2024 Session A

A1. Determine all positive integers n for which there exist positive integers a, b, and c satisfying
$$2a^n + 3b^n = 4c^n.$$

Answer: $n = 1$ only. For $n = 1$, the equation is satisfied by $(a, b, c) = (1, 2, 2)$.

Solution 1: Consider $n > 1$. If d is the greatest common divisor of a, b, c, so $a = dx$, $b = dy$, $c = dz$, then x, y, z satisfy the same equation and we can assume that the greatest common divisor is 1. We see that $2 \mid 3b^n$, so $2 \mid b$. Letting $b = 2b_1$, the equation becomes
$$2a^n + 3 \cdot 2^n b_1^n = 4c^n.$$
Since $n \geq 2$, we have that $4 \mid 2a^n$, so $2 \mid a^n$ and $2 \mid a$. Setting $a = 2a_1$, we get that
$$2^n(2a_1^n + 3b_1^n) = 4c^n$$
and $2^{n-2} \mid c^n$. Since we assumed the greatest common divisor of a, b, c is 1, we must have that $2 \nmid c$. Thus, we must have $n = 2$.

Then
$$2a^2 + 3b^2 = 4c^2,$$
and so $a^2 + c^2 = 3a^2 + 3b^2 - 3c^2$ is divisible by 3. Considering all possible cases for remainders of a and c by division by 3, we see that a^2 has remainder 0 or 1, and c^2 has remainder 0 or 1. Thus, both a^2 and c^2 must have remainder 0, so $3 \mid a$ and $3 \mid c$. Writing $a = 3a_2, c = 3c_2$ we have $b^2 = 3(4c_2^2 - 2a_2^2)$, so $3 \mid b$, contradicting the assumption that a, b, c have common divisor 1.

Solution 2: To prove that there are no solutions for $n \geq 2$, assume to the contrary that there is such a solution. Let d be the greatest common divisor of a, b, and c, and let $x = a/d$, $y = b/d$, and $z = c/d$. Then $2x^n + 3y^n = 4z^n$, and the greatest common divisor of x, y, and z is 1. In particular, at least one of x, y, and z is odd. Since $3y^n = 4z^n - 2x^n$ is even, y is even. Then since $n \geq 2$, it follows that $2x^n = 4z^n - 3y^n$ is a multiple of 4, so x is even too, whence z is odd. If $n \geq 3$, we then have the contradiction that $2x^n + 3y^n$ is a multiple of 8, but $4z^n$ is not. If $n = 2$, we can write $2(x/2)^2 + 3(y/2)^2 = z^2$. It follows that $y/2$ is odd. Since all odd squares are congruent to 1 modulo 8, we have $2(x/2)^2 \equiv z^2 - 3(y/2)^2 \equiv 1 - 3 \equiv 6 \pmod 8$, which is impossible.

A2. For which real polynomials p is there a real polynomial q such that

$$p(p(x)) - x = (p(x) - x)^2 q(x)$$

for all real x?

Answer: Only $p(x) = \pm x + c$ for c a constant.

Solution 1: Let $f(x) = p(x) - x$ and let d denote its degree. Then the desired property is equivalent to $f(x + f(x)) + f(x) = [f(x)]^2 q(x)$. By the Taylor series expansion of f at x,

$$f(x + f(x)) = f(x) + f'(x)f(x) + \frac{f''(x)}{2}\,[f(x)]^2 + \cdots + \frac{f^{(d)}(x)}{d!}\,[f(x)]^d.$$

Thus, the factorization exists if and only if $2f(x) + f'(x)f(x) = [f(x)]^2 r(x)$ for some polynomial r, which in turn is equivalent to $f(x) = 0$ or $2 + f'(x) = f(x)r(x)$. The factorization holds when $d = 0$, hence when $p(x) = x + c$. If $d > 0$, then $2 + f'(x)$ has degree $d - 1$, a contradiction, unless $2 + f'(x) = 0$, which is equivalent to $p(x) = -x + c$.

Solution 2: Let $r(x) = p(x) - x$, then $p(x) = x + r(x)$ and the equation becomes

$$r(r(x) + x) + r(x) = r(x)^2 q(x)$$

Let $r(x) = c_n x^n + \cdots + c_1 x + c_0$ be its expansion in monomials. Then

$$\begin{aligned} r(r(x)+x) &= \sum_{k=0}^{n} c_k (r(x)+x)^k = \sum_{k=0}^{n} c_k \sum_{i=0}^{k} \binom{k}{i} x^i (r(x))^{k-i} \\ &= r(x)^2 \left(\underbrace{\sum_{k=2}^{n} c_k \sum_{i=0}^{k-2} \binom{k}{i} x^i r(x)^{k-i-2}}_{q_1(x)} \right) + r(x) \left(\sum_{k=1}^{n} c_k k x^{k-1} \right) + \sum_{k=0}^{n} c_k x^k \\ &= r(x)^2 q_1(x) + r(x) r'(x) + r(x). \end{aligned}$$

Thus, the original equation is equivalent to

$$r(x)^2 q_1(x) + r(x)r'(x) + 2r(x) = r^2(x) q(x).$$

Thus, either $r(x)$ is identically 0 (so $p(x) = x$) or

$$r'(x) + 2 = r(x)(q(x) - q_1(x)).$$

If the degree of $r(x)$ is not 0, then $\deg(r'(x) + 2) = \deg(r(x)) - 1 < \deg(r(x)(q(x) - q_1(x)))$, unless $q(x) - q_1(x) = 0$. Thus, either $\deg(r(x)) = 0$, so $p(x) = x + c$, or $q(x) = q_1(x)$ and $r'(x) = -2$, so $r(x) = -2x + c$, so $p(x) = -x + c$.

Plugging the two possibilities in the original equation we see the following. For $p(x) = x + c$, we have $2c = c^2 q(x)$, so all real values for c give a solution with a constant polynomial $q(x)$. For $p(x) = -x + c$ we have $-(-x + c) + c - x = (-2x + c)^2 q(x)$, so $q(x) = 0$ gives a solution for all c.

The solutions are thus $p(x) = -x + c$ for any real c or $p(x) = x + c$ for any real c.

A3. Let S be the set of bijections

$$T\colon \{1,2,3\} \times \{1,2,\ldots,2024\} \to \{1,2,\ldots,6072\}$$

such that $T(1,j) < T(2,j) < T(3,j)$ for all $j \in \{1,2,\ldots,2024\}$ and $T(i,j) < T(i,j+1)$ for all $i \in \{1,2,3\}$ and $j \in \{1,2,\ldots,2023\}$. Do there exist a and c in $\{1,2,3\}$ and b and d in $\{1,2,\ldots,2024\}$ such that the fraction of elements T in S for which $T(a,b) < T(c,d)$ is at least $1/3$ and at most $2/3$?

Answer: Yes.

Solution 1: We consider the more general situation where the set of bijections S is $T : \{1,\ldots,m\} \times \{1,2,\ldots,n\} \to \{1,2,\ldots,mn\}$ satisfying the given inequalities, where

$$m, n \geq 2$$

. In the problem we have $m = 3, n = 2024$. To simplify the notation, we switch to the probabilistic formulation: we are choosing elements T uniformly at random from S and considering the probability that $T(a,b) < T(c,d)$, which is equal to the proportion of such bijections T. By symmetry, if m and n are exchanged, then $Pr[T(a,b) < T(c,d)]$ becomes $Pr[T(b,a) < T(d,c)]$.

Consider $Pr[T(2,1) < T(1,2)]$. If $Pr[T(2,1) < T(1,2)] \in [\frac{1}{3}, \frac{2}{3}]$, then let $(a,b) = (2,1)$ and $(c,d) = (1,2)$, and we are done. If not, then without loss of generality we can assume $Pr[T(2,1) < T(1,2)] < \frac{1}{3}$ (if instead $Pr[T(2,1) < T(1,2)] > \frac{2}{3}$, then we can exchange m and n to get $Pr[T(1,2) < T(2,1)] > \frac{2}{3}$, in which case $Pr[T(2,1) < T(1,2)] = 1 - Pr[T(1,2) < T(2,1)] < \frac{1}{3}$). Our goal now is to show that $Pr[T(2,1) < T(1,j)] \in [\frac{1}{3}, \frac{2}{3}]$ for some $j > 2$.

Let $S_i = \{T \in S : T(2,1) = i\}$ and let $q_i = |S_i|/|S|$, i.e. the probability that the bijection T has $T(2,1) = i$. If $T(2,1) = i$, we must have $T(1,j) = j < T(2,1)$ for $j \leq i-1$ and $T(1,j) > i = T(2,1)$ for $j \geq i$. So $T(2,1) < T(1,j)$ is equivalent to $j \geq T(2,1)$, and summing over all possibilities for the value $T(2,1) = 2,\ldots,j$ we have

$$Pr[T(2,1) < T(1,j)] = q_2 + q_3 + \cdots q_j.$$

In particular, $q_2 = Pr[T(2,1) < T(1,2)] < \frac{1}{3}$. Note also that $q_2 + \cdots + q_{n+1} = 1$ since these are the only possibilities for $T(2,1)$.

Claim. *We have that* $q_2 \geq q_3 \geq \cdots \geq q_{n+1}$.

Proof. We see that there is an injection $\phi : S_{i+1} \to S_i$, given by $\phi(T)(1,i) = i+1$, $\phi(T)(2,1) = i$, and $\phi(T)(r,s) = T(r,s)$ for all other (r,s). □

Finally, let $k = \max\{j : q_2 + \cdots + q_j < \frac{1}{3}\}$, which exists by the above bounds. Further, since $q_{n+1} \leq q_2 < \frac{1}{3}$, we have $q_2 + \cdots + q_n = 1 - q_{n+1} > \frac{2}{3}$, so $k < n$. By maximality we must have that $\frac{1}{3} \leq q_2 + \cdots + q_{k+1}$, and since $q_{k+1} \leq q_2 < \frac{1}{3}$, we must also have $q_2 + \cdots + q_{k+1} = (q_2 + \cdots + q_k) + q_{k+1} < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Then $(a,b) = (1,k+1)$ and $(c,d) = (2,1)$ are the desired pairs.

Solution 2: The answer is yes, in particular for $a = 2$, $b = 2024$, $c = 3$, $d = 2023$. Think of the domain of T as a grid with 3 rows and 2024 columns, with rows and columns numbered as for a matrix. The greatest value of T, namely 6072 must be in the bottom row and the rightmost column, so $T(3, 2024) = 6072$. Similarly, the value 6071 can only be immediately above or immediately to the left of $(3, 2024)$, so either $T(2, 2024) = 6071$ or $T(3, 2023) = 6071$. Thus, the number n_1 of T for which $T(2, 2024) < T(3, 2023)$ is the number of T for which $T(3, 2023) = 6071$, and the number n_2 of T for which $T(2, 2024) > T(3, 2023)$ is the number of T for which $T(2, 2024) = 6071$. To prove that $n_1/(n_1 + n_2)$ is between 1/3 and 2/3, it suffices to prove that n_1/n_2 is between 1/2 and 2.

For integers $k \geq \ell \geq m \geq 0$, let $D_{k,\ell,m} = (\{1\}\times\{1, 2, \ldots, k\}) \cup (\{2\}\times\{1, 2, \ldots, \ell\}) \cup (\{3\}\times\{1, 2, \ldots, m\})$. Notice that $D_{2024,2024,2024}$ is the domain of T in the problem statement. More generally, think of $D_{k,\ell,m}$ as a left-justified grid with k elements in the first row, ℓ elements in the second row, and m elements in the third row. Let $N(k, \ell, m)$ be the number of bijections $T : D_{k,\ell,m} \to \{1, 2, \ldots, k + \ell + m\}$ for which $T(i, j) < T(i + 1, j)$ and $T(i, j) < T(i, j + 1)$ whenever both sides of the inequality are defined. Notice that $N(2024, 2024, 2024)$ is the number of elements in S, and so is $N(2024, 2024, 2023)$, since $T(3, 2024)$ is determined by the conditions on T. Furthermore, $n_1 = N(2024, 2024, 2022)$ and $n_2 = N(2024, 2023, 2023)$.

The bulk of the remainder of the solution is to derive a formula for $N(k, \ell, m)$. The greatest value of T, namely $k + \ell + m$, must occur at the right end of a row and only if this row extends beyond the bottom row, so either $T(1, k) = k + \ell + m$ or $T(2, \ell) = k + \ell + m$ or $T(3, m) = k + \ell + m$. If $k > \ell > m > 0$, then all three cases are possible, and if $k = \ell$ then the largest element cannot be at $(1, k)$ etc. Each case can be reduced to a domain with one fewer element, resulting in the identity

$$N(k, \ell, m) = N(k - 1, \ell, m) + N(k, \ell - 1, m) + N(k, \ell, m - 1), \qquad (*)$$

where we assume that the term is 0 if the arguments are not in weakly decreasing order. Furthermore, considering the conventional values $N(0, 0, 0) = 1$ and $N(k, \ell, m) = 0$ if k, ℓ or $\ell < m$ as "boundary conditions", (*) recursively determines $N(k, \ell, m)$ for all $k \geq \ell \geq m \geq 0$ with $k + \ell + m > 0$.

We will express $N(k, \ell, m)$ as a linear combination of trinomial coefficients, which satisfy a similar recursion. Let

$$F(p, q, r) = \frac{(p + q + r)!}{p!q!r!}$$

for nonnegative integers p, q, r, and extend F to the integers by defining $F(p, q, r) = 0$ if $p < 0$ or $q < 0$ or $r < 0$.

Lemma. *For $(p, q, r) \neq (0, 0, 0)$,*

$$F(p, q, r) = F(p - 1, q, r) + F(p, q - 1, r) + F(p, q, r - 1).$$

Proof. For nonnegative p, q, r with $p + q + r > 0$, the desired equality is equivalent to

$$\frac{(p + q + r - 1)!}{p!q!r!}(p + q + r) = \frac{(p + q + r - 1)!}{p!q!r!}p + \frac{(p + q + r - 1)!}{p!q!r!}q + \frac{(p + q + r - 1)!}{p!q!r!}r.$$

If $p < 0$ or $q < 0$ or $r < 0$, then the desired equality is equivalent to $0 = 0$. □

Claim. *For $k \geq \ell \geq m \geq 0$ or $k+1 = \ell \geq m \geq 0$ or $k \geq \ell+1 = m \geq 0$ or $k \geq \ell \geq m+1 = 0$,*

$$\begin{aligned} N(k, \ell, m) = {} & F(k, \ell, m) + F(k + 2, \ell - 1, m - 1) + F(k + 1, \ell + 1, m - 2) \\ & - F(k + 1, \ell - 1, m) - F(k, \ell + 1, m - 1) - F(k + 2, \ell, m - 2). \end{aligned}$$

Proof. The claimed expression for $N(k,\ell,m)$ satisfies the recursion (*) for $k \geq \ell \geq m \geq 0$ and $k+\ell+m>0$ as an immediate consequence of the Lemma, since $k+\ell+m>0$ ensures that none of the triples on which F is being evaluated are $(0,0,0)$. It remains to verify the "boundary conditions".

If $k+1=\ell \geq m \geq 0$, then substituting $\ell = k+1$ into the claimed expression and using the fact that $F(p,q,r)=F(q,p,r)$ makes all the terms cancel out, yielding the required boundary value 0 in this case. Similarly, if $k \geq \ell+1 = m \geq 0$, then substituting $m=\ell+1$ and using the fact that $F(p,q,r)=F(p,r,q)$ makes all the terms cancel out. If $k \geq \ell \geq m+1=0$, then all terms in the claimed expression are 0. Finally, if $k=\ell=m=0$, first term is 1 and all other terms are 0, yielding the required value 1 in this case. □

For $k \geq \ell \geq m \geq 0$, it follows that

$$\begin{aligned}\frac{(k+2)!(\ell+1)!m!}{(k+\ell+m)!}N(k,\ell,m) &= (k+2)(k+1)(\ell+1)+(\ell+1)\ell m+(k+2)m(m-1)\\ &\quad -(k+2)(\ell+1)\ell-(k+2)(k+1)m-(\ell+1)m(m-1)\\ &= (k+1-\ell)\left((k+2)(\ell+1)-(k+\ell+2)m+m(m-1)\right)\\ &= (k+1-\ell)(\ell+1-m)(k+2-m).\end{aligned}$$

Then for $k \geq 2$,

$$\frac{N(k,k,k-2)}{N(k,k-1,k-1)} = \frac{(k+2)!k!(k-1)!}{(k+2)!(k+1)!(k-2)!} \cdot \frac{1\cdot 3\cdot 4}{2\cdot 1\cdot 3} = 2\,\frac{(k-1)}{(k+1)}.$$

This fraction is between $1/2$ and 2 for all $k \geq 2$. In particular, with $k=2024$, we get that n_1/n_2 is between $1/2$ and 2, which completes the solution.

Remark. This problem is a special case of the 1/3–2/3 conjecture (`https://en.wikipedia.org/wiki/1/3-2/3_conjecture`). Solution 1 is based on an argument in [S. H. Chan, I. Pak, G. Panova, "Sorting Probability for Large Young Diagrams", Discrete Analysis 24 (2021), `https://doi.org/10.19086/da.30071`]. The final formula derived in Solution 2 for $N(k,\ell,m)$ is a special case of the "hook length formula", written in the following form:

`https://en.wikipedia.org/wiki/Hook_length_formula#Related_formulas`

A4. Find all primes $p > 5$ for which there exists an integer a and an integer r satisfying $1 \le r \le p-1$ with the following property: the sequence $1, a, a^2, ..., a^{p-5}$ can be rearranged to form a sequence $b_0, b_1, b_2, ..., b_{p-5}$ such that $b_n - b_{n-1} - r$ is divisible by p for $1 \le n \le p-5$.

Answer: Only $p = 7$.

Solution: For $p = 7$, $a = 3$ yields the sequence $1, 3, 9$, which can be reordered as $1, 9, 3$.

For $p \ge 11$, we work modulo p. Suppose $1, a, a^2, \ldots, a^{p-5}$ can be rearranged with differences between consecutive terms congruent to $r \not\equiv 0 \pmod{p}$. If two of these terms were the same modulo p, then $jr \equiv 0 \pmod{p}$ where j is the distance between their indices in the arithmetic progression. Since $j < p$, we must have $j = 0$, and so the terms are all distinct modulo p. Because $p-5 > (p-1)/2$, we conclude that a has multiplicative order $p-1$ modulo p, and so $0, 1, a, a^2, \ldots, a^{p-2}$ are distinct modulo p. Therefore, $1, a, a^2, \ldots, a^{p-5}$ must be congruent to a "segment" of the "cyclic" modulo-p arithmetic progression $0, r, 2r, \ldots, (p-1)r, 0, \ldots$. Then $0, a^{p-4}, a^{p-3}, a^{p-2}$ must be congruent to the remaining segment that completes the cycle. Since $a^{p-1} \equiv 1 \pmod{p}$, these four terms are congruent to $0, c, c^2, c^3$, where c is the residue class of a^{p-2} modulo p. Because none of c, c^2, c^3 are -1 times another, 0 must be an end of the arithmetic progression, which we may assume is the beginning. Furthermore, if we multiply by c^{-4}, we obtain another arithmetic progression using the same rearrangement of the terms $0, c^{-3}, c^{-2}, c^{-1}$ as for $0, c, c^2, c^3$. Thus, with either $d = c$ or $d = c^{-1}$, we need only consider the three orderings of $0, d, d^2, d^3$ that begin with 0 and where d precedes d^3.

The progression is $0, d, d^2, d^3$. Then $d^2 \equiv 2d \pmod{p}$ and $d^3 \equiv 3d \pmod{p}$, so $d \equiv 2 \pmod{p}$ and $8 \equiv 6 \pmod{p}$, a contradiction.

The progression is $0, d, d^3, d^2$. Then $d^3 \equiv 2d \pmod{p}$ and $d^2 \equiv 3d \pmod{p}$. Thus, $d \equiv 3 \pmod{p}$ and $27 \equiv 6 \pmod{p}$, a contradiction.

The progression is $0, d^2, d, d^3$. Then $d \equiv 2d^2 \pmod{p}$ and $d^3 \equiv 3d^2 \pmod{p}$. Thus, $d \equiv 3 \pmod{p}$ and $3 \equiv 18 \pmod{p}$, a contradiction.

A5. Consider a circle Ω with radius 9 and center at the origin $(0,0)$, and a disk Δ with radius 1 and center at $(r,0)$, where $0 \le r \le 8$. Two points P and Q are chosen independently and uniformly at random on Ω. Which value(s) of r minimize the probability that the chord $\overline{PQ}$ intersects Δ?

Answer: $r = 0$. More generally, if the larger circle has radius $\rho > 1$, then the minimum probability for $0 \le r \le \rho - 1$ occurs at (and only at) $r = 0$.

Solution 1: Consider more generally the case that Δ has center $(r\cos\theta, r\sin\theta)$. The probability $p(r)$ that $\overline{PQ}$ intersects Δ is independent of θ, so we can compute $p(r)$ by considering θ to be a random variable chosen uniformly on $[-\pi,\pi]$, independently of P and Q.

Next, let O be the origin, and let Π be the set of lines through O. Let L be the line in Π that bisects angle POQ. As the angle ray $\overrightarrow{OQ}$ makes with ray $\overrightarrow{OP}$ increases from 0 to 2π, the angle L makes with $\overrightarrow{OP}$ increases half as fast from 0 to π (this sweeps through all the lines in Π). Thus, L is uniformly distributed on Π for each fixed P. Since P is uniformly distributed on Ω, the ordered pair (P,L) is uniformly distributed on $\Omega \times \Pi$. Since P and L determine Q (specifically, Q is the reflection of P through L), we can compute $p(r)$ with respect to the independent uniform random variables P, L, and θ (instead of with respect to P, Q, and θ).

Because of the uniform distribution of θ, the probability that $\overline{PQ}$ intersects Δ is independent of L. Thus, we can fix L to be vertical and compute $p(r)$ with respect to P and θ; then $\overline{PQ}$ is the horizontal line through P. By left-right symmetry, we can compute $p(r)$ using the uniform distribution for P on the half of Ω to the right of L. Thus, let $P = (\rho\cos\varphi, \rho\sin\varphi)$ where φ is uniformly distributed on $[-\pi/2, \pi/2]$. For fixed θ, the probability that $\overline{PQ}$ intersects Δ is then the probability that $\rho\sin\varphi$ lies between $r\sin\theta - 1$ and $r\sin\theta + 1$, which is

$$\frac{1}{\pi}\left(\arcsin\left(\frac{r\sin\theta+1}{\rho}\right) - \arcsin\left(\frac{r\sin\theta-1}{\rho}\right)\right).$$

Thus,

$$p(r) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\left(\arcsin\left(\frac{r\sin\theta+1}{\rho}\right) - \arcsin\left(\frac{r\sin\theta-1}{\rho}\right)\right)d\theta.$$

It follows that

$$\begin{aligned} p'(r) &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\frac{\sin\theta}{\rho}\left(\frac{1}{\sqrt{1-(r\sin\theta+1)^2/\rho^2}} - \frac{1}{\sqrt{1-(r\sin\theta-1)^2/\rho^2}}\right)d\theta \\ &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\sin\theta\left(\frac{1}{\sqrt{\rho^2-(r\sin\theta+1)^2}} - \frac{1}{\sqrt{\rho^2-(r\sin\theta-1)^2}}\right)d\theta. \end{aligned}$$

The integrand is positive when $0 < r < \rho - 1$ and $\sin\theta > 0$, because then $(r\sin\theta-1)^2 < (r\sin\theta+1)^2 < \rho^2$. Notice that the integrand is also an even function of θ, since it is the product of two odd functions. Thus, $p'(r) > 0$ for $0 < r < \rho - 1$, and therefore $p(r)$ is minimized at $r = 0$ only.

Solution 2: Let $P = (\rho\cos\theta, \rho\sin\theta)$, where θ is uniformly distributed on $[0, 2\pi)$. Let B be a point on Δ for which $\overline{PB}$ is tangent to Δ, and let $C = (r,0)$ be the center of Δ. Then PBC is a right triangle, and since length $BC = 1$, we have

$$\sin\angle BCP = \frac{BC}{PC} = \frac{1}{\sqrt{(\rho\cos\theta - r)^2 + (\rho\sin\theta)^2}} = \frac{1}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}}.$$

Let A be the arc between the two tangent rays from P to Δ. For fixed P, the conditional probability that $\overline{PQ}$ intersects Δ is the probability that Q lies in A, which is the angle measure α of A divided by 2π. Notice that α is twice the angle between the tangent rays from P to Δ, and hence $\alpha = 4\angle BCP$. Thus, the conditional probability that $\overline{PQ}$ intersects Δ is $(2/\pi)\arcsin(1/\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta})$. It follows that the overall probability $p(r)$ that $\overline{PQ}$ intersects Δ is given by

$$\begin{aligned} p(r) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{2}{\pi}\arcsin\left(\frac{1}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}}\right) d\theta \\ &= \frac{2}{\pi^2}\int_0^{\pi} \arcsin\left(\frac{1}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}}\right) d\theta. \end{aligned}$$

Notice that $p(0) = (2/\pi)\arcsin(1/\rho)$.

Since arcsin is a convex function on the interval $[0, 1]$, Jensen's inequality implies that

$$p(r) \geq \frac{2}{\pi}\arcsin\left(\frac{1}{\pi}\int_0^{\pi} \frac{1}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}}\, d\theta\right).$$

The proof that $p(r) > p(0)$ for $0 < r \leq \rho - 1$ will be complete after we prove the following claim for such r:

$$\frac{1}{\pi}\int_0^{\pi} \frac{1}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}}\, d\theta > \frac{1}{\rho}.$$

This claim turns out to be true for $0 < r < \rho$, in fact. Let $x = r/\rho \in (0, 1)$; multiplying the inequality above by $\rho\pi$ yields the equivalent claimed inequality

$$\int_0^{\pi} \frac{1}{\sqrt{1 + x^2 - 2x\cos\theta}}\, d\theta > \pi.$$

Let $t = (\sqrt{1 + x^2 - 2x\cos\theta} - 1)/x$, so that $1 + x^2 - 2x\cos\theta = (1 + xt)^2$, and notice that t goes from -1 to 1 as θ goes from 0 to π. To change variables from θ to t in the integral above, we compute $2x\sin\theta\, d\theta = 2x(1 + xt)dt$, and

$$\begin{aligned} 2x\sin\theta = 2x\sqrt{1 - \cos^2\theta} &= 2x\sqrt{1 - \left(\frac{1 + x^2 - (1 + xt)^2}{2x}\right)^2} \\ &= \sqrt{(2x + 1 + x^2 - (1 + xt)^2)(2x - 1 - x^2 + (1 + xt)^2)} \\ &= \sqrt{(1 + x + (1 + xt))(1 + x - (1 + xt))(1 + xt + (1 - x))(1 + xt - (1 - x))} \\ &= \sqrt{(2 + x + xt)x(1 - t)(2 - x + xt)x(1 + t)} = x\sqrt{1 - t^2}\sqrt{(2 + xt)^2 - x^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\pi} \frac{1}{\sqrt{1 + x^2 - 2x\cos\theta}}\, d\theta &= \int_{-1}^{1} \frac{1}{1 + xt} \cdot \frac{2(1 + xt)}{\sqrt{1 - t^2}\sqrt{(2 + xt)^2 - x^2}}\, dt \\ &= \int_{-1}^{1} \frac{2}{\sqrt{1 - t^2}\sqrt{(2 + xt)^2 - x^2}}\, dt. \end{aligned}$$

Let $f_x(t) = (2+xt)^2 - x^2$, so that the integrand above can be written $2f_x(t)^{-1/2}/\sqrt{1-t^2}$. Since the function $y \mapsto y^{-1/2}$ on the positive real numbers is convex,

$$\frac{f_x(t)^{-1/2} + f_x(-t)^{-1/2}}{2} \geq \left(\frac{f_x(t) + f_x(-t)}{2}\right)^{-1/2} = (4 + x^2t^2 - x^2)^{-1/2} > \frac{1}{2}$$

for $0 < x < 1$ and $-1 < t < 1$. Thus,

$$\int_{-1}^{1} \frac{2f_x(t)^{-1/2}}{\sqrt{1-t^2}}\, dt = \int_0^1 \frac{2(f_x(t)^{-1/2} + f_x(-t)^{-1/2})}{\sqrt{1-t^2}}\, dt > \int_0^1 \frac{2}{\sqrt{1-t^2}}\, dt = \pi$$

as claimed.

A6. Let $c_0, c_1, c_2, \dots$ be the sequence defined so that

$$\frac{1-3x-\sqrt{1-14x+9x^2}}{4} = \sum_{k=0}^{\infty} c_k x^k$$

for sufficiently small x. For a positive integer n, let A be the n-by-n matrix with i,j-entry c_{i+j-1} for i and j in $\{1,\dots,n\}$. Find the determinant of A.

Answer: $10^{(n^2-n)/2}$.

Solution 1: More generally, let

$$F(x) = \frac{1-\alpha x-\sqrt{(1-\alpha x)^2-4\beta x}}{2\beta} = \sum_{k=0}^{\infty} c_k x^k.$$

We show that the determinant of the $n \times n$ matrix defined as in the problem statement is $(\beta(\alpha+\beta))^{(n^2-n)/2}$. When $\alpha = 3, \beta = 2$, we get the problem statement.

By the quadratic formula, $F(x)$ is a root of

$$\beta F(x)^2 + (\alpha x - 1)F(x) + x = 0.$$

From its definition, observe that $c_0 = F(0) = 0$. Examining the coefficient of x^n in the functional equation, we find

$$c_n = \begin{cases} 1, & \text{if } n = 1, \\ \alpha+\beta, & \text{if } n = 2, \\ (\alpha+2\beta)c_{n-1} + \beta \sum_{k=2}^{n-2} c_k c_{n-k}, & \text{if } n > 2. \end{cases}$$

(We use the convention that a sum with strictly decreasing limits of summation is 0.)

Thus, the 1-by-1 matrix has determinant 1 and the 2-by-2 matrix has determinant

$$(\alpha+2\beta)(\alpha+\beta)\cdot 1 - (\alpha+\beta)^2 = \beta(\alpha+\beta).$$

We proceed by induction; assume the claim for some $n \ge 2$ and consider the $(n+1)$-by-$(n+1)$ matrix.

From row $n+1$, subtract $\alpha+2\beta$ times row n and βc_{n+1-k} times row k for rows $k = 2,\dots,n-1$. The entry in row $n+1$, column j is now

$$c_{n+j} - (\alpha+2\beta)c_{n+j-1} - \beta\sum_{k=2}^{n-1} c_{n+1-k}c_{j+k-1} = c_{n+j} - (\alpha+2\beta)c_{n+j-1} - \beta\sum_{k=2}^{n-1} c_k c_{n+j-k}$$
$$= \beta \sum_{k=n}^{n+j-2} c_k c_{n+j-k} = \beta \sum_{k=1}^{j-1} c_{k+n-1}c_{1+j-k}.$$

Next, reduce rows $n, n-1, \dots, 3$ similarly. Finally, subtract $\alpha+\beta$ times row 1 from row 2, so the entry in the jth column of row 2 is now

$$\beta \sum_{k=1}^{j-1} c_k c_{1+j-k}.$$

At this point, column j of rows 2 through $n+1$ is the column vector

$$\beta \sum_{k=1}^{j-1} \begin{bmatrix} c_k c_{1+j-k} \\ c_{k+1} c_{1+j-k} \\ \vdots \\ c_{k+n-2} c_{1+j-k} \\ c_{k+n-1} c_{1+j-k} \end{bmatrix} = \beta \sum_{k=1}^{j-1} c_{1+j-k} \begin{bmatrix} c_k \\ c_{k+1} \\ \vdots \\ c_{k+n-2} \\ c_{k+n-1} \end{bmatrix}.$$

Therefore, the entire first column is now the standard basis vector $\mathbf{e}_1$. The determinant of the reduced matrix (which is the same as the determinant of the original matrix) is then the determinant of its lower right n-by-n submatrix. Pull the factor β^n out of the determinant of the submatrix. Noting that, for $j \geq 3$,

$$\sum_{k=1}^{j-2} c_{1+j-k} \begin{bmatrix} c_k \\ c_{k+1} \\ \vdots \\ c_{k+n-2} \\ c_{k+n-1} \end{bmatrix}$$

is in the span of the columns $2, \ldots, j-1$ of this submatrix, we may reduce its columns from left to right, yielding $c_2 = (\alpha + \beta)$ times the original n-by-n matrix. Therefore, the determinant for $(n+1)$-by-$(n+1)$ matrix is $\beta^n(\alpha+\beta)^n$ times that for the n-by-n matrix, completing the induction.

Remark. Letting A_n denote the n-by-n matrix, the row and column reductions above can be summarized as follows:

$$\begin{bmatrix} 1 & & & & & & & \\ -(\alpha+\beta) & 1 & & & & & & \\ 0 & -(\alpha+2\beta) & 1 & & & \mathbf{0} & & \\ 0 & -\beta c_2 & -(\alpha+2\beta) & 1 & & & & \\ & & & & & \ddots & & \\ 0 & -\beta c_{n-2} & -\beta c_{n-3} & \dots & -\beta c_2 & -(\alpha+2\beta) & 1 & \\ 0 & -\beta c_{n-1} & -\beta c_{n-2} & \dots & -\beta c_3 & -\beta c_2 & -(\alpha+2\beta) & 1 \end{bmatrix}$$

$$\cdot A_{n+1} \cdot \begin{bmatrix} 1 & 0 & 0 & & & 0 \\ & 1 & -c_3/c_2 & -c_4/c_2 & \dots & -c_n/c_2 & -c_{n+1}/c_2 \\ & & 1 & -c_3/c_2 & \dots & -c_{n-1}/c_2 & -c_n/c_2 \\ & & & \ddots & & & \\ & & \mathbf{0} & & & & \\ & & & & & 1 & -c_3/c_2 \\ & & & & & & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|ccc} c_1 & c_2 & \dots & c_{n+1} \\ \hline 0 & & & \\ \vdots & & \beta(\alpha+\beta)A_n & \\ 0 & & & \end{array}\right].$$

Solution 2: We will show that $A = LDL^t$ where $[L_{i,j}]_{i,j=0}^{n-1}$ is a lower triangular matrix with 1's on the diagonal and $[D_{i,j}]_{i,j=0}^{n-1}$ is a diagonal matrix with $D_{k,k} = 10^k$. (Here we start the indexing at 0.) Then $\det A = \det(L)^2 \det(D) = 1 \cdot 10^{0+1+\cdots+(n-1)} = 10^{\binom{n}{2}}$.

Let $F(x) = \frac{1-3x-\sqrt{1-14x+9x^2}}{4} = x + 5x^2 + O(x^3)$. We have that $(4F(x)+3x-1)^2 = 1 - 14x + 9x^2$, so

$$2F(x)^2 + (3x-1)F(x) + x = 0.$$

Let $f(u) := F(u)/u = 1 + 5u + O(u^2)$ and $g(u) := \frac{F(u)-u}{5u} = u + O(u^2)$, and define $L_{i,j}$ for $i,j = 0,1,\dots$ as the coefficients of the expansion of

$$\ell(u,v) = \sum_{i,j\geq 0} L_{i,j} u^i v^j = \frac{f(u)}{1-vg(u)} = f(u)\left(1 + vg(u) + v^2 g(u)^2 + \cdots\right).$$

We see that $L_{i,j} = 0$ for $i < j$ and $L_{i,i} = 1$. Consider $B = LDL^t$ where $D_{k,k} = d^k$ for $k \geq 0$ and all other entries of D are zero. Then the entries of B are

$$B_{i,j} = \sum_{k\geq 0} L_{i,k} D_{k,k} L_{j,k}.$$

(Only a finite number of terms in the sum above are nonzero, because L is lower triangular.) Denote by $[z^k]H(z)$ the coefficient of z^k in the expansion of $H(z)$. Set

$$\begin{aligned}
b(u,v) &= \sum_{i,j\geq 0} B_{i,j}u^iv^j = \sum_{k\geq 0}[w^kt^k]\sum_{i,j,r,s\geq 0} L_{i,r}u^iw^rd^rL_{j,s}v^jt^s \\
&= \sum_{k\geq 0}[w^kt^k]\ell(u,dw)\ell(v,t) = f(u)f(v)\sum_{k\geq 0}[w^kt^k]\sum_{r,s\geq 0} d^rw^rg(u)^rt^sg(v)^s \\
&= f(u)f(v)\sum_{k\geq 0} d^kg(u)^kg(v)^k = \frac{f(u)f(v)}{1-dg(u)g(v)}.
\end{aligned}$$

The lower-triangular property $L_{i,j}=0$ for $i<j$ implies that $B_{i,j}$ depends only on values of $L_{i,k}$, $D_{k,k}$, and $L_{j,k}$ with $k\leq i$ and $k\leq j$. Thus, the equation $B=LDL^t$ holds also for the finite matrices $[B_{i,j}]_{i,j=0}^{n-1}$, $[L_{i,j}]_{i,j=0}^{n-1}$, and $[D_{i,j}]_{i,j=0}^{n-1}$. The following claim proves the desired decomposition for A, and finishes the solution.

Claim. *For $d=10$, we have $A_{i+1,j+1}=B_{i,j}$.*

Proof. Define $A_{i,j}=c_{i+j-1}$ for all $i,j=1,2,\ldots$, and let

$$\begin{aligned}
a(u,v) &= \sum_{i,j\geq 0} A_{i+1,j+1}u^iv^j = \sum_{i,j\geq 0} c_{i+j+1}u^iv^j \\
&= \sum_{r\geq 0} c_{r+1}(u^r+u^{r-1}v+\cdots+v^r) = \sum_{r\geq 0} c_{r+1}\frac{u^{r+1}-v^{r+1}}{u-v} = \frac{F(u)-F(v)}{u-v},
\end{aligned}$$

since $c_0=0$.

Next, consider

$$\begin{aligned}
a(u,v)-b(u,v) &= \frac{F(u)-F(v)}{u-v} - \frac{f(u)f(v)}{1-10g(u)g(v)} \\
&= \frac{F(u)(1-10g(u)g(v))-uf(u)f(v)-F(v)(1-10g(u)g(v))+vf(u)f(v))}{(u-v)(1-10g(u)g(v))}.
\end{aligned}$$

We have that $F(u)=\frac{F(u)-u}{2F(u)+3u}$, and so the first half of the numerator above is

$$\begin{aligned}
F(u)(1-10g(u)g(v))-uf(u)f(v) &= F(u)(1-10g(u)g(v)-f(v)) \\
&= \frac{(F(u)-u)}{2F(u)+3u}\left(1-2\frac{(F(u)-u)(F(v)-v)}{5uv}-\frac{F(v)}{v}\right) \\
&= \frac{(F(u)-u)}{(2F(u)+3u)(5uv)}(5uv-2F(u)F(v)+2vF(u)+2uF(v)-2uv-5uF(v)) \\
&= \frac{(F(u)-u)}{(2F(u)+3u)(5uv)}(3uv-2F(u)F(v)+2vF(u)-3uF(v)) \\
&= \frac{(F(u)-u)(F(v)-v)(-2F(u)-3u)}{(2F(u)+3u)(5uv)} \\
&= -\frac{(F(u)-u)(F(v)-v)}{5uv}.
\end{aligned}$$

The second half of the numerator is the negative of the first half, with u and v interchanged, so performing the same manipulations on the second half verifies that it cancels with the first half. Thus, $a(u,v)=b(u,v)$ and the claim is proved. □

Remark. This problem was inspired by the determinants of Hankel matrices used to count tilings of the Aztec diamond (`https://en.wikipedia.org/wiki/Aztec_diamond`).

2024 Session B

B1. Let n and k be positive integers. The ith row and jth column of an n-by-n grid of squares contains the number $i+j-k$. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly $1, 2, \ldots, n$?

Answer: It is possible if and only if $n = 2k - 1$.

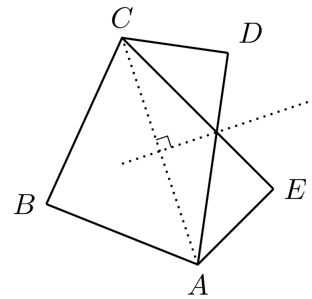
Solution: Suppose that it is possible to select such squares, and let their coordinates be $(i, w(i))$, where $w : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijection. We must have that the sum of the entries in those squares is $1 + 2 + \cdots + n = \binom{n+1}{2}$. On the other hand, the sum is equal to

$$\sum_{i=1}^{n}(i + w(i) - k) = \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} w(i)\right) - kn = 2\left(\sum_{i=1}^{n} i\right) - kn = 2\binom{n+1}{2} - kn,$$

Thus, we must have $kn = \binom{n+1}{2}$, so $k = (n+1)/2$, and $n = 2k - 1$ has to be odd.

To exhibit a possible construction for these values, let $w(i) = k + i - 1$ for $i = 1, \ldots, k$ and $w(i) = i - k$ for $i = k+1, \ldots, n$, so that $w(1), w(2), \ldots, w(n) = k, k+1, \ldots, n, 1, \ldots, k-1$. The value in the square $(i, w(i))$ is $2i - 1$ for $i = 1, \ldots, k$ (the odd numbers $1, 3, \ldots, n$), and $2(i-k)$ for $i = k+1, \ldots, n$ (the even numbers $2, 4, \ldots, n-1$).

B2. Two convex quadrilaterals are called *partners* if they have three vertices in common and they can be labeled $ABCD$ and $ABCE$ so that E is the reflection of D across the perpendicular bisector of the diagonal $\overline{AC}$. Is there an infinite sequence of convex quadrilaterals such that each quadrilateral is a partner of its successor and no two elements of the sequence are congruent?



Answer: No.

Solution 1: Let $Q_0, Q_1, \ldots, Q_6$ be a sequence of convex quadrilaterals such that Q_n and Q_{n+1} are partners for $0 \le n \le 5$. We will prove that Q_0 and Q_6 are congruent, unless Q_{n-1} and Q_{n+1} are congruent for some $1 \le n \le 5$.

In the notation of the problem statement, we'll say that $ABCD$ and $ABCE$ are partners with respect to diagonal $\overline{AC}$. A quadrilateral can have two different partners with respect to the same diagonal, but these partners are congruent to each other, because one partner is the reflection of the other about the perpendicular bisector of the diagonal. Thus for $1 \le n \le 5$, either Q_{n-1} and Q_{n+1} are congruent or they are partners of Q_n with respect to different diagonals of Q_n. Hereafter, we assume that the latter is true for all $1 \le n \le 5$.

Label the vertices of Q_n as $W_nX_nY_nZ_n$ in such a way that the three vertices in common between Q_n and Q_{n+1} are assigned the same letters. Without loss of generality, we can assume for $0 \le n \le 5$ that Q_n and Q_{n+1} are partners with respect to $\overline{W_nY_n}$ if n is even and with respect to $\overline{X_nZ_n}$ if n is odd. For each such Q_n, this allows two possibilities for Q_{n+1} that are congruent to each other, so without loss of generality we can assume that $W_{n+1} = W_n$ and $Z_{n+1} = Z_n$.

Since Q_0 is convex, its diagonals intersect, and in particular they are not parallel, so neither are their perpendicular bisectors. Let P be the intersection of their perpendicular bisectors, and let P be the origin of a polar coordinate system. Then W_0 and Y_0 are equidistant from P, and so are X_0 and Z_0. Let the coordinates of these vertices be $W_0 = (r, \alpha)$, $X_0 = (s, \beta)$, $Y_0 = (r, \gamma)$, $Z_0 = (s, \delta)$.

To form Q_1, we reflect X_0 across the perpendicular bisector of $\overline{W_0Y_0}$ to get X_1. The angle that the bisector makes with respect to P is $\frac{\alpha+\gamma}{2}$, reflecting X_0 to X_1 then gives an angular coordinate for X_1 as $2(\frac{\alpha+\gamma}{2} - \beta) + \beta$. Since P is on this perpendicular bisector, it is equidistant from X_0 and X_1, so we have $X_1 = (s, \alpha + \gamma - \beta)$, while $Y_1 = Y_0 = (r, \gamma)$. Notice also that X_1 and $Z_1 = Z_0$ are equidistant from P, so P is on the perpendicular bisector of diagonal $\overline{X_1Z_1}$. Continuing in this manner, keeping in mind that $W_n = W_0$ and $Z_n = Z_0$, we calculate

$$\begin{aligned} X_2 &= (s, \alpha + \gamma - \beta), & Y_2 &= (r, \alpha + \delta - \beta); \\ X_3 &= (s, \alpha + \delta - \gamma), & Y_3 &= (r, \alpha + \delta - \beta); \\ X_4 &= (s, \alpha + \delta - \gamma), & Y_4 &= (r, \beta + \delta - \gamma); \\ X_5 &= (s, \beta), & Y_5 &= (r, \beta + \delta - \gamma); \\ X_6 &= (s, \beta), & Y_6 &= (r, \gamma). \end{aligned}$$

In particular, Q_0 and Q_6 are congruent (though they would not necessarily coincide with each other if we had chosen partners so that W_n or Z_n changed at some step).

Solution 2: We say that an ordered quintuple (w, x, y, z, θ) "represents" a convex quadrilateral $ABCD$ if $w = AB$, $x = BC$, $y = CD$ and $z = DA$, and θ is the sum of the interior angles at B and D. (By relabeling its vertices, a quadrilateral can be represented by more than one quintuple.) We claim that two convex quadrilaterals that can be represented by the same quintuple must be congruent. Before we prove this claim, we explain why it solves the problem.

For partners $ABCD$ and $ABCE$, the interior angles at D and E are the same, and $CD = AE$ and $DA = EC$. Thus, if (w, x, y, z, θ) represents $ABCD$ using the labeling of the previous paragraph, then (w, x, z, y, θ) represents $ABCE$. Furthermore, if (w, x, y, z, θ) represents $ABCD$ using a different labeling, then $ABCE$ can be represented by some permutation of w, x, y, z followed by θ. (Notice that θ might be the sum of the interior angles at A and C in this representation, but since the sum of all four interior angles of a quadrilateral is always 2π, the value of θ is the same for $ABCD$ and $ABCE$ in this case too.) By induction, for an infinite sequence of convex quadrilaterals in which each is a partner of its successor, if (w, x, y, z, θ) represents the first member of the sequence, then each member of the sequence can be represented by a quintuple that is some permutation of w, x, y, z followed by θ. Since there are only a finite number of such permutations, two members of the sequence can be represented by the same quintuple.

To prove the claim, we again use the labeling of the first paragraph. Let ϕ be the interior angle at B; then the interior angle at D is $\theta - \phi$. By the law of cosines, both of the following expressions equal AC^2:

$$w^2 + x^2 - 2wx\cos\phi = y^2 + z^2 - 2yz\cos(\theta - \phi).$$

Since $ABCD$ is convex, both ϕ and $\theta - \phi$ lie between 0 and π, so the left side of the equation above is a strictly increasing function of ϕ, and the right side is a strictly decreasing function of ϕ. Thus, there can be only one value of ϕ that achieves equality. The values of w, x, y, z, θ, ϕ determine triangles ABC and CDA up to congruence, and therefore they determine $ABCD$ up to congruence.

Solution 3: We prove that the number of noncongruent quadrilaterals in such a sequence of convex quadrilaterals cannot exceed 12.

Observe that $\triangle ACD \cong \triangle CAE$. Thus, two convex partners have the same set of four side-lengths, the same area, and the same sums for the two pairs of opposite angles.

In quadrilateral $ABCD$, let $\rho = \angle A$ and $\sigma = \angle A + \angle C$, $w = AB$, $x = BC$, $y = CD$, $z = DA$. The areas of the sequence of quadrilaterals all equal

$$\frac{1}{2}wz\sin\rho + \frac{1}{2}xy\sin(\sigma - \rho) = \frac{wz - xy\cos\sigma}{2}\sin\rho + \frac{xy\sin\sigma}{2}\cos\rho.$$

Since the area is positive, the coefficients of $\sin\rho$ and $\cos\rho$ cannot both be zero. By convexity, $0 < \rho < \pi$, so $\sin\rho > 0$. Thus, for the derivative of the area with respect to ρ to be 0, we must have

$$\cot\rho = \frac{xy\sin\sigma}{wz - xy\cos\sigma},$$

which holds for as most one of the possible values of ρ. Then for given w, x, y, z, σ, the area takes on any particular value for at most two values of ρ. Knowing ρ determines BD; hence determines $\angle ABD$, $\angle ADB$, $\angle CBD$, and $\angle CDB$; hence determines $\angle ABC$ and $\angle ADC$.

More generally, after a sequence of convex partnerships, there is always an angle adjacent to the side of length w that is one of the opposite pair of angles that sum to σ. Let the side lengths be, in order, w, s_1, s_2, s_3, where s_1, s_2, s_3 is a permutation of x, y, z, and σ is the sum of the angle ρ between w and s_3 and the angle between s_1 and s_2. As in the previous paragraph, the values of $w, s_1, s_2, s_3, \sigma, \rho$ determine the quadrilateral up to congruence, and for given w, s_1, s_2, s_3, σ, there are at most two values of ρ that make the area of the quadrilateral equal to the area of $ABCD$. This yields at most $3! \cdot 2 = 12$ noncongruent quadrilaterals.

Remark. Solution 3 yields twice as many possibilities as Solution 1 because, in fact, there is only one possible value of ρ for given w, x, y, z, σ. This follows from an argument similar to the last paragraph of Solution 2, requiring that angles ρ and $\sigma - \rho$ yield the same value for the length of diagonal BD.

B3. Let r_n be the nth smallest positive solution to $\tan x = x$, where the argument of tangent is in radians. Prove that

$$0 < r_{n+1} - r_n - \pi < \frac{1}{(n^2+n)\pi}$$

for $n \geq 1$.

Solution: Set $d_n = r_{n+1} - r_n - \pi$. Because $\frac{d}{dx}(\tan x - x) = \sec^2 x - 1 \geq 0$ where the derivative exists, with equality only at integer multiples of π, for each period of $\tan x$ the function $\tan(x) - x$ is increasing and has a unique root. Thus, $n\pi < r_n < (n+1/2)\pi$ and $d_n < \pi/2$. Since $\tan(r_n - n\pi) = \tan r_n = r_n < r_{n+1} = \tan r_{n+1} = \tan(r_{n+1} - (n+1)\pi)$, we have $r_n - n\pi < r_{n+1} - (n+1)\pi$, and hence $d_n = r_{n+1} - (n+1)\pi - (r_n - n\pi) > 0$. Then $0 < d_n < \pi/2$, and in particular $d_n < \tan d_n$. By the formula for the tangent of a difference, we have

$$\begin{aligned} d_n < \tan(d_n) = \tan(r_{n+1} - (r_n + \pi)) &= \frac{\tan(r_{n+1}) - \tan(r_n + \pi)}{1 + \tan(r_{n+1})\tan(r_n + \pi)} \\ &= \frac{r_{n+1} - r_n}{1 + r_{n+1}r_n} = \frac{\pi + d_n}{1 + r_{n+1}r_n} \end{aligned}$$

Isolating d_n, we find

$$d_n < \frac{\pi}{r_{n+1}r_n} < \frac{\pi}{(n+1)\pi \cdot n\pi} = \frac{1}{(n^2+n)\pi}.$$

B4. Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from the set $\{1, \dots, n\}$, and let

$$a_{n,k+1} = \begin{cases} a_{n,k} + 1, & \text{if } m_{n,k} > a_{n,k}; \\ a_{n,k}, & \text{if } m_{n,k} = a_{n,k}; \\ a_{n,k} - 1, & \text{if } m_{n,k} < a_{n,k}. \end{cases}$$

Let $E(n)$ be the expected value of $a_{n,n}$. Determine $\lim_{n\to\infty} E(n)/n$.

Answer: $\dfrac{1 - e^{-2}}{2}$.

Solution 1: Let $p_{n,k}(j)$ denote the probability that $a_{n,k} = j$ and let $E(n,k)$ denote the expected value of $a_{n,k}$. When $a_{n,k} = j$, the expected value of $a_{n,k+1} - a_{n,k}$ is

$$1 \cdot \frac{n-j}{n} + 0 \cdot \frac{1}{n} - \frac{j-1}{n} = \frac{n+1-2j}{n}.$$

Therefore,

$$\begin{aligned} E(n, k+1) &= E(n,k) + \sum_{j=1}^{n} \frac{n+1-2j}{n}\, p_{n,k}(j) = E(n,k) + \frac{n+1}{n} - \frac{2}{n}\, E(n,k) \\ &= \frac{n+1}{n} + \frac{n-2}{n}\, E(n,k). \end{aligned}$$

Iterating from $E(n,0) = 1$, we find

$$E(n,n) = \left(\frac{n-2}{n}\right)^n + \frac{n+1}{n} \sum_{k=0}^{n-1} \left(\frac{n-2}{n}\right)^k = \left(\frac{n-2}{n}\right)^n + \frac{n+1}{n} \cdot \frac{1 - \left(\frac{n-2}{n}\right)^n}{2/n}.$$

Observing that

$$\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = e^{-2},$$

we conclude

$$\lim_{n\to\infty} \frac{E(n)}{n} = \lim_{n\to\infty} \frac{E(n,n)}{n} = \frac{1-e^{-2}}{2}.$$

Solution 2: Let $E_k(d)$ be the expected value of $a_{n,n}$ given that $a_{n,k} = d$. Note that $E_n(d) = d$. We seek $E(n) = E_0(1)$. We have the recursion

$$E_k(d) = \frac{d-1}{n} E_{k+1}(d-1) + \frac{1}{n} E_{k+1}(d) + \frac{n-d}{n} E_{k+1}(d+1).$$

We can prove two lemmas by (downward) induction on k:

Lemma 1: For $k \leq n$, $E_k(d+1) - E_k(d)$ is independent of d for $d \leq n-1$.

Proof: This is true for $k = n$. Now suppose it is true for $k+1$; let $c_{k+1} = E_{k+1}(d+1) - E_{k+1}(d)$. Then subtracting two terms of the recursion gives

$$\begin{aligned} E_k(d+1) - E_k(d) &= \frac{n-d-1}{n} E_{k+1}(d+2) - \frac{n-d-1}{n} E_{k+1}(d+1) \\ &\quad + \frac{d-1}{n} E_{k+1}(d) - \frac{d-1}{n} E_{k+1}(d-1) \\ &= \frac{n-d-1}{n} c_{k+1} + \frac{d-1}{n} c_{k+1} \\ &= \left(1 - \frac{2}{n}\right) c_{k+1}, \end{aligned}$$

which is independent of d.

Corollary 1: For $k \leq n$, $E_k(d+1) - E_k(d) = \left(1 - \frac{2}{n}\right)^{n-k}$.

Proof: This follows directly from the relationship between c_k and c_{k+1} from the above proof.

Lemma 2: For $k \leq n$, $E_k(d) + E_k(n+1-d) = n+1$.

Proof: This is true for $k = n$. Now suppose it is true for $k+1$. Then

$$\begin{aligned} E_k(d) + E_k(n+1-d) &= \frac{d-1}{n} E_{k+1}(d-1) + \frac{1}{n} E_{k+1}(d) + \frac{n-d}{n} E_{k+1}(d+1) \\ &\quad + \frac{n-d}{n} E_{k+1}(n-d) + \frac{1}{n} E_{k+1}(n+1-d) + \frac{d-1}{n} E_{k+1}(n+2-d) \\ &= \frac{d-1}{n}(n+1) + \frac{1}{n}(n+1) + \frac{n-d}{n}(n+1) \\ &= n+1, \end{aligned}$$

by the inductive hypothesis (and pairing the first/sixth, second/fifth, and third/fourth terms of the expansion).

Corollary 2: $\sum_{d=1}^{n} E_k(d) = \frac{n(n+1)}{2}$.

Proof: $2 \sum_{d=1}^{n} E_k(d) = \sum_{d=1}^{n} E_k(d) + \sum_{d=1}^{n} E_k(n+1-d) = n(n+1)$, so the result follows.

The rest of the proof is algebra. We have

$$\frac{n(n+1)}{2} = \sum_{d=1}^{n} E_0(d) = \sum_{d=1}^{n} \left(E_0(1) + \left(1 - \frac{2}{n}\right)^n (d-1) \right) = nE_0(1) + \left(1 - \frac{2}{n}\right)^n \frac{n(n-1)}{2},$$

so we get

$$\frac{E(n)}{n} = \frac{E_0(1)}{n} = \frac{n+1}{2n} - \frac{n-1}{2n} \left(1 - \frac{2}{n}\right)^n.$$

The limit of this expression is $\frac{1}{2} - \frac{1}{2e^2}$.

B5. Let k and m be positive integers. For a positive integer n, let $f(n)$ be the number of integer sequences $x_1, \ldots, x_k, y_1, \ldots, y_m, z$ satisfying $1 \le x_1 \le \cdots \le x_k \le z \le n$ and $1 \le y_1 \le \cdots \le y_m \le z \le n$. Show that $f(n)$ can be expressed as a polynomial in n with nonnegative coefficients.

Solution 1: For a given z, the number of sequences is the number of ways to put k balls in z (labeled) boxes, i.e. $\binom{k+z-1}{k}$, and m balls in z boxes. Summing over z, the number of sequences is

$$p_{k,m}(n) = \sum_{z=1}^{n} \binom{k+z-1}{k}\binom{m+z-1}{m} = \sum_{i=0}^{n-1} \binom{i+k}{k}\binom{i+m}{m}.$$

We may assume $k \ge m$ and proceed by induction on m, beginning with $m = 0$. First, by the hockey-stick identity,

$$p_{k,0}(n) = \sum_{i=0}^{n-1} \binom{i+k}{k} = \binom{n+k}{k+1} = \frac{(n+k)(n+k-1)\cdots n}{(k+1)!},$$

the latter expression showing this is a polynomial in n of degree $k+1$ and the coefficients are nonnegative.

Next, we find a recursion, again using the hockey-stick identity,

$$\begin{aligned}
p_{k,m+1}(n) &= \sum_{i=0}^{n-1} \binom{i+k}{k}\binom{i+m+1}{m+1} = \sum_{i=0}^{n-1} \binom{i+k}{k} \sum_{j=0}^{i} \binom{j+m}{m} \\
&= \sum_{j=0}^{n-1} \binom{j+m}{m} \sum_{i=j}^{n-1} \binom{i+k}{k} = \sum_{j=0}^{n-1} \binom{j+m}{m} \left(\binom{n+k}{k+1} - \binom{j+k}{k+1} \right) \\
&= \binom{n+k}{k+1}\binom{n+m}{m+1} - \sum_{j=0}^{n-1} \binom{j+m}{m} \left(\binom{j+k+1}{k+1} - \binom{j+k}{k} \right) \\
&= \binom{n+k}{k+1}\binom{n+m}{m+1} - p_{k+1,m}(n) + p_{k,m}(n). \qquad (1)
\end{aligned}$$

This shows, by induction on m, that $p_{k,m}(n)$ is a polynomial in n. From

$$\frac{i+k+1}{k+1} = \frac{m+1}{k+1} \cdot \frac{i+m+1}{m+1} + \frac{k-m}{k+1},$$

for $k \ge m$, we deduce

$$\begin{aligned}
p_{k+1,m}(n) &= \sum_{i=0}^{n-1} \frac{i+k+1}{k+1} \binom{i+k}{k}\binom{i+m}{m} \\
&= \frac{m+1}{k+1} \sum_{i=0}^{n-1} \frac{i+m+1}{m+1} \binom{i+k}{k}\binom{i+m}{m} + \frac{k-m}{k+1} \sum_{i=0}^{n-1} \binom{i+k}{k}\binom{i+m}{m} \\
&= \frac{m+1}{k+1}\, p_{k,m+1}(n) + \frac{k-m}{k+1}\, p_{k,m}(n).
\end{aligned}$$

Substituting for $p_{k+1,m}(n)$ in (1) yields the weighted average

$$p_{k,m+1}(n) = \frac{k+1}{k+m+2}\binom{n+k}{k+1}\binom{n+m}{m+1} + \frac{m+1}{k+m+2}\,p_{k,m}(n),$$

completing the induction.

Solution 2: Without loss of generality, assume that $k \le m$. For a given value of z, there are $\binom{z+k-1}{k}$ sequences $x_1, x_2, \ldots, x_k$ that meet the condition, because the corresponding sequences $x_1+1, x_2+1, \ldots, x_k+k$ are in one-to-one correspondence with subsets of k elements in $\{2, 3, \ldots, z+k\}$. Similarly, there are $\binom{z+m-1}{m}$ possibilities for $y_1, \ldots, y_m$. Thus,

$$f(n) = \sum_{z=1}^{n} \binom{z+k-1}{k}\binom{z+m-1}{m} = \frac{1}{k!m!}\sum_{z=1}^{n}\prod_{j=0}^{k-1}(z+j)^2\prod_{j=k}^{m-1}(z+j).$$

Here and below, an empty product (for example, the second product above if $k=m$) should be interpreted as the number 1.

In the summation above, each term is a degree $k+m$ polynomial in z. Thus, $f(n)$ can be expressed as a linear combination of sums of the form $1^\ell + 2^\ell + \cdots + n^\ell$, where ℓ goes from 0 to $k+m$. For each ℓ, this sum can be expressed as a degree $\ell+1$ polynomial in n (with rational coefficients), which is a well-known fact. Thus, $f(n)$ can be expressed as a degree $k+m+1$ polynomial $p(n)$. It remains to show that the coefficients of this polynomial are nonnegative.

For all real t, we have the identity

$$k!m!(p(t+1) - p(t)) = \prod_{i=1}^{k}(t+i)^2\prod_{i=k+1}^{m}(t+i),$$

since both sides are polynomials, and the identity is true for all positive integers t. Applying the identity for $t=0$ yields $p(0)=0$. Then, applying the identity for $x=-1,-2,\ldots,-m$ yields $0=p(0)=p(-1)=\cdots=p(-m)$ (we call this equation Property I). Also, differentiating the identity and substituting $x=-1,-2,\ldots,-k$ yields $p'(0)=p'(-1)=\cdots=p'(-k)$ (we call this equation Property II).

We claim that Properties I and II, and the fact that p has degree less than $k+m+2$, uniquely determine p up to a multiplicative constant. Indeed, if polynomials p and q with degree less than $k+m+2$ both satisfy Properties I and II, then so does each linear combination $ap+bq$. Choose a and b such that $ap'(0)+bq'(0)=0$. Then $ap+bq$ has double roots at $0,-1,\ldots,-k$ and single roots at $-k-1,\ldots,-m$, for a total of $k+m+2$ roots. Since the degree of $ap+bq$ is less than its number of roots, it must be identically zero.

Next, we will construct a polynomial of degree less than $k+m+2$ with Properties I and II that is not identically zero, and conclude that p is a constant multiple of it. The approach is similar to Lagrange interpolation. For $j=0,1,\ldots,k$, let

$$q_j(t) = (t+j)\prod_{\substack{i=0\\ i\ne j}}^{k}\frac{(t+i)^2}{(-j+i)^2}\prod_{i=k+1}^{m}\frac{t+i}{-j+i}.$$

Notice that q_j has degree $k+m+1$, and that $0 = q_j(0) = q_j(-1) = \cdots = q_j(-m)$. Notice also that by the product rule,

$$q_j'(t) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{(t+i)^2}{(-j+i)^2} \prod_{i=k+1}^{m} \frac{t+i}{-j+i} + (t+j) \prod_{\substack{i=0 \\ i\neq j}}^{k} (t+i) r_j(t)$$

for some polynomial r_j. Thus, $q_j'(-i) = 0$ for $i = 0, 1, \ldots, k$ and $i \neq j$, while $q_j'(-j) = 1$. It follows that $q_0 + q_1 + \cdots + q_k$ has Properties I and II, and hence that $p = c(q_0 + q_1 + \cdots + q_k)$ for some real number c. Notice that $c > 0$ because the t^{k+m+1} coefficient of p must be positive. Each q_j is a polynomial with nonnegative coefficients; thus, so is p.

Remark: This problem was inspired by Corollary 3.4 in [Luis Ferroni, "On the Ehrhart Polynomial of Minimal Matroids", Discrete & Computational Geometry 68 (2022), 255–273, `https://doi.org/10.1007/s00454-021-00313-4`].

B6. For a real number a, let $F_a(x) = \sum_{n\geq 1} n^a e^{2n} x^{n^2}$ for $0 \leq x < 1$. Find a real number c such that

$$\lim_{x\to 1^-} F_a(x)e^{-1/(1-x)} = 0 \quad \text{for all } a < c, \text{ and}$$
$$\lim_{x\to 1^-} F_a(x)e^{-1/(1-x)} = \infty \quad \text{for all } a > c.$$

Answer: $c = -1/2$.

Solution: Recall that $\ln x \leq x - 1$ for all positive x, and thus also $\ln(1/x) \leq 1/x - 1 = (1-x)/x$. For $0 < x < 1$, it follows that

$$\frac{1}{\ln x} \geq \frac{1}{x-1} = -\frac{1}{1-x} = -\frac{x}{1-x} - 1 \geq -\frac{1}{\ln(1/x)} - 1 = \frac{1}{\ln x} - 1.$$

Thus, for $0 < x < 1$,

$$e^{1/(\ln x)} \geq e^{-1/(1-x)} \geq \frac{e^{1/(\ln x)}}{e}.$$

Thus, replacing $e^{-1/(1-x)}$ with $e^{1/(\ln x)}$ does not affect whether the limit in question is 0, or whether it is ∞.

Let $z = -1/(\ln x)$, so that $x = e^{-1/z}$. Then

$$F_a(x)e^{1/(\ln x)} = \sum_{n=1}^{\infty} n^a e^{-z+2n-\frac{n^2}{z}} = \sum_{n=1}^{\infty} n^a e^{-\frac{(n-z)^2}{z}}.$$

Notice that $z \to \infty$ as $x \to 1^-$. For $z \geq 4$, so that $\sqrt{z} \leq z/2$, consider the portion of the sum for which $z - \sqrt{z} \leq n \leq z + \sqrt{z}$, which is equivalent to $(n-z)^2/z \leq 1$. There are at least $z + \sqrt{z} - (z - \sqrt{z}) - 1 = 2\sqrt{z} - 1$ terms in this portion, and since $z/2 \leq n < 2z$, we obtain the bound $n^a \geq 2^{-|a|} z^a$. Thus,

$$F_a(x)e^{1/(\ln x)} \geq \sum_{z-\sqrt{z}\leq n\leq z+\sqrt{z}} n^a e^{-\frac{(n-z)^2}{z}} \geq (2\sqrt{z} - 1)2^{-|a|} z^a e^{-1}.$$

If $a > -1/2$, this lower bound approaches ∞ as $z \to \infty$, so $F_a(x)e^{1/(\ln x)} \to \infty$ as $x \to 1^-$.

For $a < -1/2$, write $F_a(x)e^{1/(\ln x)} = S_1(z) + S_2(z)$ where

$$S_1(z) = \sum_{1\leq n\leq z/2} n^a e^{-\frac{(n-z)^2}{z}}; \quad S_2(z) = \sum_{n>z/2} n^a e^{-\frac{(n-z)^2}{z}}.$$

Since both sums are nonnegative, it suffices to show that each has an upper bound that approaches 0 as $z \to \infty$

We bound $S_1(z)$ above by the number of terms in the sum (which is as most $z/2$) times an upper bound on each term. Since $a < 0$, we have $n^a \leq 1$, and since $(n-z)^2/z \geq z/4$ for $n \leq z/2$, we have $S_1(z) \leq (z/2)e^{-z/4}$. Thus, $S_1(z) \to 0$ as $z \to \infty$.

Since $a < 0$, in $S_2(z)$ we can bound n^a above by $(z/2)^a$. We write $S_2(z) = s_0 + s_1 + s_2 + \cdots$ where s_k includes the terms in $S_2(z)$ for which $k \leq (n-z)^2/z \leq k+1$. Then each term in

s_k is at most $(z/2)^a e^{-k}$, and since $z - \sqrt{(k+1)z} \le n \le z + \sqrt{(k+1)z}$, there are at most $z + \sqrt{(k+1)z} - (z - \sqrt{(k+1)z}) + 1 = 2\sqrt{k+1}\sqrt{z} + 1$ terms in s_k. Thus,

$$S_2(z) \le \sum_{k=0}^{\infty} (2\sqrt{k+1}\sqrt{z} + 1)(z/2)^a e^{-k}.$$

Since $\sum_{k\ge 0} \sqrt{k+1}e^{-k} < \infty$, this upper bound approaches 0 as $z \to \infty$ if $a < -1/2$. Therefore, $F(x)e^{1/(\ln x)} \to 0$ as $x \to 1^-$, completing the proof.

Remark. This problem was inspired by Proposition 3.2 in [K. Bringmann, C. Jennings-Shaffer, K. Mahlburg , "On a Tauberian theorem of Ingham and Euler–Maclaurin summation", The Ramanujan Journal 61 (2023), 55–86, `https://doi.org/10.1007/s11139-020-00377-5`]. A similar argument shows that the limit in this problem exists for $a = -1/2$, and is equal to $\sqrt{\pi/e}$. An outline of the proof, using the notation of the solution above, follows. First, by multiple applications of L'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to 1^-} \left(-\frac{1}{\ln x} - \frac{1}{1-x} \right) &= -\lim_{x\to 1^-} \frac{1 + (\ln x)/(1-x)}{\ln x} = -\lim_{x\to 1^-} \frac{1/(x(1-x)) + (\ln x)/(1-x)^2}{1/x} \\
&= -\lim_{x\to 1^-} \frac{1 - x + x\ln x}{(1-x)^2} = -\lim_{x\to 1^-} \frac{\ln x}{-2(1-x)} = -\lim_{x\to 1^-} \frac{1/x}{2} = -\frac{1}{2}.
\end{aligned}$$

Thus, the limit of the exponential of the expression above is $e^{-1/2}$.

Next, make the change of variables $u = (n-z)/\sqrt{z}$ and $n = z + \sqrt{z}u$ to write

$$F_{-1/2}(x)e^{1/(\ln x)} = \sum_{n=1}^{\infty} \left(z + \sqrt{z}u\right)^{-1/2} e^{-u^2}.$$

This is a Riemann sum, using intervals of length $1/\sqrt{z}$, for

$$\int_{(1-z)/\sqrt{z}}^{\infty} \sqrt{z}\left(z + \sqrt{z}u\right)^{-1/2} e^{-u^2}\,du.$$

Notice that for fixed u, the integrand approaches e^{-u^2} as $z \to \infty$. The remainder of the proof is to justify that the integral approaches $\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$, and that the limit of the Riemann sums is the limit of the integrals.