k+1] \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [\vdots] \\
\text{Key zero} \rightarrow & 0 & 1 & 2 & \cdots & k-1 & k & [k+1] \\
\cline{2-8}
 & [1] & [2] & [3] & [\cdots] & [k] & [k+1] & [k+2]
\end{array}$$

Observe that if we add 1 to each of the entries $0, 1, \ldots, k$ in the row of the key zero, the conditions are still satisfied, and N increases by 1 while S increases by $k+1$. Maximality implies that $S/N \geq k+1$, contradicting $S/N \leq (k+1)/2$ above.

B5 Let p be a prime number greater than 3. For each $k \in \{1, \dots, p-1\}$, let $I(k) \in \{1, 2, \dots, p-1\}$ be such that $k \cdot I(k) \equiv 1 \pmod{p}$. Prove that the number of integers $k \in \{1, \dots, p-2\}$ such that $I(k+1) < I(k)$ is greater than $p/4 - 1$.

Solution #1: Extend the definition of I by taking $I(0) = I(p) = 0$. Next, for each $k = 1, 2, \dots, p$, let $g(k) \in \mathbb{Z}$ be defined as the element of $\{1, 2, \dots, p-1\}$ congruent to $I(k) - I(k-1) \bmod p$. Let $h(k) = g(1) + g(2) + \cdots + g(k)$. So, we have $0 < h(1) < h(2) < \cdots < h(p)$; $h(k)$ is congruent to $I(k) \bmod p$; and, for $k < p-1$, the interval $(h(k), h(k+1))$ contains at most one multiple of p, and contains such a multiple precisely if $I(k+1) < I(k)$. We thus need to show that there are at least $\lceil p/4 - 1 \rceil$ multiples of p in the interval $(0, h(p-1))$, or equivalently, $h(p-1) \geq p\lceil p/4 - 1 \rceil$.

Consider the values g can take. We have that $g(1) = 1$. For $k \in \{2, \dots, p-1\}$ we have that $g(k) = c$ if and only if $(k-1) - k \equiv ck(k-1)$ in $\mathbb{Z}/p\mathbb{Z}$. Then, since $c \neq 0$, the equation is equivalent to $(2k-1)^2 \equiv 1 - 4c^{-1}$ in $\mathbb{Z}/p\mathbb{Z}$, so has at most 2 solutions for each c, and for $c = 4$ there is exactly one solution. Consequently,

$$\begin{aligned} h(p-1) &= g(1) + g(2) + \cdots + g(p-1) \\ &\geq 1 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 1 \cdot 4 + 2 \cdot 5 + \cdots + 2 \cdot (p-1)/2 \\ &= 1 + 2\left(\frac{\frac{p-1}{2} \cdot \frac{p+1}{2}}{2}\right) - 4 = \frac{p^2 - 1}{4} - 3, \end{aligned}$$

When $p = 4q + 1$, we have $h(p-1) \geq q(4q+2) - 3 \geq (4q+1)q = p\lceil p/4 - 1 \rceil$ as long as $q \geq 3$.

When $p = 4q + 3$, we have $h(p-1) \geq (q+1)(4q+2) \geq (4q+3)q = p\lceil p/4 - 1 \rceil$ for all $q \geq 0$.

It remains to check that for $q = 1$, i.e. $p = 5$, the claim holds. We directly compute the sequence I as: $1, 3, 2, 4$, with one descent at $k = 2$, which is certainly greater than $5/4 - 1 = 1/4$.

Solution #2: Say a *step* is a subsequence of two consecutive values $I(k), I(k+1)$, and call it an *upstep* or a *downstep* according as $I(k+1)$ is greater or less than $I(k)$. For $t = 2, 3, \dots, p-2$, consider the two substrings

$$I(t-1), \quad I(t), \quad I(t+1) \quad \text{and} \quad I(I(t)-1), \quad I(I(t)), \quad I(I(t)+1). \qquad (1)$$

We claim that of the four steps illustrated here, at least one is a downstep. Observe that

$$I(t-1) + I(I(t)-1) \equiv \frac{1}{t-1} + \frac{1}{\frac{1}{t} - 1} \equiv -1 \pmod{p},$$

implying that

$$I(t-1) + I(I(t)-1) = p - 1.$$

Similarly,

$$I(t+1) + I(I(t)+1) \equiv \frac{1}{t+1} + \frac{1}{\frac{1}{t} + 1} \equiv 1 \pmod{p},$$

and since neither of the summands on the left is 0 mod p, we must have

$$I(t+1) + I(I(t)+1) = p+1.$$

Now assuming that all four steps in (1) are upsteps, we have

$$\begin{aligned} 4 &\leq [I(t+1) - I(t-1)] + [I(I(t)+1) - I(I(t)-1)] \\ &= [I(t+1) + I(I(t)+1)] - [I(t-1) + I(I(t)-1)] \\ &= (p+1) - (p-1) = 2, \end{aligned}$$

a contradiction. So (1) involves at least one downstep. Now, as t ranges through the $p-3$ values $2, 3, \ldots, p-2$, any particular step appears at most four times, namely, at most once in each of the four slots in (1). So the actual number of downsteps is at least $(p-3)/4 > p/4 - 1$, as desired.

Comment: Further reasoning along the lines of Solution 1 leads to the exact count. For $c \neq 4$, the involution $c \mapsto 4 - c$ maps $1 - 4c^{-1}$ to its inverse, so that one obtains the value c if and only if one obtains $4 - c$. Unless $c \in \{1, 2, 3\}$, the paired values of c's that occur add to $p + 4$. We obtain $c \in \{1, 3\}$ for $k > 1$ if and only if -3 is a square mod p. We obtain $c = 2$ if and only if -1 is a square mod p. In terms of the Legendre symbol, a bit of additional computation shows that the number of decreases in $I(k)$ equals

$$\frac{p - 4 - 2\left(\frac{-3}{p}\right) - \left(\frac{-1}{p}\right)}{2}.$$

B6 Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Find the largest real constant r such that there exists a function $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$g(n+1) - g(n) \geq \big(g(g(n))\big)^r$$

for all $n \in \mathbb{N}$.

Solution: We can achieve $r = 1/4$ by taking $g(n) = n^2$: then the left side of the inequality is $(n+1)^2 - n^2 = 2n+1$ while the right side is $(n^4)^{1/4} = n$. We will show that no higher r is possible. So suppose $r > 1/4$ is achieved, and seek a contradiction.

Because all $g(n)$ are positive integers, we have $\big(g(g(n))\big)^r \geq 1^r$, so $g(n+1) - g(n) \geq 1$ for all n, hence $g(n) \geq n$ by induction. Now define a sequence $b_0, b_1, b_2, \dots$ by $b_0 = 1$ and $b_{k+1} = rb_k^2 + 1$. Since $rb_k^2 - b_k + 1 = \left(r - \frac{1}{4}\right) b_k^2 + \left(\frac{1}{2} b_k - 1\right)^2 \geq r - 1/4$, we have $b_{k+1} \geq b_k + (r - 1/4)$ for each k, so the sequence (b_k) grows without bound.

We claim that for each k, there exists a positive constant $c_k \leq 1$ such that $g(n) \geq c_k n^{b_k}$ for all n. The base case $k = 0$ is already established, with $c_0 = 1$. If the claim holds for k, then for each $n \geq 1$ we have

$$g(n+1) - g(n) \geq \big(g(g(n))\big)^r \geq (c_k g(n)^{b_k})^r \geq (c_k^{b_k+1} n^{b_k^2})^r = c_k^{r(b_k+1)} n^{rb_k^2},$$

from which

$$g(n+1) \geq c_k^{r(b_k+1)} \sum_{m=1}^{n} m^{rb_k^2} \geq \frac{c_k^{r(b_k+1)}}{rb_k^2 + 1} n^{rb_k^2+1} \geq c_{k+1}(n+1)^{rb_k^2+1} = c_{k+1}(n+1)^{b_{k+1}},$$

where $c_{k+1} = c_k^{r(b_k+1)}/((rb_k^2+1)2^{rb_k^2+1})$ (and the last inequality uses $n \geq (n+1)/2$). Noting that $c_{k+1} \leq 1$, we also have $g(1) \geq c_{k+1} 1^{b_{n+1}}$, thus $g(n) \geq c_{k+1} n^{b_{k+1}}$ for all n, and the claim is established.

Now, since b_k is unbounded, for k large enough, we have $rb_k > 1$. Fix some such k. Then, we have

$$g(n+1) - g(n) \geq (c_k g(n)^{b_k})^r \geq c_k^r g(n),$$

or

$$g(n+1) \geq (c_k^r + 1) g(n)$$

for each n. Thus by induction $g(n) \geq C^{n-1}$ for each n, where $C = c_k^r + 1$, i.e. the sequence $g(1), g(2), \dots$ grows exponentially.

However, once n is large enough that $g(n) \geq n+2$, we have

$$g(n+1) \geq \big(g(g(n))\big)^r \geq \big(g(n+2)\big)^r$$

and so

$$g(n+1)^{1/r} \geq g(n+2) \geq \big(g(g(n+1))\big)^r \geq (C^{g(n+1)-1})^r$$

or

$$g(n+1)^{1/r^2} \geq C^{g(n+1)-1}.$$

Since the left side is polynomial in $g(n+1)$ and the right side is exponential, this cannot hold once $g(n+1)$ is sufficiently large – a contradiction.