The 86th William Lowell Putnam Mathematical Competition 2025

A1 Let m_0 and n_0 be distinct positive integers. For every positive integer k, define m_k and n_k to be the relatively prime positive integers such that

$$\frac{m_k}{n_k} = \frac{2m_{k-1}+1}{2n_{k-1}+1}.$$

Prove that $2m_k + 1$ and $2n_k + 1$ are relatively prime for all but finitely many positive integers k.

A2 Find the largest real number a and the smallest real number b such that

$$ax(\pi - x) \leq \sin x \leq bx(\pi - x)$$

for all x in the interval $[0, \pi]$.

A3 Alice and Bob play a game with a string of n digits, each of which is restricted to be 0, 1, or 2. Initially all the digits are 0. A legal move is to add or subtract 1 from one digit to create a new string that has not appeared before. A player with no legal move loses, and the other player wins. Alice goes first, and the players alternate moves. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

A4 Find the minimal value of k such that there exist k-by-k real matrices $A_1, \ldots, A_{2025}$ with the property that $A_i A_j = A_j A_i$ if and only if $|i - j| \in \{0, 1, 2024\}$.

A5 Let n be an integer with $n \geq 2$. For a sequence $s = (s_1, \ldots, s_{n-1})$ where each $s_i = \pm 1$, let $f(s)$ be the number of permutations $(a_1, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ such that $s_i(a_{i+1} - a_i) > 0$ for all i. For each n, determine the sequences s for which $f(s)$ is maximal.

A6 Let $b_0 = 0$ and, for $n \geq 0$, define $b_{n+1} = 2b_n^2 + b_n + 1$. For each $k \geq 1$, show that $b_{2^{k+1}} - 2b_{2^k}$ is divisible by 2^{2k+2} but not by 2^{2k+3}.

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B1 Suppose that each point in the plane is colored either red or green, subject to the following condition: For every three noncollinear points A, B, C of the same color, the center of the circle passing through A, B, and C is also this color. Prove that all points of the plane are the same color.

B2 Let $f\colon [0,1] \to [0,\infty)$ be strictly increasing and continuous. Let R be the region bounded by $x=0$, $x=1$, $y=0$, and $y=f(x)$. Let x_1 be the x-coordinate of the centroid of R. Let x_2 be the x-coordinate of the centroid of the solid generated by rotating R about the x-axis. Prove that $x_1 < x_2$.

B3 Suppose S is a nonempty set of positive integers with the property that if n is in S, then every positive divisor of $2025^n - 15^n$ is in S. Must S contain all positive integers?

B4 For $n \geq 2$, let $A = [a_{i,j}]_{i,j=1}^n$ be an n-by-n matrix of nonnegative integers such that

(a) $a_{i,j} = 0$ when $i + j \leq n$;

(b) $a_{i+1,j} \in \{a_{i,j}, a_{i,j}+1\}$ when $1 \leq i \leq n-1$ and $1 \leq j \leq n$; and

(c) $a_{i,j+1} \in \{a_{i,j}, a_{i,j}+1\}$ when $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

Let S be the sum of the entries of A, and let N be the number of nonzero entries of A. Prove that

$$S \leq \frac{(n+2)N}{3}.$$

B5 Let p be a prime number greater than 3. For each $k \in \{1, \ldots, p-1\}$, let $I(k) \in \{1, 2, \ldots, p-1\}$ be such that $k \cdot I(k) \equiv 1 \pmod{p}$. Prove that the number of integers $k \in \{1, \ldots, p-2\}$ such that $I(k+1) < I(k)$ is greater than $p/4 - 1$.

B6 Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Find the largest real constant r such that there exists a function $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$g(n+1) - g(n) \geq \big(g(g(n))\big)^r$$

for all $n \in \mathbb{N}$.