

The 85th William Lowell Putnam Mathematical Competition 2024

A1 Determine all positive integers n for which there exist positive integers a , b , and c satisfying

$$2a^n + 3b^n = 4c^n.$$

A2 For which real polynomials p is there a real polynomial q such that

$$p(p(x)) - x = (p(x) - x)^2 q(x)$$

for all real x ?

A3 Let S be the set of bijections

$$T: \{1, 2, 3\} \times \{1, 2, \dots, 2024\} \rightarrow \{1, 2, \dots, 6072\}$$

such that $T(1, j) < T(2, j) < T(3, j)$ for all $j \in \{1, 2, \dots, 2024\}$ and $T(i, j) < T(i, j + 1)$ for all $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \dots, 2023\}$. Do there exist a and c in $\{1, 2, 3\}$ and b and d in $\{1, 2, \dots, 2024\}$ such that the fraction of elements T in S for which $T(a, b) < T(c, d)$ is at least $1/3$ and at most $2/3$?

A4 Find all primes $p > 5$ for which there exists an integer a and an integer r satisfying $1 \leq r \leq p - 1$ with the following property: the sequence $1, a, a^2, \dots, a^{p-5}$ can be rearranged to form a sequence $b_0, b_1, b_2, \dots, b_{p-5}$ such that $b_n - b_{n-1} - r$ is divisible by p for $1 \leq n \leq p - 5$.

A5 Consider a circle Ω with radius 9 and center at the origin $(0, 0)$, and a disk Δ with radius 1 and center at $(r, 0)$, where $0 \leq r \leq 8$. Two points P and Q are chosen independently and uniformly at random on Ω . Which value(s) of r minimize the probability that the chord \overline{PQ} intersects Δ ?

A6 Let c_0, c_1, c_2, \dots be the sequence defined so that

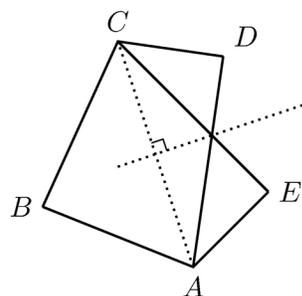
$$\frac{1 - 3x - \sqrt{1 - 14x + 9x^2}}{4} = \sum_{k=0}^{\infty} c_k x^k$$

for sufficiently small x . For a positive integer n , let A be the n -by- n matrix with i, j -entry c_{i+j-1} for i and j in $\{1, \dots, n\}$. Find the determinant of A .

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B1 Let n and k be positive integers. The square in the i th row and j th column of an n -by- n grid contains the number $i + j - k$. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly $1, 2, \dots, n$?

B2 Two convex quadrilaterals are called *partners* if they have three vertices in common and they can be labeled $ABCD$ and $ABCE$ so that E is the reflection of D across the perpendicular bisector of the diagonal \overline{AC} . Is there an infinite sequence of convex quadrilaterals such that each quadrilateral is a partner of its successor and no two elements of the sequence are congruent?



B3 Let r_n be the n th smallest positive solution to $\tan x = x$, where the argument of tangent is in radians. Prove that

$$0 < r_{n+1} - r_n - \pi < \frac{1}{(n^2 + n)\pi}$$

for $n \geq 1$.

B4 Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from the set $\{1, \dots, n\}$, and let

$$a_{n,k+1} = \begin{cases} a_{n,k} + 1, & \text{if } m_{n,k} > a_{n,k}; \\ a_{n,k}, & \text{if } m_{n,k} = a_{n,k}; \\ a_{n,k} - 1, & \text{if } m_{n,k} < a_{n,k}. \end{cases}$$

Let $E(n)$ be the expected value of $a_{n,n}$. Determine $\lim_{n \rightarrow \infty} E(n)/n$.

B5 Let k and m be positive integers. For a positive integer n , let $f(n)$ be the number of integer sequences $x_1, \dots, x_k, y_1, \dots, y_m, z$ satisfying $1 \leq x_1 \leq \dots \leq x_k \leq z \leq n$ and $1 \leq y_1 \leq \dots \leq y_m \leq z \leq n$. Show that $f(n)$ can be expressed as a polynomial in n with nonnegative coefficients.

B6 For a real number a , let $F_a(x) = \sum_{n \geq 1} n^a e^{2n} x^{n^2}$ for $0 \leq x < 1$. Find a real number c such that

$$\begin{aligned} \lim_{x \rightarrow 1^-} F_a(x) e^{-1/(1-x)} &= 0 && \text{for all } a < c, \text{ and} \\ \lim_{x \rightarrow 1^-} F_a(x) e^{-1/(1-x)} &= \infty && \text{for all } a > c. \end{aligned}$$