MATHEMATICAL ASSOCIATION OF AMERICA

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

A COMPENDIUM OF CUPM RECOMMENDATIONS

VOLUME II

A COMPENDIUM OF CUPM RECOMMENDATIONS

STUDIES

DISCUSSIONS

and

RECOMMENDATIONS

by the

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

of the

MATHEMATICAL ASSOCIATION OF AMERICA

published by
The Mathematical Association of America

PREFACE

The Committee on the Undergraduate Program in Mathematics (CUPM) was established as a standing committee of the Mathematical Association of America (MAA) in 1959.* With financial assistance from the National Science Foundation, CUPM in 1960 began to engage in several projects and activities related to improvement in the undergraduate curriculum. These projects often involved the publication of reports, which were widely disseminated throughout the mathematical community and were available from the CUPM Central Office upon request. Since a change in the funding policy of the United States government makes the continuing production and free distribution of such reports extremely unlikely, the MAA has decided to publish in permanent form the most recent versions of many of the CUPM recommendations so that these reports may continue to be readily available to the mathematical community and may conveniently be kept on the reference shelves of mathematics libraries.

This COMPENDIUM is published in two volumes, each of which has been divided into sections according to the category of reports contained therein. These CUPM documents were produced by the cooperative efforts of literally several hundred mathematicians in the United States and Canada. The reports are reprinted here in essentially their original form; there are a few editorial comments which serve to update or cross-reference some of the materials.

The editorial work for the COMPENDIUM was started by William E. Mastrocola during his term as Director of CUPM and completed after his return to Colgate University. He was assisted in the early stages by Andrew Sterrett and Paul Knopp, Executive Directors of CUPM during 1972 and 1973. Preparation of the final manuscript for the printer was the joint work of William E. Mastrocola and Katherine B. Magann. The considerable efforts of these individuals is deserving of special recognition.

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* A detailed history of CUPM can be found in an article by W. L. Duren which appeared in the American Mathematical Monthly, vol. 74, pp. 23-37.

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STATISTICS

In 1968 CUPM appointed a Panel on Statistics for the purpose of providing guidance to departments of mathematics at smaller colleges and universities on instruction in statistics. Two concerns of general interest were identified for study by the Panel: a program to prepare students for graduate study in statistics and a basic service course in statistics for students who have not studied calculus. The Panel pointed out that these two topics represent curricular extremes for statistics instruction in most undergraduate programs and that many students' program of study will lie between these extremes.

The Panel's first report, Preparation for Graduate Work in Statistics, was issued in 1971. This document describes the type of training and experiences which undergraduates contemplating graduate study in statistics ought to have. It outlines a basic one-year course in probability and statistics, indicates those mathematics courses which are valuable for pregraduate preparation in statistics, and comments on computer requirements and experience with data.

The Panel's second project involved a study of the introductory, noncalculus statistics courses which are offered by practically every college and taken by students in a wide variety of fields. Prompted by the fact that many of the existing courses are unsatisfactory for a variety of reasons, the Panel developed a set of objectives for such a course and made concrete suggestions for realizing the objectives. A detailed list of topics for a conventional course in introductory statistics, as well as some suggested alternate approaches, appears in the 1972 publication Introductory Statistics Without Calculus.

PREPARATION FOR GRADUATE WORK IN STATISTICS

A Report of

The Panel on Statistics

May 1971

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> ...Both Computer Science and Statistics have dual sources of identity and intellectual force, only one of which is mathematical; hence they are more accurately described as partly mathematical sciences. ...
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> Modern statistics could not operate without mathematics, especially without the theory of probability. Equally, it could not exist without the challenge of inference in the face of uncertainty and the stimulus of the quantitative aspects of the scientific method ... statistics is both a mathematical science and something else.
>
> ...It is true that undergraduate preparation for majors in mathematics, with its traditional emphasis on core mathematics, provides an excellent foundation of knowledge for potential graduate students in statistics. It does not, however, provide nearly enough students with either motivation to study statistics or an understanding of the extra-mathematical aspects of statistics.*

This report consists of three main sections: (1) Introductory comments on the field of statistics and its study at the graduate level; (2) A recommended undergraduate program for prospective graduate students in statistics; and (3) Implications of the recommendations for departments of mathematics and their students. Our recommendations are addressed to departments of mathematics of four-year colleges and smaller universities which have no specialized departmental programs in statistics. At these institutions, the department is unlikely to have an experienced or trained statistician although it is often called upon to offer statistics courses as a service for students majoring in other fields.

I. STATISTICS AND GRADUATE STUDY

In our modern technological society there is a continually increasing demand and necessity for quantitative information. This requires planning and skill in the collection, analysis, and interpretation of data. Statisticians deal with inherent variation in

* The Mathematical Sciences: A Report, by the Committee on Support of Research in the Mathematical Sciences of the National Research Council for the Committee on Science and Public Policy of the National Academy of Sciences, Washington, D. C. Publication 1681 (1968), pp. 84 and 157.

nature and measurement and are concerned with the planning and design of experiments and surveys, with methods of data reduction, and with inductive decision processes.

Statistics has made significant contributions to many fields, most notably to the experimental sciences, agriculture, medicine, and engineering. It has also had an important role in the development of other fields such as economics, demography, and sociology. Statistics and quantitative methods are assuming major roles in business and in the behavioral sciences, roles destined to receive more and more emphasis. The widespread use of computers in these fields increases the need for statisticians at all educational levels.

Although a demand exists in government and business for persons with only undergraduate training in statistics (especially in conjunction with training in computer technology and a subject-matter field), the attainment of competence in statistics at a professional level necessarily requires graduate study. Our recommendations deal with minimum undergraduate preparation for this study. It is generally agreed that broad knowledge of mathematics is required to proceed with graduate work; currently about two thirds of all graduate students in statistics were undergraduate mathematics majors.* Advanced study in some field (physical, biological, or social science) in which data play an important role is also very helpful.

A significant part of graduate study in statistics is the attainment of a sound understanding of advanced mathematics and the theory of statistics and probability, since these are necessary for research in statistics and also for competent consultation on applications of statistics. The consultant seldom encounters textbook applications and is regularly required to modify and adapt procedures to practical problems.

Graduate study in statistics, at both the M.A. and Ph.D. degree levels, is available in a substantial number of universities. According to a survey published in The American Statistician (October, 1968), there are in the U. S. and Canada approximately 85 departments which offer undergraduate degrees in statistics. There are approximately 160 departments at 110 universities which offer programs leading to graduate degrees in statistics or subject-matter fields having a statistics option. These graduate programs of study lead to frontiers in both theory and application. The M.A. degree provides suitable qualifications for many positions in industry and government, often as a consultant or team member in research and development, as well as for positions in teaching at junior colleges.

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* Aspects of Graduate Training in the Mathematical Sciences, Vol. II, page 62, 1969. A report of the Survey Committee of the Conference Board of the Mathematical Sciences, 2100 Pennsylvania Avenue, N.W., Suite 834, Washington, D. C. 20037.

It is also useful for persons who will pursue advanced degrees in fields such as psychology and education in which statistical methodology plays a significant role. The Ph.D. degree provides additional preparation for research and teaching careers in universities as well as in government and industrial organizations.

II. THE RECOMMENDED PROGRAM OF STUDY

Our recommendations for undergraduate courses are subdivided into four areas: (A) probability and statistics, (B) mathematics, (C) computing, and (D) other requirements. In this section we describe recommended courses in each of these areas.

A. Probability and Statistics Requirements

We recommend that students take at least a one-year course in probability and statistics and gain experience with real applications of statistical analysis.

1. Probability and Statistics Course (6 semester hours)

A description of this course (Mathematics 7) can be found in Commentary on A General Curriculum in Mathematics for Colleges, page 79.

2. Experience with Data

We believe that students should have experience with realistic examples in the use of the statistical concepts and theory of the key course. They should work with real data, consider the objectives of the scientific investigation that gave rise to these data, study statistical methods for answering relevant questions, and consider the interpretation of the results of statistical analyses. These goals are not easy to achieve, but various approaches are discussed below.

An appropriate course is not likely to be already available in a department of mathematics. The typical one-semester precalculus elementary statistics course, usually serving students from departments other than mathematics, does not fulfill the goals that we recommend.

Courses in research methodology or applied statistics in other subject matter areas may meet, at least in part, some of the objectives of exposing students to realistic statistical problems. For example, courses in biological statistics, research methods in behavioral science, economic statistics or econometrics, survey methods, sociological statistics, etc., can meet our intended objective if they are offered by practicing scientists who are familiar with the way data are generated, the complexities they usually exhibit, and the methods that help in their analysis. Although such a course will provide coverage of some topics in elementary statistical methods, it is important that the course be more than a catalog of methods. The student should see how some one field of science generates experimental data and copes with uncertainty and variability. Insight gained from relatively few examples or ideas can be considerably more valuable for the student than information obtained by covering a large number of separate topics. Modern computers could be useful in such a course.

If a faculty member with training and experience in applied statistics is available to the mathematics department, he can devise a data analysis course. This course could be based on a number of data sets of interest to the student, and it could use books on statistical methods as reference material for the appropriate statistical techniques. Basic texts on statistical methods which contain a variety of examples of applications of statistical methods include:

Dixon, Wilfrid J. and Massey, F. J., Jr. Introduction to Statistical Analysis, 3rd ed. New York, McGraw-Hill Book Company, 1969.

Fisher, R. A. Statistical Methods for Research Workers, 13th ed. New York, Hafner Publishing Company, 1958.

Guttman, Irwin and Wilks, Samuel S. Introductory Engineering Statistics. New York, John Wiley and Sons, Inc., 1965.

Li, C. C. Introduction to Experimental Statistics. New York, McGraw-Hill Book Company, 1964.

Natrella, Mary G. Experimental Statistics, Handbook 91. U. S. Department of Commerce, National Bureau of Standards, 1966.

Snedecor, George W. and Cochran, W. G. Statistical Methods. Ames, Iowa, Iowa State University Press, 1967.

Walker, Helen M. and Lev, Joseph. Statistical Inference. New York, Holt, Rinehart and Winston, Inc., 1953.

Wallis, Wilson A. and Roberts, Harry. Statistics: A New Approach. New York, Free Press, 1956.

Wine, Russell L. Statistics for Scientists and Engineers. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964.

Winer, B. J. Statistical Principles in Experimental Design. New York, McGraw-Hill Book Company, 1962.

Examples of data sources and/or statistical critiques of major scientific investigations are:

Tufte, Edward R. The Quantitative Analysis of Social Problems. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970.

Kinsey, A. C.; Pomeroy, W. B.; Martin, C. E. Sexual Behavior in the Human Male. Philadelphia, Pennsylvania, W. B. Saunders Company, 1948.

Report on Lung Cancer, Smoking, and Health. Public Health Bulletin 1103. Superintendent of Documents, U. S. Government Printing Office.

Cochran, William G.; Mosteller, Frederick; Tukey, John W. Statistical Problems of the Kinsey Report. Washington, D. C. American Statistical Association, 1954.

"The Cochran-Mosteller-Tukey Report on the Kinsey Study: A Symposium." Journal of the American Statistical Association, 50 (1955), p. 811.

Cutler, S. J. "A Review of the Statistical Evidence on the Association Between Smoking and Lung Cancer." Journal of the American Statistical Association, 50 (1955), pp. 267-83.

Brownlee, K. A. "Statistics of the 1954 Polio Vaccine Trials." Journal of the American Statistical Association, 50 (1955), pp. 1005-1014. (An invited address on the article "Evaluation of 1954 Field Trial of Poliomyelitis Vaccine: Summary Report." Poliomyelitis Vaccine Evaluation Center, University of Michigan, April 12, 1955.)

The following books use experimental and survey data to illustrate statistical concepts and techniques:

Bliss, Chester I. Statistics in Biology. New York, McGraw-Hill Book Company, 1967.

Cox, David R. Planning of Experiments. New York, John Wiley and Sons, Inc., 1958.

Davies, Owen L. Design and Analysis of Industrial Experiments, 2nd rev. ed. New York, Hafner Publishing Company, 1956.

Ferber, Robert and Verdoorn, P. J. Research Methods in Economics and Business. New York, The Macmillan Company, 1962.

Stephan, Frederick F. and McCarthy, Phillip J. Sampling Opinion--An Analysis of Survey Procedures. New York, John Wiley and Sons, Inc., 1958.

Youden, William J. Statistical Methods for Chemists. New York, John Wiley and Sons, Inc., 1951.

When an experienced applied statistician is not available, the mathematics department may be able to develop a seminar with the assistance of faculty members in other disciplines. Typical experimental areas in a discipline may be discussed and illustrated with data from on-going faculty research or from student term projects.

Another opportunity to provide students with exposure to problems in data analysis is through laboratories associated with the key course in statistics. In such a laboratory, students may be asked to analyze data sets, followed by class discussion. Alternatively, term project assignments incorporating study, review, and critique of statistical studies with major data sets could be utilized. Still another opportunity would be through projects developed for independent study following the key course.

The important element in this recommendation is that the student obtain understanding of the role played by statistical concepts in scientific investigations and be motivated to continue the study of statistics.

3. Additional Courses

We recommend that additional courses in probability and statistics be offered to follow the key course whenever possible. Such courses could explore in detail a few topics which were omitted or treated lightly in the key course, e.g., analysis of variance, experimental design, regression, nonparametric methods, sampling, sequential analysis, multivariate methods, or factor analysis.

Other subjects which could serve as useful enrichment are stochastic processes, game theory, linear programming, and operations research. Courses in these subjects would be useful to students with specialized interests and also would help widen the knowledge and capabilities of the prospective graduate student in statistics.

B. Mathematics Requirements

We recommend that students take at least a complete 9-12 semester hour sequence in calculus, a course in linear algebra, and a course in selected and advanced topics in analysis.

1. Beginning Analysis (9-12 semester hours)

This sequence includes differential and integral calculus of one and several variables and some differential equations. It is desirable that prerequisites for calculus, including a study of the elementary functions and analytic geometry, be completed in secondary school.

For detailed course descriptions, we refer the reader to Commentary on A General Curriculum in Mathematics for Colleges, page 44. This beginning analysis sequence is adequately described by GCMC's courses numbered 1, 2, and 4. We think it important to note that elementary probability theory is a rich source of illustrative problem material for students in this analysis sequence.

2. Elementary Linear Algebra (3 semester hours)

This course, which may be taken by students before they complete the beginning analysis sequence, includes the following topics: solution of systems of linear equations (including computational techniques), linear transformations, matrix algebra, vector spaces, quadratic forms, and characteristic roots. An outline for such a course (Mathematics 3) can be found on page 55.

3. Selected Topics in Analysis (3 semester hours)

The GCMC course Mathematics 5 is not particularly appropriate for statistics students, and it is recommended that a course including the special topics listed below be offered for these students in place of Mathematics 5.

Such a course should give the student additional analytic skills more advanced than those acquired in the beginning analysis sequence. Topics to be included are multiple integration in n dimensions, Jacobians and change of variables in multiple integrals, improper integrals, special functions (beta, gamma), Stirling's formula, Lagrange multipliers, generating functions and Laplace transforms, difference equations, additional work on ordinary differential equations, and an introduction to partial differential equations.

It is possible that the suggested topics can be studied in a unified course devoted to optimization problems. Such a course, at a level which presupposes only the beginning analysis and linear algebra courses and which may be taken concurrently with a course in probability theory, would be a valuable addition to the undergraduate curriculum, not only for students preparing for graduate work in statistics but also for students in economics, business administration, operations research, engineering, etc. Experimentation by teachers in the preparation of written materials and textbooks for such a course would be useful and is worthy of encouragement.

4. Additional Courses

The following courses from Commentary on A General Curriculum in Mathematics for Colleges are not required but are desirable as choices for students who wish to have more than minimal preparation. A strong course in real variables is especially recommended for students interested in working for the Ph.D. in statistics.

Mathematics 6M.	Introductory Modern Algebra.
Mathematics 8.	Numerical Analysis.
Mathematics 10.	Applied Mathematics. [See also the CUPM report Applied Mathematics in the Undergraduate Curriculum.]
Mathematics 11-12.	Introductory Real Variable Theory.
Mathematics 13.	Complex Analysis.

In general, the stronger a student's background in undergraduate mathematics, the better prepared he is for graduate work in statistics. If faculty members with special interests and competence are available, additional courses or seminars in the areas mentioned on page 466 would be valuable additions to the curriculum for a student interested in advanced work in statistics.

C. Computing Requirements

We recommend that students be familiar with a modern highspeed computer and what it can do, and that they have some actual experience analyzing data which are sufficiently obscure to require the use of a computer.

It is quite clear that the computer is becoming increasingly important to almost every academic discipline; in addition, it is an integral part of business, government, and even everyday affairs. These are strong enough reasons for every student who receives a baccalaureate degree to be acquainted with the computer and its potential. However, it is even more important that someone who intends to be a professional statistician know and understand the modern computer. Anyone who trains in statistics, who will handle data or work with and advise people who handle data must have a certain minimum competence in the use of a computer. While it is true that increased competence will be developed as the need arises and that some of the more sophisticated applications can be learned while doing graduate work, it is recommended that a student who comes into a graduate program in statistics begin his training in this area at the undergraduate level.

The most desirable way to be sure that sufficient competence is acquired would be through taking a regular course in computing such as Introduction to Computing, the course C1 described in the CUPM report Recommendations for an Undergraduate Program in Computational Mathematics [page 563]. It is, of course, possible to obtain an acceptable level of competence by attending the lectures or informal courses given in many computing centers, supplemented by sufficient additional computing experience. A student should understand and be able to use one of the major programming languages. He should learn enough of the nomenclature and characteristics of a computer to be able to stay abreast of the developments that will surely occur during his working lifetime. He should be made aware of and have some experience with library programs that are available. And, perhaps most important for our special purposes, students should have experience with actual data, with the numerical analysis and statistical problems they generate, and with the use of the computer for simulation. Some of this experience could be obtained by the techniques recommended in Section A2 above. The course CM1 of the above-mentioned report on computational mathematics [page 551] is also based on simulation techniques and would therefore be appropriate for students of statistics.

Finally, we suggest that in the next decade the availability of computers will change many subject-matter areas. If the statistician is to be an effective consultant in these areas, he must be aware of the way in which the computer is shaping the disciplines with which he will be associated.

D. Other Requirements

Statistics deals with the drawing of inferences from data and, in its applications, involves the statistician in working jointly with subject-matter specialists in the framing of relevant questions, developing appropriate methodology for drawing inferences, and assisting in the analysis of final results. Whether a person will principally do research in statistical methodology, teach statistical applications, or consult on statistical applications, knowledge of one or more areas of application and an understanding of the nature of statistical problems in them is highly desirable.

Undergraduate preparation for work in statistics should therefore include study of a variety of areas of application, with one studied in some depth. This will insure that the student, upon graduation, will have an acquaintance with fundamental concepts in a variety of areas and technical competence to a moderate extent in at least one of the physical, life, or social sciences. Courses selected for study of a field in depth may include a statistical or research methodology course offered by that field, in which the student will develop an understanding of data collection and data handling problems. The student's adviser may be particularly helpful in identifying such a course. Students who wish to undertake graduate study in specialized areas of statistics, such as econometrics or biostatistics, will find it desirable to take at least some advanced work in these areas as undergraduates.

III. IMPLICATIONS OF THE RECOMMENDATIONS

We are aware that at the present time most mathematics departments have few advanced courses in statistics available and few, if any, trained statisticians on their staffs. Because of the late date when a student may discover the field of statistics, he may not have time to elect many of our recommended courses even if these are available. Finally, some of the courses he takes will be designed not only for prospective graduate students but for students with other majors and interests as well.

Were these and other limitations not present, we would expect our recommended program and the mathematics department to serve the needs of undergraduate students by not only imparting to them knowledge of the field of statistics but also by enabling them to discover their abilities and interests and, if appropriate, by arousing their interest in graduate study in statistics or a related field. But limitations do exist, and it is not to be expected that all of our recommendations can be implemented quickly or that all the needs of students will be met by the programs in mathematics departments. It is difficult to meet all student needs under the best of circumstances. Realistically it is to be expected that severe limitations in time and facilities will prevent the student from obtaining a well-rounded understanding of the subject at the undergraduate level, but at least he can be exposed to some of the basic ideas in statistics. Perhaps most important, a program designed to achieve limited goals with relatively few courses, even if it falls short of the full program we recommend, can arouse the interest of students in statistics and related fields.

In the light of the recommendations in this and other CUPM reports (especially GCMC) dealing with courses in probability, statistics, and related areas, <u>it is highly desirable, and we recommend that each department of mathematics review its course offerings so as to establish appropriate courses in probability and statistics and arrange that these courses be staffed by a person competent in these fields</u>. Ph.D.s in statistics are increasingly available to four-year colleges and smaller universities. With the establishment of courses in probability and statistics taught by a person competent in these fields, the mathematics department can serve the needs of prospective graduate students in statistics by (1) arousing interest in and demonstrating the nature of the field of statistics; (2) giving students an acquaintance with statistics, its theory, its applications, its traditions, even some of its open problems, and its relation to other fields such as probability, pure and applied mathematics, computer science, and operations research; (3) counseling students as to courses, curricular choices, and graduate and career opportunities in probability, statistics, pure and applied mathematics, computer science, and operations research.

In creating this report, the Panel on Statistics confined its attention to recommending undergraduate programs for students who intend to do graduate work in statistics. In the development of appropriate recommendations, however, it became apparent that a broad program of study in the mathematical sciences was emerging, a program suitable, we believe, not only for graduate study of statistics but also for graduate study in the quantitative aspects of the social sciences and business and in newer areas such as operations research and computer science. The recommendations developed, therefore, can form the basis for an innovative degree program in the mathematical sciences different from the traditional programs in pure or applied mathematics.

The recommended program accomplishes important secondary objectives. These include: (1) A decision by students on the nature of their future graduate study can be made at a later point in the undergraduate program. (2) Mathematically gifted students are exposed to a wider range of potential careers than is presently the case. (3) The possibility is created for a substantial emphasis in the mathematical sciences to be used as part of an undergraduate major, not only in mathematics and statistics but also in other departments and in interdisciplinary programs.

The needs and opportunities for an innovative undergraduate program in the mathematical sciences are great. It can provide options in computer science, applied mathematics, econometrics, operations research, statistics, probability, and pure mathematics--all built around a solid core of training in mathematics. Early exposure to the concepts and possibilities of a variety of these options can lead to better choices of areas of concentration later on. A curriculum within this framework, combining mathematics, statistics, computing, and at least one field of application, has great potential for continued study in various graduate programs as well as value as a terminal degree program. The student may proceed from such an undergraduate program to advanced study in statistics, operations research, econometrics, psychometrics, demography, or computer science. He also will have excellent qualifications for advanced work in sociology, political science, business, urban planning, or education. If the undergraduate program in mathematical science is a terminal one, the student will have employment opportunities in computing, business, industry, and government, with qualifications to meet many social needs.

The minimum preparatory program outlined in Section II can be supplemented in a variety of ways with additional work leading to undergraduate majors in (1) mathematics, (2) statistics, (3) computational mathematics [see Recommendations for an Undergraduate Program in Computational Mathematics, page 528], (4) other fields (e.g., psychology, political science, economics, sociology, engineering, linguistics, business administration--especially management sciences, biological and physical sciences).

INTRODUCTORY STATISTICS WITHOUT CALCULUS

A Report of

The Panel on Statistics

June 1972

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PREFACE

Colleges and universities offer a great variety of courses in introductory statistics with no calculus prerequisite. Courses of this kind are frequently offered by several departments within a single institution, and therefore this report should be of interest to instructors in many departments, not merely to those in mathematics and statistics.

Section I contains a discussion of the background of the report and some review of past and current approaches to the general introductory course. Section II contains recommended objectives for the course and a discussion of their implications. A conventional course is developed in detail in Section III, followed by recommendations for alternative approaches in Section IV. Sections V through VII contain suggestions for improving the effectiveness of the course, useful in all of the various approaches to it. Selected bibliographies are included in each section, and a list of additional resource materials appears at the end of the report.

The Panel is indebted to many colleagues for their participation in a fact-finding conference and for their thoughtful comments on a draft version of this report.

I. INTRODUCTION

This report is concerned with a general introductory statistics course without a calculus prerequisite, which is typically a one-semester or one-quarter course offered at the sophomore or junior level in college. For many students this is a terminal course, although some students may elect additional courses in statistics or in research methods. In four-year colleges and smaller universities it is often taught by the mathematics department as a service to other departments.

An introductory statistics course without a calculus prerequisite is often required of students majoring in many different fields, such as business administration, psychology, sociology, forestry, and industrial engineering. In addition, this course serves as an elective subject for other students. An understanding of statistical concepts is important for students in any subject where data play an important role. Knowledge of basic concepts also permits students to use data more effectively in making everyday decisions as citizens and consumers, and it might stimulate them to learn more about statistics in order to obtain the competence needed for research and analysis in their major fields of interest. Some students' interest might even be aroused sufficiently by this introductory course to encourage them to prepare for a program in statistics at the graduate level.

Concern for a noncalculus-based introductory statistics course has been frequently expressed. Many studies have been conducted to consider how to make this course a worthwhile, challenging intellectual enterprise that provides students with some understanding of basic statistical concepts. (See, for instance, "Interim Report of the Royal Statistical Society Committee on the Teaching of Statistics in Schools." Journal of the Royal Statistical Society, Series A, 131 (1968), pp. 478-497.)

One point of interest has been the proliferation of introductory statistics courses that has taken place on most large campuses. These courses may differ not only in content but also in emphasis and method of approach. Some of the factors responsible for this are the desire by different disciplines to have courses tailored to their needs, the varying backgrounds of persons teaching statistics, and the different objectives for the basic statistics course.

Proliferation in itself is not necessarily undesirable and indeed may be desirable if introductory courses directed to specific disciplines serve to motivate the basic concepts of statistics more effectively or if substantially different objectives are being served. However, no widespread satisfaction has been expressed by professional statisticians concerning any of the introductory statistics courses that have been developed. These courses have been criticized as follows:

1) Their emphasis is mathematical or probabilistic without providing sufficient insights into statistical concepts.

2) They provide little insight into the variety and usefulness of the applications of statistics.

3) They are too technique-oriented, overemphasizing computation and underemphasizing the fundamental ideas underlying statistical reasoning.

4) They give insufficient attention to drawing statistical inferences from real data.

5) They fail to provide intellectual stimulation.

6) They fail to allow for the "symbol shock" suffered by some students.

7) They provide inadequate motivation for an interest in stochastic (random) models and fail to differentiate them from the deterministic models with which students are more familiar.

The discussion and recommendations that follow are designed to counter some of these criticisms through suggestions on course objectives, alternative approaches, curricular content, increased motivation, computer use, data sets, and student-generated data.

II. RECOMMENDED OBJECTIVES AND THEIR IMPLICATIONS

1. Recommended Objectives of Introductory Statistics Courses

When one considers the variety and extent of the demands for an "ideal" first course in statistics, one recognizes the impossibility of having any one course come even close to the ideal. No one course can (1) serve as an appreciation course so that students can understand the underlying ideas of statistical methodology and statistical inference, (2) discipline students to think quantitatively and to appraise data critically, (3) give students the facility to analyze data for everyday problems, (4) train students to understand probabilistic models and their uses in a variety of situations, and (5) enable students to master the basic techniques of statistical methodology and to use these techniques flexibly in their own applications.

Hence, it is not possible for the Panel to prescribe an ideal introductory statistics course. The special needs of the departments which require this course, the abilities and interests of the instructor, and the different characteristics of the students will influence the nature of the course. However, the following guidelines are recommended by the Panel:

a. The introductory statistics course must have limited objectives. Otherwise, it is likely that none of the objectives will be met adequately.

b. The primary objective of the introductory statistics course should be to introduce students to variability and uncertainty and to some common concepts of statistics; that is, to methods such as point and interval estimation and hypothesis testing for drawing inferences and making decisions from observed data.

c. A secondary objective of the introductory statistics course should be to teach the student some common statistical formulas and terms and some of the widely used statistical techniques; e.g., the t-test.

The primary objectives may be met in many ways. Much of the report considers the conventional introductory course but with a view to highlighting desired emphases and utilizing a variety of approaches to enhance interest and motivation. Utilization of computers, data sets, experiments, and demonstrations for greater clarity and motivation are discussed. The report also contains suggestions for implementing the primary objective through less conventional approaches, e.g., nonparametric, decision-theoretic, and problem-oriented approaches. The intent of much of the report is to suggest possible ideas for syllabus and presentation; the instructor must seek that combination suited to his needs and those of his students.

The Panel recommends the primary objective of the introductory statistics course, stated above, because an understanding of the basic statistical concepts is essential to an intelligent and flexible use of statistical techniques. The use of techniques without understanding of concepts can be dangerous. If the introductory statistics course is largely devoted to concepts, students can obtain additional knowledge of statistical techniques in two basic ways. They can take a second-level statistics course which focuses on a variety of statistical techniques or a course in a subject in which specialized statistical techniques are introduced in a context in which they are used.

2. Implications of the Recommended Objectives

A number of implications follow from the declared objectives.

a. Since the main objective of the course is understanding of basic statistical concepts, it follows that proofs and extensive manipulations of formulas should be employed sparingly. While statistics utilizes these, its major focus is on inferences from data.

b. The course should not dwell on computational techniques. Rather, the amount of computation and whether it is done on a high-speed computer, a desk calculator, or by hand should be determined by the extent to which it helps the students to understand the principles involved.

c. While probabilistic concepts are essential for an understanding of statistical inference, probability theory should not constitute a dominant portion of the course.

d. In order to illustrate the application of statistical methodology in making inferences, the course must be data-oriented and must incorporate analysis of real-world data.

Three critical dimensions of the introductory statistics course are inferential philosophy, mathematics, and data analysis. Among these components we believe that the emphasis should be strongly on inferential concepts and data analysis, and less on mathematical elements.

3. Recommended Topics to be Covered

As we have stated earlier, no single introductory statistics course will be suitable for all situations. Within the framework of the objectives for the introductory statistics course just recommended, a wide variety of courses can be designed. In Sections III and IV we consider several different courses which could meet the stated objectives: (1) a more or less standard statistics course, (2) a decision-theory oriented statistics course, (3) a statistics course embodying the nonparametric approach, and (4) a statistics course utilizing a problem-oriented approach.

Despite the variety of possible approaches, most courses will include the elements of probability, the binomial and normal distributions, the distinction between sample and population, descriptive statistics (such as mean, median, variance, standard deviation, and frequency distribution), and statistical inference. The study of inference will treat hypothesis testing, point estimation, and confidence interval estimation and will include the use of the t-statistic.

We present two lists: one includes important topics from which a limited selection should be made since no single course can reasonably be expected to cover a large fraction of these topics; second is a list of topics which should be avoided unless they are used as fundamental pedagogical tools or are logically essential (such as Bayes' theorem for a Bayesian approach).

a. Important topics from which a selection should be made

1. Probability: sample space, mutually exclusive events, independent events, conditional probability, random variable (expected value, variance, standard deviation)

2. Samples: frequency distribution, histogram, ogive, percentiles, mean, median, variance, standard deviation, mode, range

3. Distributions: normal, binomial, Poisson, exponential, rectangular, geometric

4. Sampling theory: Law of Large Numbers, Central Limit Theorem

5. Estimation: point estimates, confidence intervals

6. Hypothesis testing: alternative and null hypotheses, power function; errors of types I and II, significance levels, one-tail and two-tail tests

7. One-sample tests: for the mean of a normal distribution (t-test), for the proportion of a binomial distribution

8. Two-sample tests: t-test, Mann-Whitney test, sign test

9. Chi-square and contingency tables

10. Regression and correlation

11. Analysis of variance

12. Decision theory: minimax and Bayes' strategies, admissibility

b. Topics to be avoided (except when they serve as useful teaching devices)

1. Combinatorics

2. Bayes' theorem

3. Partial regression

4. Sequential analysis

5. Maximum likelihood and likelihood ratio

6. Compendia of variations of a given procedure like the t-test

7. Inference on variances using chi-square distributions (because they involve nonrobust procedures)

III. A CONVENTIONAL COURSE IN INTRODUCTORY STATISTICS

An approach to the teaching of elementary statistics whose popularity is reflected in the most widely used textbooks may be called the conventional approach. It is characterized in part by a logical development in which basic tools are developed slowly and in some detail before serious statistical problems are attacked.

Because of the popularity of the conventional approach and the existence of a wide variety of texts oriented toward this approach, it is the one most likely to be used in two-year and four-year colleges and in smaller universities in the near future. We therefore believe it is wise to focus primarily on this kind of course, particularly since significant modifications and enrichments can make it a worthwhile intellectual experience.

This traditional approach has potential defects, as noted earlier. Typically, the major ideas of statistical inference are introduced too late and in haste, and they are often illustrated by examples which are not compelling. Frequently, courses of this kind suffer from the inclusion of irrelevant concepts and excessive mathematical derivations. Finally, the attempt to break up the subject into small digestible bits may cause the student to miss the unifying concepts of statistical inference. The course outline and comments which follow attempt to provide guidance on topics and suggestions for avoiding these defects.

1. Course Outline

A recommended conventional course outline, which permits early treatment of the ideas of statistical inference and which stresses concepts rather than a proliferation of statistical techniques is given in this section. The suggested pace has been indicated by assigning a number of hours to each group of topics. A standard semester contains 42 to 48 class meetings, and we arbitrarily allowed 36 hours of class time for the presentation of new material; the slack time that we have left provides for tests, review, etc. More detailed suggestions of what to mention only briefly and what to emphasize are included in the Comments, Section 2.

CONVENTIONAL COURSE OUTLINE

	Topics	Lectures
0.	Introduction	1
1.	Statistical Description Frequency distributions, cumulative frequency distributions; measures of location and variation	3
2.	Probability Concept; sample space; addition theorem; marginal probability; conditional probability; multiplication theorem; independence	3
3.	Random Variables and Probability Distributions Concepts; simple discrete univariate probability distributions; expectation and variance of discrete random variables; functions of discrete random variables; mean and variance of functions of discrete random variables	2
4.	Special Probability Distributions Binomial probability distribution; continuous probability distributions; normal probability distribution	2

5. Sampling Distributions — 3

Random sampling; mean and variance of sum of independent random variables; sampling distribution of mean; Central Limit Theorem; sampling distribution of proportion

6. Estimation of Population Proportion — 4

Point estimation of population proportion; confidence interval for population proportion based on large samples

7. Tests Concerning Population Proportion — 5

Formulating hypotheses; statistical decision rules; types of errors; power of a test; construction of one-sided and two-sided tests (small and large samples)

8. Inferences Concerning Population Mean — 4

Point estimation of population mean; properties of estimators: unbiasedness, consistency, efficiency; confidence intervals and one- and two-sided tests of hypotheses for the mean of a population whose variance is also unknown, based on small and large samples

9. Additional Topics — 4-9

Selection from the following:

(a) Inferences concerning differences of two population means and proportions
(b) Inferences concerning population variance
(c) Chi-square and contingency tables
(d) Regression and correlation
(e) Analysis of variance
(f) Nonparametric methods
(g) Survey sampling
(h) Quality control
(i) Bayesian methods
(j) Decision theory

The number of lectures assigned to topics 1-8 is a minimal number. Use of computers, case discussions, and student-generated data will require additional time. Since these activities will vary from course to course, we have indicated a number of lectures for each topic without considering such activities. Consequently, the number of lectures available for additional topics usually will be less than the maximum of nine. We do recommend, however, that at least four lectures be devoted to one or more of the additional topics.

2. Comments on the Course Outline

General. Since the above outline contains only a list of topics, it does not recognize four important ingredients that we believe will enhance the conventional approach.

a. Continuing motivation of the student through examples and problems that he finds interesting and important. Data sets and student-generated data are discussed in Sections V and VII.

b. Use of computers to help develop basic concepts such as the Central Limit Theorem as well as to remove the drudgery of statistical calculations. Use of computers is discussed in Section VI.

c. Emphasis on the basic ideas underlying statistical inference and a demonstration of how these recur from one application to another. This requires skillful teaching and demands that the teacher see the broad picture of statistical inference.

d. Student use of programmed learning materials for developing mastery of important but routine topics. We believe it is wasteful to use precious class time having students practice constructing frequency distributions, calculating measures of location and variation, using binomial and normal tables, etc. Class time is better used for explanation and discussion, not for routine calculations and drill. Examples of self-help programmed materials are:

[1] Elzey, Freeman F. A Programmed Introduction to Statistics. Belmont, California, Brooks/Cole Publishing Company, 1966.

[2] Gotkin, Lassar G. and Goldstein, Leo S. Descriptive Statistics: A Programmed Textbook, vols. 1 and 2. New York, John Wiley and Sons, Inc., 1965.

[3] McCollough, Celeste and Van Atta, Loche. Statistical Concepts: A Program for Self-Instruction. New York, McGraw-Hill Book Company, 1963.

[4] Whitmore, G. A., et al. Self-Correcting Problems in Statistics. Boston, Massachusetts, Allyn and Bacon, Inc., 1970.

The suggested time allocation in the above outline makes it clear that, in our view, the early material should be covered quickly so that statistical inference can be reached in the first half of the course.

A nonexhaustive, illustrative list of texts fitting the pattern of the conventional course follows.

Illustrative List of Texts for an Introductory Course:

[5] Alder, Henry L. and Roessler, Edward B. Introduction to

Probability and Statistics, 4th ed. San Francisco, California, W. H. Freeman and Company, 1968.

Reviewed in Journal of the American Statistical Association, 64 (1969), p. 675. The first edition is reviewed in The American Mathematical Monthly, 68 (1961), p. 1018.

[6] Freund, John E. Modern Elementary Statistics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

Reviewed in Journal of the American Statistical Association, 62 (1967), p. 1504.

[7] Guenther, William C. Concepts of Statistical Inference. New York, McGraw-Hill Book Company, 1965.

Reviewed in Journal of the American Statistical Association, 61 (1966), p. 529.

[8] Hoel, Paul G. Elementary Statistics, 3rd ed. New York, John Wiley and Sons, Inc., 1971.

Third edition is reviewed in Journal of the American Statistical Association, 67 (1972), p. 497.

[9] Huntsberger, David V. Elements of Statistical Inference, 2nd ed. Boston, Massachusetts, Allyn and Bacon, Inc., 1967.

[10] Mendenhall, William. Introduction to Statistics, 2nd ed. Belmont, California, Wadsworth Publishing Company, Inc., 1967.

Second edition is reviewed in School Science and Mathematics, 67 (1967), p. 754.

[11] Walker, H. M. and Lev, J. Elementary Statistical Methods, 3rd ed. New York, Holt, Rinehart and Winston, Inc., 1969.

First edition is reviewed in Journal of the American Statistical Association, 54 (1959), p. 699.

Following is a list of texts which may be helpful to the instructor as resource materials.

Illustrative List of References for an Introductory Course:

[12] Blackwell, David. Basic Statistics. New York, McGraw-Hill Book Company, 1969.

Reviewed in Journal of the American Statistical Association, 65 (1970), p. 1398, and in The American Mathematical Monthly, 77 (1970), p. 662.

[13] Dixon, Wilfrid J. and Massey, F. J. Introduction to Statistical Analysis, 3rd ed. New York, McGraw-Hill Book Company, 1969.

Reviewed in Journal of the American Statistical Association, 65 (1970), p. 456. The second edition is reviewed in The American Mathematical Monthly, 64 (1957), pp. 685-686.

[14] Hodges, J. L., Jr. and Lehmann, E. L. Basic Concepts of Probability and Statistics, 2nd ed. San Francisco, California, Holden-Day, Inc., 1970.

Reviewed in Journal of the American Statistical Association, 65 (1970), p. 1680. The first edition is reviewed in The American Mathematical Monthly, 72 (1965), p. 1050.

[15] Natrella, Mary G. Experimental Statistics, Handbook 91. U. S. Department of Commerce, National Bureau of Standards, 1966.

[16] Neyman, J. First Course in Probability and Statistics. New York, Henry Holt and Company, 1950.

Reviewed in Journal of the American Statistical Association, 46 (1951), p. 386.

[17] Savage, I. Richard. Statistics: Uncertainty and Behavior. Boston, Massachusetts, Houghton Mifflin Company, 1968.

Reviewed in Journal of the American Statistical Association, 64 (1969), p. 1677.

[18] Snedecor, George W. and Cochran, W. G. Statistical Methods, 6th ed. Ames, Iowa, Iowa State University Press, 1967.

Reviewed in Applied Statistics, 17 (1968), p. 294.

[19] Wallis, W. Allen and Roberts, Harry V. Statistics: A New Approach. New York, The Macmillan Company, 1956.

Reviewed in Journal of the American Statistical Association, 51 (1956), p. 664.

We turn now to topic-by-topic comments on the outline. These comments contain occasional references to books in the above lists.

Topic 0. Introduction

This lecture should be devoted to a discussion of the nature and importance of statistics, including the difference between the inferential nature of statistics and the deductive nature of mathematics. The first lecture should illustrate the existence of statistical problems in everyday life in order to emphasize that statistics is problem-oriented.

Source material which may be helpful includes:

[20] Careers in Statistics. The Committee of Presidents of the American Statistical Association, the Institute of Mathematical Statistics, and the Biometric Society. The American Statistical Association, 806 15th Street, N. W., Washington, D. C. 20005.

[21] Kruskal, William, ed. Mathematical Sciences and Social Sciences. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1970.

[22] The Mathematical Sciences: A Collection of Essays. Edited by The National Research Council's Committee on the Support of Research in the Mathematical Sciences (COSRIMS) with the collaboration of George A. W. Boehm. Boston, Massachusetts, MIT Press, 1969.

[23] Sills, David L., ed. International Encyclopedia of the Social Sciences. New York, The Macmillan Company and the Free Press, 1968. Listings under:
Statistics
Statistical analysis
Survey analysis

[24] Tanur, Judith, et al., eds. Statistics: A Guide to the Unknown. San Francisco, California, Holden-Day, Inc., 1972.

[25] Wallis, W. Allen and Roberts, Harry V. Statistics: A New Approach. New York, The Macmillan Company, 1956.

Topic 1. Statistical Description

General comments. The discussion should begin with a problem that will motivate the need for statistical data and their analysis through frequency distributions and descriptive measures. A good problem may be used to give an overview of the course raising questions relating to later topics. Instructors will be able to find problems for which data of local interest from registrars' records are appropriate. Examples: Are student entrance test scores higher today than they were five years ago? Is student family economic status higher today than it was five years ago? Are college-age students today taller than they were 30 years ago? Do college aptitude test scores for men differ from those for women?

Our suggestion departs from the usual approach in which data are provided without the context of a problem, and a frequency distribution or a variety of descriptive measures are calculated for their own sake. The suggested approach emphasizes problems of inference from the start; the instructor should not hesitate to raise questions about inferential problems that will be considered later in the course.

A comparison of a frequency distribution with an expected, theoretical distribution, such as the birth pattern distribution in R. A. Fisher's Statistical Methods for Research Workers* (p. 67), is useful for introducing ideas of statistical inference. So is a bimodal distribution, the problem being whether it is made up of two different groups, each with a different pattern of variation.

Specific suggestions.

a. Class-generated data, from which one or several frequency distributions are constructed, often interest students.

b. In discussing frequency distributions, one should emphasize the variation inherent in the phenomenon under study and the pattern of this variation. Examples of different phenomena (e.g., life of light bulbs, number of calculators out of order per day) should be utilized to demonstrate widespread presence of variation.

c. Data for constructing frequency distributions should be simple, so that little time is spent on developing class limits.

d. The discussion of frequency polygon and histogram can be used to distinguish between continuous and discrete variables.

e. In taking up the descriptive measures of location (mean, median) and of variation (standard deviation, range), alternate computational forms of the mean and standard deviation should be called to the student's attention, but no derivations should be given. A useful means of pointing out the applicability and limitations of these measures is through a comparison of two or more frequency distributions (e.g., income distribution for 1970 and 1971).

f. Class time is better devoted to discussion of the meaning and limitations of various measures of location and variation than to drilling students on their calculation.

g. Mean and median should be explained as alternate descriptive measures of location, neither of which is perfect for all situations.

Topic 2. Probability

General comments. The way in which introductory probability is presented can vary greatly. It is a broad topic, intensive examination of which leads to deep philosophical problems. The approach of

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* [26] Fisher, R. A. Statistical Methods for Research Workers, 13th ed. New York, Hafner Publishing Company, 1958.

Blackwell [12] in the first three chapters is elementary. The concept of area is used when extended ideas are required. The basic laws of probability are developed, making effective use of tables and diagrams. A major advantage of the presentation is that it stays consistently at a modest mathematical level without disruptive digressions.

A more complete exposition of finite probability theory is presented by Hodges and Lehmann in [14]. Chapters 1 and 4 contain material appropriate for an introduction to probability in the elementary statistics course. We suggest this reference as additional reading to augment the material in Blackwell.

An interesting aspect of the modern emphasis in probability is that of assigning personal or subjective probabilities to events. For an elementary introduction, Savage [17] seems appropriate. The discussion includes a great deal of gambling terminology which may not appeal to all students. At the end of the chapter there are many stimulating notes and problems that are used to extend the theory and to show examples of probabilistic reasoning occurring in everyday life. Hodges and Lehmann [14] have a short section on subjective probability beginning on page 127.

Specific suggestions.

a. Since only three lectures are devoted to probability, discussion must be restricted to the major concepts, and proofs will need to be kept to a minimum. Only finite sample spaces should be used for explaining the probability concepts.

b. The relative frequency interpretation is a useful way to motivate the concept of probability. The equally-likely notion should not be given major emphasis, though it may be helpful if extended to real-life situations such as taste experiments or genetic problems.

c. Whether set notation is to be used depends on the mathematical training of the students. At any rate, the allotted time in the suggested outline does not permit the teaching of set theory or even of basic set concepts. The teacher may, however, wish to point out that probability theory is based on set theory.

d. In teaching statistical independence, the conditional definition:

If $P(A|B) = P(A)$, then A and B are independent,

is easier for the student to understand than the product definition:

If $P(AB) = P(A)P(B)$, then A and B are independent.

e. The small amount of time allotted to probability makes it imperative that teachers not get bogged down in combinatorial probability problems. Simple, though realistic, examples should be used to illustrate the probability concepts covered.

Topic 3. Random Variables and Probability Distributions

General comments. Random variables, probability distributions, and expectations are treated in fairly standard ways in most texts. A simple presentation is given by Blackwell [12], but his presentation is very brief.

Specific suggestions.

a. Do not give a formal definition of random variables; a statement that "the value of a random variable is a number determined by the outcome of an experiment that varies from trial to trial" should be adequate.

b. Introduce probability functions and cumulative distribution functions in terms of simple examples with a small number of points and associated probabilities. Functions of random variables also should be considered only in terms of such simple examples.

c. Stay on the intuitive level. Many definitions and results are intuitively reasonable for most students; for instance:

$$E(X) = \Sigma x_i P(X = x_i)$$

$$E(kX) = kE(X)$$

$$\sigma(kX) = k\sigma(X), \quad k > 0$$

d. Use simple everyday examples, with three or four outcomes only; e.g., the number of calculators needing repair in a group of four calculators; the number of males in families of five persons.

e. The mean and variance of a linear function of a random variable should be mentioned and their usefulness illustrated by an example, but formulas should not be derived.

Topic 4. Special Probability Distributions

General comments. Student participation can be developed both through access to a computer on an individual basis and through elementary class exercises involving coin tossing, tables of random normal deviates, and exercises of this kind. The results of student exercises can be combined to demonstrate sampling variation over the various samples obtained individually by students, thus serving as an introduction to Topic 5, Sampling Distributions.

Specific suggestions.

a. State the conditions for a probability distribution to be binomial but do not derive its formula.

b. Formulas for the mean and variance of a binomial distribution should be given and illustrated by a simple example but should not be derived.

c. Tables of binomial probabilities should be used to demonstrate characteristics of binomial distributions such as skewness. Students should not be asked to calculate binomial probabilities repeatedly.

d. Use a variety of realistic examples. Do not confine examples to coin tossing; e.g., use the number of defectives in a sample from a shipment of parts or the number of voters favoring an issue in an opinion poll.

e. The transition to a continuous random variable, for which area represents probability, can be facilitated by considering a histogram and shrinking the width of the classes.

f. The normal distribution can be introduced as:

(i) an approximate description of many real-life phenomena, such as weights of the contents of cans of fruit or the Scholastic Aptitude Test scores of students;

(ii) an approximation to certain discrete distributions, such as the binomial distribution or the distribution of the sum of the digits on three dice.

The text used will influence the approach taken, but both notions should be introduced.

g. In discussing the normal distribution, one should emphasize the variety of shapes encountered for different values of μ and σ and the method for transforming any normal random variable to a standard normal one.

h. Realistic examples should be used for the normal distribution. Students will probably be interested in SAT scores, for which the mean is 500 and the standard deviation is 100.

i. Unless the text makes the continuity correction an integral part of the normal approximation to the binomial distribution, this topic should not be taken up or should be mentioned only briefly.

Topic 5. <u>Sampling Distributions</u>

<u>General comments</u>. Computers are effective in demonstrating sampling variability and sampling distributions. Exercises can be

developed for many distributions where the computer takes repeated samples of fixed size and, for example, provides information on the distribution of the sample mean through plots and histograms. The student learns of the nature of sampling variation and can proceed through variation in sample sizes to obtain insight into the inherently increased stability of sampling distributions as sample sizes are increased. The student can be given insight thereby into the behavior of the variance of a mean and into the concepts of the Central Limit Theorem.

Specific suggestions.

a. The concept of the sampling distribution of the mean can be explained initially by considering a small finite population and sampling with or without replacement with a small sample size ($n = 2$ or 3). This permits the exact sampling distribution of the mean to be developed by enumeration. The mean and variance of this sampling distribution can then be studied.

b. Since the Central Limit Theorem should only be stated and not proved, evidence of its operation will need to be given to the student. Some texts contain exact sampling distributions for different sample sizes or the results of sampling experiments. Printouts of computer runs simulating sampling distributions for different sample sizes can also be distributed to students and discussed. If a computer is not available on campus, printouts could be obtained from a computer located elsewhere.

These approaches are helpful but, in our opinion, are not as effective as having students participate in sampling experiments. A simple experiment is to sample a rectangular distribution, either from a table of random numbers, by drawing chips from a bowl, or by computer. If a computer is used, it will also be easy to sample other kinds of populations. Sampling a moderately skew population may help convince students of the Central Limit Theorem in the absence of symmetry. Indeed, the use of several populations (e.g., rectangular, exponential) can demonstrate to the student that the rapidity with which the sampling distribution of $(\bar{x} - \mu) \div (\sigma/\sqrt{n})$ approaches a normal distribution as n increases depends on the population from which the samples are selected.

c. It should be pointed out to students that the sample proportion is a mean, and hence the Central Limit Theorem applies directly to sample proportions.

Topic 6. Estimation of Population Proportion

General comments. If the teacher prefers, he can take up estimation of the means of continuous populations and tests concerning them before he discusses inferences on population proportions. Inferences on proportions are, in our opinion, somewhat simpler to present; e.g., binomial tables can be used to derive the power of tests. Hence, we place inferences on proportions before inferences on means of continuous populations.

Also, testing hypotheses can be taken up before estimation. We recommend taking up statistical estimation first because we believe it is easier for the student to understand than hypothesis testing.

The most common interpretation of a confidence interval has been that of a random interval which contains a fixed parameter with given probability over repeated applications. This point of view has been presented traditionally in most texts. Through use of subjective probability and Bayes' theorem, one can interpret a confidence interval as a fixed interval (for given observed data) which contains a random parameter with given posterior probability. Savage [17] puts forward both concepts (see page 209 and page 260) and points out basic differences in philosophy.

<u>Specific suggestions</u>.

a. The large sample confidence interval for the population proportion p can be developed readily by relying on an extension of the Central Limit Theorem which states that $(\hat{p} - p) \div \sqrt{\hat{p}(1 - \hat{p})/n}$ is approximately normal for large sample size n, $\hat{p}$ being the observed proportion.

b. An experiment with repeated sampling of a known population, setting up a confidence interval each time, can be helpful in conveying the idea of the confidence coefficient and in illustrating that the location and width of the confidence interval varies from sample to sample. This can be done either by computer or by class-generated samples. Such an experiment is particularly desirable since it will provide evidence of the working of the extension of the Central Limit Theorem.

c. Realistic examples, such as estimation of the proportion of voters favoring a candidate or the proportion of persons in the labor force who are unemployed, will be helpful in illustrating the pervasiveness of estimation problems. The subject of sample surveys in general may be discussed in connection with statistical estimation.

d. The discussion of confidence intervals should consider the usefulness of the particular confidence interval obtained. For instance, in a close election race involving two candidates, a confidence interval for the proportion of voters favoring one of the candidates which ranges from .45 to .53 may not be useful. Discussion of several such examples can help the student to recognize that different problems may call for different levels of precision.

e. The determination of the confidence coefficient should be discussed in general terms. Several examples may be used to illustrate that important problems, such as estimating the unemployment rate for purposes of determining national economic policy, call for higher confidence coefficients than do less important problems.

f. The determination of the sample size required to yield a confidence interval of sufficiently small width for a given confidence coefficient provides a useful means of introducing the student to the notion that statistical investigations should be planned in advance.

g. In discussing interval estimation of the population proportion for small sample sizes, reference should be made to tables or to the Clopper-Pearson charts which appear in many texts and books of tables. See, for example,

[27] Owen, Donald B. Handbook of Statistical Tables. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

Topic 7. Tests Concerning Population Proportion

General comments. This subject can be discussed either by considering small sample sizes first and then large sample sizes, or vice versa. An advantage of beginning with small sample sizes is that power calculations can readily be made by using widely available binomial tables. A disadvantage is that the power obtained for small sample sizes will be relatively poor so that the student may feel that the statistical test is almost useless. If small sample sizes are taken up first, it is incumbent upon the instructor to note early that many practical problems require much larger sample sizes.

An advantage of beginning with large sample sizes is that many problems can be treated much more realistically. A disadvantage is that power calculations will be more tedious.

Neyman [16] gives a complete, systematic analysis of several stages involved in testing hypotheses (pp. 268-271).

Specific suggestions.

a. A basic difficulty for most students is the proper formulation of the alternatives H_0 and H_1 for any given problem and the consequent determination of the proper critical region (upper tail, lower tail, two-sided).

b. Practice with problems containing a verbal statement of the situation and requiring the students to develop H_0 and H_1 can be most helpful. Indeed, students should be competent in formulating the alternatives H_0 and H_1 and in designating the appropriate type of critical region before they begin the detailed calculations for determining the exact critical region.

c. Realistic problems should be used when the student is asked to formulate the appropriate alternatives H_0 and H_1.

d. Students can often be helped by introducing the critical region in terms of the statistic $\hat{p}$, the sample proportion. For instance, if:

$$H_0 : p \leqq p_o$$

$$H_1 : p > p_o,$$

the student can readily see that the appropriate decision rule is of the form:

$$\text{if } \hat{p} \leqq A, \text{ accept } H_0,$$

$$\text{if } \hat{p} > A, \text{ accept } H_1,$$

where A is the cut-off point to be determined. It can then be explained how to determine A if, for example, the probability of concluding H_1 when $p = p_o$ is to be α.

The standardized statistic $(\hat{p} - p_o) \div \sqrt{p_o(1 - p_o)/n}$ for determining the critical region is a confusing way to introduce the student to testing hypotheses. After the student gains some familiarity with testing hypotheses, the standardized statistic then is more easily used.

e. In developing the properties of a statistical decision rule for testing hypotheses, attention should not only be given to the power curve of the decision rule but also to the error curve which shows the probabilities of error directly. Many students understand the implications of a decision rule for testing more easily through the error curve than through the power curve [e.g., if $H_0 : p \leqq p_o$, $H_1 : p > p_o$, then $P(\text{Error}) = P(\text{Accept } H_1 \mid p)$ for $p \leqq p_o$ and $P(\text{Error}) = P(\text{Accept } H_0 \mid p)$ for $p > p_o$)].

f. When discussing the two-sided test, one should explain the correspondence between the confidence interval for the population proportion and the testing approach.

g. When small sample sizes are considered first, the power of a test and the error probability for a given population proportion should be obtained using binomial tables rather than by making actual calculations. Binomial tables can then be used to set up a decision rule with specified control on type I and/or type II errors. At this point a transition can be made to the case of large samples and the use of the normal approximation.

h. Students often do not understand how the p_o level is determined in an actual problem. This situation can be clarified by illustrating a variety of situations, for instance:

(i) In deciding whether a company should accept or reject an incoming shipment, p_o may be the break-even proportion of defective items for which it is equally costly to accept or reject a shipment.

(ii) In an experiment to determine whether a person has extrasensory perception by having him indicate whether coins flipped in a different city are heads or tails, p_o may be .5, the level expected if the person were guessing and no ESP were present.

These examples illustrate two commonly encountered situations, namely those for which p_o is the break-even proportion and those for which p_o is determined by theoretical considerations as the level where no effects, or no effects of practical significance, are present.

i. Discussion of the determination of the probability of a type I error (size of test) α may usefully be combined with the problem of how to determine sample size, as follows:

(i) For given n and α, find the probability of type II error β at an appropriate value of p. If β is satisfactory, use the given n and α. If β is too high, increase the sample size. If no increase in sample size is possible, raise α until a suitable balance between α and β is found.

(ii) If, for given n and α, β turns out to be smaller than necessary, the sample size may be reduced or α lowered.

j. Sampling experiments to demonstrate the behavior of a given decision rule for different levels of p may be a helpful supplement for many students.

Topic 8. Inferences Concerning Population Mean

Specific suggestions.

a. Some major properties of estimators (unbiasedness, consistency, and efficiency) could be discussed here, but only briefly. If they are discussed, we recommend that it be done here rather than in Topic 6. We feel that students should first become acquainted with the general concept of statistical estimation before they become concerned with properties of point estimators.

b. In discussing the small-sample confidence interval based on the t-distribution, emphasis should be placed on the robustness (insensitivity of properties of the procedure to departures from the assumption of normality) of the t-statistic and the consequent wider applicability of this confidence interval to populations which are not exactly normal.

c. In developing inferences concerning population means, students should not be asked to perform extensive calculations to find the sample mean and standard deviation.

d. Sampling experiments for estimation and testing with small sample sizes can be helpful in several ways:

(i) When sampling a normal population, the experiment will give the student further confidence in the t-distribution and will illustrate once again the meaning of the confidence coefficient or the level of significance and power.

(ii) When sampling a population moderately different from a normal population (e.g., a rectangular population), the experiment will illustrate the robustness of the t-statistic.

e. We recommend bypassing the case of known σ unless inferences concerning the population mean are considered before inferences concerning the population proportion. The reason for this recommendation is that the case of known σ does not arise often in practice and only serves to introduce an unnecessary repetitive element for the student. Reliance should be placed on $(x - \mu) \div (s/\sqrt{n})$ being approximately distributed as t for small n for populations not departing excessively from a normal population and being approximately distributed as a standard normal variable for larger n for almost all populations.

If there is time available to discuss planning the sample size or to determine the power of the test, we recommend the use of charts such as those in Guenther [7]. For the case of unknown σ these charts could be entered by using two values of σ within which the standard deviation is expected to fall, thereby obtaining bounds on the power or sample size.

Topic 9. Additional Topics

Students in the social sciences might benefit most from topics a, c, d, f (see page 481); in the physical and biological sciences, from topics a, c, d, e; in management science, from topics g and h; in economics, from topics d and h; in education, from topics a, d, f.

IV. SOME ALTERNATE APPROACHES

In this section we discuss three alternate approaches to the introductory statistics course. Each approach provides great potential for innovation, and some instructors may wish to try one or several of these in lieu of the conventional approach. For two of the alternate approaches (decision theory, nonparametric), a number of possible textbooks are available. Since the textbook used will have a major effect in the determination of the precise contents, the order of presentation, and the amount of time to be devoted to the various topics, no effort is made here to present outlines. Instead, we discuss the principles underlying the use of the approaches. This will serve to indicate potential advantages and disadvantages and to provide a foundation in terms of which the instructor may interpret various texts that are now and will later become available.

No text designed specifically for the third approach (problem-oriented) is available, to the best of our knowledge. The instructor wishing to implement this approach must be prepared for substantial developmental efforts.

1. A Decision Theory Course in Introductory Statistics

Basic elements of the course. Decision theory is a formulation of statistical problems in which the statistician has available a choice of actions, the consequences of which depend on an unknown state of nature. To help decide on an appropriate action, one must perform an experiment which will yield relevant data to help determine the state of nature. Since the data generally depend not only on the state of nature but also on chance, uncertainty appears in two places--the effect of chance, i.e., random variation, and the initial ignorance of the state of nature.

A decision theory course in introductory statistics should include the following concepts.

> Concepts: Philosophical principles of decision-making under uncertainty; averages and measures of variability; probability and expectation; utility; Bayes' strategies and posterior probabilities; the parameters of distributions relevant for optimal actions; testing hypotheses, significance levels; estimation, confidence intervals.

The development of these ideas requires the presentation and use of (1) elementary properties of probability and mathematical expectation, (2) sets and functions on a relatively simple level, and (3) properties of convex sets such as the separating hyperplane theorem.

Traditional statistical techniques of handling data such as histograms, cumulative frequency polygons, means and standard deviations are not vital to the decision theory approach. Their introduction has considerable potential pedagogic value, however, in preparing the student for the analogous probabilistic concepts of density, cumulative distribution function, and expectation. The standard statistical methods in estimation, hypothesis testing, and confidence intervals are not easily blended with this approach.

Results of decision theory have implications in business and other real-life situations where decisions must be made in the face of uncertainty, and a wise choice must consider consequences of the available actions.

Advantages and disadvantages of the decision theory approach. One major advantage of the decision theory approach over traditional approaches lies in the problem-solving orientation. Each example considered involves a problem in decision-making which requires a good and sensible answer.

The decision theory approach should appeal to students who are interested in the philosophical foundations of scientific inference and who are curious about the rationale for coping with random variation and uncertainty. Such students would find the decision theory approach natural and easy to follow. It would be esthetically pleasing for the student who likes mathematics and enjoys the application of basic ideas such as sets, functions, and convexity to clarify nontrivial results in inference. (It is possible to present these results without, or with a minimum of, formal derivations.) For the student who anticipates further serious study of statistics, this course can serve as a valuable complement to his other work in statistics.

This course will not serve for a student who is expected to learn some of the major tools of statistics in the first course. Although the course can present an exciting view of statistics and scientific inference, the decision theory point of view gives a limited view of statistics and does little to acquaint the student with actual statistical practice; it does not give him experience with data analysis.

Another limitation of the decision theory approach is that it tends to involve an overformulation of statistical problems in the sense that the statistician is presumed to know precisely the set of available actions, the set of possible states of nature, the consequences of the actions, and all the relevant probability distributions. Little allowance is made for the possibility of error in specifying the model. No provision is made for learning from data.

While a minimum of mathematical background is required, the student who is very weak in mathematics will fail to enjoy some of the esthetic values of this approach and may find that coping with the elementary arithmetic and algebra distracts him from the main ideas.

Finally, the opportunities to apply the ideas and methods of such a course in other academic work are relatively few, although such opportunities seem to be increasing steadily as decision-making becomes used more and more in diverse fields such as business, engineering, and medicine.

Illustrative List of Texts and References:

Aitchison, John. Choice Against Chance: An Introduction to Statistical Decision Theory. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970.

Reviewed in The Australian Journal of Statistics, 13 (1971), p. 123.

Chernoff, Herman and Moses, Lincoln E. Elementary Decision Theory. New York, John Wiley and Sons, Inc., 1959.

Reviewed in Journal of the American Statistical Association, 55 (1960), p. 291, and in The American Mathematical Monthly, 67 (1960), p. 487.

Hadley, G. Introduction to Probability and Statistical Decision Theory. San Francisco, California, Holden-Day, Inc., 1967.

Reviewed in Journal of the Royal Statistical Society, Series A, 131 (1968), p. 437.

Lindgren, B. W. Elements of Decision Theory. New York, The Macmillan Company, 1971.

Lindley, D. V. Making Decisions. New York, John Wiley and Sons, Inc., 1971.

Raiffa, Howard. Decision Analysis: Introductory Lectures on Choices Under Uncertainty. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

Reviewed in Journal of the American Statistical Association, 64 (1969), p. 1668.

Schlaifer, R. Probability and Statistics for Business Decisions: An Introduction to Managerial Economics Under Uncertainty. New York, McGraw-Hill Book Company, 1959.

Reviewed in Journal of the American Statistical Association, 54 (1959), p. 813.

2. A Nonparametric Course in Introductory Statistics

Basic elements of the course. Using nonparametric methods it is possible to introduce substantial problems in inference very early in the course, with impressive solutions which can be easily comprehended. The preparation required before beginning standard nonparametric methods consists mainly of some work in rather elementary discrete probability. No time need be devoted to the usual development of histograms and the computation of means and standard deviations. With the early introduction of meaningful problems, the discussion of sampling distributions is interwoven in the development of statistical inference in a way which leads to ready understanding by the student. It is possible to avoid a digression into combinatorics by presenting simple illustrations and relying on appropriate tables.

A nonparametric course in introductory statistics should include the following concepts.

Concepts: Probability, probability distributions, hypothesis testing, estimation, two-sample tests, chi-square tests and contingency tables, correlation, robustness.

While it is most desirable that the introductory course using a nonparametric approach concentrate on nonparametric methods, it is possible toward the end of the course to introduce the student to some parametric statistics. If the instructor wishes to cover some parametric topics, only a few should be taken up. To be in line with our recommended objectives, the course should not become technique-oriented; one should avoid making the course a compendium of nonparametric and parametric methods.

Advantages and disadvantages of the nonparametric approach. The use of nonparametric methods provides several potential advantages. With the availability of appropriate tables, the detailed working of numerical examples, regarded by many as essential to a thorough grasp of principles as well as techniques, is simple and brief. The methods are easily interpreted and have natural and sensible justifications. The methods have the advantage of robustness. Thus the student is quickly introduced to useful and simple techniques which have wide applicability. This exposure to statistical concepts throughout the course provides the student with a foundation for a good understanding of standard parametric procedures.

For the student who is required to take statistics to satisfy the demands of a major field in which statistics is used, this course is likely to provide an introduction to some useful nonparametric methods at an elementary level.

A major disadvantage of this approach is that students will have little or no exposure to those parametric methods needed for later work in other subjects. Another disadvantage is that current books in nonparametric statistics do not focus on either point or interval estimation. Also, there are few texts at the elementary level.

Illustrative List of Texts for an Introductory Course:

Conover, W. J. Practical Nonparametric Statistics. New York, John Wiley and Sons, Inc., 1971.

Reviewed in Journal of the American Statistical Association, 67 (1972), p. 246.

Kraft, Charles H. and van Eeden, Constance. A Nonparametric Introduction to Statistics. New York, The Macmillan Company, 1968.

Reviewed in Journal of the American Statistical Association, 66 (1971), p. 223, and in The American Mathematical Monthly, 77 (1970), p. 207.

Noether, Gottfried E. Introduction to Statistics: A Fresh Approach. Boston, Massachusetts, Houghton Mifflin Company, 1971.

Reviewed in Journal of the American Statistical Association, 67 (1972), pp. 496-497, and in The Mathematics Teacher, 64 (1971), p. 630.

Other Selected References in Nonparametric Statistics:

Bradley, James V. Distribution-Free Statistical Tests. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1968.

Reviewed in Journal of the American Statistical Association, 64 (1969), p. 1090.

Gibbons, J. Nonparametric Statistical Inference. New York, McGraw-Hill Book Company, 1971.

Hájek, Jaroslav. Nonparametric Statistics. San Francisco, California, Holden-Day, Inc., 1969.

Reviewed in the Australian Journal of Statistics, 13 (1971), p. 123.

Noether, Gottfried E. Elements of Nonparametric Statistics. New York, John Wiley and Sons, Inc., 1967.

Reviewed in Journal of the American Statistical Association, 63 (1968), p. 728.

Siegel, Sydney. Nonparametric Statistics for the Behavioral Sciences. New York, McGraw-Hill Book Company, 1956.

Reviewed in Journal of the American Statistical Association, 52 (1957), p. 384, and in The American Mathematical Monthly, 64 (1957), pp. 690-691.

3. A Problem-oriented Course in Introductory Statistics

Basic elements of the course. Envisioned here is a relatively radical departure, in the direction of a "case study" approach, from any of the courses described previously. Such a problem-oriented approach, which some may advocate for a second rather than a first course, is not entirely new. R. A. Fisher's classic Statistical Methods for Research Workers [26], which had a profound influence on a generation of scientists, is in large part problem-oriented. Although that book attempts to build up sophistication and complexity gradually, there are many places where a previously uninformed reader would find imposing gaps. Nevertheless, the audience, consisting largely of people with meaningful problems upon which to apply the methods and with the professional backgrounds to evaluate how sensible the conclusions were, found this book to be an invaluable guide. The level of maturity demanded was considerably higher than can be expected of students taking a first course in statistics. On the other hand, these students will have help from a teacher and other students.

We propose a modification of Fisher's approach. Let meaningful and nontrivial problems be presented (initially with a suggested solution, later without one). The detailed examination of the problems and solutions should evoke questions which lead to a discussion of fundamental concepts and methods. The order of the concepts to be discussed will depend on the problem and on the questions concerning a proposed solution. Generally, the order will not produce a simple and systematic course development of concepts. The student will need to be intelligent enough to have a quick grasp of new concepts, to recognize a reasonable approximation to a sensible solution, and to be satisfied with a nonsystematic development of the course.

The practice of scientists in presenting their work is to remove many of the untidy traces of false starts and preliminary studies which led to final clear-cut results. But a major part of the excitement of research lies in these lost traces, and much of this work involves statistics in its formal and informal aspects. It is to be hoped that a case study approach would spark this kind of excitement and interest on the part of the students and also lead to an understanding of statistical practices.

In a course of this kind, the computer and computer simulations can play an important role. When one inquires about the properties of a procedure, it is possible to use a computer simulation to see how the procedure works. The output of a simulation can also serve as a source of valuable data for further problems. Results of simulations can be reported by printouts (especially where there is no access to a computer). When adequate computer facilities are available, however, students can be encouraged to make their own independent investigations.

To illustrate the problem-oriented approach and to show how different cases are useful for different configurations of concepts, we present two examples. As these examples will make clear, the envisioned problem-oriented course differs sharply from present courses. In most current textbooks, problems are used simply to provide examples with which to illustrate the method under study. Often the problems are artificial and noncompelling, and the intellectual challenge to the student is limited to the study of how to carry out the details of the method correctly rather than of what method to apply and how well it works in the problem.

The problem-oriented course will not be easy to teach. As the need for more material on statistical concepts arises, assignments will have to be made in supplementary texts, programmed materials, or handouts developed by the instructor.

Example 1: The Peach Crop. A cling peach crop has been insured against frost damage by a growers' association. A frost occurs and a court must adjudicate disagreement between the growers and the insurance carrier on the amount of damages.

One possible way for deciding the compensation to be paid to growers is to examine each of the 26,793 trees in the orchards belonging to members of the association. An impartial expert could assess the damage to each tree. If he assesses the peaches on a tree as 15% damaged, the damage would be $37 \times .15 = 5.50$ dollars, where 37 dollars is the estimated average value of the peaches on undamaged trees. Theoretically, all 26,793 trees in the orchards could be examined in order to obtain a total damage figure.

Clearly, it is unreasonable to examine all the trees. Indeed, the cost of doing so would far exceed the value of the crop. A statistical consultant proposed examining a sample of 100 trees randomly selected from among all 26,793 trees in the orchards and using the average damage to the crop per tree in this sample as an estimate of the average damage per tree in the total population. His advice was taken, and the sample resulted in damages of 5.50, 7.10, 9.90, ..., 1.57, with a mean of 6.31. When this mean is multiplied by the number of trees in the orchards, an estimate of $6.31 \times 26,793 = \$169,063.83$ is obtained for the value of the damages suffered by the growers.

The growers are concerned about this procedure. They wish to recover the full damages caused by the frost, but they are anxious to avoid being criticized by the court for using the wrong method of estimating the damages to the peaches. They ask, "Is the above estimate the correct value for the damages to the peach crop?" No, it is not. It is an estimated damage; it is impractical to obtain the actual data needed to compute the "correct" damage. "The law is not clear on how to treat estimates. Would a mathematician say that this is the correct estimate?" This is not a mathematical problem in the sense that there exists a "correct" way to select and analyze the data. It is a statistical problem. The question should be whether the procedure used to arrive at this estimate is a reasonable procedure for an unbiased statistician. It seems reasonable, but to answer this question in a meaningful way, we must know how an estimate obtained by this method is related to the unknown "damage" and how estimates from alternative methods would compare with this.

Up to this point the concepts of random sample and average have been used. These terms can and probably should be explained in detail after some of the more pressing problems are discussed. It is necessary now to point out that the estimate is random and to illustrate how the estimate would have varied if the experiment had been repeated 10 times. It would be valuable to consider the consequence of selecting different sample sizes and the cost of sampling. The necessity of dealing with variability is now clear, and the Central Limit Theorem can be hinted at or discussed briefly.

An analysis of the data from simulation would use histograms, means, and standard deviations. These could be presented in a matter-of-fact way. Instructions on how to carry out the presentations and analyses need not be given immediately if these concepts are easily enough appreciated in this specific context.

At this point one can extend the problem in several directions. One can make it more realistic by adding the fact that the orchards are located in two different river valleys and that the location and elevation of the orchards can be used as a basis for stratified sampling. This raises the point that the more one knows, the better one can do; it raises the question of how to profit from vague information.

Alternatively, one can discuss the advantages of randomization to avoid hidden bias.

Suppose that, due to a clerical error, it was reported that there were 700 trees in a certain orchard, whereas the actual figure was 70. The statistician might not wish to decrease his original estimate accordingly, because the 630 nonexistent trees had had a chance of being sampled and could have yielded a 0. An adjustment would make his estimate biased, and he would not understand the sampling properties of such <u>ex post facto</u> procedures. On the other hand, if he were to testify in court, his opinion might be criticized if the estimate were not adjusted.

Another question that may arise is how much the insurance carrier should pay for damage to the crop if they are confident that it is between $150,000 and $190,000. Should the carrier pay $2,000 for increased sampling which is likely to reduce the length of the confidence interval to $10,000? Should one regard a payment of $170,000 in the first case as a gamble in which the insurer risks as much as $20,000? Would the $2,000 qualify as a prudent expense to reduce the amount at risk from $20,000 to $5,000? What is a reasonable trade-off?

It is apparent that this simplified example has enormous potential for illustrating statistical concepts in a context which is meaningful and interesting. Indeed, there is some danger of dwelling too long on one example, as students may become bored with it.

Example 2: Death Takes a Holiday. Do famous people, people whose birthdays are likely to be celebrated publicly, put off dying until after their birthdays? To answer this question, David Phillips studied the birth and death days of over 1,200 famous people (Tanur, Judith, *et al.*, eds. Statistics: A Guide to the Unknown. [23]). To make the classification less tedious, he examined only the birth and death months.

There are two conflicting hypotheses. One states that there is no relation between birth and death months. According to this hypothesis, someone born in April is as likely to die in November as someone born in December. The alternate hypothesis states that indeed there is a relation between birth and death months, that someone born in December is less likely to die in November than someone born in April.

Phillips does not present a formal test of these hypotheses in his paper. He shows that for four different sets of data, the month before birth is less often a death month and the four months after birth are more often death months. The consistency of this result in all four sets of data is interpreted as increasing the plausibility of the alternative hypothesis, specialized somewhat to include the four-month death rise after the birth month.

What concepts are raised by this problem? The student is introduced to two-way contingency tables (month of birth and month of death) and the use of observed proportions as estimates of probabilities. Confidence intervals can be introduced to assess whether an observed proportion is consistent with a theoretical expectation. The idea of testing hypotheses and the use of the chi-square test enter naturally, as does the notion that the chi-square test is an all-purpose test whose power can be improved upon in the presence of a sufficiently specific alternative hypothesis. This point gives one an opportunity to discuss problems about designing hypotheses after studying the data. Finally, there is the problem of describing independence carefully and indicating how this hypothesis differs from a uniform distribution in the length of time between birth and death months.

This problem emphasizes the need to supplement informal heuristics by a sound theoretical framework which serves to guarantee that plausible conclusions are sound. Further, this problem is not closed. Several additional questions are open to study. The death rate in the birth month is also high. Should this enter the analysis? What would a detailed study of death dates within a month of the birth date show, and can this be studied effectively in view of the potentially small samples involved? Does the ability of modern medicine to prolong life for a few days if necessary have an effect on the death rate very near the birth date? There seem to be low death rates four and six months before the birth month. Does this mean anything?

<u>Advantages and disadvantages of the problem-oriented approach</u>. Some major advantages of the problem-oriented approach are:

a. Discussion of meaningful and nontrivial problems evokes interest and raises fundamental statistical questions.

b. Sensible, even if only partial, answers to realistic questions provide reinforcement for understanding of key concepts and motivation for further study.

c. The opportunity to use computer simulations is valuable and may help to motivate students.

d. A feeling for the excitement of research is derived when the traces of false starts and preliminary studies are not erased.

Some major disadvantages of the problem-oriented approach are:

a. This approach leads to the use of methods and ideas that have not been carefully digested in advance. There will be some need to operate at levels of less than complete understanding. The uninvolved and unmotivated student may find it difficult to salvage anything from such a course.

b. Live problems are often very complicated. It will be difficult but necessary to simplify without throwing away too much of the essence.

c. While this approach should be exciting to those who are involved, it may be frustrating to students who expect to be told what to do and how, and who find open-ended questions or nonresolved philosophical issues difficult to tolerate.

d. In this approach, concepts and methods arise in context and not in a systematic framework for review by the student without undue repetition.

e. The heuristic approach suggested here sometimes leads to incorrect results or paradoxes. When these are corrected, the student may feel insecure. How is he to know when his reasoning is sound without being told by the teacher? Some rigorous follow-up may be necessary.

f. Problems that seem easy may require a good deal more maturity from the students than we think. It may be difficult to transform unmotivated students, required by their major departments to take a statistics course, into participants in statistical inquiry.

Selected Sources for Cases:

Brownlee, K. A. "Statistics of the 1954 polio vaccine trials." Journal of the American Statistical Association, 50 (1955), pp. 1005-1014. (An invited address on the article "Evaluation of 1954 field trial of poliomyelitis vaccine: Summary Report." Poliomyelitis Vaccine Evaluation Center, University of Michigan, April 12, 1955.)

Cochran, William G.; Mosteller, Frederick; Tukey, John W. Statistical Problems of the Kinsey Report. American Statistical Association, 806 15th St., N.W., Washington, D. C. 20005, 1954.

Coleman, James S., et al. Equality of Educational Opportunity. Washington, D. C., U. S. Government Printing Office, 1966. (A set of Correlation Tables, separately bound, is also available for the use of research workers.)

Cutler, S. J. "A review of the statistical evidence on the association between smoking and lung cancer." Journal of the American Statistical Association, 50 (1955), pp. 267-83.

Heermann, Emil F. and Braskamp, Larry A. Readings in Statistics for the Behavioral Sciences. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1970.

Kinsey, A. C.; Pomeroy, W. B.; Martin, C. E. Sexual Behavior in the Human Male. Philadelphia, Pennsylvania, W. B. Saunders Company, 1948.

Mosteller, Frederick, et al., eds. Statistics by Example, Part III, Detecting Patterns. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1973.

Mosteller, Frederick, et al., eds. Statistics by Example, Part IV, Finding Models. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1973.

Moynihan, Patrick. The Negro Family: The Case for National Action. Washington, D. C., U. S. Government Printing Office, 1965. Reprinted in Rainwater, Lee and Yancy, William L. The Moynihan Report and the Politics of Controversy. Cambridge, Massachusetts, MIT Press, 1967.

Peters, William. Readings in Applied Statistics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

Report on Lung Cancer, Smoking, and Health. Public Health Bulletin 1103, Superintendent of Documents, U. S. Government Printing Office.

Steger, Joseph A., ed. Readings in Statistics for the Behavioral Sciences. New York, Holt, Rinehart and Winston, Inc., 1971.

Tanur, Judith, et al., eds. Statistics: A Guide to the Unknown. San Francisco, California, Holden-Day, Inc., 1972.

"The Cochran-Mosteller-Tukey Report on the Kinsey Study: A Symposium." Journal of the American Statistical Association, 50 (1955), p. 811.

Tufte, Edward R. The Quantitative Analysis of Social Problems. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970.

V. USE OF DATA SETS

The use of problem material involving real data in teaching the introductory course in statistics overcomes some of the recognized defects of the traditional systematic course. By focusing attention on problems involving data which have intrinsic interest for the student, it is possible to introduce important statistical ideas and concepts in a context that will serve to facilitate student understanding. Use of examples and cases from real life can also be of great help in motivating students. Further, they can serve as vehicles for illustrating the application of statistical concepts and methods in a variety of settings.

Data sets useful for the introductory statistics course can be small or large. Small data sets can be extracted from investigations in a variety of fields (e.g., sample surveys of voters or consumers, federal statistical reports, annual business reports, and the like), or they can be generated by various class experiments or investigations. They are characterized by limited amounts of data suitable for illustrating a particular statistical concept or technique. An example is the annual earnings of a company during the past ten years, which might be used for comparing average earnings in the first five years with those in the second five years.

Large data sets are characterized by large volumes of interrelated data. For example, a data set on voting behavior may contain information about a variety of personal characteristics of the voter (e.g., age, sex, income, education, marital status, attitudinal data) as well as a variety of voting behavior (e.g., voting behavior in elections during the past five years at national, state, and local levels). Another example of a large data set is medical information about a segment of the population (e.g., age, sex, height, weight, blood pressure, cholesterol level, etc.). Large data sets are often available on tapes and can be stored in local computers for ease of access by students.

For the introductory statistics course that features a systematic development of statistical concepts, large and small data sets are used in essentially similar fashion. For example, to illustrate the construction of a confidence interval, one might use data on consumer food expenditures or on blood pressure of patients. These data could be given to the student in the form of a small data set, or he could be asked to perform a limited task of analysis with these data. (This is in distinction to the use of large data sets as a means of developing understanding of statistical concepts unsystematically in a problem-oriented fashion.)

Use of problem material based on real data sets requires that the instructor devote class time to discussion. The setting of the problem should be discussed initially so that students understand it clearly. After the statistical analysis has been completed, the meaning and interpretation of the results should be discussed, as well as any further analyses that might be required for investigation of the problem.

The use of data sets is not restricted to illustrating the application of particular statistical methods. They can also be used as a meaningful vehicle in sampling experiments. The data set for this purpose would be viewed as a population, and repeated samples would be selected from it by a table of random numbers or by computer. An advantage of this use of data sets over sampling from artificial populations is that real data sets permit the students to interpret the results more meaningfully. For example, the demonstration that the sampling distribution of the sample mean becomes more concentrated with increasing sample size is more meaningful for many students when they can interpret the numbers involved in real terms, e.g., as incomes or blood pressures.

The use of data sets requires computational effort. Although we strongly believe that the use of real data sets adds a valuable dimension to the introductory statistics course, we also strongly believe that students should not engage in computational drudgery. Consequently, adequate computational facilities need to be provided if large data sets are to be employed. If these facilities are not available, small data sets with simplified numbers should be employed.

Experience suggests that students find data sets most interesting when they come from an area of interest to them. Thus, if the introductory statistics course is taught in a number of sections, it would be desirable to set up one or more sections for social science students, one or more sections for physical science students, and one or more sections for biological science students. Cases, examples, and data sets from these subjects can then be used so that students will be motivated by illustrations from situations with which they are familiar. If the introductory statistics course is taught in only one section, examples and cases should be selected that are simple and easy to understand by most of the students in the class.

Data sets can be obtained from diverse sources. One major source, containing demographic, sociological, economic, and many other kinds of data, is:

U. S. Bureau of the Census. Statistical Abstract of the United States, 1970. Washington, D. C., U. S. Government Printing Office, 1970.

Data on consumer behavior and voting preferences, and other types of sample survey data are available from various survey research centers. A list of sources for such data sets is:

Social Science Data Archives in the United States, 1967. Council on Social Science Data Analysis, 605 West 115th Street, New York 10025.

Three books sponsored jointly by the National Council of Teachers of Mathematics and the American Statistical Association contain a wealth of statistical examples and applications:

Mosteller, Frederick, et al., eds. Statistics by Example, Part III, Detecting Patterns. Reading, Massachusetts, Addison-Wesley Publish-Company, Inc., 1973.

Mosteller, Frederick, et al., eds. Statistics by Example, Part IV, Finding Models. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1973.

Tanur, Judith, et al., eds. Statistics: A Guide to the Unknown. San Francisco, California, Holden-Day, Inc., 1972.

In addition to using data sets furnished by the instructor, students can obtain data sets of their own from their particular field of interest, or the entire class can develop data by class-organized demonstrations and experiments (see Section VII) or by surveys conducted by members of the class.

VI. USE OF COMPUTERS

A computer can serve three broad roles in the implementation of the goals of an introductory statistics course. It can: (1) clarify certain key ideas of the course, (2) perform routine numerical calculations, and (3) facilitate more active student participation in the development of statistical concepts. Each of these roles is illustrated below. The question of whether a student should learn computer programming inevitably arises in discussions of the role of computers in education. It is this Panel's view that so much is demanded in the ordinary introductory statistics course, that the taking of additional time to teach programming is not justified. This Panel does not feel that the learning of programming will enable the student to obtain sufficiently greater analytical skills to warrant taking this time away from other topics. There may be reasons in the context of a student's entire curriculum for his being taught programming. If so, a decision must be made in the context of the curriculum as to which course should contain the teaching of programming.

We turn now to illustrations of the various ways in which computers can be used to clarify important statistical concepts. The long-run relative frequency interpretation of probability can be illustrated by having a computer repeatedly simulate an experiment which results in "success" or "failure" on each trial and print the relative frequencies of success after 10, 20, 30, ... trials. The experiment should not be such a trivial one that students can find the probability by the examination of a few equally likely outcomes.

One of the primary objectives of this introductory course is to introduce students to the notions of variability and uncertainty; these concepts can be illustrated in a number of ways on the computer. The notion of a confidence interval, for example, can be brought to life with a computer. A large number of 80% confidence intervals obtained from samples of size 20 from a binomial distribution with $p = 0.6$ are easily obtained on a computer and the proportion of intervals containing 0.6 determined. The confidence level should be low enough so that students will see cases where $p = 0.6$ does not lie within the interval. This demonstration can be linked to the testing of statistical hypotheses and can provide a sound basis on which to discuss that topic.

Properties of estimators can also be studied on the computer by obtaining sampling distributions of estimators, e.g., distributions of $\frac{\Sigma(x-\bar{x})^2}{n}$, $\frac{\Sigma(x-\bar{x})^2}{n-1}$, $\frac{\Sigma(x-\bar{x})^2}{n-2}$ for fairly small n, to illustrate what is meant by an unbiased estimator. Variances of different estimators also can be studied by examination of sampling distributions generated by the computer.

An additional benefit can result from the use of random number generators in an introductory course. The necessity of knowing that pseudo-random digits generated by a computer possess characteristics expected of random digits presents an opportunity to introduce the study of statistical tests such as the chi-square test and tests of runs.

A second role of the computer in an introductory statistics course is to perform routine numerical calculations that are required in the analysis of the problems. This Panel believes that the computational aspects of the subject should be held to a minimum, consistent with the clarification of statistical ideas and principles and with the motivation of students by the introduction of interesting, contemporary problems. If an analysis of a situation that interests an instructor and his class requires computational assistance, then by all means some computational aid should be used, whether it be from an electronic computer or a desk calculator. Students expect a course to contain a touch of realism, and they are frequently disappointed if they do not attain some experience with handling large amounts of data. Students should learn the procedures for calling up a statistical package because that knowledge is often used by those who analyze data. Three widely used statistical packages are:

Dixon, W. J., ed. Biomedical Computer Programs, University of California Publications in Automatic Computation #2, Second Edition. Los Angeles, California, University of California Press, 1970.

Hogben, David, et al. OMNITAB II User's Reference Manual. NBS Technical Note 552, U. S. Department of Commerce. Washington, D.C., U. S. Government Printing Office, 1971.

Nie, Norman H., et al. Statistical Package for the Social Sciences. New York, McGraw-Hill Book Company, 1970.

Even when major reliance for computation is placed on a large electronic computer, there are occasions when the understanding of a concept or a computational technique requires that students perform computations on a desk calculator or by hand. One or two simple problems illustrating the Student t-test, for example, should require students to calculate sample variances. Experience indicates that it is unwise to assume that students always understand what computations are required with raw data and know that statistical analysis is not always based on large amounts of data requiring the use of a computer.

A computer can also serve instructors who wish to break away from presenting all the ideas of the course through formal lectures. We refer to this third function of computers as their "interactive" role. For example, the instructor need not lecture students as to what constitutes a "large" sample in order to be able to apply the Central Limit Theorem; instead, an empirical investigation can be undertaken. First, the instructor can have the students guess minimum values of n for which sample means tend to be normally distributed when sampling from a number of populations with different characteristics. The computer can then be used to generate sampling distributions of sample means for different sample sizes, thereby providing a basis for refined estimates of minimum n and ultimately enhancing the students' understanding of the Central Limit Theorem and the degree to which the shape of the population density and sample size affect the shape of the sampling distribution. Working with data of these kinds, students usually raise new questions. In the case cited, students naturally want to know how to decide whether their sampling distributions are normal, and the instructor has the motivation for a new investigation or at least a mention of inferential problems. Another way of encouraging student participation through interaction with the computer is the selection of n and α to obtain a reasonable power curve for testing hypotheses about binomial p. Students also can be asked to select those independent variables in a set of variables which are important in a regression analysis; economists in most colleges and universities can provide the instructor with suitable data for this purpose.

The interactive role of the computer is enhanced by the existence of terminals and a sufficient number of screens in the classroom, but this equipment is not essential. Students can be required to make investigations of the sort illustrated above as homework exercises using the available computer facility.

In the use of a computer in any of its three functions, it is desirable to give the student a feeling of control over the operations of the computer. This can be done by requiring the student to specify parameters, desired output, and the like. For instance, in a sampling experiment the student might specify the population to be sampled, the sample size, and the number of trials, as well as the nature of the output (e.g., frequency distribution, histogram). Some students also benefit by writing simple programs using subroutines of basic statistical operations.

The printout of a computer run will be of interest to the student to the extent that he is interested in the original problem. Hence, our earlier comments on the importance of interesting and meaningful data sets are equally relevant whether the calculations are performed on a computer or by the student.

Even when a computer is not available, students can still realize some of its benefits. For instance, an instructor can obtain a complete set of computer printouts for his students. Students can use this set of printouts in the same fashion as a laboratory manual. At appropriate times the instructor can refer to one of the printouts and explain how a particular analysis is carried out with the computer, explain the types of information given on the computer printout, and indicate how this information would be used for analysis of the data. In this way the students can obtain some of the benefits of learning how a computer can assist in statistical analysis and also obtain computer illustrations of some of the basic statistical concepts such as sampling distributions without actually having access to a computer. Dixon discusses the use of a "side inch" of computer printout in Review of the International Statistical Institute, 39 (1971), pp. 315-339. The "side inch" can be obtained for $3.00 by writing to Professor W. J. Dixon, Department of Biomathematics, School of Medicine, University of California, Los Angeles, California 90024.

The use of computers in teaching introductory statistics can be expensive. In addition to the costs of the equipment, the use of computers requires much planning by the instructor, as well as substantial monitoring during the course. A modern programmable minicomputer provides a relatively inexpensive way of performing a wide variety of statistical operations.

The impression has been gained by many instructors using computers in teaching introductory statistics that their use is of interest to students and motivates them. Despite their wide use, however, only limited formal evaluation of the effectiveness of computers in the introductory statistics course has been carried out to date. Undoubtedly, the coming years will provide important information on the best way in which computers can be used in introductory statistics courses as well as on their cost-effectiveness as teaching and learning devices.

None of the above mentioned uses of computers are to be confused with computer-assisted instruction. This method of instruction is expensive for use in the introductory course and, when compared with a well written programmed learning text, it may not be economically justified at this time. That is not to say, however, that technology will not advance sufficiently far in the next few years to make computer-assisted instruction an important alternative to existing modes of instruction.

Selected References:

Buford, Roger L. Statistics: A Computer Approach. Columbus, Ohio, Charles E. Merrill Publishing Company, 1968.

Carmer, S. G. and Cady, F. B. "Computerized data generation for teaching statistics." The American Statistician, 23 (1969), pp. 33-35.

Computers in Undergraduate Education: Mathematics, Physics, Statistics, and Chemistry. Proceedings of a Conference sponsored by the National Science Foundation and conducted at the Science Teaching Center of the University of Maryland, College Park, 1967.

Development of Materials and Techniques for the Instructional Use of Computers in Statistics Courses. Department of Statistics, University of North Carolina, Chapel Hill, North Carolina, 1971.

Foster, F. G. and Smith, T. M. F. "The computer as an aid in teaching statistics." Applied Statistics, 18 (1969), pp. 264-270.

Freiberger, W. and Grenander, U. A Short Course in Computational Probability and Statistics, Applied Mathematical Sciences, Volume 6. New York, Springer-Verlag New York Inc., 1971.

Lohnes, P. R. and Cooley, W. W. Introductory Statistical Procedures: With Computer Exercises. New York, John Wiley and Sons, Inc., 1968.

Milton, Roy and Nelder, John, eds. Statistical Computation. New York, Academic Press, Inc., 1969. The following articles:

1. "Computer-assisted instruction in statistics," William W. Cooley, pp. 337-347.

2. "Computers in the teaching of statistics: Where are the main effects?" David L. Wallace, pp. 349-361.

3. "Time sharing and interactive statistics," P. M. Britt, et al., pp. 243-265.

Proceedings of a Conference on Computers in the Undergraduate Curricula. The University of Iowa, Iowa City, Iowa, 1970. The following articles:

1. "Use of computers in undergraduate statistics instruction," Mark I. Appelbaum and Donald Guthrie, pp. 2.1-2.3.

2. "The computer as an instructional tool for the statistics course," Young O. Koh, pp. 2.4-2.17.

3. "Using the computer in basic statistics courses," Richard L. Wikoff, pp. 2.18-2.24.

4. "Use of computers in teaching statistics to engineering students," Armit L. Goel, pp. 2.25-2.31.

5. "SAMDS: A program to generate empirical sampling distributions of the mean," Henry G. Garrett, pp. 2.32-2.38.

6. "Computer-assisted teaching of experimental design: The development of a 'Master Experimenters' Program," Albert R. Gilgen and Michael A. Hall, pp. 2.39-2.46.

<u>Proceedings of the Second Annual Conference on Computers in the Undergraduate Curricula</u>. Dartmouth College, Hanover, New Hampshire, 1971. The following articles:

1. "Use of computers in statistical instruction," Richard N. Schmidt, pp. 437-443.

2. "Inclusion of explanatory material in computer programs for statistical analysis," Jolayne Service, pp. 444-455.

3. "STATLAB, a simple programming system for the statistics laboratory," G. F. Atkinson, pp. 456-462.

<u>Review of the International Statistical Institute</u> (2 Oostduinlaan, The Hague, Netherlands.) Vol. 39, Number 3 (1971). The following articles:

1. "Notes on available materials to support computer-based statistical teaching," W. J. Dixon, pp. 257-286.

2. "Side Inch," W. J. Dixon, pp. 315-339.

3. "On using a conversational mode computer in an intermediate statistical analysis course," D. Quade, pp. 343-345.

4. "Development of materials and techniques for the instructional use of computers in statistics courses," D. Quade, pp. 361-362.

5. "Survey sampling in a computerized environment," T. E. Dalenius, pp. 373-397.

Schatzoff, Martin. "Application of time-shared computers in a statistics curriculum." <u>Journal of the American Statistical Association</u>, 63 (1968), pp. 192-208.

Sterling, T. and Pollack, S. "Use of the computer to teach introductory statistics." Communication of the Association of Computing Machinery, 9 (1966), pp. 274-276.

VII. EXPERIMENTS, SIMULATIONS, DEMONSTRATIONS, AND TEACHING AIDS

Since planning of experiments and analysis of actual data are major concerns of most practicing statisticians, it would seem that one might use relatively simple experiments to develop basic statistical concepts, to obtain data for future analysis, and to give some experience in designing an experiment. There are arguments against this approach. It can use up a considerable amount of time for the individual student, for the class, and especially for the teacher. It can sometimes be frustrating. The arguments for such an approach center about the personal involvement of the student, his increase in interest, and his discovery of important principles or relationships. There is considerable evidence that such an approach can be successful. Practically all of the basic training courses in statistical quality control used experiments and demonstrations to introduce statistical concepts, and these uses were judged successful. Two helpful references on these uses are:

Olds, E. G. and Knowler, L. A. "Teaching statistical quality control for town and gown." Journal of the American Statistical Association, 44 (1949), pp. 213-230.

Scherwin, R. L. "Teaching aids for statistics and quality control." Industrial Quality Control, 23 (1967), pp. 654-660.

1. Demonstrations and Verification Experiments

For out purposes we would like to distinguish among several types of experiments. The first type we shall designate as a "demonstration." The student is given an assignment for which the instructor has the theoretical solution. If the student also knows or guesses the expected outcome, then this approximates the typical high school science experiment which simply illustrates a law stated or derived in the text book. Requiring students to toss a coin 100 times and to record the total number of heads would be a typical demonstration. The students know that the probability is close to ½ and most of them also know that one does not really expect 50 heads and 50 tails.

An assignment that the student count the number of rolls of a die until all six digits occur or count the number of digits in a random digit table until all ten digits are found is another type of experience. The instructor, with his knowledge of the geometric distribution and the theorems on sums of independent random variables, may know the mean and variance of the two distributions. He also may know that the distributions are skewed. The student does not have this knowledge and in this first course is unlikely to acquire it. To the student this assignment is a venture into the unknown. On the other hand, the instructor, while he may be surprised by an individual result, is quite sure that he will not be surprised by the entire set of results. We will call an activity such as this a "verification experiment."

There are many demonstrations and verification experiments from which the students may discover or better understand basic statistical concepts. The student might be asked to toss a coin 64 times, recording his results (0 or 1) in a 4 by 16 table, and to find the average for each column, for each row, and for the entire table. These results could be used to justify statements about the distribution of the sample mean as n increases. The student might be asked to draw samples, using a sampling paddle, from a box of beads or balls of at least two different colors. Bowls of numbered chips might be used, made up to represent different distributions, either of different types, such as rectangular, triangular, exponential, and normal, or of one type with different parameters. The student might draw small samples from a population, construct a confidence interval for each sample, and determine the proportion of confidence intervals which include the population mean or proportion. In general, the distribution of any sample statistic might be motivated by a demonstration or experimental verification. To a great extent, the value of such activities depends upon the student's interest and enthusiasm. In any case, the activities should be used with moderation.

Selected References:

Berkeley, Edmund C. Probability and Statistics--An Introduction Through Experiments. New York, Science Materials Center, 1961. This book describes some 27 experiments. It is written to accompany a kit.

Dixon, Wilfrid J. and Massey, Frank J., Jr. Introduction to Statistical Analysis. New York, McGraw-Hill Book Company, 1969. Most chapters have a set of class exercises based on random number tables, bead drawings, etc.

Harrison, R. D. "An activity approach to the teaching of statistics and probability." (in three parts) Mathematics Teaching, 34 (1966), pp. 31-38; 35 (1966), pp. 52-61; 36 (1966), pp. 57-65.

Malpos, A. J. Experiments in Statistics. Edinburgh, Scotland, Oliver and Boyd, Ltd., 1969.

2. Open Experiments

An "open experiment" is quite different from a demonstration or an experimental verification. Neither the instructor nor the student is able to predict the outcome. As an extreme example, let us consider a different version of the coin-tossing demonstration. The class, in some manner unknown to the instructor, will split into two groups of equal size. Each member of the first group will actually toss a coin 100 times and record the outcomes (0 or 1) in order. Each member of the second group will write out a sequence of 100 numbers (0 or 1) without tossing a coin, in such a fashion that the sequence, in his opinion, simulates the results of actual coin tossing. Using statistical tests (which may be developed over the term), the instructor will attempt to identify in which group each sequence belongs. The teacher might classify a sequence as group two if the results come too close to 50-50, or if there are too many runs, or if the longest run is suspiciously small, or if the cumulative fraction of heads stays too close to $n/2$. But now the experimental situation is completely reversed; the students know the answers and the teacher is the discoverer.

Actual classroom experiments of the open type can cover a wide range of difficulty. A relatively simple one is to ask the class the probability of getting a head on a coin if the coin is spun instead of tossed. First, one must specify the essential conditions. What kind of coin? How long a spin? If the coin is spun by holding it with the finger of one hand and flicking it with a finger of the other hand, does the initial position of the coin make a difference?

Some apparently simple experiments can suddenly develop complications. Consider the experiment of empirically determining the approximate probability that a thumbtack will fall point up. Obviously one must first specify at least the particular make of thumbtack. But does one need to specify the type of surface it falls on? And can one speed up the experiment by taking ten thumbtacks in a container and shaking them up? In one class, two students shook the thumbtacks in a china cup with a gentle lateral motion and then poured them on a table. Result--almost 100% points up! At this point another student decided to investigate the effect of different grades of sandpaper for the receiving surface and discovered an interesting regression problem.

An open experiment that can display individual differences is the "coin shove" experiment (see the Jowett reference, below). On some smooth poster board 30 inches long, draw a target line (designated 0) approximately 6 inches from one end. Also draw lines an inch apart (designated +1, +2, ...; -1, -2, ...), parallel to the target line. Then draw a starting line approximately 6 inches from the other end of the poster board. A trial of this experiment consists of shoving a coin from behind the starting line so that it stops as close to the target line as possible. The purpose of the experiment is to study the frequency distribution of scores that result from n repeated trials. Many questions can be investigated:

Is there a significant improvement in the scores if 5 practice crials are allowed? Is there a significant difference between the abilities of individuals? Is it reasonable to assume independent trials? Can the scores of individuals be improved by allowing the use of a slingshot-type coin shooter? At what value of n is student interest replaced by boredom?

Selected References:

"Interim Report of the R. S. S. Committee on the Teaching of Statistics in Schools." Journal of the Royal Statistical Society, Series A, 131 (1968), pp. 478-497.

Jowett, G. H. and Davies, H. M. "Practical experimentation as a teaching method in statistics." Journal of the Royal Statistical Society, Series A, 123 (1960), pp. 10-35.

Mosteller, Frederick, et al. Probability with Statistical Applications, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970. Manual describes a variety of experiments, demonstrations, and teaching aids. Pages 449 to 454 of the text suggest a set of projects for highspeed computers.

Rade, Lennart, ed. The Teaching of Probability and Statistics. New York, John Wiley and Sons, Inc., 1970.

3. Simulations

Another type of laboratory experience might be classified a "simulation." Here there may exist a theoretical solution to the problem, but it may not be at the competence level of the class. One example might be a simplified epidemic where each day 20 persons line up in random order. Suppose that of the $n = 20$ persons, $x = 2$ persons are contagious, $y = 3$ persons are immune, and $z = 15$ are susceptible. A susceptible person who stands next to a contagious person catches the disease. The duration of the disease is one day, so that a contagious person is immune the next day. To simulate the epidemic, shake a container having two red beads labeled C for contagious, three yellow beads labeled I for immune, and 15 white beads labeled S for susceptible, and pour the beads in a trough or a long-stemmed funnel. (A computer may be used instead of beads.) Suppose the result for the first trial is:

S S C S S I C S S I S S S S S I S S S S

Then on the next trial we would have three infectious persons (places 2, 4, 8), five immune persons (places 3, 6, 7, 10, 16), and 12 susceptible persons. Variables which might be investigated are the number of trials until the epidemic is ended, the number of susceptible left at the end of the epidemic, and the largest number of contagious persons at any time. The simulation study can be enlarged by investigating the behavior of the epidemic for various values of x, y, and n.

Another simulation is the server problem. Customers arrive at integral values of time t from 1 to n (for example, let $n = 50$). The probability of a customer's arriving at any time t is a constant p (for example, $p = 1/5$). The service time for a customer is a constant c less than $1/p$ (for example, $c = 4$). At the end of n time intervals the gates are closed but the customers in line must still be served, and of course the persons serving them must be paid overtime. For equipment one can use a die, a pack of playing cards, a table of random digits, or a computer. The first decision that needs to be made concerns what data to record. After one class tried the experiment, they decided that they should record: number of arrivals, total service time, idle time, overtime, sum of the delay times, and sum of squares of delay times. This last number they named the "riot index."

Films

A number of films pertaining to probability, statistics, and quality control are available. Of those known to the Panel, the film "Random Events" of the PSSC Physics Series seems most suitable for the kind of course we are considering.

Selected Films:

1.	Probability and Statistics 25 minutes B & W 1957	ASSN Films 347 Madison Avenue New York, New York 10036
2.	Mathematics and the River 19 minutes Color 1959 Horizons of Science Series	Educational Testing Service 20 Nassau Street Princeton, New Jersey 08540
3.	Probability and Uncertainty 56 minutes 1965	Educational Service, Inc. 40 Galen Street Watertown, Massachusetts 02172
4.	Mean-Median-Mode 13 minutes Color, B & W 1966	McGraw-Hill Text Films 330 West 42nd Street New York, New York 10018
5.	Probability 12 minutes Color 1966	McGraw-Hill Text Films 330 West 42nd Street New York, New York 10018
6.	Matter of Acceptable Risk 30 minutes B & W 1967	Indiana University Audio-Visual Center Bloomington, Indiana 47401
7.	Random Events 31 minutes B & W 1962 PSSC Physics Series	Modern Learning Aids 315 Springfield Avenue Summit, New Jersey 07901

8.	How's Chances 30 minutes B & W 1957 Westinghouse	Association Films 347 Madison Avenue New York, New York 10017
9.	It's All Arranged 30 minutes B & W 1957 Westinghouse	Association Films 347 Madison Avenue New York, New York 10017
10.	Photons 19 minutes B & W 1960 PSSC Physics Series	Modern Learning Aids 315 Springfield Avenue Summit, New Jersey 07901
11.	Predicting Through Sampling 10 minutes Color 1969	Bailey Films Associates 11559 Santa Monica Blvd. Los Angeles, California 90025
12.	The Probabilities of Zero and One 11 minutes Color 1969	Bailey Films Associates 11559 Santa Monica Blvd. Los Angeles, California 90025
13.	Probability, An Introduction 9 minutes Color 1969	Bailey Films Associates 11559 Santa Monica Blvd. Los Angeles, California 90025

Visual Aids and Other Materials

There are available a variety of visual aids and other helpful materials. Various manufacturers of games offer large dice, roulette wheels, chuck-a-luck cages, etc. Biased and mismarked dice are available, although it is sometimes difficult to locate a source.

Selected Sources:

a. Lightning Calculator Company, Box 6192, St. Petersburg, Florida 33736.

1. Quincunx or Galton Board. A device for generating a binomial or normal distribution. This model features an adjustable outlet which enables one to shift the population mean, a sliding control which enables one to drop from one to five beads at a time, and a control for examining small samples before they are accumulated into a large sample; completely self-enclosed so that the beads cannot drop out.

2. Sampling Demonstrators. Consists of six different colors of beads, plastic container, and sampling paddles. Available in two different models, one with 2000 beads and three sampling paddles, a second with 1000 beads and two sampling paddles.

3. Control Chart Simulator. Device consisting of various distribution patterns, dowel rods for indicating limits of variability, etc., and frame. Reverse side has set of horizontal wires with beads to demonstrate observations over time, either individual or grouped.

b. Ray R. Lilly, 30 Lilly Road, Wanaque, New Jersey 07465.

1. Sampling demonstrators, plastic balls in various colors and lot sizes, sampling paddles. Available in different models or to specification.

2. Demonstration Board. One side consists of various models for distributions, with templates (attachable) to show distribution of average for $n = 5$ and $n = 20$. Also a model to show change in variability. Dowel rods show specification limits, control limits, modified control limits, etc. Reverse side consists of one set of horizontal wires for distribution over time of either individuals or subgroups, plus overlay set of wires to demonstrate distribution of medians.

3. Three-dimensional models. One consists of a peg-board with plastic rods and beads to demonstrate multivariable control charts, bivariate distributions, etc. A second has a set of sliding channels with pegs, which can be used to demonstrate correlation, regression, etc.

c. E. S. Lowe Company, Inc., 200 Fifth Avenue, New York, New York 10010. Manufacturer of games and accessories. Items such as large dice (up to 4 inches), roulette wheels, bingo games, chuck-a-luck, miniature slot machines, etc.

d. Hunter Spring Company, Lansdale, Pennsylvania 19446. Normal frequency distribution template. For use in illustration work. Small template with $\sigma = 10$ mm.

e. Tell Manufacturing Company, 200 South Jefferson Street, Orange, New Jersey 07050. Manufacturer of plastic beads and balls in various size ranges from 6 mm to 23 mm diameter. Some 13 different colors. Smaller sizes in units of 100.

f. Walco Products Company, 37 West 37th Street, New York, New York 10018. Manufacturer of wooden beads in various shapes and sizes. Spherical beads come in sizes from 3 mm to 20 mm. Cylindrical and barrel shapes up to 1 inch in height. Small beads must be ordered in units of 1000, larger in units of 100.

g. Quality Service Foundation, Weena 700, Rotterdam, Netherlands.

1. Polyhedra numbered 0 to 9 for generating random digits or selecting random samples.

2. Drafting triangle with two normal templates for distribution of individuals and distribution of averages. (n = 5)

h. Japanese Standards Association, Ginza Higashi 6-1, Chuo-ku, Tokyo, Japan. Icosahedron numbered 0 to 9 for generating random digits or for selecting random samples.

Course Organization

Inclusion in the introductory statistics course of the varied motivational and instructional activities, such as the use of data sets, computers, sampling experiments, and films, is time-consuming. The instructor may therefore wish to add another class meeting each week to permit more extensive use of this pedagogy.

One method of including these activities is by scheduling them in class when they naturally arise. An advantage of this approach is that the student can understand the activity in its context. A disadvantage is that many of the activities, such as sampling experiments, are time-consuming and therefore may disrupt the continuity of the overall development.

Another approach is to schedule formal laboratory periods from time to time, during which these activities take place. Then the development of the course material is not disrupted, but the timing of events can not always be made to coincide with the class developments.

Learning Resources Centers

Demonstrations, verifications, experiments, and simulations require physical facilities and organization. The Panel has been encouraged to learn of various institutions that have developed learning resources centers where students can listen to tapes, view films, find programmed material for remediation, carry out experiments, and plan their own group investigations. Such a center provides an opportunity for small group discussions both at the planning stage of an investigation and at the analysis stage. For some of the experiments discussed, such as the server problem or the epidemic problem, it is advisable to have students work in teams of two to four, with each team repeating the experiment a small number of times. The results can then be pooled. For other experiments there will need to be serious consideration given to the question of pooling. For the coin shove experiment, for instance, the question of pooling results will raise serious problems and can lead to a discussion of individual and group differences.

Organizing students into small groups to carry out investigations can be an effective means of stimulating student interactions and can thereby aid the learning process. Student teams should be reasonably small so that each student can participate effectively.

The teacher may meet with each team from time to time or be available for assistance when a team requires it.

Some learning resources centers have a time-shared console or are adjacent to a computation facility having such consoles. The Panel suggests that mathematics departments offering statistics courses consider the establishment of such a facility.

ADDITIONAL RESOURCE MATERIALS

Folks, LeRoy. "Some prior probabilities on the future of statistics." The American Statistician, 24 (1970), pp. 10-13. References cited in the Folks article:

1. "Some aspects of the teaching of statistics." John Wishart, 1939. JRSS, 102, pp. 532-564.

2. "The teaching of statistics." Harold Hotelling, 1940. Annals of Math. Stat., pp. 457-471.

3. "On the future of statistics." M. G. Kendall, 1942. JRSS, 105, pp. 69-91.

4. "The teaching of statistics." Institute of Mathematical Statistics Committee on the Teaching of Statistics, 1948. Annals of Math. Stat., 19, pp. 99-115.

5. "The teaching of statistics." John Wishart, 1948. JRSS, A, 111, pp. 212-229.

6. "Why statistics?" P. C. Mahalanobis, 1948. Sankhyā, 10, pp. 195-228.

7. "The future of data analysis." J. W. Tukey, 1962. Annals of Math. Stat., 33, pp. 1-67.

8. "On the future of statistics--a second look." M. G. Kendall, 1968. JRSS, A, 131, pp. 182-194.

Kirk, Roger E., ed. Statistical Issues: A Reader for the Behavioral Sciences. Belmont, California, Brooks/Cole Publishing Company, 1972.

Klein, Morris. Mathematics in Western Culture. New York, Oxford University Press, Inc., 1953. The following articles:

1. "The mathematical theory of ignorance: The statistical approach to the study of man," pp. 340-358.

2. "Prediction and probability," pp. 359-375.

3. "Our disorderly universe: The statistical view of nature," pp. 376-394.

Mathematical Thinking in Behavioral Sciences (Readings from The Scientific American). San Francisco, California, W. H. Freeman and Company, 1968. The following articles:

1. "Chance," A. J. Ayer, October , 1965.

2. "What is probability?" Rudolph Carnap, September, 1953.

3. "Subjective probability," John Cohen, November, 1957.

4. "Probability," Mark Kac, September, 1964.

Newman, James R., ed. World of Mathematics, 4 vols. New York, Simon and Schuster, Inc., 1956. The following articles:

Vol. 2, Part VII: The Laws of Chance

1. "Concerning probability," Pierre Simon de Laplace, p. 1325.

2. "The red and the black," Charles Sanders Peirce, p. 1334.

3. "The probability of induction," Charles Sanders Peirce, p. 1341.

4. "The application of probability to conduct," John Maynard Keynes, p. 1360.

5. "Chance," Henri Poincaré, p. 1380.

Vol. 3, Part VIII: Statistics and the Design of Experiments

1. "Foundations of vital statistics," John Graunt, p. 1421.

2. "First life insurance tables," Edmund Halley, p. 1437.

3. "The law of large numbers," Jacob Bernoulli, p. 1452.

4. "Sampling and standard error," L. C. Tripett, p. 1459.

5. "On the average and scatter," J. C. Moroney, p. 1487.

6. "Mathematics of a lady tasting tea," Sir Ronald A. Fisher, p. 1512.

7. "The vice of gambling and the virtue of insurance," George Bernard Shaw, p. 1524.

Preparation for Graduate Work in Statistics, 1971. CUPM, P. O. Box 1024, Berkeley, California 94701.

Review of the International Statistical Institute. 2 Oostduinlaan, The Hague, Netherlands. Vol. 39, Number 3 (1971). The following articles:

1. "New techniques of statistical teaching: Opening remarks," pp. 253-256.

2. "The first course in statistical methods and the use of teaching aids," K. Austwick, J. Hine, G. B. Wetherill, pp. 287-306.

3. "Post college continuing education activities in statistics," J. Stuart Hunter, pp. 307-311.

4. "Comments about a general audience TV course on statistics," J. Hemelrijk, pp. 312-314.

5. "The joint American Statistical Association-National Council of Teachers of Mathematics Committee on the Curriculum in Statistics and Probability," F. Mosteller, pp. 340-342.

6. "Operation of the Centre for Applied Statistics in Medicine and Medical Biology," D. Schwarz, pp. 346-347.

7. "The teaching of statistics," T. Yoshizawa, pp. 348-350.

8. "New techniques of statistical teaching," H. C. Hamaker, pp. 351-360.

9. "The jackknife," F. Mosteller, pp. 363-368.

10. "Round table meeting: Recommendations," pp. 369-372.

The American Statistician. 805 15th Street, N.W., Washington, D. C. 20005. "The Teacher's Corner" frequently contains articles on topics suitable for an introductory statistics course.

The Statistician. 55 Park Lane, London, W 1, England. See Vol. 18, Number 3, 1968:

The Teaching of Statistics

1. "Innovation in teaching," D. T. Page, p. 207.

2. "Teaching management statistics by programmed instruction," D. Kitchen, pp. 209-226.

3. "Practical classes in statistics," H. M. Davies, pp. 227-236.

Varberg, Dale E. "The development of modern statistics," The Mathematics Teacher. Part 1, 56 (1963), pp. 252-257; Part 2, 56 (1963), pp. 344-348.

COMPUTING

During the decade beginning in 1962, CUPM made a continuing effort to advise college mathematics departments on curricular matters related to the tremendous growth in the use of the computer and the pervading influence which the computer has come to exert on society. Initial steps in this direction were taken by the Panel on Physical Sciences and Engineering, which issued its Recommendations on the Undergraduate Mathematics Program for Work in Computing* in 1964. Taking account of the significant changes which had recently occurred in the relationship of mathematics to computing and to computing machines, the Panel proposed a program designed to prepare students whose careers were likely to be intimately connected with highspeed computing. The program included reference to three types of courses: (1) mathematics courses of a general nature which should be available for the prospective specialists in computer science; (2) technical courses in computer science; and (3) an introductory course in computer science.

Two years later CUPM commissioned R. W. Hamming of Bell Telephone Laboratories, Inc., to prepare a monograph on Calculus and the Computer Revolution.* Published in 1966, this book describes and illustrates briefly some aspects of computing as they are related to the beginning calculus course.

A task force was appointed in 1966 for the purpose of advising CUPM on a future course of action with regard to computing. The task force suggested the creation of a Panel on Computing, which would work closely with various computing organizations and would have several charges related to the impact of the computer on mathematics education. Such a panel was appointed in 1967.

One of the Panel's projects was to gather and disseminate information regarding the use of computers in introductory calculus courses. A newsletter entitled "Calculus With Computers,"* issued in 1969, contained general observations and summaries of statements from various institutions which had instituted computer-oriented calculus courses.

The Panel's primary aim was to develop a systematic approach to the impact of computers on undergraduate mathematics programs, rather than to address itself to the training of computer scientists per se. (The latter topic had already been considered by the Association for Computing Machinery in its report Curriculum 68--Recommendations for Academic Programs in Computer Science.) The Panel formulated a specific undergraduate program in computational mathematics, combining courses in mathematics, computer science, and computational mathematics--complete with course outlines and suggestions for implementation. This course of study is presented in the 1971 publication

* Not included in this COMPENDIUM

Recommendations for an Undergraduate Program in Computational Mathematics. The main concern of this report is for the education of mathematicians who wish to know how to use and to apply computers.

The report of the Panel on Computing attacked a significant problem: the need for new, innovative courses directly concerned with computational mathematics and computer science. Remaining to be considered, however, was another important question: How should the computer affect traditional mathematics courses? To study this question and related points, CUPM in 1971 appointed a Panel on the Impact of Computing on Mathematics Courses to succeed the Panel on Computing. The new Panel's investigations culminated in the publication of Recommendations on Undergraduate Mathematics Courses Involving Computing in 1972. This document includes outlines for lower-division courses in elementary functions, calculus, discrete mathematics, and linear algebra with stress on algorithms, approximations, model building, and the nature of the entire problem-solving process.

RECOMMENDATIONS FOR AN UNDERGRADUATE PROGRAM

IN

COMPUTATIONAL MATHEMATICS

A Report of

The Panel on Computing

May 1971

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PREFACE

During the last two decades the development of computers has helped to stimulate the dramatic increase and diversification in the applications of mathematics to other disciplines. In the belief that the time is appropriate for a systematic approach to the impact of computers on undergraduate mathematics programs, the CUPM Panel on Computing presents this report.

> Our basic recommendation is that mathematics departments should experiment with innovative undergraduate mathematics programs which emphasize the constructive and algorithmic aspects of mathematics, and which acquaint students with computers and with the uses of mathematics in computer applications.

A specific undergraduate program in computational mathematics is proposed. This is not a program in computer science, nor is it a minor modification of the traditional undergraduate mathematics major. It is, rather, a program in the mathematical sciences that combines courses in mathematics, computer science, and computational mathematics. It can be used as a basis for further specialization in any of several areas, including computer science, or mathematics, or one of the areas of application of mathematics.

1. Philosophy and Aims of the Program

Since publication of the 1964 CUPM report Recommendations on the Undergraduate Mathematics Program for Work in Computing, computer science has developed as a separate area of study. More and more colleges and universities are establishing computer science departments, and the number of students enrolled in computer science programs is increasing rapidly. The need for separate curriculum studies in this new area was recognized by the Association for Computing Machinery, and in 1968 its Curriculum Committee on Computer Science published a report entitled Curriculum 68--Recommendations for Academic Programs in Computer Science. This widely acclaimed report is still regarded as giving a good description of curricula in computer science. Its recommended minimal mathematics preparation is about equivalent to that usually required of students in the physical sciences and engineering.

More recently, three trends have become noticeable. First, there appears to have developed a strong tendency on the part of computer science programs to minimize prerequisite requirements in traditional mathematics, particularly analysis, and also to underemphasize or even to disregard most areas of scientific computing.

Second, many disciplines, including in particular the biological, social, and behavioral sciences, have become increasingly mathematical, giving rise to a need in these fields for expanded education in mathematics and in scientific computing. Finally, the computer has begun to have a direct effect upon mathematics courses themselves. New courses, particularly in computationally-oriented applied mathematics, are being introduced into many mathematics curricula, and traditional courses are being modified and taught with a computer orientation. As an example of the latter we cite only the teaching of calculus. Approximately 100 schools now offer a course in calculus using the text Calculus, A Computer Oriented Presentation, published by the Center for Research in College Instruction in Science and Mathematics. Other computer calculus projects were reported in the 1969 CUPM Newsletter, "Calculus with Computers," now out of print.

These three trends all indicate a need for the mathematics community to accept a responsibility for mathematical or scientific computing and to broaden educational opportunities toward a more encompassing "mathematical science" in which students may explore the areas of overlap between pure and computational mathematics, as well as computer science. There is thus a need for innovative undergraduate programs which provide for a wide range of options, different opportunities for graduate study, and a variety of future careers.

A new view of mathematics as a mathematical science in the above sense raises many curricular questions, to which several CUPM panels have begun to address themselves. In particular, a need arose for reappraisal of the already-cited 1964 report. Such a reappraisal is desirable if for no other reason than that a large number of all undergraduate mathematics majors are likely to find themselves later in some computer-related field.

The present report is the result of such a reappraisal by the CUPM Panel on Computing. From the outset it was evident that the aims of this report should be different from those of the earlier work, since its intended audience is different. The present report does not address itself to the training of computer scientists. Instead, its concern is for the education of mathematicians who will know how to use and to apply computers. Programs in computational mathematics necessarily have different objectives than do programs in computer science.

In accordance with our previous remarks, the mathematics program presented here is intended to be a departure from the traditional undergraduate mathematics curriculum. It should not be regarded, however, as a replacement for that curriculum, but rather, together with it, as one of several equally valid options for students of the mathematical sciences. It should meet the needs of students who plan to enter careers in scientific computing or who wish to enroll in graduate programs in computationally-oriented applied mathematics. With some suitably selected options during the senior year, a continuation in many computer science graduate programs should be possible. With other options, a continuation in pure mathematics should also be possible. At the same time, several of the courses included in the program meet the mathematical needs of students in other disciplines and may also be appropriate for prospective secondary school mathematics teachers.

The program proposed here is presented in a spirit of open experimentation, not as a final product. In its design the Panel has been neither as conservative nor as radical as it might have been. For instance, a conservative approach might be to combine a list of suitable mathematics courses of a traditional nature with a complementary list of computer science courses. This is easily accomplished in an institution having both a mathematics and a computer science department, but it leads to a large number of required courses and provides for little or no interaction between the two parts of the program. At the other extreme stands a curriculum in which computing has been completely integrated with the mathematical material, either by the introduction of new courses or by the repackaging of old ones.

In designing its program the Panel has taken a path somewhere between the extremes indicated above. Several new computer-oriented mathematics courses are described here; at the same time, some standard computer science and mathematics courses are included and, in particular, no recommendations are made concerning the redesigning of standard mathematics courses, such as the calculus, to include computer use. Where they are available, such computationally-oriented basic mathematics courses could be ideal components of this program, but their definition still requires considerable study and experimentation. The Panel felt that such a study on its part would serve only to divert its attention from its main concern, namely, the description of a new curriculum in computational mathematics for the undergraduate mathematics major which can be implemented in many institutions without excessive cost or delay.

In this latter connection the Panel believes that its program can be offered even by smaller colleges having suitable access to educational computing equipment, with only modest additions to their mathematics staffs. More specifically, through the junior year, the new computationally-oriented mathematics courses recommended here number only four. These, together with the three basic and relatively standard computer science courses, could be handled by the equivalent of one mathematician interested in applied mathematics with an emphasis on computing and numerical analysis and one specialist in computer science. The remaining core courses can be taught by the other members of the mathematics department. Clearly, this small staff could offer only a few of the additional courses listed in this report as possible electives, but the Panel believes that even such a minimal program would be desirable for many students.

2. Recommendations and Brief Course Descriptions

For a major undergraduate program in Computational Mathematics we recommend a basic core curriculum of 12 one-semester courses: five in mathematics, four in computational mathematics, and three in computer science. We will refer to these courses in the sequel, respectively, by the symbols M1, M2, M3, M4, M5, CM1, CM2, CM3, CM4, C1, C2, and C3.

Each of the courses carries 3 credits; at the same time it is desirable that some of the computer-oriented courses include a scheduled laboratory period for which additional credit may be awarded. As described below, this sequence can be handled in three years, leaving the senior year for electives, also set forth below.

2.1 Basic Component

Before describing the 12 courses in the Basic Component, it may be instructive to illustrate one way of imbedding them into the first three undergraduate years. In the chart on page 534, arrows indicate the "prerequisite structure," i.e., the dependency of each course on those which precede it. Notice that two courses are recommended for each semester. Mathematical progress within the program is not different from that in standard programs. If the student wishes to switch to pure mathematics after sampling the eight core courses of the first two years, it will be a simple matter for him to do so with no loss of mathematical pace. It should also be noted that of the CM and C courses, three are taught in the first semester and four in the second semester of each year. This part of the program could easily be handled by the equivalent of two teachers in a small college where multiple sections are unlikely.

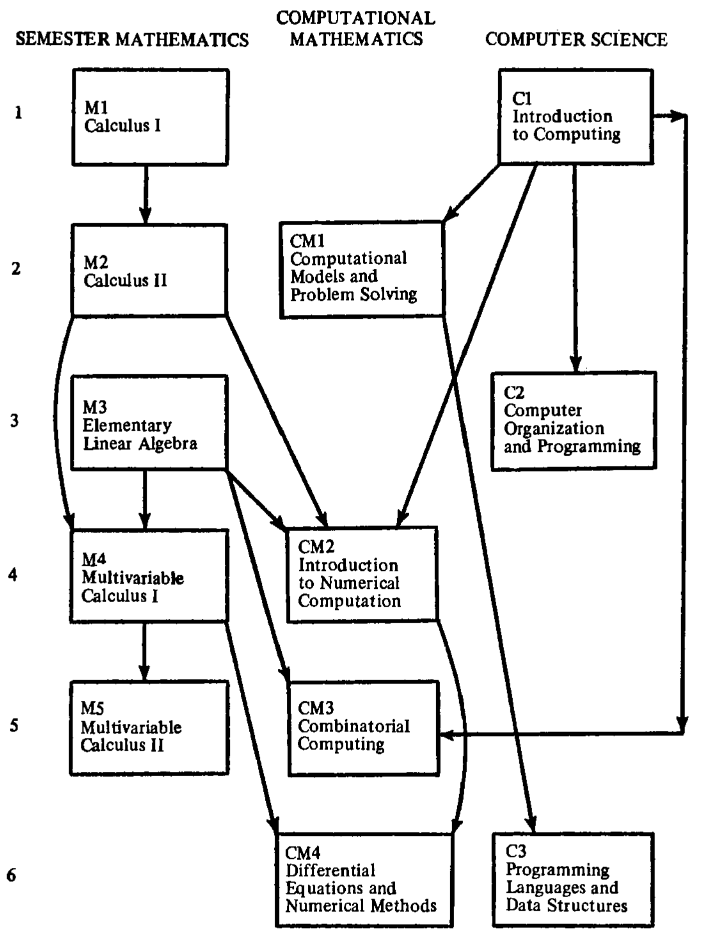
Let us now describe these 12 courses briefly, leaving detailed course outlines and references for Section 4.

a) Mathematics courses

These five courses are described in the CUPM document Commentary on A General Curriculum in Mathematics for Colleges, page Incidentally, the committee which produced the Commentary has already noted that there is little need for M3 to require M1-M2 as explicit prerequisites. This fact has been observed in the chart.

M1	Calculus I
M2	Calculus II
M3	Elementary Linear Algebra
M4	Multivariable Calculus I
M5	Multivariable Calculus II

CHART SHOWING ONE WAY OF IMBEDDING THE BASIC COMPONENT INTO THE FIRST THREE UNDERGRADUATE YEARS. ARROWS INDICATE THE PREREQUISITE STRUCTURE.



b) Computational Mathematics

These courses constitute the heart of our program. While their spirit is mathematical, computing plays an important role in each. The courses CM1 and CM3 are novel in character, while CM2 and CM4 are intended to replace the traditional first courses in Numerical Analysis and Ordinary Differential Equations. In the initial phase of implementing this program the traditional versions of these courses could be used temporarily in place of CM2 and CM4, thereby allowing the faculty to concentrate first on the development of the new courses CM1 and CM3.

CM1. <u>Computational Models and Problem Solving</u>

Prerequisite: C1

The purpose of this course is to introduce students early in their programs to a wide variety of different computer applications. This is to be accomplished mainly through the construction and interpretation of computational models for several interesting and worthwhile practical problems from various disciplines, including the biological and behavioral sciences as well as the physical sciences and mathematics.

The spirit in which the course is presented is of utmost importance. The applications discussed in the course should be reasonably realistic and comprehensive, and the students should become aware of the very serious difficulties and limitations that can arise. Questions should be raised about the validity of models, the effect of numerical errors, the significance of statistical results, the need for data verification, the difficulties in testing programs, documentation, etc. Whenever possible, the basic mathematical aspects of the different models should be discussed in general and related to the computational results. However, since the course is intended for freshmen or sophomores, no attempt can be made to enter into any deeper analysis of specific mathematical questions. With a proper balance between the computational and mathematical points of view, the course should provide the students not only with an appreciation of both the potential and limitations of computer applications but also with an interest in learning more about the many relevant areas of mathematics.

The outline included in Section 4 places special emphasis on the use of computational models for the simulation of random and nonrandom processes, although a few numerical and nonnumerical computer applications are also included. The latter types of problem will be considered in more detail in the subsequent courses CM2 and CM3.

It should be noted that this course may also be of considerable value and interest to students outside the present program.

CM2. Introduction to Numerical Computation

Prerequisites: C1, M2, M3

This first course in numerical analysis may be taken in the sophomore year. Since it is based on as little as one year of analysis, the emphasis should be more on intuition, experimentation, and error assessment than on rigor. The methods considered should be amply motivated by realistic problems. It is better to treat a few algorithms thoroughly than to be exhaustive in the number of algorithms considered. Students should be expected to program and run a number of problems on a computer, and considerable time should be spent analyzing the results of such runs. In particular, the analysis of roundoff and discretization errors, as well as the efficiency of algorithms, should be stressed.

Topics should include the solution of linear systems, the solution of a single nonlinear equation, interpolation and approximation (including least squares approximation), differentiation and integration, and elements of the numerical solution of eigenvalue problems.

CM3. Combinatorial Computing

Prerequisites: C1 and M3

Combinatorial computing is concerned with the problem of how to carry out computations with discrete mathematical structures. It bears to combinatorial (discrete, finite) mathematics the same relationship that numerical analysis bears to analysis. Numerical analysis is much more widely known and much better developed than combinatorial computing. However, there are many reasons to believe that within the next decade combinatorial computing will rival numerical analysis in its importance to computer users. In fact, outside of the traditional areas of applications of mathematics to the physical sciences, discrete mathematical structures may occur more frequently than continuous ones, and even in large problems in the physical sciences data-handling considerations lead quickly to questions in combinatorial computing.

This course is intended as an introduction to the emerging field of combinatorial computing. Its objectives are (1) to acquaint students with certain types of problems which occur frequently when problems are formulated in combinatorial terms, so that they are able to recognize them when they encounter them in disguise, and (2) to teach students certain important concepts and proven techniques which experience has shown to be useful in solving many combinatorial problems, particularly on a computer.

Typical topics to be covered in the course are the representation of integers, sets, and graphs; counting and enumeration techniques; sorting and searching methods; and selected problems and algorithms in graph theory. Students should be expected to write programs for various algorithms and to experiment with their application to appropriate problems.

CM4. Differential Equations and Numerical Methods

Prerequisites: CM2 and M4

This course is intended to replace the more traditionally oriented course in differential equations in which the focus is often on nonconstructive developments. It has the objective of introducing the student to key concepts underlying the qualitative understanding of differential equations as well as to methods for constructing their approximate solutions. It is intended for the junior year. The historical development of the subject is closely related to the physical and engineering sciences; nevertheless, it is recommended that examples from biology, economics, and other fields be chosen where possible, so as to draw upon a student's intuitive understanding of the processes illustrated. Some further suggestions for such material can be found in the CUPM report Applied Mathematics in the Undergraduate Curriculum, page 705.

As a result of this course the student should have confidence in his ability to develop an approximate solution of a differential equation, be able to discuss the basic qualitative behavior of the solution, and have an appreciation of the importance of analytic methods in furthering his understanding of the subject.

Typical topics should include a discussion of simple linear equations, the initial value problem for the first-order equation $y' = f(x,y)$ and some methods for its numerical solution, a basic introduction to first-order systems and their applications including plane autonomous systems, and finally some topics relating to boundary value problems.

c) Computer Science

The following three courses represent certain modifications of several of the basic courses in Curriculum 68. All three courses should not consist simply of lectures but should also incorporate a scheduled laboratory period.

C1. Introduction to Computing

Prerequisite: College admission

This first course in computing has by now become standard in many institutions. The 1964 CUPM report Recommendations on the Undergraduate Mathematics Program for Work in Computing recommended a particular version of this course, and the corresponding course B1 in

Curriculum 68 has been widely referenced. The course serves several purposes:

(1) To develop an understanding of the concept of an algorithm and of the algorithmic formulation of methods for the solution of problems on a computer.

(2) To train the student in the use of at least one algorithmic programming language and to introduce him to the basic structural aspects of such languages.

(3) To acquaint the student with the basic characteristics and properties of computers.

For the program proposed here the stress of the course should be on problem solving by computer. Accordingly, the student should be assigned a number of different problems both of the numerical and non-numerical type, including at least one larger project.

C2. Computer Organization and Programming

Prerequisite: C1

The purpose of this course is to provide the student with a basic introduction to the structure and organization of digital computers and to the use of assembly language programming systems, without becoming involved in a too-detailed discussion of computer hardware or assembly language programming.

The course proposed here is in part similar to the course B2 in Curriculum 68 with the addition of some topics from the course I3 in the same report. However, unlike those courses, it has primarily a survey character. Typical topics include computer structure, assembly languages, data representation, addressing techniques, elements of logic design, discussion of the principal units of a digital computer, systems software, and a survey of contemporary computers.

C3. Programming Languages and Data Structures

Prerequisite: CM1

This course is intended to introduce the student to some of the elements of programming languages as well as to certain important techniques of organizing and linking together information stored in a computer. Topics covered in the course include the basic structure of algorithmic languages, tree and list structures in a computer, string manipulation, data structure and storage allocation, and basic aspects of languages and grammars. The students should become acquainted with at least two different-level languages, such as a string manipulation language and an advanced algorithmic language.

The course covers a number of topics from the ACM courses I1 and I2 but is otherwise novel in character. Some instructors may find it desirable to use CM3 as a prerequisite; this would be similar in spirit to the approach of the ACM recommendations. But it is equally conceivable to introduce C3 as a prerequisite for CM3, allowing a much wider range of computational assignments in the latter course.

2.2 Elective Component

Given the Basic Component described above, and depending on the student's particular interests, there are several ways to round out a good major program. Broadly speaking, possible technical electives can be grouped under the following six--somewhat overlapping--categories, not necessarily in order of importance:

a) Mathematics
b) Probability and statistics
c) Computationally-oriented mathematics
d) Other applied mathematics
e) Computer science
f) Other disciplines

The specific courses listed here under each of these headings are not meant to exhaust all possibilities; clearly, there are various other choices and variations. If the Basic Component of the program has been completed during the first three years, the elective courses will--most probably--be concentrated during the senior and part of the junior year. But other arrangements of the Basic Component are also possible, thereby allowing for a distribution of elective courses throughout most of the undergraduate program.

a) Mathematics

Several of the courses offered as part of the standard mathematics curriculum can serve as electives for a computational mathematics program. This involves, in particular,

Introductory Real Variable Theory (Mathematics 11-12 of GCMC)
Complex Analysis (Mathematics 13)
Introductory Modern Algebra (Mathematics 6M)
Linear Algebra (Mathematics 6L)
Introduction to Mathematical Logic

The Basic Component, augmented by a year course in real variables and a year course in algebra, would constitute minimally adequate preparation for graduate study in mathematics. These additions could easily be achieved in the senior year.

The standard introductory course in ordinary differential equations has not been mentioned here again since it was replaced by CM4.

A beginning course in partial differential equations is included in subsection c) below.

b) Probability and Statistics

Statistical computations represent a large percentage of scientific computing work in many disciplines. Accordingly, the Panel believes that a good introduction to probability and statistics is highly important to students in a program of the kind discussed here. In fact, it may be very desirable to require such an introduction of all students in the program.

The Panel recommends a one-year combination of probability and statistics with M4 as a prerequisite. The first semester should provide an introduction to probability, with the second covering suitable topics from statistics. Courses like this are already offered in many schools, and recommendations about the material to be covered have been set forth by the CUPM Panel on Statistics in <u>Preparation for Graduate Work in Statistics</u>, page 459.

For the purposes of a computational mathematics program it may be highly desirable to integrate computational aspects directly into these courses. But in line with the approach taken in this report, the Panel did not wish to make any such specific recommendations at this time.

c) Computationally-oriented Mathematics

The courses grouped under this subheading are similar to CM1-CM4; that is, their spirit and content are mathematical, but computing plays an important role in each. Accordingly, it is most desirable that a program in computational mathematics include at least some additional courses of this nature.

From among the variety of possible topics the Panel decided to select five course areas which appear to be fairly representative.

<u>Numerical linear algebra</u>

This course covers the description and analysis of some of the principal computational methods in linear algebra. It uses CM2 and M3 as prerequisites and could replace the standard advanced linear algebra course for students in this program. Typical topics might include a thorough discussion of elimination methods and of Wilkinson's backward error analysis, iterative methods for large linear systems and the corresponding basic convergence results, and methods for solving eigenvalue-eigenvector problems. The various topics should be motivated and illustrated by means of different applications.

Courses like this have become almost standard in many institutions. The course material can be found, for example, in parts of the following texts:

Forsythe, George E. and Moler, Cleve B. Computer Solution of Linear Algebraic Systems. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

Householder, Alston S. The Theory of Matrices in Numerical Analysis. Waltham, Massachusetts, Blaisdell Publishing Company, Inc., 1964.

Noble, Ben. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

Varga, Richard S. Matrix Iterative Analysis. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

Wilkinson, James H. The Algebraic Eigenvalue Problem. New York, Oxford University Press, Inc., 1965.

Applied modern algebra

The purpose of this course is to introduce the student to the discrete algebraic structures most commonly used in applications. It is intended to replace the standard modern algebra course (Mathematics 6M of GCMC) for those students who are concerned with applications of algebra rather than with algebra as pure mathematics. Whereas the topics are in general not intended to be treated in depth, the treatment should be adequate enough in each case to enable the student to read independently in more complete expositions. Accordingly, the presentation should include formal definitions and proofs of fundamental theorems, but at the same time there should be considerable emphasis on practical applications.

While courses on applied and computational linear algebra have become reasonably common, the same cannot be said about courses on applied modern algebra. Moreover, at present there exists essentially only one text on this topic, namely,

Birkhoff, Garrett and Bartee, Thomas C. Modern Applied Algebra. New York, McGraw-Hill Book Company, 1970.

This book contains material for a full year course. A one-semester course on the senior level with a prerequisite of CM3 might begin with a review of set algebra and an introduction to semigroups and groups and some of their applications. Then the stress could be placed on partially ordered sets, lattices and Boolean algebra, and their applications in switching algebra and logic. Another approach would be to play down Boolean algebra and to stress rings and fields, including, in particular, polynomial rings and finite fields, and their applications to coding theory.

Optimization

Optimization problems arise frequently in scientific computer applications. This includes problems from the entire area of mathematical programming as well as from optimal control theory, calculus of variations, and from parts of combinatorics. A one-semester introductory course in optimization problems, with CM2 and M4 as prerequisites, is therefore a highly desirable elective in a program of this kind.

Such a course--not stressing computational aspects--was described in the CUPM report Mathematical Engineering--A Five Year Program, page 649.* It begins with a discussion of specific examples of typical optimization problems from the various cited fields, and continues with an introduction to convexity and n-space geometry, Lagrange multipliers and duality, and the Simplex method. Then it turns to some combinatorial problems and to elements of the classical calculus of variations and of control theory. In a more computationally-oriented version of the course it appears to be desirable to delete the latter three topics and to present instead an extended coverage of the numerical aspects of linear programming, as well as a discussion of transportation problems. The course could then end with an introduction to numerical methods for convex programming problems. The student would be assigned computational projects involving some of the many available library subroutines; in fact, an important by-product of the course in this form might be to familiarize the students with the extensive computational effort that has already been spent in connection with mathematical programming techniques.

There is an extensive list of available references relating to this course. Without attempting to be comprehensive, we mention only the following books:

Berge, Claude and Ghouila-Houri, A. Programming, Games and Transportation Networks. New York, John Wiley and Sons, Inc., 1965.

Dantzig, George B. Linear Programming and Extensions. Princeton, New Jersey, Princeton University Press, 1963.

Hadley, George F. Linear Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

Hadley, George F. Nonlinear and Dynamic Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1964.

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* See also the CUPM reports Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists [page 628] and Applied Mathematics in the Undergraduate Curriculum [page 705].

Künzi, Hans P.; Tzschach, H.; Zehnder, C. Numerical Methods of Mathematical Optimization with ALGOL and FORTRAN Programs. New York, Academic Press, Inc., 1968.

Polak, E. Computational Methods in Optimization. New York, Academic Press, Inc., 1971.

Partial differential equations and numerical methods

The general aim of this course is to survey the standard types of partial differential equations, including, for each type, a discussion of the basic theory, examples of applications, classical techniques of solution with remarks about their numerical aspects, and finite difference methods. By necessity, most proofs of existence and uniqueness theorems and of the properties of the numerical methods are to be omitted.

A course of this kind--based on CM4 and M5--requires in general two semesters, and even then it will be very demanding of the students at the senior level. Typical topics include first-order equations and the elements of the theory of characteristics for linear and quasilinear equations; linear second-order equations in two variables; classification; canonical forms; a discussion of the wave, diffusion, and Laplace equations; and a survey of some topics about other equations. For a description of a one-year course on partial differential equations--not stressing numerical methods--see also the CUPM report Mathematical Engineering--A Five Year Program, page 649.

There do not appear to be any entirely appropriate texts for this course. The following are some possible titles:

Ames, William F. Numerical Methods for Partial Differential Equations. New York, Barnes and Noble, 1970.
Probably too difficult as a text for a first undergraduate course, but valuable as a reference for the course.

Berg, Paul W. and McGregor, James L. Elementary Partial Differential Equations. San Francisco, California, Holden-Day, Inc., 1966.
Elementary introductory text, but does not emphasize numerical methods.

Forsythe, George E. and Wasow, Woolfgang R. Finite Difference Methods for Partial Differential Equations. New York, John Wiley and Sons, Inc., 1960.
Important reference for numerical methods.

Mitchell, A. R. Computational Methods in Partial Differential Equations. New York, John Wiley and Sons, Inc., 1969.

Weinberger, Hans F. A First Course in Partial Differential Equations. Waltham, Massachusetts, Blaisdell Publishing Company, 1965.
Introductory text which places special consideration on physical applications.

Introduction to applied functional analysis

The purpose of this course is to present some of the basic material of elementary functional analysis as it is of use and importance in numerical and applied mathematics. With a prerequisite of CM2 and M5, the course includes an introduction to metric spaces, the contraction mapping theorem and various of its applications, normed linear spaces, linear and nonlinear operators, the differential calculus on normed spaces, applications to iterative processes such as Newton's method, minimization techniques for nonlinear functionals on Banach spaces, and, if time permits, some discussion of the relationships between functional analysis and approximation theory.

By necessity, the material has to be presented from a geometrical and intuitive viewpoint rather than in a formal and abstract manner. Some of the results should be explored further by applying them to specific computational problems; here team projects may be very appropriate.

The following are some texts which cover parts of the material mentioned above:

Collatz, Lothar. Functional Analysis and Numerical Mathematics. New York, Academic Press, Inc., 1966.
Survey of many of the interactions between the two fields.

Davis, Philip J. Interpolation and Approximation. Waltham, Massachusetts, Blaisdell Publishing Company, 1963.
For the connections to approximation theory.

Dieudonné, Jean. Foundations of Modern Analysis. New York, Academic Press, Inc., 1969.
For the differential calculus on normed linear spaces.

Goffman, Casper and Pedrick, George. First Course in Functional Analysis. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1965.
For much of the basic material; not numerically oriented.

Goldstein, Allen A. Constructive Real Analysis. New York, Harper and Row, Publishers, 1967.
For minimization methods.

Kantorovich, L. V. and Akilov, G. P. Functional Analysis in Normed Spaces. Elmsford, New York, Pergamon Press, Inc., 1964.
Contains a detailed discussion of Newton's method.

Kolmogoroff, A. N. and Fomin, S. V. Elements of the Theory of Functions and Functional Analysis, vol. I. Baltimore, Maryland, Graylock Press, 1957.

Schechter, Martin. Principles of Functional Analysis. New York Academic Press, Inc., 1971.

In some of these courses it may be desirable to add a second semester in order to provide a more extended coverage of the material. This applies also to CM4, where a second semester is probably very desirable for many students.

The Panel believes that students in a program such as this might benefit by being able to deepen their knowledge in graph theory and combinatorics beyond the material covered in CM3. A course in this area is described in Applied Mathematics in the Undergraduate Curriculum, page 734.

d) Other Applied Mathematics Courses

As discussed in the beginning, a main aim of this program is to provide the student with a basic understanding of the application and use of computers in the solution of scientific problems. Accordingly, it will be most important that the student acquire a certain familiarity with at least some of the many applications of mathematics and with mathematical model building.

A number of suitable topics for an applied mathematics course are discussed in Applied Mathematics in the Undergraduate Curriculum, page 705 . Outlines for some courses in physical mathematics are described in the CUPM report Mathematical Engineering--A Five-Year Program, page 649. From this latter report we mention, in particular, the following courses:

ME3	Mechanics
ME8	Electromagnetics
ME9	Thermodynamics and Statistical Mechanics
OR2	Operations Research
OR3	Systems Simulation
OM3	Celestial Mechanics
OM4	Orbit Theory
CT2	Control
CT4	Linear Systems
CT7	Information Theory

It should be stressed that for the purposes of this program the specific topics covered in any of these courses are not as important as the applied mathematical spirit, that is, the emphasis on model building, on analysis of the model, and on interpretation of the results.

It should also be noted that by listing these courses separately from those in the previous subsection we do not mean to imply that little or no computational work is to be involved here. In fact, in many of these courses computer applications might prove to be of considerable value and might strengthen the student's understanding of the interrelationship among scientific problems, mathematical models for them, and numerical methods for finding approximate solutions of these models.

e) Computer Science

In an institution with an ongoing computer science program, many of the courses offered as part of that program can serve well as possible technical electives in this curriculum. We mention, in particular, the following courses described in Curriculum 68:

I4	Systems Programming
I5	Compiler Construction
I6	Switching Theory
I7	Sequential Machines
A1	Formal Languages and Syntactic Analysis
A2	Advanced Computer Organization
A4	System Simulation
A5	Information Organization and Retrieval
A7	Theory of Computability

Some introductory courses in mathematical logic covering topics from I7, A1, and A7 are also offered by many mathematics departments and may serve as possible electives.

Finally, it may be of interest to note that with the addition of three or four courses, such as I4 and I5 or A1, to the Basic Component, a student would meet more than the minimal requirements in Curriculum 68 for an undergraduate major in computer science. Such additions could easily be achieved in the senior year.

f) Other Disciplines

This last and yet by no means least important subgroup of possible electives concerns courses in any of the disciplines outside of mathematics which are sources of mathematical computing problems. The Panel firmly believes that an understanding of the ideas, principles, and methods of at least one such area is a basic ingredient of the education of a computational mathematician and hence that any student in this program should take at least some suitable courses in another discipline. It should be stressed that this need not be the traditional introductory physics sequence, but that beginning courses in the engineering, biological, behavioral, or social sciences might be equally appropriate. The specific type and number of courses depends in each case on what is available, the field selected, and the student's depth of interest.

3. Implementation of the Program

3.1 Staff

It was stated earlier that the program of the Basic Component could be carried out by a mathematics department with the equivalent of one faculty member interested in numerical analysis and computing, and one in computer science. Such a department could offer some of the elective courses as well, depending on the interests of its members. A year course in Probability and Statistics is already taught in many colleges, and courses in Modern Applied Algebra are beginning to appear in addition to, or as replacements for, the usual courses in Abstract Algebra. Courses resembling those in Optimization or Applied Functional Analysis are also offered by many colleges.

Thus, while the entire program could not be offered except in an institution with several faculty members in applied and numerical mathematics as well as in computer science, much of it--and especially the Basic Component--may be possible in a small college with an expanded mathematics department as described above, provided a computer is available.

This leaves, of course, the question of staffing the computing facility itself, which in turn depends strongly on the nature of that facility. In most cases, such a facility requires the supervision of at least one professional manager or director, who in turn may be capable of teaching the necessary computer science courses in this program. Besides this person, many colleges have found that the problem of staffing the computing laboratory can be solved in part, or even completely, through the students themselves. One of the virtues of the computer as an instructional device is the personal involvement that it demands of and readily receives from the students. They learn quickly for the most part and teach one another very effectively. They serve well in many jobs associated with the operation of the computer facility. To bring them formally into the teaching process is sensible and rewarding.

3.2 Facilities

Apart from dealing with the arrays of desk calculators which have served statistical laboratories in the past, mathematics departments have not faced the wide variety of problems connected with the incorporation of laboratory work into their academic programs. The implementation of this program necessarily requires careful planning and maintaining of proper laboratory facilities. Because of sustained increases in costs of education, college administrations are understandably hesitant to incur major new expenditures. The following discussion is directed toward helping to clarify or distinguish among various factors which might characterize a computational facility suitable to this program.

The principal ways of incorporating computer use into an educational program can be characterized as follows:

1) Discussion of computational results obtained directly or indirectly by the instructor.

2) Student use of computers outside the classroom in a batch mode. Here, typically, programs are collected and submitted to be run together on a computer, without the possibility of further interaction from the originator. Very often the input is in the form of punched cards.

3) Student use of computers outside the classroom in a time-sharing mode. Here either simple teletypewriters or more elaborate character- and graphical-display devices are in open communication with the processor. A user can input his program almost instantaneously and, by executing or modifying it at will, he is able to interact in an experimental manner with the computational process.

4) Use of time-shared classroom display facilities to integrate the presentation of the theoretical and computational aspects of the course material.

5) Use of special laboratories having dedicated computers (i.e., reserved solely for this use) for part or all of the meetings of the class in order to integrate computational work directly into the instructional process.

6) Use of special laboratories for computer-aided instruction.

At present, the most frequently used approaches are those under 1), 2), and 3); for this program, 1) by itself is not satisfactory. Accordingly, we shall focus our discussion primarily upon the use of the batch mode 2) or the time-sharing mode 3).

No matter which type of computational service is chosen, the most essential points appear to be that it must be reliable, responsive to fluctuating student demands during a semester, and capable of allowing the student to complete assignments in a reasonable time span. In line with this, a complete dependence on slack-hour use of a computer owned by local industry, the shared use of campus equipment dedicated primarily to accounting and administrative functions, or the "generous" gift of an outdated computer will generally prove unsatisfactory.

For most of the requirements of this program, computational services in the batch mode can be entirely satisfactory, effective, and at the same time economical. One of the critical factors is then the "turnaround time" between the submission of input and the return of the output to the originator. Since the completion of a problem by a student may require four to eight, or even more, machine runs, a turnaround time that allows at least two runs during a normal day appears to be rather desirable. (With the aid of multi-processing systems, it is possible to achieve a turnaround time of a few minutes or less for short student runs.) Besides the turnaround time, another controlling parameter in batch service is the availability of ancillary equipment for producing and handling punched cards. Here queues easily develop which are not readily reduced without considerable cost. It may be hoped that this latter problem will be alleviated considerably by the development of less expensive mark-sensing or character-reading devices.

Time-sharing services have much to recommend them. However, their costs are generally higher than those of acceptable batch services. Moreover, they can also lead to considerable queueing problems if not enough consoles are available to the students. The critical parameter is the maximal number of terminals which can be sustained by the particular computer system without a significant degradation of the response time.

The repertoire of available computer languages is an important consideration for any computational service. For many of the requirements of this program, one scientific language such as FORTRAN, BASIC, ALGOL, PL/1, or APL is sufficient. In general, however, it is desirable that the student gain experience with more than one language, and in certain courses, such as C3 or CM3, additional languages such as SNOBOL are particularly important. In several courses, including, for instance, CM1, CM2, or CM4, plotting and display facilities could also play a useful role. Indeed, here a versatile time-shared classroom display system of the type mentioned under 4) might be ideal and could completely determine the character of the courses. However, more modest services can be completely successful.

Broadly speaking, the computational services required by this kind of program can be provided in one or a combination of the following ways:

1) Use of off-campus computing facilities

2) Participation in an educational computer network

3) Operation of a campuswide educational computer facility

4) Operation of separate computer laboratories by different departments

Except under special circumstances, exclusive dependence on the first of these approaches is, in the long run, not very satisfactory. However, certain supplementary off-campus computer services, if reliable and economical, can provide highly advantageous solutions to enriching more modest services available on the campus

At present, educational computer networks have been established in only a few geographical locations. The organization of these networks ranges from fairly loose mutual assistance groups to highly organized hardware networks. Either time-sharing or batch-processing services can be provided--sometimes both. Clearly, the access to a large central computer with a massive program library, large memory, fast central processor, and large systems and programming staffs represents a considerable advantage. On the other hand, logistical and communication problems, lack of control, etc., may turn out to be very detrimental for a participant college. Nevertheless, the possibility of joining such a network when feasible certainly deserves proper consideration.

Probably the most common approach toward meeting the educational computer needs of a college or university is the establishment of a centralized, campuswide academic computing center. Such a center will serve its expected purpose only when operated by an adequate staff in an efficient, professional manner; this is a point too often overlooked.

In recent years many small and medium-sized computers have been marketed at relatively low prices. This has made it possible for many institutions to have separate computers of varying sizes for individual departmental use. The assured availability of a specialized service to the department is, of course, one of the greatest advantages of this approach. It also allows the development of special laboratories of the type mentioned under 5) above. On the other hand, the computational work possible on these machines is severely limited by their size, and for more sophisticated tasks additional computer services are often needed.

The actual costs of a computing facility depend upon many factors, including the desired quality of the service, the intended group of users, the specific type of equipment selected, local physical facilities, and the corresponding staff needs. The Panel therefore decided not to include here any cost estimates for the facilities needed in this program. Some data on such costs are given, for example, in recent reports of the Southern Regional Education Board and the American Council on Education.*

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* See Guidelines for Planning Computer Centers in Universities and Colleges and Computers in Higher Education, both publications of the Southern Regional Education Board, 130 Sixth Street, N.W., Atlanta, Georgia 30313. See also Computers on Campus, American Council on Education, One Dupont Circle, Washington, D. C. 20036, and "A Survey of Computing Costs," CRICISAM Newsletter 3, September, 1971, pp. 2-5.

4. Detailed Course Outlines

In this section we present outlines for the seven courses CM1, CM2, CM3, CM4 and C1, C2, C3. They are intended to suggest topics which might be included in these courses and should not be interpreted as check lists of required material. Where appropriate, suggested numbers of lectures to be spent on the various topics are included in the descriptions. These lectures total approximately 36 hours for a semester course; this leaves room for examinations, reviews, and lectures on supplementary technical material.

CM1. Computational Models and Problem Solving

Prerequisite: C1

Depth of treatment for the topics outlined below will vary with the interest of instructor and students, and lecture hours are therefore not assigned to any of the topics. It is recommended that a small number of fairly substantial projects be required in this course, rather than a larger number of smaller problems. Some of the material is suitable for group projects.

Detailed Outline

Statistical calculations

- Tabulation of data
- Calculation of means and variances
- Least squares fitting of straight lines
- Intuitive meaning of randomness
- Random number generators
- Tests of generators (e.g., chi-square)

Simulation of random processes

- Queues, inventories, random walks, etc.
- Discussion of statistical significance (confidence intervals)
- Games such as blackjack and bingo
- Monte Carlo calculations

Simulation of nonrandom processes

- Simple hypothetical computer
- Approximations to physical, economic, and biological processes
- Discussion of errors in such approximations
- Deterministic games such as nim

Other nonnumerical problems

- Enumeration
- Searching and sorting

Connectivity of graphs, shortest paths
Text editing
Elementary computer graphics
Handling arithmetic expressions

Sample Problems

1. Develop a program for the least squares fitting of straight lines to given data. The program should input pairs of values (x_i, y_i) and output the values of (a,b) where $y = ax + b$ is the best fit. Is the same line obtained when the values of x and y are interchanged? Show how your program can be used to fit curves given by $y = ab^x$ or $y = ax^b$ by taking the logarithms of each side of these equations. Are the results the same as those obtained by a true least squares fit of these curves without taking logarithms?

2. Write a program to generate 1000 pseudo-random numbers and calculate the chi-square statistic that is associated with 10 equal subintervals of the interval in which the random numbers are supposed to be uniformly distributed. If the numbers are random, the value of this statistic should exceed 16.9 with a probability of only 5 per cent. On the basis of this test, have you any reason for doubting the usefulness of your generator?

3. One relatively simple game of solitaire begins with a deal of nine cards, face up. If any two of these cards have the same face value, they are covered with two new cards, also face up. The last step is repeated until the deck has been exhausted except for one card, in which case the dealer has won the game, or until there are no more pairs showing, in which case the dealer has lost. Write a program to simulate this game and use it to determine an approximation to the probability of winning. How reliable do you believe the approximation to be?

4. Describe a model of cars moving through a highway toll station, and write a program to simulate the process. Use it to find approximations to the average delay and show how this delay depends on traffic density. Discuss the main limitations of your model. Assuming that one has a good model, what further limitations are there in the results obtained from any such simulation?

5. Write a program to simulate a game of blackjack and use it to compare different strategies. (This problem can be used as the basis for a group project.)

6. Describe a simple hypothetical computer and write a program to simulate its behavior. The description of the machine should be carefully documented so that any potential user will be able to determine exactly what the machine will do in every conceivable circumstance.

7. A man starts at the southwest corner of a field and runs north at 15 feet per second. His dog starts at the southeast corner,

200 feet from where the man starts, and runs directly towards his master at the rate of 40 feet per second. Calculate an approximation to the dog's path and to the time taken by the dog to catch his master. Compare this time with the time required if the shortest path had been taken.

8. Suppose that the adjacency matrix for a graph is given, along with two of its nodes. Write a program that will determine whether or not there is a path between the two nodes. Develop a second program for the same task, but based on a distinctly different algorithm, and compare the relative merits of the two different programs.

9. Develop a program for right-justifying text material. Input to the program should be a paragraph of text, and the corresponding output should be the same paragraph properly justified. (This problem can be expanded into a more substantial project on text editing by including additional features such as section headings and paging.)

10. A package of programs is to be developed for producing sequences of pictures. The pictures are to be output on a printer and must therefore be relatively simple, but the basic ideas are similar to those needed for computer-produced movies. (This can be a good group project. Once agreement is reached on how to represent the data, members of the group can be assigned separate tasks, such as developing subprograms for input, output, moving, shrinking, and rotating pictures.)

<u>Bibliography</u>

Most introductory books on computer programming contain material on computer applications. Some of these texts are cited in the outline of course C1 below. The following texts are primarily concerned with computer application problems suitable for this course:

Barrodale, Ian; Ehle, Byron L.; Roberts, F. D. K. <u>Elementary Computer Applications in Science, Engineering, and Business</u>. New York, John Wiley and Sons, Inc., 1971.

Gruenberger, Fred and Jaffray, George. <u>Problems for Computer Solution</u>. New York, John Wiley and Sons, Inc., 1965.

Hull, Thomas E. and Day, David D. F. <u>Computers and Problem Solving</u>. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970. (in particular, Part 2)

CM2. Introduction to Numerical Computation

Prerequisites: C1, M2, M3

Each of the major topics in the course should be amply motivated by introducing applications from the physical and social sciences. A consideration of electrical networks or input-output systems in economics leads, for instance, to linear systems; vibration problems from mechanics or Markov processes provide examples for eigenvalue problems; root-locus problems arise in several areas of engineering; observational data collected in practical experiments lead to a consideration of interpolation and least squares techniques. A selection of problems can be found, for example, in the book by Carnahan, Luther, and Wilkes (see bibliography).

During the course the students should solve a number of problems on the computer. Some of these should involve programming of the simpler algorithms and others should make use of library subroutines.

Detailed Outline

Introduction (2 lectures)

Number representation on a computer
Computer arithmetic
Discussion of the various types of errors

Linear systems of equations (9 lectures)

Gaussian elimination and the LU factorization
Partial and complete pivoting
Example of ill-conditioning
Discussion of ways for detecting ill-conditioning
The Wilkinson backward error result and its implications (no proofs)
Iterative improvement
Iterative methods with simple convergence criteria (no proofs)

Solution of a single nonlinear equation (6 lectures)

Successive approximation
The Point of Attraction Theorem and its implications
Discussion of the rate of convergence
Newton's method and the simplified Newton method
Secant method and method of false position
Stopping criteria for iterations
Extension of Newton's method to two equations in two unknowns
Roots of polynomials
Sturm sequences
Example of ill-conditioning of the roots of a polynomial

Interpolation and approximation (6 lectures)

- Lagrange interpolating polynomial
- Error term for an interpolating polynomial
- Newton forward and backward difference polynomials
- Piecewise polynomial interpolation
- Least squares approximation, including numerical problems associated with the normal equations and orthogonal polynomials and their use in least squares
- Chebychev economization of power series

Numerical differentiation and integration (6 lectures)

- Error in differentiating the interpolating polynomial
- Differentiation by extrapolation to the limit
- Integration formulas based on interpolating polynomials and the associated error terms
- Romberg integration
- Gaussian quadrature formulas
- Adaptive methods

The eigenvalue problem (6 lectures)

- Direct root-finding methods such as Muller's or the secant method
- The power method for the dominant eigenvalue
- Subdominant eigenvalues by the inverse iteration method
- The Householder-Givens method for symmetric matrices (without proofs)

Bibliography

Carnahan, Brice; Luther, H. A.; Wilkes, James O. Applied Numerical Methods. New York, John Wiley and Sons, Inc., 1969.
Primarily as a source of problems.

Conte, Samuel D. Elementary Numerical Analysis: An Algorithmic Approach. New York, McGraw-Hill Book Company, 1965.

Fox, Leslie and Mayers, D. F. Computing Methods for Scientists and Engineers. New York, Oxford University Press, Inc., 1968.

Fröberg, Carl E. Introduction to Numerical Analysis, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

Henrici, Peter K. Elements of Numerical Analysis. New York, John Wiley and Sons, Inc., 1964.

McCracken, Daniel D. and Dorn, William S. Numerical Methods and FORTRAN Programming. New York, John Wiley and Sons, Inc., 1964.

Stiefel, E. L. An Introduction to Numerical Mathematics. New York, Academic Press, Inc., 1963.

Wendroff, Burton. First Principles of Numerical Analysis. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

CM3. Combinatorial Computing

Prerequisites: C1 and M3

The material listed here may be more than can be covered properly in one semester. Since many topics are rather independent of each other, an instructor can make his own selection of what to exclude. For this reason no breakdown into the number of lectures for each topic was included. Students are expected to implement some of the algorithms on the computer and also to experiment with relevant library subroutines. For this computational work, it may be desirable to assign team projects rather than to let every student proceed on his own.

Detailed Outline

The machine tools of combinatorics

- Integers and their representation, including radix, modulo, and factorial representation (and its use in indexing over permutations), monotonic vector representation (and its use in indexing over combinations and partitions)
- Sets and their representation, including bitstring and index representation
- Some aspects of list processing and storage organization, including representation of variable length sequences, one- and two-way lists, tree structures, free storage, and garbage collection

Enumeration and counting

- Enumeration techniques, such as backtrack and sieve methods
- Counting techniques, including recurrence relations and techniques for solving them, Pólya's counting formula

Sorting

- Internal sorting; insertion, selection, and enumeration methods
- External sorting; long-sorted subsequences, merging, distribution sorting

Searching

- Searching in a linearly ordered set, including hash-coding or scatter storage techniques, Fibonacci search
- Trees and their use in ordering sets, rooted trees and their properties, representation of trees, methods of traversing trees, internal and external path length, optimal and near optimal search trees

Heuristic search, game trees, minimax evaluation, pruning, static evaluation functions, backing up uncertain values

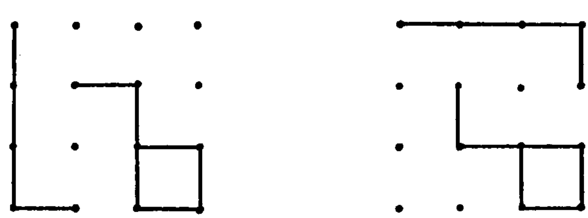
Graph algorithms

Some concepts from graph theory, such as graphs, directed graphs and their representation, paths, trees, circuits and cutsets
Connectedness and shortest path problems, including various related algorithms
Flow problems, max-flow and min-cut theorem, Ford-Fulkerson algorithm
Spanning trees, and algorithms for finding them
Graph isomorphisms
Planarity of graphs

<u>Sample Problems</u>

1. In how many different ways can one color the six faces of a cube which may be freely rotated with two colors? [Topics: counting, group of transformations, Pólya's theorem]

2. An integrated circuit manufacturer builds chips with 16 elements arranged in a 4 x 4 array as shown below. To realize different circuits all patterns for interconnecting the elements are needed. Direct interconnections are made only between horizontally or vertically adjacent elements, e.g., as shown below:



(Closed loops do not usually occur, but this is ignored here for simplicity's sake.) To deposit interconnections on the chip a photo-mask of the interconnection pattern is needed. Notice that the same photo-mask will do for the two interconnection patterns shown above. How many photo-masks are required in order to lay out all possible interconnection patterns on these chips?

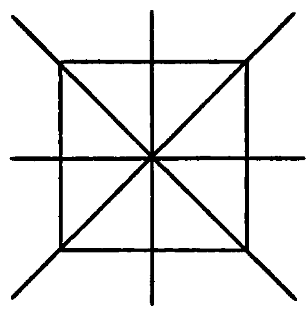
a) Carefully define the permutation group involved.

b) Solve the problem using Burnside's lemma alone.

c) Solve the problem using Pólya's counting formula.

[Topics: counting, group of transformations, Pólya's theorem]

3. List all the essentially different ways in which eight queens can be placed on a chessboard so that no two are on the same row, column, or diagonal. Two ways of placing queens are essentially different if they cannot be transformed into each other by a rotation of the board or by reflection on any of the axes shown in the figure:



4. Assume a large deck of N punched cards is dropped on the floor, but fortunately each card contains a unique sequence number from 1 to N which indicates its position in the deck. After the cards have been picked up, the deck is not in complete disorder; it contains long runs of cards in proper order. Discuss what sorting techniques can be considered to sort the deck as efficiently as possible. What standard sorting techniques would definitely be inefficient in this case? [Topics: linear order, expected number of comparisons, sorting algorithms]

5. a. Prove that every positive integer A has a unique representation $a_1, a_2, \ldots, a_n$ which satisfies the conditions

(i) $A = a_1 \cdot 1! + a_2 \cdot 2! + \ldots + a_n \cdot n!$;

(ii) $0 \leqq a_i \leqq i$ for $i = 1, 2, \ldots, n$;

(iii) $a_n \neq 0$.

Let the factorial representation for zero be $a_1 = 0$, so that $0 = 0 \cdot 1!$.

b. Devise an algorithm for adding 1 to a number in factorial representation.

c. Devise algorithms for adding and subtracting two numbers in factorial representation.

d. For fixed $N \geq 1$, there are $(N+1)!$ numbers whose factorial representation $a_1, a_2, \ldots, a_n$ has $n \leq N$. From this fact and from the uniqueness of the factorial number representation proved in (a), derive the identity:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + N \cdot N! = (N+1)! - 1$$

e. The factorial number representation is useful in enumerating permutations. This can be done in many ways. The technique discussed below is called the Derangement Method of M. Hall.

Let $P = (i_0, i_1, \ldots, i_N)$ be a permutation of the $N + 1$ integers $0, 1, \ldots, N$. For $j = 1, 2, \ldots, N$ define:

a_j = (the number of integers $< j$ which occur to the right of j in permutation P)

As an example, the permutation $P = (2, 0, 1)$ yields $a_1 = 0$, $a_2 = 2$.

By considering $a_1, \ldots, a_N$ to be the factorial representation of an integer A, we have set up a correspondence between the $(N+1)!$ permutations of the integers $0, 1, \ldots, N$ and the $(N+1)!$ numbers with factorial representation $a_1, \ldots, a_n$ $(n \leq N)$.

(e_1) Prove that this correspondence is 1:1.

(e_2) Devise an algorithm which constructs the permutation associated with an integer A from the factorial representation of A.

6. Devise an algorithm for finding shortest paths in a graph with weighted nodes. The length of a path is defined to be the sum of the weights of all nodes which lie on the path.

Consider the following three variations of the problem:

a. paths between two given nodes

b. paths between one given node and all other nodes

c. paths between all pairs of nodes

[Topics: shortest paths, wave propagation algorithm]

Bibliography

There are several good books on combinatorial mathematics in general and on graph theory in particular, but there appears to be none which is written from the point of view proposed here, of emphasizing the computational aspects of algorithms for solving combinatorial problems.

The book that comes closest to this point of view is

Beckenbach, Edwin F., ed. Applied Combinatorial Mathematics. New York, John Wiley and Sons, Inc., 1964.

Much useful material on computational and programming aspects of algorithms, combinatorial ones in particular, can be found in:

Knuth, Donald E. The Art of Computer Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc.
- Vol. 1. Fundamental Algorithms, 1968.
- Vol. 2. Seminumerical Algorithms, 1969.
- Vol. 3. Sorting and Searching, 1971-72.

The following references are not intended to be exhaustive by any means, but simply to point to a few papers which are typical of those which concentrate on computational aspects of combinatorics.

The machine tools of combinatorics

Hall, Marshall, Jr. and Knuth, Donald E. "Combinatorial analysis and computers." American Mathematical Monthly, 72 (1965), pp. 21-28.

Lehmer, Derrick H. "The machine tools of combinatorics." In Beckenbach, Edwin F., ed. Applied Combinatorial Mathematics. New York, John Wiley and Sons, Inc., 1964.

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Enumeration and counting

Golomb, Solomon W. and Baumert, Leonard D. "Backtrack programming." Journal of the Association for Computing Machinery, 12 (1965), pp. 516-524.

Lehmer, Derrick H. "The sieve problem for all-purpose computers." Mathematical Tables and Other Aids to Computation, 7 (1953), pp. 6-14.

Swift, J. D. "Isomorph rejection in exhaustive search techniques." Proceedings of Symposia in Applied Mathematics, 10. Combinatorial Analysis, pp. 195-200. Providence, Rhode Island, American Mathematical Society, 1960.

Walker, R. J. "An enumerative technique for a class of combinatorial problems." Proceedings of Symposia in Applied Mathematics, 10. Combinatorial Analysis, pp. 91-94. Providence, Rhode Island, American Mathematical Society, 1960.

Searching

Hibbard, Thomas N. "Some combinatorial properties of certain trees with applications to searching and sorting." Journal of the Association for Computing Machinery, 9 (1962), pp. 13-28.

Morris, Robert. "Scatter storage techniques." Communications of the Association for Computing Machinery, 11 (1968), pp. 38-44.

Peterson, W. W. "Addressing for random access storage." IBM Journal of Research and Development, 1 (1957), pp. 130-146.

Graph algorithms

Corneil, D. G. and Gottlieb, C. C. "An efficient algorithm for graph isomorphism." Journal of the Association for Computing Machinery, 17 (1970), pp. 51-64.

Dijkstra, E. W. "A note on two problems in connexion with graphs." Numerische Mathematik, 1 (1959), pp. 269-271.

Edmonds, Jack. "Paths, trees and flowers." Canadian Journal of Mathematics, 17 (1965), pp. 449-467.

Gottlieb, C. C. and Corneil, D. G. "Algorithms for finding a fundamental set of cycles for an undirected linear graph." Communications of the Association for Computing Machinery, 10 (1967), pp. 780-783.

Lee, C. Y. "An algorithm for path connections and its applications." Institute of Radio Engineers Transactions on Electronic Computers, EC-10 (1961), pp. 346-365.

Moore, Edward F. "The shortest path through a maze." Proceedings of the International Symposium on the Theory of Switching, pp. 285-292. Cambridge, Massachusetts, Harvard University Press, 1959.

Warshall, Stephen. "A theorem on Boolean matrices." Journal of the Association for Computing Machinery, 9 (1962), pp. 11-12.

CM4. Differential Equations and Numerical Methods

Prerequisites: CM2, M4

Throughout this course it is desirable to introduce problems which lead to the types of equations considered at the time. Excellent sources include circuit theory, mechanical systems, biological systems, particle dynamics, and economics. Numerical methods are to be introduced early in the course both to illustrate the qualitative behavior of solutions and to motivate uniqueness and existence arguments. In considering these methods the student should be made aware of the effects of discretization--and roundoff errors--and of stability. The students are expected to write some programs for various methods and to use existing library subroutines for others.

Detailed Outline

Origin and examples of differential equations (2 lectures)

- Sample (deterministic and nondeterministic) problems from the physical, social, and biological sciences, including predator-prey model
- Difference equations, including examples of different equations leading to the same differential equation

Simple linear equations (4 lectures)

- $y' = f(x)$, $y' = ay + f$, $y'' = ay' + by + f$
- Representation of solutions by indefinite integrals and special functions
- Direction fields
- Qualitative behavior of solutions
- Uniqueness and continuous dependence on initial data
- Consequences of linearity
- Approximation by Taylor series
- Polygon method
- Trapezoidal approximation
- Equivalence of second-order equations to first-order systems
- Introduction to first- and second-order difference equations and their elementary properties

The first-order equation $y' = f(x,y)$ (9 lectures)

- Graphical treatment, polygon method
- Relation to integral equations, Picard iteration
- Quadrature methods
- Picard existence and uniqueness theorem with proof
- Statement of Peano existence theorem
- Nonuniqueness examples
- Discussion of continuous dependence on initial data
- Power series solution and numerical methods
- Runge-Kutta methods
- Predictor-corrector methods
- Discussion on consistency and convergence (without proofs)

First-order systems of equations (8 lectures)

- Redevelopment for first-order systems--using vector notation--of the major results about single first-order equations
- Review of matrix results, similarity transformations, series for exp(At) and semigroup properties
- Vector space of solutions of $y' = Ay$, the adjoint solution

Representation of solutions of nonhomogeneous problems
Stiff systems

Plane autonomous systems (7 lectures)

Numerical exploration of $y' = ax + by + f(x,y)$, $x' = cx + dy + g(x,y)$
Poincaré phase plane and critical solutions
Critical points and concepts of stability
Numerical comparison of linear and nonlinear equations
The Lienard equations
Liapounov's ideas
Exploration of predator-prey model

Two-point boundary value problems (6 lectures)

Exploration of the linear second-order equation with mixed boundary conditions by shooting techniques
Discretization and methods for solving the resulting equations
Extensions to nonlinear equations

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Birkhoff, Garrett and Rota, Gian-Carlo. Ordinary Differential Equations. Boston, Massachusetts, Ginn and Company, 1962.
Selected topics.

Daniel, James W. and Moore, Ramon E. Computation and Theory in Ordinary Differential Equations. San Francisco, California, W. H. Freeman and Company, 1970.

Henrici, Peter. Discrete Variable Methods in Ordinary Differential Equations. New York, John Wiley and Sons, Inc., 1962.

Keller, Herbert B. Numerical Methods for Two-Point Boundary Value Problems. Boston, Massachusetts, Ginn and Company, 1968.
Advanced discussion of material on two-point boundary value problems.

Lapidus, Leon and Seinfeld, John H. Numerical Solution of Ordinary Differential Equations. New York, Academic Press, Inc., 1971.

C1. Introduction to Computing

Prerequisite: College admission

As stated in Section 2, this course should be oriented toward problem solving with computers. Accordingly, it is important that, throughout the course, different types of problems are considered and appropriate algorithms for their computational solution are designed and discussed. In particular, it is essential that both numerical and nonnumerical applications are presented. The problems should be reasonably interesting and realistic, and some should be open-ended, requiring a certain effort to identify what is required and how the solution is to be obtained. At least one major project leading to a completely verified and documented program should be included.

The course can serve to introduce many traditional mathematical ideas from a different point of view (e.g., subroutines and functions, induction and recursion, etc.). Such identifications should be strengthened where possible.

The course should be organized so that students can write small computer programs almost immediately. This may be accomplished by representing algorithmic processes from the outset both by flowcharts and programming languages.

The following outline is for a one-semester course meeting three times each week for lectures. In addition, it is generally advisable to schedule a regular weekly laboratory period of at least two hours. No lecture hours were assigned since the need for proper sequencing of programming assignments often demands that certain topics are either interchanged or distributed throughout the course.

<u>Detailed Outline</u>

Problems, algorithms, and programs

Typical problems and mathematical models
Concept of an algorithmic process
Flowcharts
Basic structure and properties of algorithms
Concept of a program
How computers execute programs
Elements of a higher-level programming language

Basic programming

Number and character representation
Constants and variables
Principal syntactic statements of the language
Functions, subroutines, and complete programs
Elements of the system being used
Libraries
Program testing and documentation

Errors and approximations

The approximate character of mathematical models
Truncation and roundoff error
Verification of algorithms
Error conditions and messages
Techniques for algorithm testing
The idea of numerical stability

Data structures

Discussion of a variety of problems leading to different data structures such as vectors, arrays, strings, trees, linked structures

Basic manipulation of the different structures

Advanced topics

Further details of the programming language

Aspects of compilers

Basic structure of an operating system

Aspects and organization of computer systems

Survey of computers, languages, and systems

Historical developments, discussion of different language types, aspects of systems programs, new developments

Bibliography

Arden, Bruce W. An Introduction to Digital Computing. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.
A good reference for the instructor.

Cole, R. W. Introduction to Computing. New York, McGraw-Hill Book Company, 1969.

Forsythe, Alexandra I.; Kennan, Thomas A.; Organick, Elliott I.; Stenberg, Warren. Computer Science: A First Course. New York, John Wiley and Sons, Inc., 1969.
This is a text for a high school course but may be appropriate for this course.

Galler, Bernard A. The Language of Computers. New York, McGraw-Hill Book Company, 1962.
A good reference for the instructor.

Gruenberger, Fred. Computing: An Introduction. New York, Harcourt Brace Jovanovitch, Inc., 1969.

Hull, Thomas E. Introduction to Computing. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

Hull, Thomas E. and Day, David D. F. Computers and Problem Solving. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970.
Part I of this text emphasizes material appropriate for this course.

Kemeny, John G. and Kurtz, Thomas E. Basic Programming. New York, John Wiley and Sons, Inc., 1967.
An introduction to programming with applications.

Rice, J. K. and Rice, J. R. Introduction to Computer Science: Problems, Algorithms, Languages, Information and Computers. New York, Holt, Rinehart and Winston, Inc., 1969.

Walker, Terry M. and Cotterman, William W. An Introduction to Computer Science and Algorithmic Processes. Boston, Massachusetts, Allyn and Bacon, Inc., 1970.

C2. Computer Organization and Programming

Prerequisite: C1

This course includes computational projects in assembly language programming. However, in line with the survey character of the course, care should be taken not to involve the students in a too-detailed discussion of assembly languages or of computer hardware. A scheduled laboratory period is desirable.

Detailed Outline

Computer structure and machine language (2 lectures)

- Fundamentals of computer organization, including registers, arithmetic units, memory, I/O units, and their interdependence
- Description of typical single-address machine instructions
- Programs as sequences of machine instructions and their execution

Introduction to symbolic coding and assembly systems (5 lectures)

- Mnemonic operation codes
- Labels, symbolic address
- Literals
- Pseudo operations
- General construction of assemblers
- Simple examples and exercises using a locally available assembler

Digital representation of data (3 lectures)

- Bits, fields, words
- Character representation
- Radix representation of numbers, radix conversion, representation of integers, floating point, and multiple precision numbers in binary and decimal form
- Variable length data

Addressing (2 lectures)

- Absolute addressing, indexing, indirect addressing, relative addressing
- Zero-, one-, two-, three-address instruction formats
- Address transformations

Machine organization to implement addressing structures
Character- versus word-oriented machines

Logic design (5 lectures)

Elements of Boolean algebra
AND, OR, NOT logic gates
Implementation of Boolean functions
Encoders and decoders
Descriptive discussion of clocked circuits, flip-flops, registers, shift registers, accumulators, counters, timing chains

Arithmetic units (3 lectures)

Serial versus parallel arithmetic
Implications of choice of radix
Design of a simple arithmetic unit
Design of half-adder and adder
Algorithms for multiplication and division

Instruction units (3 lectures)

Instruction fetch and decoding
Program sequencing
Branching
Subroutine calls
Interrupts
Control and timing logic
Micro-programming as a means of implementing control units

Storage units (3 lectures)

Structure of core memory
Typical memory bus structure
Memory overlap, protection, relocation, and paging
Word versus character organizations
Types of bulk memories
Descriptive discussion of stack memories, associative memories, read-only memories, and virtual memory schemes

Input-output systems (3 lectures)

Direct memory access I/O
I/O channels and controllers, multiplexers
Characteristics of various types of input/output devices
Relation of I/O system to control unit and main memory
Input/output programming
Buffering and blocking
Interrupts
Problems of error detection and correction in data transmission

Systems software (4-5 lectures)

- Operating systems
- Input/output packages
- Assemblers, loaders
- Interpreters, compilers
- Utility programs and libraries

Survey of contemporary computers (3-6 lectures)

A survey of contemporary computers emphasizing a variety of machine organization. Typical topics: large versus small computers; single register, multiple register, and stack machines; unorthodox machines. Discussion of possible implementation of high-level programming language statements on typical computers.

Bibliography

Bell, C. G. and Newell, A. Computer Structures. New York, McGraw-Hill Book Company, 1970.
Survey of computer organizations. Source of material for the survey of contemporary computers.

Chu, Yaohan. Digital Computer Design Fundamentals. New York, McGraw-Hill Book Company, 1962.
A somewhat dated reference on logic design.

Gear, C. William. Computer Organization and Programming. New York, McGraw-Hill Book Company, 1969.
Reference on assembly language programming.

Gschwind, H. W. Design of Digital Computers: An Introduction, 5th ed. New York, Springer-Verlag New York, Inc., 1970.
Text on computer design and organization, slightly engineering-oriented.

Hellerman, H. W. Digital Computer System Principles. New York, McGraw-Hill Book Company, 1967.
Uses Iverson notation, directed toward IBM equipment, especially S/360.

Knuth, Donald E. The Art of Computer Programming. Volume 2, Seminumerical Algorithms. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.
Reference for a mathematical treatment of computer arithmetic (Chapter 4).

McCluskey, E. J. Introduction to the Theory of Switching Circuits. New York, McGraw-Hill Book Company, 1965.
Reference for basic switching theory.

Nashelsky, Louis. Digital Computer Theory. New York, John Wiley and Sons, Inc., 1966.
A paperback containing a survey of many of the topics covered in this course.

C3. Programming Languages and Data Structures

Prerequisite: CM1

Detailed Outline

Structure of algorithmic languages (8 lectures)

- Review of basic program constituents of the language introduced in C1
- Introduction to the elements of ALGOL or PL/1
- Informal syntax and semantics of simple statements in that language
- Backus normal form
- Grouping of statements and block structure of programs
- Scopes, local and nonlocal quantities
- Functions and procedures
- Formal and actual parameters
- Binding time of program constituents
- Simple recursive procedures
- Concept of a stack
- Simulation of recursions as iterations using stacks

Arithmetic statements (4 lectures)

- Brief discussion of graphs and trees
- Tree diagrams of arithmetic expressions
- Informal discussion of precedence hierarchies
- Infix, prefix, postfix notation
- Translation between infix and postfix notation
- Evaluation of expressions in postfix notation

Trees and lists in a computer (8 lectures)

- Types of data nodes and linkages
- List names, list heads, sublists
- Multilinked lists
- Stacks as list structures with usage discipline
- Representation of trees as special cases of lists
- Accessing, insertion, deletion, and updating in trees
- Traversal schemes for trees
- Application to the generation of machine code from expression trees

String manipulation (7 lectures)

- Introduction to a string manipulation language such as SNOBOL

Data declarations in such a language
Recursive algorithms in such languages
Applications to formal differentiation of expressions

Data structures and storage allocation (3 lectures)

Storage allocation for algorithmic language structures such as independent, nested blocks, strings, arrays, etc.
Procedures using run-time stacks
Storage allocation for string manipulation languages

Some aspects of languages and grammars (6 lectures)

Syntax, semantics, and pragmatics of programming languages
The concept of a formal grammar
Production notation
Discussion of Chomsky's classification of grammars
Discussion of computability, undecidability
Syntax and semantics of arithmetic statements
Precedence and operator precedence grammars
Syntactic specification of procedures, blocks, and statements
Formal semantics corresponding to syntactic specifications

Bibliography

Genuys, F., ed. Programming Languages. New York, Academic Press, Inc., 1968.

Harrison, M. C. Data Structures and Programming. Courant Institute of Mathematical Sciences, New York University, 1970.

Knuth, Donald E. The Art of Computer Programming. Volume I, Fundamental Algorithms. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.
Important presentation of data structures.

Rosen, Saul, ed. Programming Systems and Languages. New York, McGraw-Hill Book Company, 1967.
Contains, among other things, a discussion of SNOBOL and a comparison of list processing languages.

Sammet, Jean E. Programming Languages: History and Fundamentals. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.
A comprehensive survey of languages.

Wegner, Peter. Programming Languages, Information Structures, and Machine Organization. New York, McGraw-Hill Book Company, 1968.
An approach to programming languages as information structures.

RECOMMENDATIONS ON UNDERGRADUATE MATHEMATICS COURSES INVOLVING COMPUTING

A Report of

The Panel on the Impact of

Computing on Mathematics Courses

October 1972

TABLE OF CONTENTS

1. Preface

The growing influence of modern electronic computing in many fields of knowledge has contributed to a dramatic increase and diversification in the application of mathematics to other disciplines. No longer are the uses of mathematics confined exclusively to the physical sciences and engineering; they are found with increasing frequency in the social, behavioral, and life sciences as well. Correspondingly, the use of the computer has led to different requirements for the solution process in mathematics itself. Theory construction and model building have assumed a different dimension; in addition to knowing existence theorems, the user of mathematics must know constructive methods for solving problems, and he must have the means to ascertain the efficiency as well as the correctness of these methods.

These developments have created new challenges with regard to revision of the undergraduate mathematics curriculum. The basic curriculum should reflect the contemporary points of view associated with computer application in mathematics; it should acquaint the students with the newly developed methods of solving standard problems and also introduce them to the host of problems which have arisen in the past few years.

There now appears to be growing recognition that more consideration should be given to the potential impact of the computer on the basic undergraduate courses which serve not only potential mathematicians and computer scientists but many other students as well. The present report is the result of the first study of this problem by CUPM.

Although a consensus about the role of computers in the basic mathematics curriculum has not yet evolved, we believe there is an urgent need for experimentation in this area. This report presents ideas for such experimentation by proposing changes in various basic mathematics courses and by suggesting some new courses which are designed to take advantage of the presence of computers.

As is the case with all CUPM reports, these recommendations must be regarded as general suggestions which will need to be adapted to local circumstances and revised in the light of subsequent experience. Nevertheless, mathematics departments should immediately concern themselves with the ideas outlined in this report so that they can prepare their students for the uses of mathematics in the context of the availability of computers.

2. Premises of this Study

We suggest four ways in which the computer can influence undergraduate mathematics education:

(i) Computing can be introduced into traditional mathematics courses;

(ii) New courses in computationally-oriented mathematical topics can be designed;

(iii) The entire curriculum can be modifed to integrate computing more fully into the student's program;

(iv) Computers and computer-related devices can be used as direct aids to mathematical instruction.

This report addresses itself to (i) and certain aspects of (ii). As for (iii), some possible curriculum restructuring relating to computing has been discussed in the CUPM report Recommendations for an Undergraduate Program in Computational Mathematics. Finally, (iv) is a very broad area which would require a separate study of its own; we include only a brief discussion of some related topics in the final section of the report.

We do not suggest that all mathematics instruction be modified along any of these lines. In recent years it has become appropriate to speak of the mathematical sciences in a broad sense rather than of mathematics in the more familiar, narrower sense. This situation indicates a need for different avenues within mathematics education; the introduction of computer-oriented material should therefore be regarded as a development parallel to the standard curriculum which interacts with the standard curriculum at a number of places.

Nearly every student taking mathematics courses can benefit from some computer-oriented mathematics instruction. The use of computers is beginning to pervade all phases of life in our society, and in most disciplines, including mathematics, there is a need for students to become familiar with some aspects of computing. Many mathematics departments have observed that well over half of their undergraduate majors enter computer-related careers or graduate programs after graduation. Clearly, these students would benefit considerably from computer-oriented courses and curricula. Computer-oriented courses also serve all those students from other disciplines who are interested in learning more mathematics in order to solve problems from their own fields. Modern applied mathematics has a strong computer orientation; when students enter this field, those whose education stresses concern for computational problems have a decided advantage over those who are familiar only with theoretical results. Finally, the growing trend toward introducing computers in high schools will require that prospective teachers learn about the interaction between the computer and mathematics.

Although the content and objectives of computational material differ considerably for various groups of students, computing provides all of them an unusual opportunity for active participation. For this reason the motivational aspects of computing are significant for most students, and the value of such motivation should not be underestimated.

A more substantive objective must be to select course material and approaches so as to reflect the actual influence of computing on mathematics. The recommendations which follow are based on the premise that any program which seeks to reflect this influence should stress four points--namely, algorithms, approximations, model building, and the nature of the entire problem-solving process.

Algorithms. The modern computer's development as a general-purpose problem-solving system derives not so much from its arithmetic capabilities but from its ability to handle logical and nonnumerical problems. From a mathematical viewpoint this has led to a greater emphasis on the construction and analysis of algorithms for the actual solution of mathematical problems rather than only on the proof of the existence of solutions. Stressing the algorithmic aspects forces the student to state both the problem and the method of solution in precise and unambiguous terms. It fosters his ability to organize and formulate logically an attack on a problem as well as to recognize and clarify the assumptions he is making in order to solve the problem.

Approximations. In most analysis courses numerical algorithms are more prevalent than nonnumerical ones. This leads to questions of error or, more generally, to questions about the quality of the approximations produced by the algorithm. If an algorithm produces an answer, some statement is needed to relate this answer to a solution of the original mathematical problem. If a process appears to converge, there is a need to prove that the process converges as well as to determine how rapidly it converges. If a method is bound to be applicable to certain input data, it is necessary to establish what happens when changes are introduced in these data. Clearly, in undergraduate courses these questions can rarely be answered satisfactorily but the student should acquire a concern for them and an appreciation of their importance.

Model Building. An important part of every real application of mathematics is the recognition and formulation of a satisfactory mathematical model of the given nonmathematical problem. Developing the student's skill in this process should be an objective of every course involving computer applications.

The Problem-Solving Process. Modeling, the development of algorithms, the study of the approximations used, and the computation and interpretation of results are all principal steps in the process of solving a problem on a computer. It is important to stimulate in the student an understanding of this process viewed as a whole by discussing and assigning the complete solution of appropriate simple problems.

The four points raised here--namely, the stress on algorithms, the development of a concern for the quality of approximations, the emphasis on model building, and a general emphasis on the entire problem-solving process--should be considered as general objectives in the student's program. In those undergraduate courses which involve a limited amount of computing, little more can be done than to illustrate the importance of these points and to instill in the student an intuitive understanding and concern for them. This requires a careful selection of the computational topics to be discussed and of the problems to be assigned. In other words, without underestimating the motivational value of computing, we believe that the introduction of computational material into a mathematics course should go beyond merely illustrating a mathematical concept. It should at least provide a definite answer to a specific question or problem in order to give the student deeper insight into the theory, the model, and the algorithms used.

Implementation and Precautions. There are wide variations in the extent and type of computer use which can be introduced into a traditional mathematics course. Such variations arise both from differences in computing facilities and from differences in the instructors' opinions of how essential these uses can and should be for the course. Where computing facilities are readily available or where courses are modified extensively to emphasize the four objectives discussed above, the trend is often to use the computer frequently and in a matter-of-fact manner. This means that computer-related material is presented throughout the course and that computer problems are assigned as a regular part of the homework; the student is expected to master these problems in order to have a coherent understanding of the subject. Where computing facilities are not so readily available or where computer-related material enters the course only in a secondary, supportive role, the trend is to consolidate computer use into the solution of a number of relatively substantial problems and to expect the student to apply his mathematical knowledge to these problems, but not to demand mastery of these problems for a coherent picture of the course. In this case, extra credit is sometimes given for the computer component of a course.

In whatever way the computer is used, there are a number of precautions which ought to be observed. Primarily, one should neither misuse nor overuse the computer. The computer is certainly misused when one is not mathematically honest about what it can or cannot do. For example, a computer can approximate a limit, but it cannot "compute" one or verify its existence, nor can it "test" a function for continuity. Specific examples of overuse of the computer are harder to provide, but it can be recognized when computing begins to crowd mathematical material out of the course or when students become bored by it. Overuse of the computer can result when the excitement of the new approach obscures the principal purpose: to teach mathematics. It is primarily the algorithmic approach together with the other three objectives, rather than the actual use of a computer, which will help to advance this purpose. Many points about algorithms can be made without using a computer at all; three-digit arithmetic can be used to discuss approximations and roundoff error, and model-building and problem-solving expertise can also be gained from judiciously chosen paper-and-pencil problems. Also, students need not program every algorithm they encounter, and experiments with preprogrammed algorithms can often provide more insight than the lengthy drudgery of debugging a complicated program. It is up to the individual instructor to maintain a proper balance between the use of the computer and the other components of the course.

3. Basic Courses

3.1 Introduction

In this section we discuss five one-semester beginning undergraduate courses which include an emphasis on computing. As in all CUPM reports, the outlines given here are not meant to be prescriptive but are intended to extend the exposition of our ideas in the previous section by giving suggestions and possible approaches for implementing them.

For reference purposes we begin with a list of brief catalog descriptions for these courses. The descriptions do not include any programming requirements; these are discussed in Section 5.2.

MC-0. Elementary Functions and Problem Solving. [Prerequisite: College admission] Basic computer programming, elementary functions, matrix operations. These topics are to be motivated by, and applied to, practical problems.

MC-1. Calculus I with Computer Support. [Prerequisite: MC-0 plus trigonometry, or equivalent mathematical background] Differential and integral calculus of the elementary functions with associated analytic geometry, supported by computer applications.

MC-2. Calculus II with Computer Support. [Prerequisite: MC-1] Techniques of integration, introduction to multivariate calculus, and elements of differential equations, supported by computer applications.

MC-DM. Discrete Mathematics. [Prerequisite: No specific course prerequisite, but see page 590] Concepts and techniques in discrete mathematics that find frequent applications in computing problems.

MC-3. Algorithmic Elementary Linear Algebra. [Prerequisite: MC-0 or equivalent background] An introduction to matrix and vector algebra in n dimensions with an emphasis on algorithmic aspects.

The four courses MC-0 through MC-3 represent computer-oriented versions of the courses Mathematics 0 through 3 in the 1972 CUPM report Commentary on a General Curriculum in Mathematics for Colleges (GCMC Commentary). In the case of MC-0 and MC-3, the material in the GCMC Commentary courses was considerably modified and rearranged in order to introduce a fairly strong emphasis on computation. In MC-1 and MC-2, on the other hand, the purpose and the outline have remained essentially the same as for traditional courses; the emphasis on what is taught, however, has shifted along with the shift in the kinds of applications that are possible with the computer. The remaining course MC-DM represents a new development. It has a strong algorithmic flavor and introduces material of considerable importance in many computer applications. The course may not only supplement the standard curriculum but could also serve well as a first mathematics course for students from many disciplines.

Ideally, a student entering any of these computer-related courses other than MC-0 should have at least a rudimentary knowledge of programming. Since this is, at present, an unrealistic requirement, several possible alternatives are suggested in Section 5.2. Since these alternatives depend strongly on local circumstances, no further mention of them is made in the outlines. Only in the case of MC-0 is some time allotted to introduce certain elementary computer concepts.

In each of the following outlines, the suggested pace is indicated by assigning a number of hours to each group of topics. A standard semester contains 42 to 48 class meetings, and we follow the GCMC Commentary in allowing approximately 36 hours for discussion of new material; the remaining time can be devoted to tests, reviews, etc.

3.2 Course Outlines

MC-0. Elementary Functions and Problem Solving

[Prerequisite: College admission] The aim of this freshman-level course is to teach students ways to approach problems in the physical, natural, and social sciences and to equip them with some fundamental mathematical and computational tools for the solution of these problems. Typical problems are concerned with measurement and prediction: given a process such as a factory producing steel, traffic moving on a city street, or a shifting population, it is desired to predict future properties of the process on the basis of past measurements. The approach taken in the course is first to have students model and simulate specific processes using a computer and then look for functional relationships between various aspects of these processes, e.g., between time and the total output of steel, between traffic light timing and traffic density, or between past and future population distributions. The objectives of this approach are to create an understanding of modeling through approximation and simulation and a feeling for the types of questions asked about models and functions.

Studies of elementary functions, computational techniques, and matrix operations are interwoven in the course and are used to illustrate and motivate one another. Depending upon the selection and treatment of particular topics, this course may serve either as a refresher course for students going on to calculus or as a terminal course for students who intend to take only one course in mathematics. However, this course alone probably will not prepare students adequately for a one-year sequence in calculus, since it does not contain the topics from trigonometry which they will need. The orientation towards applications and the emphasis on computing should make the course attractive to many students who might otherwise avoid more traditional mathematics courses. There are no prerequisites, as instruction in the use of a computer is integrated with the rest of the course.

COURSE OUTLINE

1. Introduction. (6 hours) Number representation, algorithms, elements of programming, functions, relations.

2. Linear and quadratic functions. (8 hours) Simulations involving constant and accelerated rates of change, graphs of linear and quadratic functions, zeros, maxima, minima, applications.

3. Linear programming. (4 hours) Linear functions of two variables, linear inequalities, maxima, minima, applications.

4. Matrix operations. (6 hours) Representations of tabular data, subscripts, matrix and vector operations, simultaneous equations, applications.

5. Algebra of functions. (6 hours) Algebraic operations on functions, polynomial and rational functions, maxima, minima, zeros, inverses, composition.

6. Exponential and logarithmic functions. (6 hours) Simulations of exponential growth, properties of exponents, logarithms as inverses of exponentials.

COMMENTARY

1. Introduction. Simple mathematical concepts can be introduced or reviewed in the context of teaching the rudiments of programming in a computer language such as APL, BASIC, or FORTRAN. For a start, students should learn to use arithmetic, branching, and simple looping statements; other techniques, such as subroutines, can be considered later as the need for them arises. Machine arithmetic can be contrasted with ordinary arithmetic, with examples of roundoff error being given.

Functions should be introduced as single-valued rules of association, with the relationships between the inputs and outputs of computer programs providing many examples of both numeric and nonnumeric functions. Questions of scaling which arise in the development of a simple program for graphing functions can be used as a bridge to the next section on linear functions.

2. Linear and quadratic functions. In studying rates of change, the student can first write a computer program to model a situation involving constant change. After this model has been used to motivate a study of linear functions, the computer program can be modified by the addition of a single statement to model constant acceleration, thereby motivating a study of quadratic functions; later, the added statement can be changed to have the program model more complicated rates of acceleration (e.g., a bouncing ball or exponential growth). Questions about zeros, maxima, and minima should be raised and answered to ascertain properties of models, functions, and graphs. In this way the study of linear and quadratic functions provides a framework for later material in the course.

In addition to using the computer for simulation, one can stress the algorithmic aspects of graphing by using programs to compute the slopes of lines or the zeros of a quadratic function by the quadratic formula. Zeros, and in particular square roots, can also be approximated by the bisection method.

3. Linear programming. The study of linear functions leads naturally to a study of linear programming in two dimensions. Boundary conditions lead to a consideration of linear inequalities and to

the solution of simultaneous equations in two unknowns in order to determine constraint regions. Cost functions can be introduced as functions from vectors in those regions to numbers, and the location of the maxima and minima of these functions at the vertices of regions can be demonstrated by drawing level curves.

4. Matrix operations. As a further example of the applicability of linear methods, models of population movement can be studied. One can introduce a vector V to represent the population distribution at a given time and a matrix M to represent the percentage redistribution of population over a year's time. Matrix and vector multiplication can be motivated by writing a computer program to model population movement over a number of years and observing that $M^n V$ is the population distribution after n years. Finally, this model can be used to motivate the solution of simultaneous linear equations or the inversion of a matrix to find the equilibrium distribution.

5. Algebra of functions. By associating functions with subroutines which compute them, one can motivate a general discussion of the domains and ranges of functions, as well as of algebraic operations on functions. Applied to polynomials, this leads naturally to the rational functions. In order to answer standard questions about these functions, one can discuss numerical techniques such as bisection and hill-climbing for locating zeros, maxima, and minima. The inverse of a function can be found by computing the zeros of translated functions.

6. Exponential and logarithmic functions. Computations involving population growth, interest rates, or radioactive decay lead to a study of exponential functions. The logarithm can be computed by the method developed in Section 5, and its properties can be established from the properties of exponentials.

REFERENCES

No presently available text is suitable for this course. While some of the references listed below contain material that can be used in the course, no text develops computing and mathematics together along the lines suggested by units 1 and 2. The approaches to computing in two of the references bracket the suggested approach:

Vogeli, *et al.*, is generally too elementary and does not use computing in a substantial way, while Higgins presumes too much prior experience both in computing and in mathematics. The remaining references are programming texts which contain some examples appropriate for the course.

Barrodale, Ian; Ehle, Byron; Roberts, F. D. K. *Elementary Computer Applications in Science, Engineering, and Business*. New York, John Wiley and Sons, Inc., 1971.

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Vogeli, Bruce R.; Prevost, Fernand; Gilbert, Glenn; Carroll, Edward. *Algebra One*. Morristown, New Jersey, Silver Burdett Company, 1971.

MC-1 and MC-2. *Calculus I and II with Computer Support*

[Prerequisite for MC-1: MC-0 and trigonometry, or equivalent background; prerequisite for MC-2: MC-1] The introductory courses on calculus appear to be those mathematics courses in which the use of the computer has been most popular. One reason for this is the fact that many concepts and methods in the calculus have a practical flavor which can be enhanced by introducing computing. While the motivational value of computational work plays a considerable role, the student can also handle more realistic problems using the computer as a tool and with it learn to appreciate more fully the power and usefulness of calculus.

There are at this time no firm guidelines as to how the computer should be introduced into calculus courses; many radically different experiments have been conducted and are still being carried on. We believe that at present a practical and rather attractive approach is to use the computer to support courses which are more or less traditional in the selection and sequencing of the material. Hence the courses described below are, at least in outline, identical with the courses Mathematics 1 and 2 in the GCMC Commentary, and their basic purpose remains essentially the same--namely, that of being an intuitive, yet sound, introduction to limits in various forms, such as derivatives, integrals, or sums of series, along with applications of several types, such as maximum-minimum problems or questions leading to integrals.

We do not believe that it is enough to teach the calculus courses more or less as usual and to assign computer projects as supplements to the course. Such an approach does not take full advantage of the interplay between theoretical and algorithmic ideas. The new courses must be taught in a different manner, if only because computing can provide much useful motivation for the calculus. Moreover, the emphasis on what is taught should shift towards the kind of realistic applications that are possible with the computer. In these ways the computer can be used to support the presentation of material rather than merely to supplement it.

The commentaries below indicate ways in which this supportive role of the computer can be accomplished. For the most part these commentaries are meant to supplement rather than replace those in the GCMC Commentary.

COURSE OUTLINE FOR MC-1

1. Introduction. (4 hours) Review of the function concept. Function evaluation and graphing on a computer.

2. Limits, continuity. (3 hours) Limit and approximation defined intuitively. Derivatives as examples. Definition of continuity, types of discontinuity, Intermediate Value Theorem. Computational applications involving the bisection method and showing effects of truncation and roundoff error.

3. Differentiation of rational functions; maxima and minima. (5 hours) Computational projects involving search algorithms for finding extrema. Newton's method.

4. Chain rule. (3 hours) Include derivatives of functions defined implicitly, inverse function and its derivative. The algorithmic aspect of functional composition.

5. Differentiation of trigonometric functions. Higher derivatives. (3 hours)

6. Applications of differentiation. (3 hours) Tangent as "best" linear approximation. Approximations using differentials. Additional extremal problems. Related computer applications.

7. Intuitive introduction to area. (2 hours) Computational approximation of areas of regions under a curve.

8. Definite integral. (3 hours) Simple quadrature rules and their applications.

9. Indefinite integrals, Fundamental Theorem. (4 hours)

10. Logarithmic and exponential functions. (3 hours) Computer problems involving exponential growth or decay using Euler's method.

11. Applications of integration. (3 hours)

COMMENTARY ON MC-1

1. Introduction. The computer can be used to evaluate functions, thereby considerably extending the kinds of functions a student can handle and, indeed, even recognize as functions (e.g., functions with piecewise or algorithmic definitions can be evaluated numerically even though they may not be expressible algebraically). Students should recognize that the relation between the input and the output of a computer program can define a function; such an awareness can be used later to demonstrate the existence of various interesting functions.

A good computing facility would enable the student to experiment with functions--and also their graphs when they can be drawn--in much the same way as he can work with simple functions when he has only pencil and paper. A graphing program should be provided or developed, and students should become reasonably familiar with it, so that it can be used to motivate later topics in terms of graphs.

2. Limits, continuity. Many kinds of calculations help to motivate the need for a precise definition of limit. Such calculations arise in practical attempts to approximate limits of functions or rates of change. While the student will sense that successive approximations are approaching a limit, he will also discover that the limitations of numerical approximations due to truncation or roundoff errors prevent him from calculating that limit exactly. This awareness should be used to motivate the need for mathematical proofs of the existence of limits.

The bisection method for finding zeros of continuous functions can be introduced either as motivation for or as an application of the Intermediate Value Theorem.

In this section, as well as in others, one should recall several points observed earlier concerning the use or misuse of the computer. First, one should use terminology carefully so as not to mislead students; a computer can approximate a limit, but it cannot "compute" one, nor can it "test" a function for continuity. Second, one should remember that the primary purpose of the course is still to teach calculus and that it is the algorithmic approach, and only secondarily the actual use of the computer, which advances this purpose; hence the computer does not have to be used in every conceivable situation, and many points about algorithms can be made without a computer at all.

3. Differentiation of rational functions. Nonnumerical algorithms can be recognized when they appear even though they may not be programmed. For example, formal differentiation should be recognized as a process that can be mechanized.

More realistic maximum and minimum problems can be attempted. The approach to such problems would include graphing functions, searching for extremal points, and sometimes finding zeros of derivatives. The computer increases the student's power to find zeros of functions since the bisection method or Newton's method are available when algebraic techniques fail.

4. Chain rule. Computer programs and flowcharts can be used to explain the process of functional composition and to motivate the chain rule. Information about the inverse of a function f can be obtained by finding zeros of $f(x) - a$, for various values of a.

5. Differentiation of trigonometric functions. The graphing program can be used for motivation.

6. Applications of differentiation. The limitations of numerical methods can be used to motivate the need for theorems concerning, say, the number of extremal points. For example, numerical methods may lead one to suspect that $x^2 + \cos^2(kx)$ has a unique minimum when k is slightly larger than 1, rather than two minima which are separated by a maximum at 0.

Again, more realistic maximum and minimum problems can be attempted. Newton's method for locating zeros of functions can be developed and compared with the bisection method for its rate of convergence and range of applicability.

7. Intuitive introduction to area.

8. Definite integral. The notion of the definite integral can be made concrete prior to the proof of the Fundamental Theorem so that the student need not confuse the existence of the definite integral of a function with his ability to find an antiderivative. The student can write programs to approximate definite integrals by techniques such as the trapezoidal rule. Improper integrals can be motivated in terms of programs to approximate them.

9. Indefinite integrals, Fundamental Theorem. The nature of the indefinite integral as a function of the upper endpoint can be illustrated by considering a computer program to approximate values of this function.

10. Logarithmic and exponential functions. Numerical methods can be used to discuss and sketch solutions of the differential equation $y' = ky$.

11. Applications of integration. The computer greatly increases the variety of examples which can be treated. Applications of the integral as the limit of Riemann sums, and not merely as an antiderivative, were recommended in the GCMC Commentary and can be handled much more successfully with the use of the computer. For example, in following those suggestions one can use numerical techniques to integrate the normal probability distribution or to graph a logistic curve corresponding to a differential equation $N' = (a - bN)N$ governing population growth. One can also observe the general applicability of numerical techniques as opposed to the often limited applicability of analytical techniques. For example, given experimental data concerning the acceleration of a vehicle, one can compute the values of integrals to obtain the velocity and position of that vehicle [cf. Garfunkel, Solomon. "A laboratory and computer based approach to calculus." American Mathematical Monthly, 79 (1972), pp. 282-290].

COURSE OUTLINE FOR MC-2

1. Techniques of integration. (9 hours) Integration by trigonometric substitutions and by parts; inverse trigonometric functions; quadrature formulas and computer applications; improper

integrals and numerical questions; volumes of solids of revolution.

2. Elementary differential equations. (7 hours) Elementary methods for computational solution.

3. Analytic geometry. (10 hours) Vectors; lines and planes in space; polar coordinates; parametric equations.

4. Partial derivatives. (5 hours)

5. Multiple integrals. (5 hours)

COMMENTARY ON MC-2

1. Techniques of integration. At the discretion of the instructor, less attention might be paid to techniques of formal integration in order to provide time for a study of numerical methods for approximating definite integrals. Experiments can be performed to suggest theorems about the rates of convergence of various methods. In some simple cases one might attempt to place bounds on the numerical errors due to the approximation method and to truncation and roundoff effects. In general, an applied flavor can be introduced into the calculus course by relating some of the theorems to realistic numerical processes.

Although formal integration is more complicated than formal differentiation, certain aspects, such as integration of powers of sines and cosines or the use of partial fractions, can be considered from an algorithmic point of view.

As an example of finding error bounds, consider the midpoint (or tangent) approximation

$$\int_a^b f(x)\,dx \approx h \sum_{k=1}^{n} f(a + [k - \tfrac{1}{2}]h),$$

where $h = \frac{b-a}{n}$, to the integral of a twice-differentiable function f. One can show with the aid of Taylor's theorem that the truncation error is bounded by $\frac{(b-a)h^2}{24}B$, provided that $|f''(x)| \leqq B$ for $a < x < b$. Furthermore, for suitable a, b, and h, the error in evaluating the approximation is bounded by $nhE_1 + n(n-1)E_2Fh$, which is less than $(b-a)(E_1 + nE_2F)$, where E_1 is the maximum absolute error in the computation of $f(x)$ for $a < x < b$, E_2 is a bound for $\frac{(1+r)^n - 1}{n}$ (r being the relative roundoff error bound), and $|f(x)| \leqq F$ for $a < x < b$. In the particular case $n = 128$ and $E_1 = E_2 = 1.01 \times 2^{-27}$, the midpoint approximation to $\int_1^2 \frac{dx}{x} = \log 2$ can be guaranteed to have an error of no more than 10^{-5}.

It should be observed that formal and numerical methods are not mutually exclusive alternatives, and that many problems require a combination of the two. Analytical techniques may be used to transform an integral for numerical methods. For example, the integral

$$\int_{\frac{1}{2}\pi}^{\infty} \frac{\sin x}{x}\, dx$$

is more easily handled numerically if it is first transformed to

$$\frac{4}{\pi^2} - 2\int_{\frac{1}{2}\pi}^{\infty} \frac{\sin x}{x^3}\, dx,$$

which is obtained by integrating by parts twice.

2. Elementary differential equations. The notion of a tangent field can be used to suggest numerical methods for the approximate solution of first-order differential equations. Higher-order equations can also be treated by translating them into systems of first-order equations which can then be solved numerically.

A bound on the propagated error for a simple method can be derived. With Euler's method applied to $y' = f(x,y)$, $y(x_o) = y_o$, it can be shown that the propagated error is bounded by

$$\frac{R + T}{hL} \exp[L(x_n - x_o)],$$

where R is a bound on the local "roundoff" error

$$y_n^c - y_{n-1}^c - hf(x_{n-1}, y_{n-1}^c),$$

T is a bound on the local "truncation" error

$$y(x_n) - y(x_{n-1}) - hf(x_{n-1}, y(x_{n-1})),$$

h is the step-size, L is a Lipschitz constant, and y_n^c is the computer approximation to $y(x_n)$, $x_n > x_o$. [See Gear, C. William. Numerical Initial Value Problems in Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971.]

3, 4, 5. Analytic geometry, Partial derivatives, Multiple integrals. Due to the lack of numerical methods which are both elementary and practical, the computer itself has less impact on the teaching of multivariable calculus than on single-variable calculus. However, an algorithmic approach can still be used, for example, to stress analogies with single-variable calculus or to introduce formal manipulations. If computing applications are desired, one might discuss hill-climbing techniques for finding maxima or some relatively simple method for approximating the values of double integrals.

REFERENCES

The following texts contain elementary applications of numerical methods to the calculus.

1. Sources of applications

Barrodale, Ian; Ehle, Byron L.; Roberts, F. D. K. Elementary Computer Applications in Science, Engineering, and Business. New York, John Wiley and Sons, Inc., 1971.

Dorn, William S.; Bitter, Gary G.; Hector, David L. Computer Applications for Calculus. Boston, Massachusetts, Prindle, Weber, Inc., 1972.

Hamming, Richard W. Calculus and the Computer Revolution. Boston, Massachusetts, Houghton Mifflin Company, 1968.

Hull, Thomas E. and Day, David D. F. Computers and Problem Solving. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969. Chapter 9.

2. Some calculus texts having a computational flavor

Flanders, Harley; Korfage, Robert R.; Price, Justin J. Calculus. New York, Academic Press, Inc., 1970.

Henriksen, M. and Lees, M. Single-Variable Calculus. New York, Worth Publishers, Inc., 1970.

Stenberg, Warren, et al. Calculus, A Computer Oriented Presentation, Parts 1 and 2. CRICISAM, Florida State University, 1970.

MC-DM. Discrete Mathematics

[Prerequisite: The material of this course can be taught at various levels of difficulty and sophistication in the undergraduate curriculum. Although no specific college mathematics courses are prerequisite for MC-DM, it is important that the student have ability in manipulating symbols and in using formulas.] Although an obvious goal of this course is to equip the students with some useful mathematical tools, a more important goal is to develop their ability to perceive, formulate, and solve problems that are discrete in nature. This course can be taken prior to, or concurrently with, a course in calculus. Indeed, students who are not mathematics majors might be counseled to take such a course rather than the traditional freshman calculus course. (For example, it can be argued that for social science, behavioral science, and biological science majors, a course in discrete mathematics might be more useful than a course in calculus.) There are several undergraduate-level mathematical courses that can follow this course naturally. A course in Probability and Statistics (see, for example, 4.2 below) or a course in Applied Algebra may well be popular choices. Other courses are Combinatorial Mathematics, Optimization Techniques (such as those proposed by the Panel on Applied Mathematics in the CUPM report Applied Mathematics in the Undergraduate Curriculum), Applied Logic, Graph Theory, and Computational Algorithms. Although computing facilities are not absolutely essential in such a course, they can play a very attractive supporting role. Since there is a strong algorithmic flavor throughout this course, the implementation of some computational algorithms will enhance the understanding and appreciation of the mathematical results. Also, probably in a less significant way, ideas such as graphical representation of discrete functions and solution of difference equations can be illustrated on a computer.

The number of hours specified is intended to indicate the relative emphasis for the various topics. Some instructors will find these time estimates unsuitable and will therefore need to make adjustments for their classes. However, because the material covered in this course is so new and unusual in the undergraduate curriculum, the Panel felt it would be valuable to present a wide variety of ideas for a course in discrete mathematics.

COURSE OUTLINE

1. Elementary set theory. (2 hours) Basic concepts and terminology in set theory. Subsets. Empty set. Intersection, union, symmetric difference, and complementation of sets. Venn diagrams.

2. Permutations and combinations. (4 hours) Permutation and combination of objects. Simple enumeration formulas such as that for the number of ways to select or to arrange r objects from n

objects with or without repetitions. Simple machine tools of combinatorics such as computer algorithms for generating all permutations and all combinations of a set of objects.

3. Discrete functions. (2 hours) Domain and range of a function. One-to-one and onto functions. Pigeonhole principle.

4. Manipulation of discrete functions. (2 hours) Forward and backward differences of a discrete function. Accumulated sum of a discrete function. Sum, product, convolution, and correlation of discrete functions.

5. Generating functions. (4 hours) Generating functions as alternative representations of discrete functions. Operations on discrete functions and the corresponding operations on their generating functions.

6. Difference equations. (5 hours) Linear difference equations with constant coefficients. Homogeneous solution and particular solution. Boundary conditions and undetermined coefficients. Solution of difference equations by the technique of generating functions. Simultaneous difference equations.

7. Relations. (2 hours) Cartesian product of sets. Binary relations. Reflexive, symmetric, transitive relations. Equivalence relations. Partial ordering relations. Union, intersection, and complementation of relations.

8. Graphs. (2 hours) Basic terminology in the theory of graphs. Directed graphs. Linear graphs and multigraphs. Connectedness. Paths. Graphs as representations of binary relations. Graphs as structural models.

9. Trees, circuits, and cut-sets. (4 hours) Mathematical properties of trees. Trees as structural models. Spanning trees. Circuits. Cut-sets.

10. Path problems in graphs. (5 hours) Eulerian path. Hamiltonian path. Existence of Eulerian paths and Hamiltonian paths in graphs. Physical interpretation of these notions. Shortest path algorithms and related problems.

11. Network flow problems. (4 hours) Transportation networks. Maximum-flow minimum-cut theorem. The Ford-Fulkerson algorithm for finding maximum flow.

COMMENTARY

1. Elementary set theory. The theme of this course is "discrete objects and their relationships." Consequently, the language of elementary set theory will be used throughout the course. The discussion should remain at an intuitive level, although it is quite reasonable to mention topics such as Russell's paradox which may lead to a discussion of axiomatic set theory.

2. Permutations and combinations. The discussion can begin with a determination of the number of subsets of a given set, a natural continuation of the material in Section 1. An important lesson to teach the students is that often some seemingly difficult problems may have very simple methods of solution when considered from the correct point of view.

Example: Design an algorithm for generating all r-combinations of n objects with unlimited repetitions.

Example: From all 5-digit numbers a number is selected at random. What is the probability that the number selected has its digits arranged in nondescending order?

[Answer: $\binom{10 + 5 - 1}{5}$ 10^5]

3. Discrete functions. The notion of discrete functions is introduced as an association of values (elements in the range) to objects (elements in the domain). There are numerous examples of discrete functions: coloring the faces of a polyhedron, assigning grades to students, classification of documents, etc. Point out the obvious extension to the notion of continuous function. The pigeonhole principle (also known as the shoebox argument) is a powerful technique, although it sounds extremely simple, as the following example illustrates.

Example: The integers 1, 2, 3, ..., 101 are arranged randomly in a sequence. Show that there is either a monotonically increasing subsequence or a monotonically decreasing subsequence of 11 (or more) integers.

[Solution: Let $a_1, a_2, a_3, \ldots, a_{101}$ denote a random arrangement of the integers 1, 2, 3, ..., 101. Let us label each integer a_k with a pair of numbers (i_k, j_k), where i_k is the length of a longest monotonically increasing subsequence that begins at a_k, and j_k is the length of a longest monotonically decreasing subsequence that begins at a_k. Suppose that $1 \leq i_k \leq 10$, $1 \leq j_k \leq 10$ for $k = 1, 2, \ldots, 101$. According to the pigeonhole principle, there must exist a_m and a_n which are labelled with the same pair of numbers. However, this is an impossibility because $a_m < a_n$ implies that $i_m > i_n$, and $a_m > a_n$ implies that $j_m > j_n$. (We assume that $m < n$.)]

4. Manipulation of discrete functions. The notion of the forward and backward differences of a discrete function corresponds to the notion of the derivative of a continuous function. The notion of the accumulated sum of a discrete function corresponds to the notion of the integral of a continuous function. The convolution $z(n)$ of two discrete functions $x(n)$ and $y(n)$ is defined to be

$$z(n) = \sum_i x(i)\, y(n-i).$$

The crosscorrelation function $w(n)$ of two discrete functions $x(n)$ and $y(n)$ is defined to be

$$w(n) = \sum_i x(i)\, y(i-n).$$

The autocorrelation of a function is the crosscorrelation of the function with itself.

Example: Consider the sequence $A = \{1, 1, 1, -1, 1, 1, -1\}$ as a signal transmitted by a radar transmitter. This signal is bounced back by an object whose distance from the radar is to be measured. (The distance can be determined from the elapsed time between the transmission of the signal and the arrival of the return signal.) To minimize the effect of noise interference, we want to choose a sequence so that the correlation function between the transmitted and the received signals will have a large peak value. Show that A is a good choice. [Answer: The autocorrelation function of the sequence A is $\{-1,0,-1,0,-1,0,7,0,-1,0,-1,0,-1\}$, which has a large peak value.]

5. Generating functions. The concept of the generating function of a discrete function corresponds to the concept of the Laplace transformation or the Fourier transformation of a continuous function. The sum of two discrete functions corresponds to the sum of their generating functions. The convolution of two discrete functions corresponds to the product of their generating functions.

Example: Show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Give a combinatorial interpretation (in terms of selection of objects) of this equality. [Answer: When a coin is tossed $2n$ times, there are 2^{2n} sequences of possible outcomes. Both sides of the above equality give the number of sequences of outcomes in which the number of heads occurring in the first n tosses is equal to the number of heads occurring in the last n tosses.]

6. Difference equations. Students will be better prepared for a course in differential equations after they have studied difference equations in this course. Indeed, concepts such as homogeneous solutions and particular solutions carry over directly to differential equations. Solving difference equations by the technique of generating functions corresponds to solving differential equations by the technique of Laplace transformations.

Example: A certain nuclear reaction in a system containing nuclei and high and low energy free particles is described as follows. There are two kinds of events: (i) a high energy particle strikes a nucleus, causing it to emit 3 high energy particles and 1 low energy particle, and is absorbed; (ii) a low energy particle strikes a nucleus, causing it to emit 2 high energy particles and 1 low energy particle, and is absorbed. Every free particle causes an event 1 μsec after it is emitted. If a single high energy particle is injected at time $t = 0$ into a system containing only nuclei, what will the total number of free particles in the system be at time $t = 20$ μsec? [Solution: Let a_n

denote the number of high energy particles and b_n the number of low energy particles in the system at the n^{th} μsecond. We have the simultaneous difference equations: $a_n = 3a_{n-1} + 2b_{n-1}$ and $b_n = a_{n-1} + b_{n-1}$, with the initial conditions $a_o = 1$ and $b_o = 0$. Solving these equations, we obtain $a_n + b_n = \frac{1}{2\sqrt{3}} \{(2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}\}$.]

7. Relations. Structural properties of sets of discrete objects can be described by relations. There are numerous examples of the concept of relations between objects: for instance, the relation "is the father of" is nonreflexive, nonsymmetric, and non-transitive; the relation "is the spouse of" is symmetric; the relation "is divisible by" (between integers) is a partial ordering relation.

Example: Write a computer program to determine all possible assignments of 0's and 1's to the vertices of the partial ordering diagram in Fig. 1 so that a 1 never precedes a 0.

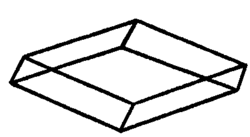


Fig. 1

8. Graphs. There are many examples from various disciplines using graphs as abstract models of structures, among which are social structures, finite state machines, PERT charts, data structures in computer programs.

Example: The inputs to an electronic combination lock are strings of 0's and 1's. The lock will be opened when the pattern 010010 appears at the end of the input string. Such a lock can be modeled graphically as in Fig. 2, where a string of 0's and 1's defines a path starting at the initial vertex.

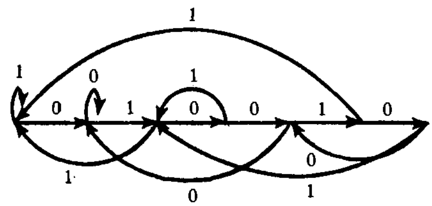


Fig. 2

9. Trees, circuits, and cut-sets. Although trees are very simple in concept, they are rich in structure and find application in many areas of study. There is enormous room for further discussion beyond the basic concepts; topics such as enumeration of trees, optimal trees (notably the Huffman algorithm for determining trees with minimum weighted path lengths), and algorithms for traversing trees may be considered.

Example: Communication links are to be built between cities. Suppose the cost of building a link between two cities is proportional to the distance between them. We want to build a set of links so that there is a path through these links between every two cities. Design a nonexhaustive algorithm that will yield a layout of minimal total cost. (This is a problem of designing an algorithm for finding a minimal spanning tree in a graph with weighted edges.)

10. Path problems in graphs. The notion of a shortest path in a graph has a clear interpretation in physical terms. If computing facilities are available, the implementation of some graph algorithms by students would be highly desirable. The discussion of Eulerian paths brings out another feature in our study of discrete structures--a simple criterion for the existence of some properties in a large class of structures. The following examples also illustrate some physical interpretations of the abstract notions of Eulerian and Hamiltonian paths.

Example: Arrange all n-digit binary numbers in such a way that the last (n-1) digits of a number are equal to the first (n-1) digits of the successive number. (This is an Eulerian path problem with application in digital engineering.)

Example: Arrange all n-digit binary numbers in such a way that two adjacent numbers differ only at one digit. (This is a Hamiltonian path problem with application in digital engineering.)

11. Network flow problems. This discussion not only exposes the students to the general problem of discrete optimization but also shows them a recursive technique in which the solution is improved in a step-by-step manner until an optimal solution is reached. This has exactly the same flavor as that of the simplex method in linear programming.

Example: Engineers and technicians are to be hired by a company to participate in three projects. The personnel requirements of these three projects are listed in the following table:

Minimal number of people needed in each project		Minimal number in each category			
		Mechanical engineers	Mechanical technicians	Electrical engineers	Electrica technicia
Project I	40	5	10	10	5
Project II	40	10	5	15	5
Project III	20	5	0	10	5

Moreover, to prepare for later expansion, the company wants to hire at least 30 mechanical engineers, 20 mechanical technicians, 20 electrical engineers, and 20 electrical technicians. What is a minimal number of persons in each category that the company should hire, and how should they be allocated to the three projects? (This problem can be formulated as a problem of finding a minimal flow in a transportation network where there is a lower bound on the flow-value in each of the edges.)

REFERENCES

The following four books would be useful in a course at the freshman-sophomore level. Berman and Fryer contains a very broad coverage of combinatorics. Kemeny, Snell, and Thompson discusses many interesting application problems. Berztiss was written mainly for computer science students. Vilenkin has a nice collection of examples and problems.

Berman, Gerald and Fryer, K. D. Introduction to Combinatorics. New York, Academic Press, Inc., 1972.

Berztiss, A. T. Data Structures. New York, Academic Press, Inc., 1972.

Kemeny, John G.; Snell, J. Laurie; Thompson, Gerald L. Introduction to Finite Mathematics, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

Vilenkin, N. Y. Combinatorics. New York, Academic Press, Inc., 1971.

The following books are more suitable for a junior-senior level course. They can also be used as references in a freshman-sophomore level course. Although these books are more advanced than the books cited above, they are quite readable even for undergraduate students.

Berge, Claude. Principles of Combinatorics. New York, Academic Press, Inc., 1971.

Berge, Claude. The Theory of Graphs and its Applications. New York, John Wiley and Sons, Inc., 1962.

Busacker, Robert G. and Saaty, Thomas L. Finite Graphs and Networks: An Introduction with Applications. New York, McGraw-Hill Book Company, 1965.

Knuth, Donald E. The Art of Computer Programming, vol. I. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

Liu, C. L. Introduction to Combinatorial Mathematics. New York, McGraw-Hill Book Company, 1968.

Ryser, Herbert J. Combinatorial Mathematics, MAA Carus Monograph 14. New York, John Wiley and Sons, Inc., 1963.

MC-3. Algorithmic Elementary Linear Algebra.

[Prerequisite: MC-0 or equivalent background] This course corresponds to Mathematics 3, "Elementary Linear Algebra," as described in the GCMC Commentary, and we refer the reader to that report for some additional comments. The differences between the two courses are mainly matters of emphasis and arrangement of topics. Whereas the course described in the GCMC Commentary stresses the algebraic and geometrical aspects of linear algebra and has a certain abstract flavor, the present course has a predominantly algorithmic viewpoint and its discussion revolves around the various ramifications of solving a system of linear equations. Throughout the course, detailed algorithms are to be presented and discussed, in flowchart or some simple step-by-step form, and the students should use these in connection with various practical problems, on a computer where possible. At the same time, in this course it is particularly important to warn the student that the algorithms are based on arithmetic with real numbers and that in a practical computation the effect of roundoff errors may lead to considerable distortions of the final result. This may be illustrated with well-chosen examples, but no attempt should be made to enter into a deeper discussion of such numerical problems.

COURSE OUTLINE

1. Introduction. (3 hours) Discussion of various practical problems involving matrices. Review of the elimination process for 2×2 and 3×3 systems of equations. Examples showing various cases of solvability of such systems.

2. Matrix algebra. (5 hours) Definition of real $n \times m$ matrices. Examples of various special forms of matrices. Transposes; symmetric and diagonal matrices. Equality, addition, and scalar multiplication. Matrix product and its properties. Computational applications of matrix algebra.

3. Vectors and geometry. (4 hours) Geometrical interpretation of 1×3 matrices. Algebraic properties in 3-space. The inner product. Euclidean length, angle, orthogonality, direction cosines. Linear combinations. Lines and planes. Projections. Vector proofs of simple geometric theorems. Matrices as linear transformations in R^2 and R^3. Geometric interpretation of one linear equation in three variables and of a 3×3 system.

4. Inverses and the row echelon form. (5 hours) Left and right inverses of an $n \times m$ matrix and relation to existence and

uniqueness of solutions of linear equations. Review of the elimination process as motivation for the elementary row operations. Elementary row operations and their formulation in terms of multiplication by elementary matrices. The algorithm for transforming a matrix to row echelon form. Equivalent systems of equations. Solvability properties of homogeneous and inhomogeneous systems using row echelon form.

5. Linear dependence and independence. (5 hours) Linear combinations of n × m matrices. Linear spaces of vectors and matrices. Subspaces. Linear dependence and independence of vectors in R^3 and in R^n, and of matrices. Examples and basic properties. Use of row echelon form to determine linear dependence or independence in R^n. Bases. Exchange algorithm. Dimension. Sum and intersection of subspaces and their dimensions.

6. Elimination. (5 hours) Algorithm for solving triangular systems. Inverses of triangular matrices. Gaussian elimination without pivoting; triangular decomposition. Pivots and the general algorithm. Backsubstitution and the solution of square linear systems. Algorithm for the computation of inverses. Numerical examples of ill-conditioning.

7. Rank. (5 hours) Linear mappings between linear spaces. Range space and null space. Relation between algebra of mappings and of matrices. Uniqueness aspect of row echelon form. Rank of a matrix. Rank of the transpose. Dimensions of null space and range space and related results.

8. Euclidean spaces. (4 hours) The inner product. Schwarz inequality. Euclidean length in R^n. Orthogonal bases. Gram-Schmidt process. Orthogonal projections. The least squares method.

9. [Optional] Abstract vector spaces. Axiomatic definition of vector space over R. Examples. Linear transformations and their algebra. The matrix of a linear transformation with respect to a given basis. Change of basis.

COMMENTARY

1. Introduction. Practical problems involving matrices abound. They may include the adjacency matrix of a street net, a simple resistive electrical network, a Markov chain example, the method of least squares, etc. [See, e.g., Noble, Ben. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969, Chapter 2.]

2. Matrix algebra. The stress here is on the algorithms of matrix algebra. Matrix multiplication can be motivated by practical examples of inner products leading to the product of a $1 \times n$ matrix by an $n \times 1$ matrix. Then the transformation of variables in linear equations readily provides a motivation of the matrix product. The examples introduced earlier can now be elaborated; for example, connectivity of a street net can be determined by forming powers of the adjacency matrix. A subroutine package for matrix algebra may be very useful for these applications.

3. Vectors and geometry. This section is rather standard. For comments we refer the reader to the GCMC Commentary.

4. Inverses and the row echelon form. In this section a basic algorithm is introduced, namely, the reduction to row echelon form; it will play a central role in the remainder of the course. Various applications are possible--for instance, determining solvability properties of a resistive electrical network.

5. Linear dependence and independence. In discussing the use of the row echelon form to determine linear dependence and independence, it is particularly important to illustrate the numerical problems which might occur when a computer is used. This can be motivated well by simple 2- and 3-dimensional examples. If time permits, the role of the exchange algorithms in linear programming can be illustrated by simple examples. [See, e.g., Stiefel, E. L. Introduction to Numerical Mathematics, translated by W. C. Rheinboldt. New York, Academic Press, Inc., 1963.]

6. Elimination. After a thorough discussion of the overall algorithm, it may be desirable to use a well-written subroutine package for computer assignments involving the solution of linear

systems arising in the practical problems introduced earlier. [See, for example, the routines given in Forsythe, George E. and Moler, C. Computer Solution of Linear Algebraic Systems. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.]

7. Rank. For some applications of rank, e.g., to chemical reactions, see Chapter 5 of the book by Ben Noble which was cited above.

8. Euclidean spaces. Again, applications abound. In particular, various problems leading to the use of the least squares method can be discussed.

REFERENCES

1. Matrices and linear algebra

Davis, Philip J. The Mathematics of Matrices. Waltham, Massachusetts, Blaisdell Publishing Company, 1965. A well-written elementary introduction to matrices.

Hohn, Franz E. Elementary Matrix Algebra, 2nd ed. New York, The Macmillan Company, 1964. An introductory text which proceeds in a manner similar to the outline above.

Noble, Ben. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969. This excellent text corresponds in spirit and approach to our outline but contains considerably more material and is, in parts, somewhat more advanced.

2. Numerical aspects

Forsythe, George E. and Moler, C. Computer Solution of Linear Algebraic Systems. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967. A brief, modern, but more advanced text on the basic numerical aspects of solving systems of linear equations.

Fox, Leslie. Introduction to Numerical Linear Algebra. New York, Oxford University Press, Inc., 1965. This text is a good source of instructive examples of error problems in numerical linear algebra.

4. Further Undergraduate Courses

In this section we discuss a rather heterogeneous group of upper-division undergraduate mathematics courses and areas affected by computing. The given list does not exhaust the possibilities and, even for the areas discussed here, there may well be other ways of incorporating the effect of computing. Clearly, at this level there is considerably more flexibility and there are probably many ways of modifying the approaches we suggest here.

For the courses in this section, the programming prerequisites are, of course, more advanced than for the previous courses; the computational facilities may also need to be more flexible. (See Sections 5.1 and 5.2). Further, the knowledge of computing and applied mathematics required by faculty members teaching these courses differs considerably from course to course. Thus for Ordinary Differential Equations (4.1) and Numerical Calculus (4.4) a knowledge of numerical analysis as well as facility in programming are absolutely essential. For Discrete Probability and Computing (4.2) reasonable programming experience in addition to a knowledge of probability is required. For Algebra Courses Influenced by Computing (4.5) a grounding in the algebraic foundations of computer sciences is needed in addition to the more usual kinds of computer expertise. Finally, for Mathematical Computer Modeling (4.3) a thorough knowledge of the applications involved is essential, of course, in addition to the programming and numerical analysis knowledge required by the selected applications.

4.1 Ordinary Differential Equations (3 semester hours)

[Prerequisites: MC-3, Mathematics 4 from the GCMC Commentary, good programming experience] How ordinary differential equations arise in practice. Separation of variables, integrating factors, variation of parameters, substitution. Equations with constant coefficients. Series solutions. Euler's method and a brief treatment of existence and uniqueness. An explicit Runge-Kutta method; a trapezoidal method for stiff systems. [For a discussion of stiff systems, see Gear, C. William. Numerical Initial Value Problems in Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971.] An introduction to boundary value problems.

The purpose of this course is basically the same as that of a more traditional course on ordinary differential equations, except that greater emphasis should be given to practical methods of solution. The most significant change is the inclusion of several carefully chosen numerical methods.

One numerical method is based on a well-established Runge-Kutta formula and is treated in enough detail to permit the writing of a reasonably effective computer program. This method is adequate for nonstiff problems, and there is no need to make more than brief reference to more complicated methods, such as multi-step methods, for these problems. However, one other method is needed for stiff systems of ordinary differential equations. A method based on the trapezoidal rule is included in this course because it is both simple and adequate. Numerical methods for boundary value problems are also included. A more detailed discussion of other numerical methods should be left to courses in numerical analysis.

This course can be followed by a second semester course covering several more advanced topics and exploiting more thoroughly various numerical methods. Topics for such a second semester may be chosen from among the following:

> Series solutions (including special functions), autonomous systems, Laplace transform, comparison theorem, eigenvalues and eigenfunctions, perturbation theory, asymptotic behavior, numerical methods, Galerkin methods, applications.

COURSE OUTLINE

1. Systems of equations. (2 hours) How ordinary differential equations arise in physical, chemical, biological, and economic problems.

2. Elementary analytic methods. (5 hours) Variables separable, e.g., in $y' = 1 - y^2$. Integrating factors, e.g., in $y' + P(x)y = Q(x)$. Substitution, e.g., in $y' = (ax + by)/(cx + dy)$. Variation of parameters. Solving equations with constant coefficients, e.g., the system $y' = ay + bz$, $z' = cy + dz$, or the higher-order equation $y'' + ay' + by = f(x)$. Introduction to series solutions.

3. Euler's method. (5 hours) Brief treatment of an existence and uniqueness theorem (perhaps without a detailed proof) of the Cauchy-Lipschitz kind, which can also be viewed as a theorem about the convergence of a simple numerical procedure. Bound on propagated error with Euler's method. Numerical examples, including a difficult one such as the Volterra equations that often arise in biological problems, e.g., $y' = 2(y - yz)$, $z' = -z + yz$.

4. More efficient numerical methods. (6 hours) Motivation of explicit Runge-Kutta formulas. A complete numerical method, including a strategy for changing step-size (see flowchart given below). Numerical examples, comparison with Euler's method. Note

the generality of the numerical method for systems of first-order equations: it can be used with nonlinear as well as linear equations; moreover, higher-order equations can be reduced to systems of first-order equations. Brief mention of more complicated multi-step methods.

5. Stiff systems of equations. (6 hours) The inability of standard methods to cope efficiently with stiff systems (e.g., with stable linear systems whose eigenvalues differ in magnitude by large factors). A complete numerical method for stiff systems based on the trapezoidal rule. Numerical examples, e.g., $y' = -101y - 100z$, $z' = y$. Compare Runge-Kutta and trapezoidal methods. Brief mention of other methods for stiff systems.

6. Boundary value problems. (10 hours) Elementary theoretical considerations, including an introduction to eigenvalues and eigenfunctions. Shooting methods. Finite difference methods. Mention of Galerkin methods.

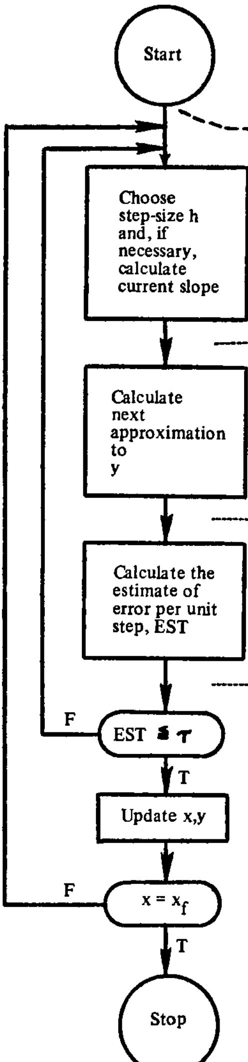
7. Limitations of numerical methods. (2 hours) Acknowledge the limitations of numerical methods and the need for their improvement. Point out the need for further analysis of solutions of differential equations, for example in the neighborhood of a singularity.

COMMENTARY

This course is intended to provide a reasonable balance between analytic and numerical methods that can be applied to problems involving ordinary differential equations. The students are expected to carry out numerical work related to applications.

This theme can be illustrated with Volterra's equations, which are mentioned above in the detailed outline. To begin with, examples of this sort are easily motivated in terms of predator-prey relationships. Then analytic methods can provide some useful information, such as existence and uniqueness of the solutions, and, with certain initial conditions, the existence of periodic solutions. But finding reasonable approximations to the solutions involves the use of numerical methods. The analytic methods are limited to relatively simple problems but help to provide an understanding for more general situations. The numerical methods are much more generally applicable, but

A Runge-Kutta method for nonstiff problems



Start with $f(x,y)$, x_0, y_0, x_f (final value of x), T (tolerance per unit step), h_{max} (maximum step-size)

$$h = \begin{cases} \min\{h_{max}, x_f - x\}, & \text{on entry} \\ \min\left\{h_{max}, x_f - x, .9\left(\frac{T}{EST}\right)^{1/4} h_{old}\right\}, & \text{otherwise} \end{cases}$$

Find slope on entry and after successful step

Use Kutta's formula

[$y_{i+1} = y_i + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$,

where $k_0 = hf(x_i, y_i)$

$k_1 = hf(x_i + ½h, y_i + ½k_0)$, etc.] twice with step-size ½h

Find approximation $\bar{y}$ by using Kutta's formula once with step-size h; then $EST = \left\| \frac{y - \bar{y}}{15h} \right\|$

Stop with x_0, y_0 replaced by x_f and the computed approximation to $y(x_f)$

they do not contribute very much to one's understanding; moreover, it is often difficult to assess their reliability.

We include in this section a flowchart for the explicit Runge-Kutta method and some comments on a trapezoidal method for stiff systems.

Notes:

1. Choosing h to be $.9(\tau/EST)^{\frac{1}{4}}$ times its previous value can be justified as follows. First of all, the exponent is $\frac{1}{4}$ because the method is a fourth-order method and the ratio $(\tau/EST)^{\frac{1}{4}}$ is asymptotically equal to the ratio of step-sizes associated with errors of τ and EST respectively. The trial step-size should be chosen to be somewhat smaller than what is determined by this ratio, and the factor .9 has been shown experimentally to be reasonably good.

2. Some modifications of the above are needed if we wish to allow $x_f < x_o$.

3. Care should be taken to avoid possible overflow in calculating τ/EST.

4. Provision could be made for using an error exit if the error test fails with h equal to a given h_{min}.

<u>A trapezoidal method for stiff systems</u>

A relatively simple method for stiff systems can be patterned on the flowchart given above. The only major change that needs to be made is to replace Kutta's formula with the trapezoidal formula

$$y_{i+1} = y_i + \frac{h}{2}(y_i' + y_{i+1}')$$

<u>and</u> to arrange for this equation to be solved by Newton's method. (The latter is required because simple iterations on this formula will not usually converge for stiff systems.)

Some minor changes are also needed. The exponent $\frac{1}{4}$ which is used in finding h must be replaced by $\frac{1}{2}$ because the trapezoidal formula is only second-order, and the factor 15 in the formula for EST must be replaced by 3 for the same reason.

REFERENCES

There is no one book which contains all the topics described in this outline. However, the following books taken together cover the material, although the last three especially contain too much for this one course; thus, topics will have to be selected.

Boyce, William E. and DiPrima, Richard C. Elementary Differential Equations and Boundary Value Problems, 2nd ed. New York, John Wiley and Sons, Inc., 1969.

Coddington, Earl A. An Introduction to Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964. Complete coverage of the theoretical aspects.

Davis, Harold T. Introduction to Nonlinear Differential and Integral Equations. New York, Dover Publications, Inc., 1960. Good treatment of practical examples, including the Volterra equations.

Gear, C. William. Numerical Initial Value Problems in Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971. Numerical methods for initial value problems, including stiff systems.

Keller, Herbert B. Numerical Methods for Two-Point Boundary-Value Problems. Boston, Massachusetts, Ginn and Company, 1968.

4.2 Discrete Probability and Computing

[Prerequisites: MC-2 and some knowledge of programming and computing procedures such as those found in Sections 2 and 3 of MC-DM] This course is intended as an introduction to the elements of probability. The main difference between it and a standard probability course, apart from the use of computing, is that, in order to get to more complex problems, less time is spent developing tools for solving simple problems. This difference is reflected in the amount of time allotted to the various units comprising the course, as well as in the fact that difficult theorems (such as the Central Limit Theorem) are to be stated without proof. However, in cases where proofs are omitted, the computer is used to provide experimental intuition for the validity of the theorems.

COURSE OUTLINE

1. Definition of a discrete probability measure; conditional probability for experiments with a finite number of outcomes. (3 hours)

2. The frequency concept of probability; fluctuation theory

illustrated by simulation; the arcsine law for the number of times in the lead. (3 hours)

3. Sums of sequences of independent random variables with common distribution; generating functions; mean; variance. (7 hours) Computational illustration of the Central Limit Thoerem. Proof of the Weak Law of Large Numbers. Illustration of the Strong Law of Large Numbers by simulation.

4. Brief discussion of probabilities on infinite spaces. (4 hours) Poisson, normal, and exponential distributions. Waiting-time problems illustrated by computer simulation.

5. Fair games (martingales). (6 hours) System theorems. Ruin probabilities. The meaning of convergence of nonnegative martingales illustrated using branching processes and other examples.

6. Finite Markov chains. (6 hours) Recurrent and absorbing chains. Use of matrix computation to write programs for basic descriptive quantities relating to Markov chains.

7. Additional topics. (7 hours) Applications of previous topics to selected problems in discrete potential theory, simulation of complex systems, or statistics.

COMMENTARY

1. Definition of a discrete probability measure. This unit represents in part a survey of material from MC-DM. Counting is restricted to permutations and combinations. Computational applications involve the properties of the binomial coefficients.

2. The frequency concept of probability. A possible computer assignment involves the discovery of the highly unintuitive arcsine law for the number of times in the lead in a penny-matching game. Once a conjecture has been established on the basis of experiments, a proof can be given using Feller's treatment based on the reflection principle. This provides an example of an easy limit theorem.

3. Sums of sequences of independent random variables with common distribution. Let $X_1, X_2, \ldots$ be a sequence of independent integer-valued random variables, and let $S_n = X_1 + \ldots + X_n$. A computer program can be used to compute

$$p_j^{(n)} = \Pr[S_n = j]$$

using only that

$$p_j^{(n)} = \sum_k p_k^{(1)} \; p_{j-k}^{(n-1)}.$$

This program may then be employed to motivate the concepts of mean and variance and to illustrate the Central Limit Theorem.

4. Brief discussion of probabilities on infinite spaces. This unit is included primarily to provide background for a later course on statistics. The discussion should be limited to distributions for a single experiment, with concepts such as the mean and variance being introduced by analogy with the finite case.

5. Fair games (martingales). Chapter 8 of Kemeny, Schleifer, Snell, and Thompson provides source material. This unit could be replaced by a unit on branching processes and generating functions.

6. Finite Markov chains. For a discussion of some related computational work, see, for example, Kemeny, John G. and Kurtz, Thomas E. Basic Programming, 2nd ed. New York, John Wiley and Sons, Inc., 1971, especially Section 16.3.

7. Additional topics. The purpose of this unit is to unify numerous applications through techniques discussed previously. For example, discrete potential theory can be applied to optimal stopping problems and to Markov decision processes; and the solution of the Dirichlet problem can be found using (a) the voltage in an electrical network, (b) the value of a stopped martingale (the Monte Carlo method), and (c) Markov chain methods. Such applications would build upon units 5 or 6 or both. Alternative or additional applications could include the simulation of complex systems (cf. Forester) or an introduction to elementary statistics.

REFERENCES

There is no single text which is suitable for the entire course. Sections of Feller and of Kemeny, Schleifer, Snell, and Thompson can be used for various units of the mathematical topics, i.e., noncomputational aspects. Freiberger gives a more advanced treatment, and Forester is an example of an application of these ideas to a real-life problem.

1. Mathematical background

Feller, William. An Introduction to Probability Theory and Its Applications, vol. 1, 3rd ed. New York, John Wiley and Sons, Inc., 1968.

Kemeny, John G.; Schleifer, A.; Snell, J. Laurie; Thompson, Gerald L. Finite Mathematics with Business Applications, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1972.

2. Some computational applications

Forester, Jay W. Urban Dynamics. Cambridge, Massachusetts, MIT Press, 1969.

Freiberger, Walter F. and Grenander, Ulf. A Short Course in Computational Probability and Statistics. New York, Springer-Verlag New York, Inc., 1971.

4.3 Experimental Development of a Course in Mathematical-Computer Modeling

A mathematical model of a phenomenon, mechanism, or process can be a system of algebraic, differential, difference, or functional equations, a stochastic process, or an abstract structure in terms of which a problem or question can be studied and can be given a mathematical solution. The usefulness of mathematical models in the physical sciences and engineering is beyond question; in many instances the models are so good that computer simulation is as accurate as any experimental measurements that can be made. The power of the computer to simulate and to compute widens the scope of acceptable models, affects the usefulness of mathematical methods, and makes possible procedures which are much different from those of the past and far superior to them.

In view of the complexity of physical phenomena which have been successfully subjected to mathematical analysis, mathematicians and scientists do not doubt that useful mathematical models can be constructed in all of the sciences. Indeed, for a long time we have witnessed a growing mathematization within the nonphysical sciences.

In all of this we are just beginning to appreciate the impact of the computer, and we are even less aware of the impact which computing and the computer will eventually have upon mathematics and pedagogy. Today our mathematical instruction is barely beyond the pencil-and-paper and chalk-and-blackboard stage; relatively few mathematicians have had experience in mathematical modeling and in effective use of the computer.

Although it seems imperative today to re-examine the content of our courses and to give our students some training in the processes by which mathematics is and can be applied, it is certainly beyond our experience at the moment to do so extensively at an elementary level. Modeling itself might best be introduced as an integral part of courses designed to teach a certain body of mathematics, but initially it might be better and easier to gain experience by experimenting with a separate course in mathematical-computer modeling at a post-calculus level. This could be a joint experimental undertaking by a number of faculty members and a few students; it should consist of the study and investigation in depth of a small number of carefully selected problems.

In selecting a problem one should take the following things into account:

(1) The problem should be easily stated. Without requiring extensive specialized knowledge or background, it should be possible to distinguish enough of the essential features in order to begin to construct some mathematical models, however crude.

(2) The problem should have mathematical content--the simpler the better at this level--which illustrates how mathematics is needed (i) to provide insight, (ii) to test the model (e.g., against a simple special case where the solution is obvious or easy), (iii) to develop a theory of the essential features of the model, and (iv) to indicate computational procedures.

(3) The problem should in some essential way require use of the computer (i) to provide insight through computer experimentation with the model or problem, (ii) to provide approximate answers and practical solutions, and (iii) to test the model and the solutions.

This does not imply that it is impossible to learn a great deal about modeling with pencil and paper, but a basic objective here is to go beyond this stage and to learn something about the uses and misuses of the computer and mathematical theory. More time needs to be spent in thinking about what goes into and what comes out of the computer than about the computation itself.

It is within the rules of the game to use mathematical or scientific results without proof, although where proofs are easily accessible and instructive they could be included. It would also be good pedagogy to consider models which are known to be poor, impractical theories and solutions, and poor numerical methods.

A SAMPLE PROBLEM

An excellent example is suggested by the work of Harold W. Kuhn. See his papers listed in the references at the end of this section; see also Courant and Robbins.

Fermat-Steiner-Weber Problem

Given n distinct points $p^1 = (x_1,y_1)$, $p^2 = (x_2,y_2)$, ..., $p^n = (x_n,y_n)$ in the plane and n positive numbers $w_1, w_2, \ldots, w_n$, find points which minimize

$$F(p) = \sum_{i=1}^{n} w_i |p - p^i| ,$$

where $p = (x,y)$ and $|p| = (x^2 + y^2)^{\frac{1}{2}}$ (thus $|p - p^i|$ is the Euclidean distance between p and p^i). This problem, posed by Fermat in the early 17th century with $n = 3$ and $w_1 = w_2 = w_3 = 1$, has had a long history and has been studied recently with renewed interest because of applications to spatial economics (optimal location of a factory, a shopping center, a hospital, a communications center, etc.).

Omitting the trivial case when the n points are collinear, we can show without difficulty that F is strictly convex, has a unique minimum which is in the convex hull of $p^1, p^2, \ldots, p^n$, and that the vanishing of a gradient (suitably defined at the vertices p^j) is a necessary and sufficient condition for a minimum.

The history and theory is interesting and provides a necessary background to the problem of finding approximate solutions numerically. The computational difficulties are nontrivial.

The following algorithm has been independently proposed at least three times:

Let

$$q^n = T(q^{n-1}),$$

where

$$T(q) = q + h(q)\ \nabla F(q),$$

$$h(q) = \left(\sum_{k=1}^{n} w_k\ |q - p^k|^{-1}\right)^{-1}$$

with $T(p^k) = p^k$ at the vertices [$h(q)$ is the harmonic mean of the distances to the vertices].

It can be shown that:

If q minimizes F, then it is a fixed point of T. If q

is a fixed point of T that is not a vertex, then q minimizes F. Either (1) $T^n(q)$ converges to a fixed point or (2) $T^j(q) = p^k$ for some j and some k.

Kuhn gives an algorithm which controls the step-size $h(q)\,\nabla F(q)$ for which he conjectures that $F(q^{n+1}) \leqq F(q^n)$. This would imply convergence. Calculations involving $n = 3$ to $n = 24$ give close approximations after seven iterations.

Outline for the Study of this Problem

1. Nonmathematical statement and discussion of an economic problem involving the optimal location of a plant, shopping center, etc.

2. Mathematical statement of the problem. Locate in the plane a point that minimizes the weighted sum of its distances to n given points in the plane.

3. History of the problem. Solution of simple cases. Simplest case (3 points, equal weights) considered by Fermat (c. 1635) in an essay on maximum and minimum problems. The more general problem with weights w_1, w_2, w_3 appears in an early book on "fluxions" by Simpson, one of the first textbooks on calculus.

4. Some mathematical theory.
   a. Existence-uniqueness.
   b. Necessary and sufficient conditions.
   c. Dual problem.

5. Computational methods. Use of the computer.
   a. As a problem in mathematical programming.
   b. A proposed algorithm and its motivation. Iterations, convergence, and fixed points.
   c. Computation of some examples.
   d. The conjecture $F(q^{n+1}) \leqq F(q^n)$. Special cases in which it can be verified.
   e. Computer tests of the conjecture.

6. A specific application. Study the problem of a good location for a large regional high school in the community.

7. Generalizations and unanswered mathematical and practical questions (research problems).

Desirable Features Illustrated by the Example

1. It is simple to describe, easily understood, explicit, interesting, and significant.

2. It has deep roots within the history of mathematics. Special cases of this problem appear as exercises in the earliest texts on "fluxions." It can be considered today in the light of new ideas, new mathematics, and computational procedures related to modern digital computers.

3. It serves to review and illustrate mathematics to which the student has been exposed: max-min, Lagrange multipliers (not required if the dual problem is omitted), simple linear algebra (analytic geometry), convergence.

4. It requires introduction at an elementary level of some new mathematics and new ideas important in mathematics and applications: convexity, duality, iteration (successive approximations), fixed points, and mathematical programming.

5. It provides an opportunity to develop a small body of theory.

6. It raises questions of computation, significant examples of which require the computer.

7. It raises a conjecture which can be proved in special cases and can be tested on the computer in more general cases.

8. It reaches the frontiers of research (generalizations to nonlinear costs, noneuclidean distance, etc., which are significant for applications).

A RECOMMENDATION

The development of individual topics, problems, exercises, etc., needed for a course of this type will require considerable work and imagination. This might be accomplished through isolated projects for independent group study with selected students, directed by an applications- and computer-oriented mathematician and a colleague representing the area of application.

Such experimental courses would be the testing ground for the development of instructional model building and are encouraged by the Panel. In the long run we believe that such model building should come in directly as a vehicle for teaching mathematics and its applications (for an example see the book by Grenander and the book by Freiberger and Grenander).

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4.4 Thoughts on a "Postponed" Calculus Course with Emphasis on Numerical Methods

In recent years many questions have been raised about the special role played by the basic calculus sequence as the first set of courses in traditional college mathematics curricula. There are many arguments for beginning with the calculus, but with the growth of computer science and the need for more mathematics in the behavioral and social sciences there are more and more arguments for postponing the calculus courses.

For those students who do not need to use the calculus in other courses until the junior or senior year, a drastically revised one-year calculus course which makes heavy use of computing and algorithmic ideas may be suitable. This course would have the Discrete Mathematics course MC-DM and a thorough knowledge of programming as prerequisites and would not be taken until the sophomore or junior year. A constructive approach to the basic concepts of the calculus would be used throughout the course and heavy emphasis would be placed on both numerical and nonnumerical algorithms. The course would contain some elementary numerical analysis, attention being paid to error analysis and degrees of approximation.

By necessity, some of the traditional topics of the calculus will have to be slighted, but the knowledge that the students will gain in being able to handle fairly complex real-world problems would certainly offset this.

The following outline should be considered as a first tentative suggestion. Given the novelty of the approach, there are very few experiences which might have been used as a guide. The material is ample for a one-year course, but no attempt is made to indicate the pace. The increased mathematical maturity of the students should make possible a faster pace than that in the usual calculus course. It should be kept in mind throughout the course that the topics are to be treated with heavy emphasis on numerical orientation.

A TENTATIVE OUTLINE

1. Numbers. A brief review of (intuitive) number concepts. Distribution of floating-point numbers on the line. Arithmetic problems with floating-point numbers. Roundoff errors. Ordering, inequalities, distances, and absolute value. All of this should be computationally oriented.

2. Sequences. Computational example of approximating the square root. Squeeze concept. Other related examples of limits. Need for irrational numbers to "fill" the number line; completeness concept. Definition of limit. Basic limit theorems (prove only a few). Squeeze theorem. Importance of error estimates. Slow and fast convergence illustrated by various examples.

3. Functions. Review of function concept (functions as mappings). Functions defined by algorithms, e.g., Horner's scheme, etc. Graphs and basic curve sketching. Arithmetic combination of functions. Geometric discussion of "near" functions and simple computational examples of approximations. Composition of functions, inverses. Monotonicity. Zeros, bisection algorithm. Uniform continuity; Intermediate Value Theorem for uniformly continuous functions using bisection algorithm.

4. Interpolation. Polynomial interpolation, undetermined coefficients, Lagrange formula, application to the solution of equations.

5. Derivatives. Limits, basic limit theorems with reference to the sequential case. Motivation and definition of the derivative. The cases x^k (small k) and $1/x$. Concept of higher derivatives. Continuity. Basic differentiation theorems. Derivative of polynomials, Horner's algorithm again. Derivative of rational functions. Linearization. Newton's method. Monotone convergence of Newton's method. Derivatives of monotone, convex, and concave functions. Chain rule. Implicit functions, inverse functions, application to $x^{1/n}$.

6. Area. Intuitive discussion of properties of area. Area of regions under monotone functions by approximations with sums of rectangles. Extension to nonmonotone functions, application to x^k, $k = 0, 1, 2, 3$.

7. Integral. Riemann sums, existence for uniformly continuous functions, basic properties. The Fundamental Theorem of Calculus. Application to the calculation of definite integrals. Substitution and integration by parts.

8. Quadrature. Review of rectangular approximation, trapezoidal rule and Simpson's rule. Integration by the Lagrange formula. Algorithmic treatment of partial fractions.

9. Differential equations. Direction fields, concept of solving first-order initial value problems. Separable case. Euler's method. Differential equations of radioactive decay, first-order logarithms. Second-order linear equations, superposition principle, harmonic motion, trigonometric functions.

10. Taylor's theorem. Mean Value Theorem, Taylor's theorem. Lagrange and integral remainder, application to error of interpolation, quadrature, l'Hôpital's rule, critical points, simple numerical methods for critical points.

11. Numerical solution of differential equations. Euler's method reviewed, trapezoidal rule, local discretization error, modified Euler's methods, Taylor's polynomial methods, idea of Runge-Kutta and multi-step methods. Brief geometric discussion of stability problems.

REFERENCES

There is no single textbook which covers the material proposed here, but parts of the following three texts may be used.

Flanders, Harley; Korfhage, Robert R.; Price, Justin J. Calculus. New York, Academic Press, Inc., 1970.

Henriksen, M. and Lees, M. Single-Variable Calculus. New York, Worth Publishers, Inc., 1970.

Stenberg, Warren, et al. Calculus, A Computer Oriented Presentation, Parts 1 and 2. CRICISAM, Florida State University, 1970.

For the numerical analysis portions, parts of various standard books on numerical methods can be used, especially for problems and applications.

4.5 Algebra Courses Influenced by Computing

At the present time it is not clear how the standard undergraduate introduction to algebra (e.g., Mathematics 6M in the GCMC Commentary) should be modified to reflect the growing influence of computers. Some knowledge of algebra is essential for an understanding of areas such as algebraic algorithms and symbol manipulation which have a strong algebraic flavor. Nonetheless, there is no consensus as to how the usual introduction to abstract algebra should be modified. In the discussion below we present brief outlines of three possible modifications, along with some sources of further information.

1. At Harvard University the Department of Applied Mathematics has taught a one-year course based on Birkhoff and Bartee, Modern Applied Algebra (New York, McGraw-Hill Book Company, 1970). Topics are selected from among the following:

> Sets and functions, relations, graphs. Finite state machines, programming languages. Monoids, groups, lattices, Boolean algebras, rings, polynomials, finite fields. Optimization and computer design, binary group codes, polynomial codes, recurrent sequences, computability.

For further details about the course, the book by Birkhoff and Bartee should be consulted.

2. Professor John Lipson of the University of Toronto has taught a modification of the one-year algebra course to advanced students in computer science for the past three years. Lecture notes for this course are expected to be available to interested parties sometime in 1973.

The principal topic in the second half of this course is a study of algebraic algorithms which incorporates recent work not readily available in the textbook literature. The following topics are considered:

> Sets, relations, functions. Examples of algebraic systems. Universal algebra. Lattices, Boolean algebra, groups, rings, finite fields. Interpolation theory, algebraic algorithms.

In addition to the usual textbooks in algebra, the following sources are used:

Berztiss, A. T. Data Structures. New York, Academic Press, Inc., 1971.

Birkhoff, Garrett. Lattice Theory, 3rd ed. Providence, Rhode Island, American Mathematical Society, 1966.

Birkhoff, Garrett and Bartee, Thomas C. Modern Applied Algebra. New York, McGraw-Hill Book Company, 1970.

Knuth, Donald E. The Art of Computer Programming. Reading, Massachusetts, Addison-Wesley Publishing Company. Vol. I, 1968; Vol. II, 1969.

3. A one-semester modification of the course described has been taught in the Department of Electrical Engineering at the Massachusetts Institute of Technology and in the Division of Applied Mathematics at Brown University. The following topics are covered:

> Sets, relations, functions, morphisms, diagram graphs and applications. Monoids, groups, lattices. Finite state machines, semantics of flow diagrams, programming languages. Rings, fields, polynomials, extension fields, finite fields.

5. Implementation

5.1 Computing Facilities

See Section 3.2 of Recommendations for an Undergraduate Program in Computational Mathematics, page 547.

5.2 Programming Requirements

The principal objective of any of the courses described in this report is to describe a mathematical subject area and applications related to it. Thus, the teaching of programming should not, by itself, be a purpose of any of these courses. Ideally, a student entering any of the lower-division courses except MC-0 should be required to have at least a beginning knowledge of one of the standard algorithmic languages implemented at his institution, as well as the ability to develop flowcharts and basic programs from a general description of a process. For the upper-division courses a more thorough familiarity with such a language and more programming expertise is required.

At present, few students entering the lower-division courses will have the corresponding programming background, although the expanding use of computers in high schools may change this picture in the future. Meanwhile, there are several alternatives that can be adopted.

If a student's schedule permits, one solution would be for him to take a one-semester introduction to computing, such as the course C1 described in the CUPM report Recommendations for an Undergraduate Program in Computational Mathematics. If this approach leads to delays in the mathematical progress, a possible alternative might be to let him take the computing course and his first mathematics course concurrently. In that case any one of the courses in Section 3 could be modified and taught in such a way that programming is not absolutely essential, although the results of the computations and the problems raised by computations would, of course, be used in the course.

Another alternative not involving a separate computing course is to teach a minimum amount of programming in supplementary lectures to those who need it during the first few weeks of the freshman courses. The time required for this depends considerably on the computing facilities available and on the language used; here computer use in a conversational mode is often particularly helpful. It is essential that the students be given ample opportunity to write and run programs of their own and to operate the necessary equipment, such as terminals or key punches. Moreover, it is important that consultants be available who can help them over their difficulties without overwhelming them with technical details. In courses where additional credit is given for the computational work, the supplementary programming lectures would, of course, take up the first few of the laboratory sessions held throughout the semester.

Which of these alternatives is the most feasible in a given situation depends not only upon the intended use of the computer in the course but also upon the nature of the available computing facilities.

As mentioned before, a few lectures in programming are not sufficient preparation for the more advanced courses. A consistent programming experience in the lower courses may, in general, enable a student to read an introductory computer science text on his own and to round out his computer knowledge in this way. Otherwise, a first computing course such as the course C1 cited earlier is certainly a natural prerequisite for the upper-division courses.

5.3 Changes in Instructional Techniques

In connection with the general topic of this report it is appropriate to review the state of teaching techniques in light of requirements for incorporating computers into the curriculum and to develop new teaching methods for bringing computational results and numerical algorithms into the classroom. The principal objective is to foster the "laboratory" atmosphere in class and to make each student feel that he is actively engaged in learning through problem solving, experimentation, and discovery.

It is important to bring the computational results into the classroom. Although thoughtful students will learn well from programming projects assigned as homework, the hurried or less thoughtful students see these assignments as chores to be done as quickly as possible. Sometimes a student will turn in a program with an error in it so gross as to make his answers meaningless. He will not have learned anything from the activity unless the instructor is able to review the work in class and exhibit the results which the problem was intended to elucidate.

The college mathematics teacher has always been at a real disadvantage when asked to make his lectures with chalk and blackboard as exciting and interesting as those of, say, his colleagues in physics who have carefully orchestrated, and often dramatic, experiments to perform in class. More than ever, though, we find chalk and blackboard inadequate for the presentation of the new material being proposed in this report; a teacher filling a board with computer results to six significant digits is likely to deter even the most energetic student! We hope that authors and publishers will address themselves to this problem and begin to develop new teaching materials for the mathematics teacher. Three possibilities are mentioned below, in order of increasing cost and complexity.

The first and most accessible teaching assist might come from sets of transparencies to be used with an overhead projector. Graphs of functions of one and two dimensions, successively "blown-up" portions of them, and computational results can all be presented. Carefully prepared overlays can give graphical results a dynamic sense. We are all familiar with the power and appeal of really good, professionally executed illustrations in textbooks. A library of transparencies of equal excellence with which a teacher could illustrate his lecture would go far toward livening up the classroom. The teacher interested in developing visual material should seek help from a media specialist.[1]

The second possibility to be considered is that of <u>videotaped or filmed presentations</u>. Here the dynamic nature of the algorithms can be well conveyed. For illustration, let us consider the concept of the definite integral. If the limit definition is phrased in an algorithmic form, the student will comprehend it best if he sees the approximating rectangles sketched, their areas added in one at a time, and the whole process repeated for a finer partition. When the partition is refined, he sees the effect of taking a larger number of smaller contributions to the integral. It is very difficult to draw accurately enough and fast enough on a blackboard to give students this sense of dynamism. Also, when animation is under consideration, it is natural to try to incorporate <u>computer-produced graphics</u> in these presentations.[2]

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1. Some information might also be obtained from the Association for Educational Communications and Technology, 1201 16th Street, N.W., Washington, D. C. 20036.

2. Advice may be obtained from Educational Development Center, 55 Chapel Street, Newton, Massachusetts 02160.

An independent reason for developing recorded presentations is that television cassette technology is reaching a stage which will allow a student to view a presentation independently, making individualized instruction a reality. A "library" of cassettes will make it possible for him to spend as much time as necessary on precisely the material that is appropriate for him. Courses could become modular in nature and it would no longer be necessary for all students to proceed in lock-step through the material. We note that freshman classes are becoming increasingly heterogeneous, both with respect to the students' capabilities and to the quality and quantity of their high school mathematics preparation. As "learning centers" with carrels containing TV screens and other audio-visual devices become increasingly common, the mathematical community should be concerned with their potential impact and usefulness.

We encourage authors who wish to prepare materials utilizing these new media to seek professional help from audio-visual specialists. Television and film offer new opportunities for innovative teaching. Simply to televise or to film traditional lectures would fail to take full advantage of the possibilities afforded by these media.

The third and most sophisticated and desirable technological solution is to have an <u>on-line terminal</u> connected to a reliable computer available at all times in the classroom. Devices are available which tap the input to a cathode ray tube display device and put the same image on one (or more) television monitors so that a large class can "participate" in the interaction.[3] If an on-line computer is used, a great deal of preliminary work is required on the part of the teacher. Numerical experiments must be chosen with great care, lest roundoff errors, the peculiarities of the computer operating system, etc., produce unanticipated results. Thus "inverting" a nearly singular matrix or "summing" an alternating series with terms alike to 6 digits using 5-digit arithmetic would obscure rather than illuminate, and could carry the teacher far deeper into the theory of computation than he ever intended to go. These problems are particularly likely to arise if a mini-computer with a small word length is used with only single precision arithmetic.

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3. For an overview of these technological developments, we recommend Ronald Blum, ed., <u>Computers in Undergraduate Science Education Conference Proceedings</u>, Commission on College Physics, College Park, Maryland, 1971 (available from American Institute of Physics, 335 East 45th Street, New York, New York 10017). See also <u>Proceedings of a Conference on Computers in the Undergraduate Curricula</u>, 1970 (available from the University of Iowa Computer Center, Iowa City, Iowa 52240), <u>Proceedings of the Second Annual Conference on Computers in the Undergraduate Curricula</u>, 1971 (available from The New England Press, Box 979, Hanover, New Hampshire 03755), and <u>Proceedings of the 1972 Conference on Computers in the Undergraduate Curricula</u>, 1972 (available from Southern Regional Education Board, 130 Sixth Street, N.W., Atlanta, Georgia 30313).

APPLIED MATHEMATICS

The importance of applications of mathematics to other areas was recognized by CUPM early in its existence. Among the original four panels were a Panel on Mathematics for the Physical Sciences and Engineering and a Panel on Mathematics for the Biological, Management, and Social Sciences, each charged with the task of making recommendations for the undergraduate mathematics program of students whose major interest lay in one of the stated fields.

The Panel on Physical Sciences and Engineering concentrated its efforts on the training of engineers and physicists, issuing its first report (Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists) in 1962. The demand for this document was so great that it was necessary to have it reprinted in 1965. Significant developments which occurred during the mid-sixties prompted the Panel to revise its recommendations and issue a new report in 1967. In the meantime this Panel had also developed CUPM's first definitive statement regarding the role of the computer in undergraduate mathematics. Its 1964 report Recommendations on the Undergraduate Mathematics Program for Work in Computing contained outlines for introductory and technical courses in computer science and a description of a program for mathematics majors planning to enter the field of computing; it is not being reproduced in this COMPENDIUM because it has been superseded by more recent CUPM documents. (See the section on COMPUTING.) Another document, Mathematical Engineering--A Five-Year Program, was issued by the Panel in 1966 to provide a means of alleviating what was then a drastic shortage of engineers having a substantial background in mathematics. Described as "a suggestion, rather than a recommendation," this report gives several outlines for options in operations research, orbit mechanics, and control theory.

The Panel on Mathematics for the Biological, Management, and Social Sciences, confronting problems which were less well defined, issued its Tentative Recommendations for the Undergraduate Mathematics Program for Students in the Biological, Management, and Social Sciences* in 1964. Primarily concerned with the mathematics curriculum for prospective graduate students in those fields, the report was meant to serve as a basis for discussion and experimentation. As a result of several issues raised in reaction to this document, CUPM decided in 1967 to concentrate on individual disciplines and, as a first step, appointed a Panel on Mathematics in the Life Sciences, charged with making recommendations for the mathematical training of the undergraduate life science student, whether or not he goes on to graduate school. The term "life science" here referred to agriculture and renewable resources, all branches of biology, and medicine. This Panel worked closely with the Commission on Undergraduate Education in the Biological Sciences, and its investigations culminated

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* Not included in this COMPENDIUM.

in the publication of Recommendations for the Undergraduate Mathematics Program for Students in the Life Sciences--An Interim Report (1970). Although it was anticipated that a final form of this report would eventually be issued, this project was never undertaken due to lack of funds.

Appointed in 1964, the CUPM *ad hoc* Subcommittee on Applied Mathematics was charged with suggesting appropriate undergraduate programs for students planning careers in applied mathematics. The Subcommittee's recommendations for such a program, together with suggestions for implementation and course descriptions, appeared in the 1966 report A Curriculum in Applied Mathematics.* During the years 1967-69 an Advisory Group on Applications kept CUPM informed on current developments in applied mathematics. The extremely rapid development of applications of mathematics, particularly in fields outside the physical sciences, together with a renewed interest in applications among mathematicians, led CUPM to appoint in 1970 a Panel on Applied Mathematics, whose duty was to reconsider some of the questions which the Subcommittee had studied earlier, and to draw up new recommendations in line with the nature and methods of applied mathematics. The Panel's suggestions, which emphasize the role of model building, are given in Applied Mathematics in the Undergraduate Curriculum (1972). This report contains detailed outlines of three options for a course in applied mathematics, each of which utilizes the model-building approach.

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* Not included in this COMPENDIUM.

RECOMMENDATIONS ON THE UNDERGRADUATE MATHEMATICS PROGRAM FOR ENGINEERS AND PHYSICISTS

A Report of

The Panel on Mathematics for the Physical Sciences and Engineering

Revised January 1967

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BACKGROUND (1962)

One reason for the current effort on the undergraduate program is the rapid change in the mathematical world and in its immediate surroundings. Three aspects of this change have a particular effect on undergraduate curricula in the physical sciences and engineering. The first is the work being done in improving mathematics education in the secondary school. Several programs of improvement in secondary school mathematics have already had considerable effect and can be expected to have a great deal more. Not only can we hope that soon most freshmen expecting to take a scientific program will have covered precalculus mathematics, but, perhaps more important, they will be accustomed to care and precision of mathematical thought and statement. Of course, not all students will have this level of preparation in the foreseeable future, but the proportion will be large enough to enable us to plan on this basis. Students with poorer preparation may be expected to take remedial courses without credit before they start the regular program.

This improved preparation obviously means that we will be able to improve the content of the beginning calculus course since topics which take time in the first two years will have been covered earlier. More than that, however, it means that the elementary calculus course will have to take a more sophisticated attitude in order to keep the student from laughing at a course in college which is less careful mathematically than its secondary school predecessors.

The second aspect of change in mathematics which confronts us is the expansion in the applications of mathematics. There is a real "revolution" in engineering--perhaps "explosion" is an even better description than "revolution," because, as it turns out, several trends heading in different directions are simultaneously visible. One is a trend toward basic science. The mathematical aspect of this trend is a strengthening of interest in more algebraic and abstract concepts. An orthogonal trend is one toward the engineering of large systems. These systems, both military and nonmilitary, are of ever-increasing complexity and must be optimized with regard to such factors as cost, reliability, maintenance, etc. Resulting mathematical interests are linear algebra and probability-statistics. A further trend, in part a consequence of the preceding two, is a real increase in the variety and depth of the mathematical tools which interest the engineer. In general, engineers are finding that they need to use new and unfamiliar mathematics of a wide variety of types.

A third factor is the arrival of the electronic computer. It is having its effect on every phase of science and technology, all the way from basic research to the production line. In mathematics it has, for one thing, moved some techniques from the abstract to the practical field; for example, some series expansion, iterative techniques, and so forth. Then too, computers have led people to tackle problems they would never have considered before, such as large systems of linear equations, linear and nonlinear programming, and

Monte Carlo methods. Many of these new techniques require increased sophistication in mathematics.

An additional factor entering from another direction must also be mentioned. Mathematicians in the United States have in recent years become much more closely involved with areas adjacent to their own research. Of the many factors which enter here, we may mention the greatly increased interest of mathematicians at all levels in education, the rapid growth of mathematical employment in industry, the spread of research and consulting contracts into the universities, and the development of a number of mathematical disciplines, such as information theory, that have many applications but are not classical applied mathematics. There is thus a real desire among mathematicians and scientists to cooperate in matters of education.

The conclusions above and the recommendations that constitute the body of this report were formulated by the Panel after extensive consultation with mathematicians, physicists, and engineers. In engineering, in particular, representatives of many fields and many types of institutions were consulted, as well as officials of the American Society for Engineering Education.* The recommendations for physicists were drawn up in close collaboration with the Commission on College Physics.

In considering the recommendations which follow, it is crucial to examine what has been our attitude toward certain ideas which inevitably occupy a central position in any discussion of mathematical education. Among these are mathematical sophistication and mathematical rigor, motivation, and intuition. Now it is a fact that mathematical rigor--by which we mean an attempt to prove essentially everything that is used--is not the way of life of the physicist and the engineer. On the other hand, mathematical sophistication--which means to us careful and clear mathematical statements, proofs of many things, and generally speaking a broad appreciation of the mathematical blocks from which models are built--is desired by, and desirable for, all students preparing for a scientific career. How does one choose what is actually to be proved? It seems to us that this is related to the plausibility of the desired result. It is unwise to give rigor to either the utterly plausible or the utterly implausible, the former because the student cannot see what the fuss is all about, and the latter because the most likely effect is rejection of mathematics. The moderately plausible and the moderately implausible are the middle ground where we may insist on rigor with the greatest profit; the great danger in the overzealous use of rigor is to employ it to verify only that which is utterly apparent.

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* Some of the results of a conference with engineers are embodied in four addresses delivered at a Conference on Mathematics in the Engineering Curriculum, held under the auspices of this Panel in March, 1961. These addresses were published in the Journal of Engineering Education, 52 (1961), pp. 171-207.

Let us turn next to the subject of motivation. Motivation means different things to different people and thus requires clarification. One aspect of motivation is concerned with the difference between mathematics and the applications of mathematics, between a mathematical model and the real world. For many engineers and physicists motivation of mathematical concepts can be supplied by formulating real situations which lead to the construction of reasonable models that exhibit both the desirability and the usefulness of the mathematical concept. Thus, motion of a particle or growth of a bacterial culture may be used as physical motivation for the notion of a derivative. It is also possible, of course, to give a mathematical motivation for a new mathematical concept; the geometric notion of a tangent to a curve also leads to the notion of derivative and is quite enough motivation to a mathematician. Since each kind of motivation is meaningful to large groups of students, we feel that both should appear wherever relevant. It is certainly a matter of individual taste whether one or both motivations should precede, or perhaps follow, the presentation of a mathematical topic. In either case, however, it is necessary to be very clear in distinguishing the motivating mathematical or physical situation from the resulting abstraction.

Physical and mathematical examples which are used as motivation, as well as previous mathematical experience, help to develop one's intuition for the mathematical concept being considered. By "intuition" we mean an ability to guess both the mathematical properties and the limitations of a mathematical abstraction by analogy with known properties of the mathematical or physical objects which motivated that abstraction. Intuition should lead the way to rigor whenever possible; neither can be exchanged or substituted for the other in the development of mathematics.

A mathematics course for engineers and physicists must involve the full spectrum from motivation and intuition to sophistication and rigor. While the relative emphasis on these various aspects will forever be a subject for debate, no mathematics course is a complete experience if any of them is omitted.

INTRODUCTION TO THE REVISION (1967)

In the five years that have elapsed since the first publication of these recommendations, several factors have emerged to affect the teaching of mathematics to engineers. The most striking of these is the widespread application of automatic computers to engineering problems. It is now a commonplace that all engineers must know how to use computers and that this knowledge must be gained early in their training and reinforced by use throughout it. We have, accordingly, included an introductory course in computer science as a requisite for all engineering students and have increased the amount of numerical mathematics in other courses wherever possible.

A second factor is the fairly general acceptance of linear algebra as part of the beginning mathematics program for all students. In the engineering curriculum this is tied in to the expansion in computing, since linear algebra and computers are precisely the right team for handling the large problems in systems analysis that appear in so many modern investigations. Five years ago there were only a handful of elementary texts on linear algebra; now treatments are appearing almost as fast as calculus books (with which they are often combined).

A development of particular interest to these recommendations is the appearance of the CUPM report A General Curriculum in Mathematics for Colleges (1965), referred to hereafter as GCMC. It is too early to judge how widely the GCMC will be adopted, but initial reactions, including those of teachers of engineering students, have been generally favorable. GCMC makes considerable use of material in the first version of these recommendations, and now we, in turn, borrow some of the courses in GCMC.

Minor changes in the content of courses and some rearrangement and changes of emphasis are the result of experience and discussions over the years.

Relatively little change has been made in the program for physicists. The only major one has been the inclusion of Introduction to Computer Science in the required courses. We do this in the conviction that all scientists (if not, indeed, all college graduates) should know something about the powers and limitations of automatic computers.

Applications of Undergraduate Mathematics in Engineering, written and edited by Ben Noble, published in January, 1967, by the Mathematical Association of America and the Macmillan Company, is based on a collection of problems assembled as a joint project of CUPM and the Commission on Engineering Education. The book has five parts: Illustrative Applications of Elementary Mathematics, Applications of Ordinary Differential Equations, Applications to Field Problems, Applications of Linear Algebra, Applications of Probability Theory.

INTRODUCTION TO THE RECOMMENDATIONS

This report presents a program for the undergraduate mathematical preparation of engineers and physicists.

Since obviously no single program of study can be the best one for all types of students, all institutions, and all times, it is important that anyone expecting to make use of the present recommendations understand the assumptions underlying them. The following comments should make these assumptions clear and also explain some other features of the recommendations.

1. This is a program for today, not for several years in the future. Programs somewhat like this are already being given at various places, and the sample courses we outline are patterned after existing ones. We assume a good but not unusual background for the entering freshman.

Five or ten years from now the situation will undoubtedly be different--in the high schools, in research, in engineering practice, and in such adjacent areas as automatic computation. Such differences will necessitate changes in the mathematics curriculum, but a good curriculum can never be static, and it is our belief that the present proposal can be continually modified to keep up with developments. However, the material encompassed here will certainly continue to be an important part of the mathematical education needed by engineers and physicists.

2. The program we recommend may seem excessive in the light of what is now being done at many places, but it is our conviction that this is the minimal amount of mathematics appropriate for students who will be starting their careers four or five years from now. We recognize that some institutions may simply be unable to introduce such a program very soon. We hope that such places will regard the program as something to work toward.

3. Beyond the courses required of all students there must be available considerable flexibility to allow for variations in fields and in the quality of students. The advanced material whose availability we have recommended can be regarded as a main stem that may have branches at any point. Also, students may truncate the program at points appropriate to their interests and abilities.

4. The order of presentation of topics in mathematics and some related courses is strongly influenced by two factors:

a. The best possible treatment of certain subjects in engineering and physics requires that they be preceded by certain mathematical topics.

b. Topics introduced in mathematics courses should be used in applications as soon afterwards as possible.

To attain these ends, coordination among the mathematics, engineering, and physics faculties is necessary, and this may lead to course changes in all fields.

5. The recommendations are, of course, the responsibility of CUPM. In cases where it seems of interest and is available, we have indicated the reaction of the groups of engineers and physicists who were consulted. For convenience we refer to them as "the consultants."

LIST OF RECOMMENDED COURSES

It is desirable that all calculus prerequisites, including analytic geometry, be taught in high school. At present it may be necessary to include some analytic geometry in the beginning analysis course, but all other deficiencies should be corrected on a non-credit basis.

The following courses should be available for undergraduate majors in engineering and physics:

1. Beginning Analysis. (9-12 semester hours)

As far as general content is concerned, this is a relatively standard course in calculus and differential equations. There can be many variations of such a course in matters of rigor, motivation, arrangement of topics, etc., and textbooks have been and are being written from several points of view.

The course should contain the following topics:

a. An intuitive introduction of four to six weeks to the basic notions of differentiation and integration. This course serves the dual purpose of augmenting the student's intuition for the more sophisticated treatment to come and preparing for immediate applications to physics.

b. Theory and techniques of differentiation and integration of functions of one real variable, with applications.

c. Infinite series, including Taylor series expansion.

d. A brief introduction to differentiation and integration of functions of two or more real variables.

e. Topics in differential equations, including the following: linear differential equations with constant coefficients and first-order systems--linear algebra (including eigenvalue theory, see 2 below) should be used to treat both homogeneous and nonhomogeneous problems; first-order linear and nonlinear equations, with Picard's method and an introduction to numerical techniques.

f. Some attempt should be made to fill the gap between the high school algebra of complex numbers and the use of complex exponentials in the solution of differential equations. In particular, some work on the calculus of complex-valued functions of a real variable should be included in items b and c.

g. Students should become familiar with vectors in two and three dimensions and with the differentiation of vector-valued functions of one variable. This material can obviously be correlated with the course in linear algebra (see below).

h. Theory and simple techniques of numerical computation should be introduced where relevant. Further comments on this point, applying to the whole program, will be found below (under course 3).

We feel that the above comments on beginning analysis sufficiently describe a familiar course. The remaining courses in our list are less generally familiar. Hence the brief descriptions of courses 2 through 12 are supplemented in the Appendix [or elsewhere in this COMPENDIUM] by detailed outlines of sample courses of the kind we have in mind.

2. Linear Algebra. (3 semester hours)

A knowledge of the basic properties of n-dimensional vector spaces has become imperative for many fields of applications as well as for progress in mathematics itself. Since this subject is so fundamental and since its development makes no use of the concepts of calculus, it should appear very early in the student's program. We recommend a course with strong emphasis on the geometrical interpretation of vectors and matrices, with applications to mathematics (see items 1-e and 1-g above), physics, and engineering. Topics should include the algebra and geometry of vector spaces, linear transformations and matrices, linear equations (including computational methods), quadratic forms and symmetric matrices, and elementary eigenvalue theory.

It may be desirable, for mathematical or scheduling purposes, to combine beginning analysis and linear algebra into a single coordinated course to be completed in the sophomore year.

For outlines of a Beginning Analysis sequence, see the courses Mathematics 1, 2, and 4 described in Commentary on A General Curriculum in Mathematics for Colleges, page 44 . The course Mathematics 3 (Elementary Linear Algebra) of the GCMC Commentary (page 55) approximates the linear algebra course described here, but does not contain the recommended material on quadratic forms and elementary eigenvalue theory. This Panel's recommended courses on functions of several variables, functions of a complex variable, real variables, and algebraic structures coincide with those of the GCMC Commentary (Mathematics 5 [alternate version], 13, 11, and 6M, respectively).

3. Introduction to Computer Science. (3 semester hours)

The development of high-speed computers has made it necessary for the appliers of mathematics to know the path from mathematical theory through programming logic to numerical results. This course gives an understanding of the position of the computer along this path, the manner of its use, its capabilities, and its limitations. It also provides the student with the basic techniques needed in order to use the computer to solve problems in other courses.

An even more important part of the path must be provided by the student's program as a whole. All the courses discussed here should contain, where it is suitable and applicable, mathematical topics motivated by the desire to relate mathematical understanding to computation. It is especially desirable that the student see the possibility of significant advantage in combining analytical insight with numerical work. Indications of such opportunities are scattered throughout the recommended course outlines.

4. Probability and Statistics. (6 semester hours)

Basic topics in probability theory, both discrete and continuous, have become essential in every branch of engineering, and in many engineering fields an introduction to statistics is also needed. We recommend a course based on the notions of random variables and sample spaces, including, *inter alia*, an introduction to limit theorems and stochastic processes and to estimation and hypothesis testing. Although this should be regarded as a single integrated course, the first half can be taken as a course in probability theory. For ease of reference we designate the two halves 4a and 4b.

5. Advanced Multivariable Calculus. (3 semester hours)

Continuation of item 1-d. A study of the properties of continuous mappings from E_n to E_m, making use of the linear algebra in course 2 , and an introduction to differential forms and vector calculus based on line integrals, surface integrals, and the general Stokes theorem. Application should be made to field theory, elementary hydrodynamics, or other similar topics, so that some intuitive understanding can be gained.

6. Intermediate Ordinary Differential Equations. (3 semester hours)

This course continues the work on item 1-e into further topics important to applications, including linear equations with variable coefficients, boundary value problems, rudimentary existence theorems, and an introduction to nonlinear problems. Much attention should be given to numerical techniques.

7. Functions of a Complex Variable. (3 semester hours)

This course presupposes somewhat more mathematical maturity than courses 5 and 6 and so would ordinarily be taken after them, even though they are not prerequisites as far as subject matter is concerned. In addition to the usual development of integrals and series, there should be material on multivalued functions, contour integration, conformal mapping, and integral transforms.

8. Partial Differential Equations. (3 semester hours)

Derivation, classification, and solution techniques of boundary value problems.

9. Introduction to Functional Analysis. (3 semester hours)

An introduction to the properties of general linear spaces and metric spaces, their transformations, measure theory, general Fourier series, and approximation theory.

10. Elements of Real Variable Theory. (3 semester hours)

A rigorous treatment of basic topics in the theory of functions of a real variable.

11. Optimization. (3 semester hours)

Linear, nonlinear, and dynamic programming, combinatorics, and calculus of variations.

12. Algebraic Structures. (3 semester hours)

An introduction to the theory of groups, rings, and fields.

13. Numerical Analysis.

14. Mathematical Logic.

15. Differential Geoemtry.

The last three courses are topics that might well be of interest to special groups of students. Their lengths and contents may vary considerably. For a sample outline of a course in Numerical Analysis, see Mathematics 8 (Introduction to Numerical Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 83.

The above list of courses is the result of careful consideration by the Panel and the consultants. The brief description given here and the detailed sample outlines found in the Appendix [or elsewhere in this COMPENDIUM], while based on the mathematical structure of the topics themselves, reflect strongly the expressed interests of engineers and physicists. We realize that the nature of the institution and the requirements of other users of mathematics as well as of the mathematics majors may influence the specific offerings.

RECOMMENDED PROGRAM FOR ENGINEERS

A. Courses to be required of all students.

1. Beginning Analysis. This recommendation needs no comments.

2. Linear Algebra. The great majority of the consultants felt that this is important material that all engineers should have during the first two years.

3. Introduction to Computer Science. Developments of the last few years make it clear that engineering is strongly dependent on a knowledgeable use of computers.

4a. Probability. All students should have at least a 3-semester-hour course in probability. The consultants agreed on the value of probability to an engineer, but there was considerable disagreement among the consultants as to the advisability of requiring it of all students. However, the members of our Panel are unanimously and strongly of the opinion that this subject will soon pervade all branches of engineering and that now is the time to begin preparing students for this development.

B. Courses recommended for students intending to go into research and development.

4b. Statistics.

5. Advanced Multivariable Calculus.

6. Intermediate Ordinary Differential Equations.

7. Functions of a Complex Variable.

The consultants agreed to the value of the material in courses 5, 6, and 7, and some preferred that it be completed within the junior year. The Panel is convinced that an adequate presentation requires a minimum of nine semester hours, which could, of course, be taken in one year if desired. The order in which courses 5 and 6 are taken is immaterial except as they may be coordinated with other courses. If they are to be presented to the students in a fixed order, the instructor may wish to adjust the time schedules and choice of topics.

C. Courses which should be available for theoretically minded students capable of extended graduate study.

8. Partial Differential Equations.

9. Introduction to Functional Analysis.

10. Elements of Real Variable Theory.

Presumably a student would take either 9 or 10 but not both; 9 is probably more valuable but 10 is more likely to be available.

11. Optimization.

D. Courses of possible interest to special groups.

12. Algebraic Structures.

13. Numerical Analysis.

14. Mathematical Logic.

15. Differential Geometry.

RECOMMENDED PROGRAM FOR PHYSICISTS

A. Courses to be required of all students.

1. Beginning Analysis.

2. Linear Algebra. Like the engineers, the physicists felt that this material is essential.

3. Introduction to Computer Science.

5. Advanced Multivariable Calculus. This course should be taken in the sophomore year if possible, and in any event no later than the first part of the junior year.

6. Intermediate Ordinary Differential Equations.

B. Additional courses, in order of preference. Students contemplating graduate work should be required to take a minimum of three to nine semester hours of these courses.

7. Functions of a Complex Variable.

9. Introduction to Functional Analysis.

4a. Probability. The value of requiring this course in the undergraduate program of all physicists is not as well established as it is for engineers.

12. Algebraic Structures.

10. Elements of Real Variable Theory.

8. Partial Differential Equations.

Appendix

DESCRIPTION OF RECOMMENDED COURSES

While we feel strongly about the spirit of the courses outlined here, the specific embodiments are to be considered primarily as samples. Courses close to these have been taught successfully at appropriate levels, and our time schedules are based on this experience. Some of these courses are sufficiently common that approximations to complete texts already exist; others have appeared only in lecture form.

2. Linear Algebra. (3 semester hours)

The purpose of this course is to develop the algebra and geometry of finite-dimensional linear vector spaces and their linear transformations, the algebra of matrices, and the theory of eigenvalues and eigenvectors.

The course Mathematics 3 (Elementary Linear Algebra) of Commentary on a General Curriculum in Mathematics for Colleges (page 55) approximates the linear algebra course which this Panel has in mind. Mathematics 3 does not, however, contain the recommended material on quadratic forms and elementary eigenvalue theory.

3. Introduction to Computer Science. (3 semester hours)

This course serves a number of purposes:

(1) It gives students an appreciation of the powers and limitations of automata.

(2) It develops an understanding of the interplay between the machine, its associated languages, and the algorithmic formulation of problems.

(3) It teaches students how to use a modern computer.

(4) It enables instructors in later courses to assign problems to be solved on the computer.

For an outline of such a course, see C1 (Introduction to Computing) in Recommendations for an Undergraduate Program in Computational Mathematics (page 563).

4. Probability and Statistics. (6 semester hours)

This is a one-year course presenting the basic theory of probability and statistics. Although the development of the ideas and results is mathematically precise, the aim is to prepare students to formulate realistic models and to apply appropriate statistical techniques in problems likely to arise in engineering. Therefore new ideas will be motivated and applications of results will be given wherever possible.

First Semester: Probability.

a. Basic probability theory. (4 lectures) Different theories of probability (classical, frequency, and axiomatic). Combinatorial methods for computing probability. Conditional probability, independence. Bayes' theorem. Geometrical probability.

b. Random variables. (5 lectures) Concept of random variable and of distribution function. Discrete and continuous types. Multidimensional random variables. Marginal and conditional distributions.

c. Parameters of a distribution. (4 lectures) Expected values. Moments. Moment-generating functions. Moment inequalities.

d. Characteristic functions. (4 lectures) Definition, properties. Characteristic functions and moments. Determination of distribution function from characteristic function.

e. Various probability distributions. (6 lectures) Binomial, Poisson, multinomial. Uniform, normal, gamma, Weibull, multivariate normal. Importance of normal distribution. Applications of normal distribution to error analysis.

f. Limit theorems. (6 lectures) Various kinds of convergence. Law of Large Numbers. Central Limit Theorem.

g. Markov chains. (4 lectures) Transition matrix. Ergodic theorem.

h. Stochastic processes. (6 lectures) Markov processes. Processes with independent increments. Poisson process. Wiener process. Stationary processes.

Second Semester: Statistics.

a. Sample moments and their distributions. (5 lectures) Sample, statistic. Distribution of sample mean. Student's distribution. Fisher's Z distribution.

b. Order statistics. (4 lectures) Empirical distribution function. Tolerance limits. Kolmogorov-Smirnov statistic.

c. Tests of hypotheses. (5 lectures) Simple hypothesis against simple alternative. Composite hypotheses. Likelihood ratio test. Applications.

d. Point estimation. (5 lectures) Consistent estimates. Unbiased estimates. Sufficient estimates. Efficiency of estimate. Methods of finding estimates.

e. Interval estimation. (6 lectures) Confidence and tolerance intervals. Confidence intervals for large samples.

f. Regression and linear hypotheses. (4 lectures) Elementary linear models. The general linear hypothesis.

g. Nonparametric methods. (5 lectures) Tolerance limits. Comparison of two populations. Sign test. Mann-Whitney test.

h. Sequential methods. (5 lectures) The probability ratio sequential test. Sequential estimation.

5. Advanced Multivariable Calculus. (3 semester hours)

For an outline of this course, see Mathematics 5 (Multivariable Calculus II--alternate version) in Commentary on A General Curriculum in Mathematics for Colleges, page 77.

6. Intermediate Ordinary Differential Equations. (3 semester hours)

The presentation of the course material should include: (1) an account of the manner in which ordinary differential equations and their boundary value problems, both linear and nonlinear, arise; (2) a carefully reasoned discussion of the qualitative behavior of the solution of such problems, sometimes on a predictive basis and at other times in an a posteriori manner; (3) a clearly described awareness of the role of numerical processes in the treatment of these problems, including the disadvantages as well as the advantages--in particular, there should be a firm emphasis on the fact that numerical integration is _not_ a substitute for thought; (4) an admission that we devote most of our lecture time to linear problems because (with isolated exceptions) we don't know much about any nonlinear ones except those that (precisely or approximately) can be attacked through our understanding of the linear ones. Thus, a thorough treatment of linear problems must precede a sophisticated attack on the nonlinear ones.

The distribution of time among items d through f cannot be prescribed easily or with universal acceptability. Only a superficial account of these topics can be given in the available time, but each should be introduced.

a. _Systems of linear ordinary differential equations with constant coefficients_. (6 lectures) Review of homogeneous and nonhomogeneous problems; superposition and its dependence on linearity; transients in mechanical and electrical systems. The Laplace transform as a carefully developed operational technique without inversion integrals.

b. _Linear ordinary differential equations with variable coefficients_. (10 lectures) Singular points, series solutions about regular points and about singular points. Bessel's equation and Bessel functions; Legendre's equation and Legendre polynomials; confluent hypergeometric functions. Wronskians, linear independence, number of linearly independent solutions of an ordinary differential equation. Sturm-Liouville theory and eigenfunction expansions.

c. _Solution of boundary value problems involving nonhomogeneous linear ordinary differential equations_. (7 lectures) Methods using Wronskians, Green's functions (introduce δ functions), and eigenfunction expansions. Numerical methods. _Rudimentary_ existence and uniqueness questions.

d. _Asymptotic expansion and asymptotic behavior of solutions of ordinary differential equations_. (3 lectures) Essentially the material on pp. 498-500 and pp. 519-527 of _Methods of Mathematical Physics_ by Harold Jeffreys and Bertha S. Jeffreys (third edition; New York, Cambridge University Press, 1956).

e. _Introduction to nonlinear ordinary differential equations_. (6 lectures) Special nonlinear equations which are reducible to linear ones or to quadratures, elliptic functions (pendulum oscillations), introductory phase plane analysis (Poincaré).

f. Numerical methods. (7 lectures) Step-by-step solution of initial value problems for single equations and for systems. Error analysis, roundoff, stability. Improper boundary conditions, discontinuities, and other pitfalls.

7. Functions of a Complex Variable. (3 semester hours)

For an outline of this course, see Mathematics 13 (Complex Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 97.

8. Partial Differential Equations. (3 semester hours)

This course is suitable for students who have completed a course in functions of a complex variable. The emphasis is on the development and solution of suitable mathematical formulations of scientific problems. Problems should be selected to emphasize the role of "time-like" and "space-like" coordinates and their relationship to the classification of differential equations. (It seems very useful to introduce the appropriate boundary conditions motivated by the physical questions and be led to the classification question by observing the properties of the solution.) The student should be led to recognize how few techniques we have and how special the equations and domains must be if explicit and exact solutions are to be obtained; he particularly must come to realize that the effective use of mathematics in science depends critically on the researcher's ability to select those questions which both fill the scientific need and admit efficient mathematical treatment. To accomplish this realization, the instructor should frequently introduce a realistic question from which he must retreat to a related tractable problem whose interpretation is informative in the context of the original question.

a. Derivation of equations. (2 lectures) The derivation of mathematical models associated with many scientific problems. Review of heat conduction to a moving medium, the flow of a fluid in a porous medium, the diffusion of a solute in moving fluids, the dynamics of elastic structures, neutron diffusion, radiative transfer, surface waves in liquids.

b. Eigenfunction expansions. (5 lectures) Eigenfunction expansions in both finite and infinite domains (Titchmarsh).

c. Separation of variables. (7 lectures) The product series solutions of partial differential equation boundary value problems.

Integral transforms such as the Laplace, Fourier, Mellin, and Hankel transforms and their use. Copious illustration of these techniques, using elliptic, parabolic, and hyperbolic problems.

d. Types of partial differential equations. (5 lectures) The classification of partial differential equations, characteristics; appropriate boundary conditions. Domains of influence and dependence in hyperbolic and parabolic problems. The use of characteristics as "independent" coordinates.

e. Numerical methods. (8 lectures) Replacement of differential equations by difference equations; iterative methods; the method of characteristics. Convergence and error analysis.

f. Green's function and Riemann's function. (9 lectures) Their determination and use in solving boundary value problems. Their use in converting partial differential equation boundary value problems into integral equation problems.

g. Similarity solutions. (3 lectures)

h. Expansions in a parameter. (3 lectures) Perturbation methods in both linear and nonlinear problems.

9. Introduction to Functional Analysis. (3 semester hours)

The purpose of this course is to present some of the basic ideas of elementary functional analysis in a form which permits their use in other courses in mathematics and its applications. It should also enable a student to gain insight into the ways of thought of a practicing mathematician and it should open up much of the modern technical literature dealing with operator theory.

Prerequisite to this course is a good foundation in linear algebra and in the concepts and techniques of the calculus of several variables. The material of this course should be presented with a strong geometrical flavor; undue time should not be spent on the more remote and theoretical aspects of functional analysis. Topics should be developed and first employed in mathematical surroundings familiar to the student. It would be very much in keeping with the intention of the course to emphasize the relationship between functional analysis and approximation theory, discussing (for example) some aspects of best uniform or best L^2 approximation to functions, and some error estimates in integration or interpolation formulas.

While some knowledge of measure theory and Lebesgue integration is needed for an understanding of this material, it is not intended that the treatment be as complete as that in a standard real analysis course. The intended level is that to be found in the treatment by Kolmogorov and Fomin (Kolmogorov, A. N. and Fomin, S. V. Elements of the Theory of Functions and Functional Analysis. Vol. 2: Measure, the Lebesgue Integral, Hilbert Space. Baltimore, Maryland, Graylock Press, 1961.) If there is additional time, students might be introduced to some of the elementary theory of integral equations, or to applications in probability theory, or to the study of a specific compact operator, or to distributions.

For an outline of such a course, see Mathematics Q (Functional Analysis) in A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, page 125.

10. Elements of Real Variable Theory. (3 semester hours)

For an outline of this course, see Mathematics 11 (Introductory Real Variable Theory) in Commentary on A General Curriculum in Mathematics for Colleges, page 93.

11. Optimization. (3 semester hours)

Attempts to determine the "best" or "most desirable" solution to large-scale engineering problems inevitably lead to optimization studies. Generally, the appropriate methods are highly mathematical and include such relatively new techniques as mathematical programming, optimal control theory, and certain combinatorial methods, in addition to more classical techniques of the calculus of variations and standard maxima-minima considerations of the calculus.

The 3-semester-hour course outlined below is planned to provide a basic mathematical background for such optimization studies. Another outline for a course in optimization, utilizing methods of programming and game theory, can be found in the report Applied Mathematics in the Undergraduate Curriculum, page 722.

a. Simple, specific examples of typical optimization problems. (3 lectures) Minimization with side conditions (Lagrange multipliers, simple geometrical example). Linear program (diet problem). Nonlinear program (least squares under inequality constraints, delay line problem). Combinatorial problem (marriage or network). Variational problem (brachistochrone). Control problem (missile). Dynamic program (replacement schedule).

b. Convexity and n-space geometry. (6 lectures) Convex regions, functions, general definition (homework: use definition to show convexity [or nonconvexity] in nonobvious cases, such as Chebychev error over simple family of functions). Local, global minima. Convex polyhedra (review matrix, scalar product geometry). Geometric picture of linear programming.

c. Lagrange multipliers and duality. (6 lectures) Classical problem with equality constraints. Kuhn-Tucker conditions for inequality constraints. Linear programs. Dual variables as Lagrange multipliers. Reciprocity, duality theorems.

d. Solution of linear programs--simplex method. (3 lectures)

e. Combinatorial problems. (6 lectures) Unimodular property. Assignment problem (Hall's theorem, unique representatives). Networks (min-cut max-flow).

f. Classical calculus of variations. (7 lectures) Stationarity. Euler's differential equation, gradient in function space. Examples, especially Fermat's principle and brachistochrones.

g. Control theory. (8 lectures) Formulation. Pontryagin's maximum principle (Lagrange multipliers again).

12. Algebraic Structures. (3 semester hours)

For an outline of this course, see Mathematics 6M (Introductory Modern Algebra) in Commentary on A General Curriculum in Mathematics for Colleges, page 68.

MATHEMATICAL ENGINEERING

A FIVE-YEAR PROGRAM

A Report of

The Panel on Mathematics for the Physical Sciences

and Engineering

October 1966

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INTRODUCTION

During a conversation at the mathematics meetings in January, 1964, the late S. S. Wilks of Princeton University pointed out the absence of any special large-scale efforts to provide technical personnel for the nation's space program. As he saw it, there was a continuing need for persons really well-trained in the mathematical sciences and able to apply their fields to the complicated engineering problems of the space effort. His aim was to provide technical manpower specifically prepared for a strong combination of mathematics and engineering--in addition to the more customary converts from a great variety of backgrounds. He felt that CUPM, with its tradition of pervasive concern for all aspects of collegiate mathematical education, might undertake a study of this problem. The present report is, in part, a response to his ideas.

A pair of meetings with representatives of NASA and the space industry confirmed the eminent need for the projected product. At the same time it became clear that many other industries, such as electronics and communication, would have an equal interest in a mathematical engineer with the same basic background but possibly somewhat different specialization. All these industries face innumerable engineering problems with a common need for extensive and sophisticated mathematical analysis. For the solution of such problems it is no longer true that the prime requirement is a good physical intuition; rather, one also needs a well-developed mathematical intuition. Thus the mathematical engineering program must involve a heavy concentration in mathematics, but with a choice of topics that will give a useful basis for applications as well as solid grounding in theory. As the Panel came to grips with this multiplicity of purposes, it developed the notion of a common core, for all mathematical engineers, of material in the basic physical sciences and, more extensively, in the mathematical sciences. The core, in turn, is complemented by a number of options, which are more specialized developments in depth and which assure that the student will be fairly well acquainted with at least one branch of engineering. Orbit mechanics, operations research, and control theory are three options which are developed in detail in the present report.

It turns out that a minimum of five years, rather than four, is needed to carry out the desired sequences in depth. It is not immediately obvious what the student's area of concentration should be called. The dual emphasis on mathematics and engineering makes either field conceivable; in fact, the program comes close to being recognizable as a master's program in applied mathematics. However, the heavy emphasis on the physical sciences, the concern in each option with the building of mathematical models, and the rather heavily prescriptive nature of the program make a realization within the engineering school more suitable. Wherever the program may appear within an institution's offerings, it should involve close cooperation between the mathematicians and the engineers.

A natural question also arises as to the possible fields of further graduate study which a student could enter on completion of the present program. It is our opinion that relatively little, if any, "remedial" work will be necessary to qualify the student for a doctoral program in applied mathematics or, depending on the particular option, in engineering science, or in industrial or electrical engineering. Whether or not a master's degree should be given for the completion of this program is a matter for the offering institution to decide. As remarked above, the content of the program is of the right order of magnitude for this degree. Other considerations (e.g., requirement of a thesis) may be deciding factors.

A number of additional remarks about the program are in order. A most important aspect is the flexibility which would automatically be built into the mathematical engineer. With such a background and with a considerable facility in making connections between the real world and mathematical models thereof, such a man could easily retrain himself, say, from space science to oceanography, if a sudden shift of present national interest should make this desirable. Secondly, the similarity in spirit of this program to the recently introduced engineering physics and engineering science programs is worth noting. The idea of these programs is to give the student a solid background of the kind of physics that would be useful in a wide variety of engineering applications, along with enough engineering subjects to impart some feeling for the kinds of problems he would encounter. There is now no question of the value of such training. In just the same way, mathematical engineering combines a solid foundation in major areas of applicable mathematics with real strength in some particular area of engineering, and experience in connecting the two. Incidentally, it should be remarked that mathematical engineering has existed for some years, much in the spirit of the present report, at several universities in the Netherlands. It seems to be a successful program from the point of view of both employment opportunity and preparation for further graduate work.

DESCRIPTION OF PROGRAM

As remarked above, the program is constructed around a core consisting of a heavy concentration of mathematics and the physical sciences. Attached to the core there may be many options, each providing motivation, application, and extension of the core material to some phase of engineering. The core is fairly well defined and will probably not vary greatly from one institution to another. The options, on the other hand, will necessarily have much local flavor both in their general subject matter and in the particular courses that compose them. The three options that we present here are thus to be regarded as samples of what can be done.

The Core

Modern engineering is built upon a three-part foundation consisting of mathematics, the physical sciences, and automatic computing. The last of these is a newcomer whose precise role and manner of development are still matters of speculation, but there is no question as to its basic importance. These three topics, then, compose the core.

The mathematical portion, which is, for this program, the most extensive, is based on Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 628. The courses recommended there as preparation for graduate work have been modified somewhat and about nine semester hours have been added, much of it as additional work in topics already begun. This gives us the following list (the initial number refers to the course outlines given in the Appendix [or elsewhere in this COMPENDIUM], and "hours" means "semester hours"):

- Calculus and Linear Algebra (12-15 hours)
1. Functions of Several Variables (3 hours)
2. Intermediate Ordinary Differential Equations (3 hours)
4. Numerical Analysis (3 hours)
5. Probability and Statistics (6 hours)
6. Complex Variables (3 hours)
7. Functional Analysis (3 hours)
10. Partial Differential Equations (6 hours)
11. Optimization (3 hours)

Discussion of the individual courses is deferred until the whole core has been described.

The resulting 42-45 hours of mathematics in the core is about the magnitude of a good undergraduate major in mathematics, but the emphasis is quite different. This program stresses analysis heavily. Indeed, the minimal treatment of algebra and geometry is perhaps the most vulnerable point of this curriculum. However, for the foreseeable future the topics included are certainly of first importance. Other mathematical topics needed in certain courses can be developed when needed to the extent required.

We recognize full well that the value of these courses depends on the spirit in which they are taught. One must keep in mind that their ultimate purpose is application, either directly or as preparation for more obviously applicable topics. Such courses as Partial Differential Equations and Optimization should lean heavily on applied problems. Computational methods should be stressed throughout. Further, as much interconnection as possible should be built into the whole program. It is planned that both the mathematics and the engineering courses should reinforce one another to an unprecedented degree.

A corollary of this requirement is that the courses should be taught by whoever is capable of doing the best job, regardless of the department he happens to be in. For the more standard and the more theoretical courses the mathematics department would be the natural place to look for teachers, but where applications are heavily stressed the best teacher may well be found in some other field.

Beyond a fairly standard 15- to 18-hour introduction to physics and chemistry, the program calls for 12 more hours in the basic sciences. Six of these are accounted for by a mechanics course, intended to be a coordinated combination of physics, mathematics, and computing. Considerable time is spent on variational methods and continuum mechanics, as well as on the standard mechanics of particles and rigid bodies.

The remaining six hours is divided between electromagnetics and thermodynamics (including statistical mechanics). Of the many possible continuations of the basic material it was felt that these two, because of their fundamental nature, their wide applicability, and their susceptibility to interesting mathematical analysis, are particularly appropriate to this program.

Modern computing facilities and the techniques for using them are still developing with bewildering rapidity, and no program fixed now will give adequate coverage for very long. We are painfully aware of these rapid developments and claim no special powers of prophecy. The proposal delineated here provides two realistic approaches to current problems of computing. A direct approach is the inclusion of two courses devoted to computation, the Numerical Analysis mentioned above and an Introduction to Computer Science [such as the course C1 in <u>Recommendations for an Undergraduate Program in Computational Mathematics</u>]. These give the principles of modern computation, including the use of a programming language, and their basic applications to mathematical problems.

Less direct, but perhaps of more ultimate importance, is the inclusion of computational methods in connection with each appropriate topic in other courses. Such topics occur in almost all the core courses, but particularly in Differential Equations (both Ordinary and Partial), Mechanics, and Optimization. It is expected that significant problems to program and run on a computer will be part of the work in these courses. Only in this way can a real understanding of the power and (especially) of the limitations of modern computing techniques be communicated.

The overall structure of the core can be seen in the table. Here each entry represents a 3-hour course.

Year		Core Courses			Electives*
I		Calculus		Physics Chemistry	6
		Calculus		Physics Chemistry	6
II		Linear Algebra		Physics Computer Science	6
		Calculus		Physics	9
III	1.	Functions of Several Variables	3. 4. 5.	Mechanics Numerical Analysis Probability	3
	2.	Ordinary Differential Equations	3. 5.	Mechanics Statistics	6
IV	6.	Complex Variables	8.	Electromagnetics	9
	7.	Functional Analysis	9.	Thermodynamics	9
V	10.	Partial Differential Equations	11.	Optimization	9
	10.	Partial Differential Equations			12

* semester hours

Of an assumed total of 150 semester hours for the five years, 45 hours are devoted to mathematics and computing and 30 to basic sciences. This leaves the remaining 75 hours for humanities electives and for additional courses in engineering, mathematics, and science. With a roughly even split this should satisfy normal requirements. The most demanding of our sample options, Control Theory, specifies only 24 hours, leaving, say, 15 hours for basic engineering and technical electives and 36 hours for the humanities.

The first two years of the program are fairly standard. For the mathematics portion we recommend the sequence of courses described in <u>Commentary on A General Curriculum in Mathematics for Colleges</u>, page 33. In addition to linear algebra and the usual elementary calculus of one or more variables this includes an introduction to differential equations.

Introductory physics and chemistry courses are at present under intensive review by professional groups; some radically new physics courses have recently appeared and other experimental programs are underway. We therefore refrain from specifying these courses in any detail but urge the reader to consult the publications of the Commission on College Physics and the Advisory Council on College Chemistry. Widespread adoption of new elementary courses could require great changes in the content, or even in the selection, of the later science courses in the core.

The introductory computer science course should include a discussion of the nature of an automatic computer and the manner in which it solves problems, an introduction to a specific computer language and its role in this process, and some practice in the actual solution of various types of problems. Either by means of this course or from supplementary instruction, students should be able to program and run simple problems early in their sophomore year.

The third year has the heaviest concentration of core courses, foundation for the more advanced material in the core and for the technical applications in the options. These courses are of fairly standard type except for Mechanics, which has been described, and Intermediate Ordinary Differential Equations. The occurrence of considerable material on differential equations in the calculus course justifies the initial adjective in the latter title and permits the course to concentrate on linear equations with variable coefficients, boundary value problems, and special functions. There is also a brief introduction to nonlinear equations.

The report Commentary on A General Curriculum in Mathematics for Colleges outlines three courses in Functions of Several Variables (Mathematics 5. Multivariable Calculus II.). The first of these has a classical vector analysis approach, while the second uses differential forms; the third is particularly suited for students in statistics. We recommend the second course, outlined on page 77, partly because the vector technique is covered in the physics courses but also because the more general approach is a valuable background for fourth-year Functional Analysis.

With the exception of the Probability and Statistics all the third-year work is closely interconnected, and considerable thought should be given to the sequence of topics so as to get the most coordination. In particular, linear algebra, numerical techniques, and the use of a computer to solve problems are ever-recurring themes in the year's work.

The fourth and fifth years of the core are fairly light, since here will come most of the work in the options. In the mathematics courses, in addition to the obvious requirements of Complex Variables and Partial Differential Equations--six hours of the latter is necessary for any sort of coverage, we have included courses in Functional Analysis and Optimization. The first of these provides an introduction to some hitherto abstract topics that are proving useful in a variety of applications, such as numerical analysis, communication theory, quantum physics, and many branches of mathematics. General metric and linear spaces, operators and functionals, with an introduction to measure theory, are the central topics.

Optimization is another introductory course, but one tied very closely to applications. Based on the notions of compactness, convexity, and Lagrange multipliers, it treats briefly the various types of mathematical programming, some combinatorial problems, and the calculus of variations.

The two science courses in the fourth year, Electromagnetics and Thermodynamics, could vary considerably in content. In any case, however, they should take advantage of the students' exceptional background in analysis, probability, mechanics, and computing to give a considerably more sophisticated treatment than could commonly be contemplated.

The selection and arrangement of the courses comprising the core represent the Panel's best judgment of the curriculum currently needed to develop the kind of highly trained but still flexible engineer described in the Introduction. Local conditions and opinions will undoubtedly suggest some changes, and future developments--for example, an upsurge of biological engineering--may call for a reappraisal of the whole program. But the basic framework of mathematics, science, and computing should still be appropriate for many years to come.

The Options

The role of the option is to provide the student with a solid acquaintance with some branch of engineering, at the same time giving background and applications of many of the subjects treated in the core. In general, serious work in the option will begin in the fourth year, following the heavy load of third-year core courses and some appropriate technical introduction. With this background the work in the option can begin, and proceed, at a higher level of sophistication than is usually possible.

As samples of what might be done we present three options, labeled, for want of better names, Operations Research in Systems Engineering, Orbit Mechanics, and Control Theory. There is nothing special about these; they simply happened to be topics of interest to some of the Panel members and consultants. Other topics of equal suitability might be, for example, Fluid Mechanics, Solid Mechanics, Electronics and Microwaves, Wave Propagation and Plasma Physics, Materials Engineering, and Nuclear Engineering.

For each of the sample options we give here a brief description of the program and an outline of its structure. Detailed syllabi of the courses are given in the Appendix.

OPERATIONS RESEARCH IN SYSTEMS ENGINEERING

Operations Research has been described as the application of mathematical methods to the solution of practical optimization problems in engineering, in business, and in government. The Operations Research Option builds onto the core those features of operations research that pertain especially to the design, development, and production of large-scale engineering systems. These require analysis of the complicated interrelationships among component and system performances, development and production costs, scheduling priorities, available manpower and facilities, and a host of other factors. Such considerations have made necessary the use of various optimization techniques, the application of probabilistic and statistical methods, the development of a highly mathematical reliability theory, Monte Carlo simulation methods, optimal control theory, linear, nonlinear and dynamic programming methods, and queueing theory. These mathematical topics provide the typical tools for operations research studies which find wide applicability in the evaluation (and comparison) of performance, programs, and policies in certain types of engineering and industrial situations.

The Operations Research Option, which has been fleshed out in some detail, represents an attempt to provide suitable training for engineers who have to cope with such problems. Building upon the 6-hour course in Probability and Statistics of the core, it provides a 6-hour course in mathematical methods of reliability engineering. The course includes both probabilistic models of reliability problems and statistical techniques of reliability estimation. The introductory optimization course of the core is supplemented by a further 6-hour course in linear programming techniques, dynamic programming, inventory and scheduling problems, queueing theory, and related topics.

The third course recommended in this option is a 3-hour course in System Simulation, which exploits the use of a computer in carrying out the analysis of such operations research activities.

Additional courses in economics, such as Economic Decision Theory, or in management science would constitute appropriate electives for certain students.

Note that the core course in Optimization has been moved into the fourth year to provide the necessary background for the fifth-year course in Operations Research.

Year	Core Courses	Option Courses	Electives
IV	6. Complex Variables	OR1. Reliability	9
	7. Functional Analysis	OR1. Reliability	
	8. Electromagnetics		
	9. Thermodynamics and Statistical Mechanics		
	11. Optimization		
V.	10. Partial Differential Equations	OR2. Operations Research	15
	10. Partial Differential Equations	OR2. Operations Research	
		OR3. Systems Simulation	

ORBIT MECHANICS

As with each of the options, the aim is to build upon the foundation supplied by the core to provide greater specialization in an aspect of mathematics of central importance in modern engineering, here space science.

The design of space vehicles; prediction, correction, and control of their space flight; transmission and evaluation of information collected in space--all such tasks place unusual new requirements on engineering skills and training. Additional problems arise from the necessity for real-time computations and corrections during space flight. Underlying all these difficulties is the problem of developing a correct physical intuition for the nature of space travel, vehicle control, and environmental conditions in space.

The Orbit Mechanics Option supplements the core courses in mechanics with substantial one-semester courses in celestial mechanics and in orbit theory. The addition of an advanced programming course and an introduction to control theory provides solid grounding for many problems of space vehicle engineering. The course in data smoothing and prediction provides training essential to the successful collection, retrieval, and interpretation of telemetered information.

Additional courses in astronomy or in space physics constitute natural electives for students in such a program.

Since some of Partial Differential Equations is needed for Advanced Numerical Analysis and Celestial Mechanics, the core course 10 must be put in the fourth year. This gives a rather heavy concentration of mathematics in the fourth year, but this could be relieved, if desired, by a further shifting of some of the other courses.

Year	Core Courses		Option Courses		Electives
IV	6.	Complex Variables	OM1.	Advanced Numerical Analysis	9
	7.	Functional Analysis			
	8.	Electromagnetics			
	9.	Thermodynamics and Statistical Mechanics			
	10.	Partial Differential Equations			
	10.	Partial Differential Equations			
V	11.	Optimization	OM2.	Advanced Programming	12
			OM3.	Celestial Mechanics	
			OM4.	Orbit Theory	
			CT2.	Control	
			CT5.	Data Smoothing and Prediction	

CONTROL THEORY

The advances in computers and in instrumentation have brought an enormous increase in the sophistication of control systems. The instruments allow us to measure rapidly and precisely many variables which were previously hard to measure, and the computer allows us to make use of all the data while it is still current. The space program has given a great impetus to control theory by bringing up a number of new problems with very strict requirements. Another aspect of many control problems is that they involve control loops which extend over great distances, thereby creating an interface problem between the control and the communications specialist.

The Control Theory Option starts in the third year with a one-semester course in circuit theory which exposes the student to the modeling problem, to some specific physical devices which he will encounter later, and to basic system concepts in simple physical situations. In the fourth year the control course will furnish the student the basic facts about control systems and the linear systems course will provide the common base for further courses in control, communications, and circuits. The fifth year includes a course on the techniques of optimization, one on advanced control, one on advanced communications, and one on information theory.

Year	Core Courses	Option Courses	Electives
IV	6. Complex Variables	CT2. Control	6
	7. Functional Analysis	CT3. Laboratory	
	8. Electromagnetics	CT4. Linear Systems	
	9. Thermodynamics and Statistical Mechanics	CT5. Data Smoothing and Prediction	
V	10. Partial Differential Equations	CT6. Advanced Control	12
	10. Partial Differential Equations	CT7. Information Theory	
	11. Optimization	CT8. Advanced Communications	

APPENDIX

Sample Outlines of the Courses

The course outlines given in this Appendix or elsewhere in the COMPENDIUM are intended in part as extended expositions of the ideas that we have in mind, in part as feasibility studies, and in part as proposals for the design of courses and textbooks. They have a wide variety of origins. Some are standard courses now given in universities and some are experiments that have never yet been tried. Most of them, however, are modifications or combinations, more or less radical, of familiar material. They have been prepared by many different persons, with a broad spectrum of interests in mathematics and related fields, and representing industrial as well as academic interests. However, all outlines were carefully scrutinized by the whole Panel and were not accepted until their value to the whole program was clear. For better or worse, this is a committee product.

It will be observed that there is considerable overlapping in some of the content of the courses, for example in Intermediate Ordinary Differential Equations and in Numerical Analysis. This is inevitable in the courses in any modern university, where most courses are taken by a variety of students in different programs and with different backgrounds. If a neater dovetailing of these courses is possible in particular cases, the contents should of course be modified accordingly.

The Core

I, II		Physics (12)		Calculus and Linear Algebra	(12-15)
		Chemistry (6)		Computer Science	(3)
III	1.	Functions of Several Variables			(3)
	2.	Intermediate Ordinary Differential Equations			(3)
	3.	Mechanics			(6)
	4.	Numerical Analysis			(3)
	5.	Probability and Statistics			(6)
IV	6.	Complex Variables			(3)
	7.	Functional Analysis			(3)
	8.	Electromagnetics			(3)
	9.	Thermodynamics and Statistical Mechanics			(3)
V	10.	Partial Differential Equations			(6)
	11.	Optimization			(3)

1. Functions of Several Variables. (3 semester hours)

For an outline of this course, see Mathematics 5 (Multivariable Calculus II--alternate version) in Commentary on A General Curriculum in Mathematics for Colleges, page 77.

2. Intermediate Ordinary Differential Equations. (3 semester hours)

For an outline of this course, see Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 643.

3. Mechanics. (6 semester hours)

The course outlined below differs from that given in certain textbooks in that the discussion of mechanics is interruped at various stages in order to deal with topics in numerical analysis; e.g., after the equations of motion are formulated, various methods for numerically integrating initial value problems are discussed and analyzed. It is assumed that the student has had the linear algebra course as well as the computer science course. Homework assignments that involve use of a computer should be made.

a. Kinematics. (8 lessons) Cartesian coordinates in Euclidean 3-space, cartesian tensors, the numerical tensors δ_{ij}, ϵ_{ijk}. Parametric equations of curves. Velocity and acceleration in cartesian coordinates, in general coordinates. Moving general coordinates and the velocity and acceleration in such coordinates. Equations of straight lines in moving general coordinates. Characterization of inertial coordinate frames.

b. Particle mechanics. (10 lessons) Equations of motion. Initial value problems for a system of ordinary differential equations, existence, uniqueness, continuous dependence on parameters and initial values. Numerical methods for integrating initial value problems, their stability.

c. Perturbation theory. (8 lessons) Physical stability. Numerical stability. Linearization of nonlinear problems.

d. Central forces. (10 lessons) Planetary orbits. Energy integrals, angular momentum integrals. Constants of motion and symmetry properties.

e. Variational principles and rigid body motion. (13 lessons) Hamilton's principle, generalized coordinates of Lagrange, canonical equations, contact transformations, partial differential equations of Hamilton and Jacobi. Rigid body motion.

f. Multidimensional variational principles. (8 lessons) Variation of multiple integrals and applications to problems in statics and dynamics of deformable bodies. Vibrating strings and membranes. Rayleigh-Ritz method. Use of polynomials to derive difference equation approximation to the boundary value differential equations that are the Euler equations of a variational principle. Numerical integration of boundary value problems on the line and in the plane.

g. Continuum mechanics. (21 lessons) Stress and strain tensors. Conservation of mass, momentum and energy. Partial differential equations describing the motion of a perfect fluid. One-dimensional isentropic motions (simple and compound waves). Numerical integration of 1-dimensional motions. Existence of shocks. Numerical integration in the presence of shocks.

4. Numerical Analysis. (3 semester hours)

For an outline of this course, see Mathematics 8 (Introduction to Numerical Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 83.

5. Probability and Statistics. (6 semester hours)

For an outline of this course, see Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 642.

6. Functions of a Complex Variable. (3 semester hours)

For an outline of this course, see Mathematics 13 (Complex Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 97.

7. Introduction to Functional Analysis. (3 semester hours)

For an outline of this course, see Mathematics Q (Functional Analysis) in A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, page 125.

8. Electromagnetics. (3 semester hours)

This course combines the essentials of classical electromagnetic theory with the foundations of applications to plasma media. It should take advantage of the preparation in mechanics, especially continuum mechanics, as well as of the fundamentals of electricity and magnetism in the introductory physics course.

a. Electrostatics. (4 lessons) Vacuum field and potential theorems. Dielectrics. Boundary conditions. Energy relations and forces.

b. Magnetic fields. (5 lessons) Current and moving charges. The vector potential. Magnetic media. Energy relations and forces.

c. Maxwell's equations. (6 lessons) The law of induction. Maxwell's equations, extended to moving media. Energy, force, and momentum relations in the electromagnetic field.

d. Wave propagation. (8 lessons) The scalar wave equation. Plane, cylindrical, spherical waves. Homogeneous isotropic media. Dispersion. Nonhomogeneous isotropic media. Rays; the geometrical optics approximation. Wave packets, including nonhomogeneous media and absorption.

e. Electromagnetic waves. (7 lessons) Free space and homogeneous isotropic media. Homogeneous plasmas. Inhomogeneous media. Anisotropic media, including plasma with magnetic field.

f. Radiation. (9 lessons) Simple radiating systems. Radiation by moving charges. Radiation in ionized gases. Synchrotron radiation.

9. Thermodynamics and Statistical Mechanics. (3 semester hours)

There is a current trend to combine the macroscopic and the microscopic aspects of thermal physics from the beginning, instead of giving a careful treatment of classical thermodynamics, with applications, as in Thermal Physics by Philip M. Morse (second edition; Menlo Park, California, W. A. Benjamin, Inc., 1969). The suggested outline follows the latter plan, as probably more appropriate as a background for varied applications.

Thermodynamics.

a. State variables and equations of state. (3 lessons) Temperature, pressure, heat, and energy. Extensive and intensive variables. Pairs of mechanical variables. The perfect gas and other equations of state.

b. The first law of thermodynamics. (4 lessons) Work, internal energy, heat. Heat capacities. Isothermal and adiabatic processes.

c. The second law of thermodynamics. (6 lessons) Heat cycles. Reversible and irreversible processes. Entropy. Applications to simple thermodynamic systems.

d. The thermodynamic potentials. (3 lessons) Internal energy, enthalpy, Gibbs and Helmholtz potentials. Examples and procedures for calculation.

e. Phrase equilibria. (3 lessons) Melting, evaporation, triple point, and critical point.

f. Chemical applications. (2 lessons) Reaction heats, electrochemical processes.

Statistical Mechanics (Equilibrium).

a. Statistical methods. (3 lessons) Random walk; probability distributions; mean values; binomial, Poisson, and Gaussian distributions.

b. Statistical description of systems of particles. (3 lessons) Ensembles, ergodic hypothesis, postulates, limiting behavior for large N, fluctuations.

c. Quantum statistics. (5 lessons) Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac distributions with applications (solids, gases, electron gas, blackbody radiation, etc.).

Microscopic Description of Nonequilibrium.

a. Elementary kinetic theory. (2 lessons)

b. Transport theory. (2 lessons) Based on Boltzmann's equation, in simplified form.

c. Brownian motion. (3 lessons) Possibly including the Fokker-Planck equation. More on random variables. Markov processes, fluctuations, irreversible processes.

10. Partial Differential Equations. (6 semester hours)

This course ordinarily occurs in the fifth year of the sequence, although with certain options (e.g., Orbit Mechanics) it should be taken in the fourth year. The material should strike a reasonable balance between the classical analytical theory of partial differential equations and modern computational aspects of the subject. For that reason, existence theorems and the like should be of the constructive type whenever possible. Further, application to problems in classical and modern physics should constantly be borne in mind. Physical models should be used both to predict results concerning the behavior of solutions to partial differential equations and to interpret phenomena revealed analytically or computationally.

a. Introduction. (6 lessons) Derivation of some equations; discussion of mathematical models, continuous dependence theorems, and relation to physics.

b. Classification and characteristics. (9 lessons) Cauchy problem for first-order equations, formulation and statement of Cauchy-Kowalewski theorem.

c. Hyperbolic equations. (12 lessons) Existence and continuous dependence for second-order equations. Riemann method. Three-dimensional wave equation. Retarded potentials. Numerical methods--finite difference schemes and stability.

d. Elliptic equations. (21 lessons) Potential theory in three dimensions with smooth boundaries. Eigenvalue problems--estimates. Numerical methods.

e. Parabolic equations. (12 lessons) Thermal potential theory. Convergence to steady state and relation to potential problems. Numerical methods and connection, in steady state, to numerical methods for elliptic problems.

f. Integral representation of solutions. (12 lessons) Green's functions. Integral equations.

g. Equations of hydrodynamics. (6 lessons) Shock phenomena, weak solutions. Numerical methods.

11. Optimization. (3 semester hours)

For an outline of this course, see Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 647. A natural successor to this course is the 6-semester-hour course Operations Research in the Operations Research Option, where linear programming techniques are developed in depth and additional topics in dynamic programming, inventory and scheduling problems, Monte Carlo simulation techniques, and queueing theory are introduced.

OPERATIONS RESEARCH IN SYSTEMS ENGINEERING OPTION

Year	Core Courses			Option Courses		
III	1.	Functions of Several Variables	(3)			
	2.	Intermediate Ordinary Differential Equations	(3)			
	3.	Mechanics	(6)			
	4.	Numerical Analysis	(3)			
	5.	Probability and Statistics	(6)			
IV	6.	Complex Variables	(3)	OR1.	Reliability	(6)
	7.	Functional Analysis	(3)			
	8.	Electromagnetics	(3)			
	9.	Thermodynamics and Statistical Mechanics	(3)			
	11.	Optimization	(3)			
V.	10.	Partial Differential Equations	(6)	OR2.	Operations Research	(6)
				OR3.	Systems Simulation	(3)

OR1. Quantitative Methods in Reliability Engineering. (6 semester hours)

The growing complexity of systems over the last two decades has inspired the development of a body of quantitative methods for developing, improving, and measuring system reliability. Since reliability is such a critical factor in the successful completion of every highly technical program, it is of value to the engineer to learn quantitative reliability theory, as presented in the following course. The mathematical and statistical methods of the course are not simply routine applications of well-known theory, but in many cases represent new developments motivated by reliability problems.

The 6-semester-hour course in Probability and Statistics of the core is an essential prerequisite for this course.

First Semester: Probabilistic Models in Reliability.

a. Failure distributions in reliability theory. (8 lessons) Typical failure laws. The exponential as the failure law of complex equipment. Monotone failure rates. Bounds for distributions with monotone failure rate. General failure rates.

b. Prediction of system reliability from a knowledge of component reliabilities. (3 lessons) Analytical methods for computing reliability exactly. Bounds on system reliability based on paths or cuts. Monte Carlo methods. Qualitative relationships for multicomponent structures.

c. Redundancy optimization. (5 lessons) Optimal allocation of redundancy subject to constraints. Optimal redundancy assuming two kinds of failure.

d. Operating characteristics of maintenance policies. (6 lessons) Renewal theory. Replacement based on age. Comparison of age and block replacement policies. Random replacement. Repair of a single unit.

e. Optimum maintenance policies. (4 lessons) Replacement policies. Inspection policies.

f. Stochastic models for complex systems. (8 lessons) Semi-Markov processes. Repairman problems. Marginal checking. Optimal maintenance policies under Markovian deterioration.

Second Semester: Statistical Reliability Theory.

a. Estimating reliability parameters assuming form of distribution known. (8 lessons) Maximum likelihood estimation in the case of normal, exponential, gamma, Weibull, and binomial distributions. Confidence and tolerance limits in these cases. Minimum variance unbiased estimation in these cases.

b. Estimating reliability parameters under physically plausible assumptions. (9 lessons) Errors resulting from incorrect assumption as to form of failure distribution. Maximum likelihood estimation assuming a monotone failure rate. Maximum likelihood estimation assuming a decreasing and then increasing failure rate. Conservative confidence and tolerance limits.

c. Estimating reliability growth. (7 lessons) Form of growth assumed known. Only monotonicity of reliability assumed. Conservative confidence limits.

d. Confidence limits on system reliability using observations on individual components. (6 lessons) Success or failure observations. Life length observations. Asymptotic methods.

e. Hypothesis testing. (9 lessons) Acceptance sampling, fixed sample size, truncated and censored sampling, sequential sampling. Accelerated life testing. Testing for monotone failure rate.

OR2. Operations Research. (6 semester hours)

The accent is on the mathematical aspects of the subject, rather than the management or industrial engineering aspects. It is assumed that time and facilities are available for a computation laboratory in connection with both semester courses. A prerequisite of an introductory computer programming course is desirable.

The first semester develops linear programming in depth, building on the preparation given in a previous one-semester course in Optimization.

First Semester: Advanced Linear Programming.

a. Review of the simplex algorithm. (4 lessons) Variations of the simplex algorithm. Degeneracy, perturbation. Revised simplex method.

b. Games and linear programs. (4 lessons) Matrix games. Equivalence of matrix games and linear programs.

c. The transportation problems. (4 lessons) Elementary transportation theory. The transshipment problem.

d. Networks and the transshipment problem. (4 lessons) Graphs and trees. Interpreting the simplex method on the network. The shortest route problem.

e. Variables with upper bounds. (3 lessons) The general case. The rounded variable transportation problem.

f. Programs with variable coefficients. (3 lessons) Wolfe's generalized program. Special cases.

g. Decomposition principle for linear programs. (9 lessons) The general principle. Decomposing multistage programs.

h. Convex programming. (4 lessons) General theory. Separable convex objectives. Quadratic programming.

i. Discrete variable problems. (4 lessons) Survey of methods. Gomory's method of integer forms.

Second Semester: Dynamic Programming and Stochastic Models.

a. Dynamic programming. (10 lessons) Principle of optimality. Multistage allocation problems. Arrow-Harris-Marschak inventory model.

b. Dynamic programming and Markov processes. (5 lessons) Discrete dynamic programming. Optimal policies with discounted returns.

c. Monte Carlo techniques. (10 lessons) Production of random variables by computer. Simulating stochastic systems on the computer.

d. Mathematical theory of queues. (14 lessons) Single server; Poisson input; exponential service. Many servers; Poisson input; exponential service. The busy period. Stochastic inventory models.

OR3. Systems Simulation. (3 semester hours)

This course examines those symbol manipulation applications of the computer that involve the numerical and logical representation of some existing or proposed system, for the purpose of experimenting with the model and of comparing methods of operating the system. The primary purpose of the computer is thus not a calculating adjunct to experimentation but is the experimental medium itself. A course in probability and statistics is a prerequisite.

a. Programming languages. (11 lessons) Special languages designed for use in simulation, such as SIMSCRIPT and GPSS. Additional study of the languages will arise in their use throughout the rest of the course.

b. Technical problems of simulation. (14 lessons) Synchronization of events, file maintenance, random number generation, random deviate sampling.

c. Statistical problems peculiar to simulation. (7 lessons) Sample size estimation, variance reducing techniques, problems of drawing inference from a continuous stochastic process.

d. Applications. (7 lessons) Queueing models; storage, traffic, and feedback systems; design of facilities and operating disciplines.

ORBIT MECHANICS OPTION

Year	Core Courses			Option Courses		
III	1.	Functions of Several Variables	(3)			
	2.	Intermediate Ordinary Differential Equations	(3)			
	3.	Mechanics	(6)			
	4.	Numerical Analysis	(3)			
	5.	Probability	(3)			
IV	6.	Complex Variables	(3)	OM1.	Advanced Numerical Analysis	(3)
	7.	Functional Analysis	(3)			
	8.	Electromagnetics	(3)			
	9.	Thermodynamics and Statistical Mechanics	(3)			
	10.	Partial Differential Equations	(6)			
V	11.	Optimization	(3)	OM2.	Advanced Programming	(3)
				OM3.	Celestial Mechanics	(3)
				OM4.	Orbit Theory	(3)
				CT2.	Control	(3)
				CT5.	Data Smoothing and Prediction	(3)

OM1. Advanced Numerical Analysis. (3 semester hours)

This course, with its emphasis on topics in partial differential equations and elementary functional analysis, demands a reasonable amount of mathematical maturity. It should be taken after the first semester of Partial Differential Equations.

a. Matrix inversion and matrix eigenvalues. (10 lessons) Review and extension of iterative methods. Jacobi, Householder, and other methods of finding eigenvalues. Ill-conditioning and error analysis.

b. Ordinary differential equations, boundary value problems, eigenvalue problems. (11 lessons) Finite difference methods, extremal principles.

c. Partial differential equations of second order. (18 lessons) Topics selected from the following: Classification, analytical solutions of well-posed problems for single equations; maximum principles for elliptic and parabolic equations, L_2--or energy--estimates as well as pointwise estimates of solutions; hyperbolic equations, domain of dependence; Fourier analysis and stability for constant coefficient equations, eigenvalues for elliptic equations, iterative methods for difference equations arising from partial differential equations.

OM2. Advanced Programming. (3 semester hours)

This course deals with various types of computer programming and serves to introduce students to the concepts involved in current work in this area. An introductory course in computer science is a prerequisite, and it is assumed that the students have considerable facility in programming with FORTRAN or ALGOL.

a. Survey. (9 lessons) Assembly systems, methods of storage allocation when using these, pseudo-orders, macros, modify and load techniques, monitor and executive systems.

b. Structure of languages. (15 lessons) Study of a particular language such as ALGOL, its ambiguities, its method of dealing with recursions and procedures. List-processing languages, compiler-writing languages.

c. Theory of compilers. (15 lessons) Nature of syntax-directed compilers, compilers for dealing with problem-oriented languages, compilers for dealing with compiler syntax languages. Discussion of the evolution of a translator from a simple language whose translator is given in machine language.

OM3. Celestial Mechanics. (3 semester hours)

This course in celestial mechanics concerns itself with the mathematical structures underlying the physical theory, and with deriving from them methods which are commonly used to attack the fundamental problems of interest to space research and technology.

a. Introduction. (6 lessons) Description of dynamical systems by means of Lagrangian functions. Lagrangian equations of motion. Ignorable coordinates, energy equation, and other elementary instances of first integrals. Liouville systems in general. Classical examples: harmonic oscillator, simple pendulum, spherical pendulum, central forces, a charged particle in an electromagnetic field, solid body. The principle of dynamical analogy.

b. Phase space. (6 lessons) Legendre duality (with a suggestion about how it is applied to derive state functions in classical thermodynamics). Transition from Lagrangian functions and Lagrangian equations to Hamiltonian functions and canonical equations. Canonical mapping: its definition, its multiplier, and its residual functions. Completely canonical mappings; canonical extensions of coordinate transformations. Generation of canonical mappings by numerical functions. Invariance with respect to the group of canonical mappings: canonical equations, Poisson brackets, Lagrange parentheses.

c. Canonical constants of a dynamical system. (6 lessons) Definition of a set of canonical constants of integration. Variation of canonical constants. The Hamilton-Jacobi equation as an algorithm for constructing sets of canonical constants. The action and angle variables. Separable Hamiltonians; Staeckel systems and Liouville systems. Applications: the problem of two bodies, the problem of two fixed centers. Normal modes of vibrations and vibrations of molecules.

d. Integrals of a dynamical system. (9 lessons) Poisson's theorem about the bracket of two integrals and its dual application to Lagrange parentheses. Integrals in involution; Liouville's theorem. Jacobi-last multiplier. Application to the motion of a solid body. Whittaker's adelphic integral. Application to the investigation of a dynamical system around the equilibrium. Isoenergetic reduction. Application to the regularization and the binary collisions in the problem of two bodies and in the restricted problem of three bodies.

e. Perturbation theory. (12 lessons) Poincaré's method of the small parameter. Birkhoff's method of iterative canonical mappings. Application to the motion of a satellite of an oblate planet.

OM4. Orbit Theory. (3 semester hours)

The purpose of this course is to offer illustrations of mathematical principles and to compel the student to master them securely by careful numerical examples. The material should be arranged to provide a nearly continuous flow of computational work for the laboratory sessions no matter at what level the course is set.

The instructor and his students should be directed to develop the topics all the way down to an efficient and reliable program in the FORTRAN or ALGOL language on an electronic computer. As most of the textbooks on the subject matter cater to computers who use logarithms or hand-operated desk-model calculating machines, special care should be taken to rearrange classical algorithms and formulas for use on an electronic computer.

Topical problems should be selected from at least these four main research areas:

a. Orbit determination. (10 lessons) The obvious reference here is P. Herget's The Computation of Orbits, published privately by the author at Cincinnati Observatory, 1948. This booklet has been updated by the author in his lectures on "Practical Astronomy" and on "Orbit Determination" given at the summer course in Space Mathematics, Cornell University, 1963.

b. Orbit analysis. (12 lessons) The question here is to gain physical information from the comparison between the orbit as it has been observed and the orbit as it has been computed from a particular mathematical model.

c. Orbit design. (7 lessons) How to produce orbits that satisfy *a priori* conditions (e.g., given initial conditions, mission requirements, optimum characteristics, etc.).

The course should limit itself to well-tried problems and should aim at producing examples where good-quality results can be reached without too much effort. This can be achieved in the restricted problem of three bodies.

After an introduction to that problem, the instructor should review two or three methods for integrating numerically the equations of motion, either in cartesian coordinates or in regularized coordinates. Then should come a development on variational equations. Thereafter the theory of characteristic exponents should be applied to the analysis of a family of periodic orbits.

d. Analytical theories. (10 lessons) How to expand on literal theory by enabling an electronic computer to handle symbol manipulations in a given algebra.

This new field promises to provide mathematicians with powerful tools to develop literal theories in an extensive set of physical problems.

CONTROL THEORY OPTION

Year	Core Courses			Option Courses		
III	1.	Functions of Several Variables	(3)	CT1.	Electric Circuits	(3)
	2.	Intermediate Ordinary Differential Equations	(3)			
	3.	Mechanics	(6)			
	4.	Numerical Analysis	(3)			
	5.	Probability and Statistics	(6)			
IV	6.	Complex Variables	(3)	CT2.	Control	(3)
	7.	Functional Analysis	(3)	CT3.	Laboratory	(3)
	8.	Electromagnetics	(3)	CT4.	Linear Systems	(3)
	9.	Thermodynamics and Statistical Mechanics	(3)	CT5.	Data Smoothing and Prediction	(3)
V	10.	Partial Differential Equations	(6)	CT6.	Advanced Control	(3)
	11.	Optimization	(3)	CT7.	Information Theory	(3)
				CT8.	Advanced Communications	(3)

CT1. Electric Circuits. (3 semester hours)

This is a basic course in circuit theory. The purpose is to teach in a precise language the fundamental facts of circuit theory while developing skills in writing and solving the circuit equations and keeping close contact with physical circuits (filters, amplifiers, digital circuits).

a. Lumped circuits. (3 lessons) Lumped circuit approximation. Kirchhoff laws, relation to Maxwell's equations. Circuit elements; including nonlinear and time-varying elements.

b. Simple circuits. (13 lessons) First- and second-order circuits: zero-input response and zero-state response; response to step, impulse, and sinusoid. Linearity and time-invariance: convolution. Impedance, phasors, frequency response, and resonance.

c. Coupling elements. (2 lessons) Coupled inductors, transformers, and dependent sources. Dependent sources as parts of models for electronic devices.

d. Power and energy. (2 lessons) Energy stored and power dissipated in elements; relation with real and imaginary part of impedance.

e. General methods of analysis. (6 lessons) Graph theory: trees, links, cut-sets, loops. Loop and cut-set analysis, mixed method. Duality. Computer programs for analysis of circuits.

f. Linear time-invariant circuits. (6 lessons) Reduction of systems of equations. Network functions: poles, zeros, gain and phase.

g. Network theorems. (3 lessons) Superposition, Thévenin, Norton, reciprocity. Careful discussion of range of applicability: comments on nonlinear and time-varying circuits.

h. Two-port description. (4 lessons) Two-port description of electronic devices: relation to their graphical characteristics. Linear time-invariant networks as two-ports. Interconnection of two-ports.

CT2. Control. (3 semester hours)

In this course the student learns some basic facts about control systems, their analytical description, and techniques of design. This course is mostly concerned with single-variable control.

a. Description of feedback systems and components. (5 lessons) Advantages and disadvantages of feedback, importance of measuring device, noise problems. Basic components, electrical, hydraulic, pneumatic. Requirements and specifications of control systems. Examples.

b. Linear time-invariant control systems. (20 lessons) Analysis, illustrated by several extensive examples, based on differential equations and integral equations (convolution). Stability, root locus, Nyquist criterion. Design of compensating networks to obtain stability and meet the specifications.

c. Sampled systems. (6 lessons) Examples of systems where the feedback data are naturally sampled periodically. Analysis of sampled systems. Stability, root locus, Jury's criterion. Design of compensating networks.

d. Nonlinear systems. (8 lessons) Local stability near equilibrium. Example of limit cycles. Stability: approximate methods describing functions. Lyapunov's second method. Application to design.

CT3. Laboratory. (3 semester hours)

The purpose of the laboratory is to insure that students connect the classroom concepts and results to physical reality and appreciate the power and limitations of experimental work. Typically, one part of the laboratory could be devoted to circuits work: behavior of linear circuits (including resonance), effects of nonlinear elements on waveform and power spectrum, some pulse circuits. The second part of the laboratory would cover control: study of a typical control system, experiments with various compensations; stability; experiments with and simulation of a nonlinear control system.

CT4. Linear Systems. (3 semester hours)

Purpose: to provide a solid foundation of concepts, facts, and techniques to be used in later courses in control, communication, and circuits.

a. Systems. (6 lessons) State as a parametrization of input-output pairs and as part of the system description. Operator point of view. State equivalence. Linear systems. Linear systems obtained by linearization of ordinary differential equation about a nominal trajectory. Examples throughout.

b. Linear systems of the form

$$\dot{x} = Ax + By$$
$$\dot{y} = Cx + Dy$$

and their discrete time analogs. (6 lessons) Time-invariant case: explicit solution by function of a matrix and Laplace and z-transforms. For simple linear operators, diagonalization, mode interpretation (including numerical techniques). Jordan form, analog computer interpretation. Time-varying case: properties of the state transition matrix. Periodic systems: Floquet theory, kinematic equivalence.

c. Impulse response and transfer functions. (12 lessons) Free use of Fourier and Laplace transforms. Superposition integral. Asymptotics of impulse response and transfer function. Minimum phase. Uncertainty principle. Group delay. Signal flow graphs.

d. Stability. (9 lessons) Characterization of stability for linear time-invariant, periodic and time-varying systems (zero-input stability: Liénard Chipart, Nyquist; bounded input implies bounded output; implications of an impulse response which is in L^1), Lyapunov method.

e. Input-output description and state equations. (6 lessons) Controllability, observability, and normality. Characterization in time-invariant and time-varying cases. Output controllability. Controllability and observability of an interconnection of systems.

CT5. Data Smoothing and Prediction. (3 semester hours)

a. Representation of functions by Fourier series and integrals. The Fourier transform in L^1 and L^2. (8 lessons)

b. Random processes: definition, examples, representations; autocorrelation, power spectrum; estimation of spectral densities. (14 lessons)

c. Linear mean-square estimation, filtering and prediction. The Wiener-Hopf equation; solution by the Wiener filter and Kalman-Bucy filter. (11 lessons)

d. Detection and parameter estimation. Application to digital communications system and radar. (6 lessons)

CT6. Advanced Control. (3 semester hours)

This course treats advanced topics in control so that students can readily read the current literature. Multiple-input multiple-output systems are included.

a. Nonlinear control. (10 lessons) Describing function. Subharmonics. Stability: Sandberg circle criterion, Popov-type criteria. Lyapunov method used as a design tool. Lyapunov method for systems with inputs. Bounds on output.

b. Adaptive control. (8 lessons) Examples of adaptive control: identification techniques and parameter adjustment. Stochastic approximation.

c. Optimum control. (21 lessons) Formulation of the problem. Examples: systems described by ordinary differential equations and difference equations. Maximum principle for differential systems. Numerical methods. Relation of maximum principle with steepest descent.

CT7. Information Theory. (3 semester hours)

a. The concept of the source and of an information measure. Desirable properties of information measure, examples of simple sources. (3 lessons)

b. Codes, their efficiency and redundancy. The efficient encoding of discrete independent sources. (4 lessons)

c. General discrete sources, Shannon's encoding theorem, the nature of written and spoken English. (4 lessons)

d. The concept of a channel, channel capacity, symmetry of a channel. (3 lessons)

e. The fundamental theorem of information theory, error-detecting and error-correcting codes, the geometric interpretation of coding problems. (17 lessons)

f. Generalization to continuous channels, channel capacity of continuous channels. (8 lessons)

CT8. Advanced Communications. (3 semester hours)

a. Review. (4 lessons) Signal and noise representations, the purpose of modulation.

b. Amplitude modulation. (7 lessons) The generation and detection of AM waves, power spectrum, single side-band and vestigial side-band transmission, effects of distortion and noise.

c. Frequency and angle modulation. (12 lessons) Generation, detection, power spectrum, effects of distortion and noise.

d. Pulse modulation. (10 lessons) Pulse amplitude modulation, pulse position modulation, pulse duration modulation. A brief introduction to pulse code modulation.

e. Design. (6 lessons) The design of optimum receivers in the presence of additive noise and fading.

RECOMMENDATIONS FOR THE UNDERGRADUATE MATHEMATICS PROGRAM FOR STUDENTS IN THE LIFE SCIENCES

An Interim Report of

The Panel on Mathematics for the Life Sciences

September 1970

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1. Introduction

The Panel on Mathematics for the Biological, Management, and Social Sciences was primarily concerned with the mathematics curriculum for prospective graduate students in those fields. Their recommendations, which were published in the 1964 CUPM report Tentative Recommendations for the Undergraduate Mathematics Program of Students in the Biological, Management, and Social Sciences (BMSS), were meant to serve as a basis for discussion and experimentation. From these discussions it became apparent that (1) some of the recommendations would be very difficult for mathematics departments to implement, (2) a different program was needed for the terminal bachelor's degree, (3) the single program presented would not seem to be ideally suited to the diverse fields included in the BMSS disciplines, especially at the advanced level. In response to these findings, CUPM decided to concentrate on individual disciplines and, as a first step, appointed the present Panel on Mathematics for the Life Sciences, charged with making recommendations for the mathematical training of all undergraduate life science students, not only pregraduate students. (Here, life sciences are taken to mean agriculture and renewable resources, all branches of biology, and medicine.)

The Panel on Mathematics for the Life Sciences has undertaken, through conferences and extensive consultation with leaders in the biological field, to learn from them what mathematics they consider to be necessary for their students. In particular, the Panel held several meetings with representatives of the Commission on Undergraduate Education in the Biological Sciences (CUEBS). After the biologists had specified the mathematics needed by students of biology, the Panel proceeded to describe mathematics courses that contain this mathematics. This report is the outcome of these consultations and studies. Finally, the Panel held a special conference in which a preliminary draft of this report was submitted to a group of biologists for comment and criticism. The discussions and conferences with the biologists have emphasized a serious fourth problem: (4) heavy requirements in chemistry, physics, and biology make it difficult for a major in the life sciences to add mathematics courses to his program.

In preparing this report, the Panel considered all four of the problems mentioned above, and it presents herewith its recommendations for a basic mathematics core for life science undergraduate majors (see Part I) and for certain more specialized studies (see Part II).

Part I describes a basic core of mathematics for all undergraduate majors in the life sciences. In Section 2 we describe the level of mathematical preparation on which this core is based. In Section 3 the recommendations for the mathematical core are stated and justified, and in Section 4 we treat some of the principles and details of implementation of this core.

The Panel feels strongly that every life science major should gain substantial experience with computers (digital, analogue, and hybrid). We feel that the time is ripe now for a detailed treatment of the role of the computer in the undergraduate program, especially as it relates to the life science student. This accounts for the detail found in Section 5.

Part II outlines certain more specialized studies which this Panel believes will be important for some students in mathematics as well as for students in the life sciences. Section 6 describes a program for undergraduate preparation for the study of biomathematics at the graduate level. A description of an upper-division course focusing on the building of mathematical models in the life sciences and some suggestions for its implementation appear in Section 7.

Certain of the courses in A General Curriculum in Mathematics for Colleges are cited frequently; their descriptions appear elsewhere in this COMPENDIUM.

I. MATHEMATICS FOR UNDERGRADUATE BIOLOGY MAJORS

2. Background of the Students

In recent years much effort has been expended to improve mathematics education in the elementary and secondary schools. Several programs of improvement in secondary schools have already had considerable effect and we hope that they will have a great deal more. In particular, we hope that mathematics courses in the secondary school will contain a judicious mixture of motivation, theory, and applications. For the purposes of our discussion it is assumed that the student is acquainted with both the algebraic and geometric aspects of elementary functions (see the description of Mathematics 0 in Commentary on A General Curriculum in Mathematics for Colleges, page 75); moreover, we assume that the student has been exposed to the idea of a set, mathematical induction, binomial coefficients, and the summation notation. Thus, our discussion applies to students in the life sciences who are prepared to begin their collegiate mathematics with a calculus course, although departments of mathematics may have to offer precalculus courses in order to prepare some students adequately for this program.

Historically, mathematics has been closely allied to the physical sciences, especially to physics. In secondary schools and in elementary undergraduate courses, applications of mathematics have traditionally been limited to the physical sciences. Therefore, it is not uncommon for students whose interests lie in other fields to enroll in a bare minimum of mathematics courses. If students are to possess the prerequisites stated above, proper counseling both in high school and in college is imperative. Students must be made aware of the doors that are closed to them in fields of the life sciences, as well as in the physical sciences and engineering, when they terminate their study of mathematics prematurely. We hope that this message will be transmitted to guidance and counseling personnel, and we urge all concerned to give attention to ways by which counseling of potential life science students can be improved in their locality.

3. The Basic Core: Recommendation and Justification

The Panel on Mathematics for the Life Sciences has considered the problem of recommending a basic core of mathematics courses for students in the life sciences. The prospective life science major, whatever his specialty or career goal, now needs more mathematics than was recognized to be the case a few years ago. As a result of its study, the Panel concludes that the mathematical core for the undergraduate life science major should include one year of calculus, some linear algebra, and some probability and statistics.

More specifically, the Panel believes that this core can be provided by the following courses: Mathematics 1 (Calculus I), Mathematics 2 (Calculus II), Mathematics 3 (Elementary Linear Algebra), and Mathematics 2P (Probability). Outlines for all of these courses can be found in Commentary on A General Curriculum in Mathematics for Colleges. In addition, we recommend that each student gain some experience in the use of an automatic computer in the first two years of study. This might come in the form of a sequence of laboratory exercises (see Section 5) in which algebraic language problems are developed and run. Institutions which do not have computation centers may be able to provide service via remote terminals or through the courtesy of nearby organizations. This permits the use of computing algorithms, lecture demonstrations, or problem assignments in biology and mathematics courses at appropriate times.

The recommendations are consistent with the findings of the two life science commissions sponsored by the National Science Foundation. In Publication No. 18 of the Commission on Undergraduate Education in the Biological Sciences, "Content of Core Curricula in Biology" (June, 1967), pp. 30-31, we find: "Fourth, we recommend that careful attention be given to relating biology courses to the background of the student in mathematics, physics, and chemistry...in mathematics, at least through the level now generally taught as calculus, ...some background in physical and organic chemistry." This recommendation clearly indicates that a full-year sequence of calculus (including multivariable calculus) should be taken by a biology major. In this same publication the curricula for biology majors at Purdue University, Stanford University, North Carolina State University, and Dartmouth College are presented. At three of these institutions, one year of calculus is required in addition to some probability and linear algebra. In the remaining institution, additional calculus is required instead of "finite mathematics" (here taken to mean basic linear algebra with applications--such as Markov chains--and combinatorial probability).

In 1967 the Commission on Education in Agriculture and Natural Resources (CEANAR) charged a committee "to recommend mathematics requirements to be met ten to fifteen years hence in undergraduate curricula for Agriculture and Natural Resources." In its report* this committee chose to state its recommended requirements almost entirely in terms of courses described in the CUPM report A General Curriculum in Mathematics for Colleges. Mathematics 1, 2P, and some computer instruction are recommended for majors in all areas covered by CEANAR. Moreover, for students majoring in technology programs, Mathematics 2 and Mathematics 7 (Probability and Statistics) are recommended; to this students majoring in science programs should add Mathematics 3 and Mathematics 4 (a third course in calculus).

There are other good reasons for recommending this core of four mathematics courses: 1, 2, 3, and 2P. First of all, these are standard mathematics courses whose broad availability should facilitate implementation. Secondly, this curriculum is flexible enough to accommodate a student who may decide to change his major. For example, if in the first year or two he enters a discipline that involves more mathematics, he will not have lost any time. Thirdly, compressing this material into a shorter three-course sequence is unwise from a pedagogical point of view. Very few students are capable of gaining even a minimal mastery of calculus in a one-semester course, and at least one semester is needed to cover a significant amount of linear algebra or of probability. Moreover, if a student eventually decides to take more advanced mathematics and still continue in the branch of life sciences he originally chose, he will have the appropriate prerequisites. In this connection we discuss in Section 7 the role of a course in the applications of mathematics to the life sciences for the research-bound student. Since such a course involves relatively advanced mathematics, it will carry certain further mathematical prerequisites beyond the core itself.

Preparation for research in certain areas of biology will demand competence in mathematics equivalent to the Master's degree level. Some biologists have even asserted that a student who has a Bachelor's degree with a major in mathematics and appropriate courses in chemistry and physics would be welcomed as a graduate student in biology, even though he had had no courses in biology. Further development of this line of thought is found below in Section 6 on biomathematics.

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* See "Undergraduate Education in the Physical Sciences and Mathematics for Students in Agriculture and Natural Sciences," pp. 32-35. This report is available from the Division of Biology and Agriculture, the National Research Council, 2101 Constitution Avenue, Washington, D. C. 20418.

Although the Panel feels that an offering to life science students of fewer than four semesters of mathematics course work will not meet the objectives laid out by the life scientists whom we have consulted, we must recognize that this amount of mathematics is more than will be accepted by some of them as a requirement for all undergraduate life science majors. We have been urged to consider what can be done with three courses. Any three-course program will lose some of the desirable features described in the last paragraphs. We present several options of three courses and point out some of the advantages and shortcomings of each:

(1) Mathematics 1, 2, 3;
(2) Mathematics 1, 3, and 2P;
(3) Mathematics 1, some appropriate interweaving of 2 and 3, and 2P;
(4) Mathematics 1, 2, and one semester of finite mathematics;
(5) An integrated three-semester sequence, specifically designed for life science students and built around finite mathematics, calculus through multivariable calculus, probability, and statistical inference.

It may be well to observe that each of the above options could lead to completion of the core in graduate school, if this were desired. This could be done with standard courses in the case of options 1 or 2 or with one or more special courses in the case of other options.

Some features of the options are highlighted in the following

	Number of special mathematics courses required	Number of core subjects omitted entirely	Full year of calculus included
Core	0	0	Yes
Option 1	0	1	Yes
Option 2	0	1	No
Option 3	1	0	No
Option 4	1	0	Yes
Option 5	3	0	No

The first column measures to some extent the extra load placed on the mathematics department by each option. The offering of each special course involves planning and coordination activities and in a small institution may have a high cost per student because of limited enrollment.

Any option involving special courses inevitably raises the possibility of requiring additional course work (or equivalent thereof) to provide the proper prerequisites for further study in mathematics. The severity of this effect can be assessed only in the context of a given institution, a given spectrum of courses, and some designated group of students at a given skill level.

Column 3 relates to the remark that offering less than a full year of experience in calculus seems to be insufficient.

The Panel feels that option 1 is the least undesirable, since 1) probability can be added by many students as an elective, 2) this sequence is easier for most mathematics departments to staff than one involving probability, and 3) many schools now offer linear algebra as an integral part of the calculus sequence. Option 2 is considerably less desirable than option 1, since it omits multivariable calculus, a topic that the Panel feels is vital to modern biology (see the previous reference to CUEBS Publication No. 18).

Options 3, 4, and 5 share the disadvantage of having no efficient continuing mathematics course which can be used to complete the core material.

Options 3 and 4 may be desirable for larger institutions in which the mathematics and biology departments can work out arrangements for an additional elective special course which would complete the core. If many departments were to adopt one of these options, graduate departments might choose to require that the core be completed in graduate school. Care must be exercised in implementing option 3 to include topics in calculus that are needed in physics and chemistry prerequisites to biology courses.

The essential feature of option 5 is to construct a three-semester sequence, illustrated by life science examples and containing essential material from the calculus, while interweaving some probability, statistics, and linear algebra in an integrated format. However, in situations where there are substantial numbers of students who have not had appropriate mathematics courses by the time they begin graduate work in biology, the departments of biology and mathematics may wish to collaborate in designing special programs. It is more important for a graduate biology student to understand the basic concepts of the mathematics he uses than to develop the computational skills needed by a physical scientist or engineer. By taking advantage of the maturity, strong motivation, and established field of interest of these students, satisfactory programs (such as option 5 above) stressing the understanding of these mathematical concepts could be designed in such a way as to require less time than the standard undergraduate courses.

Such cooperation involving another department with the mathematics department has proved successful in the past. This was particularly true in some areas of the social sciences, where the demand for such programs eventually diminished when the great majority of entering graduate students came with a sufficient mathematical background. The research needs of some biology departments have motivated them to hire biomathematicians, who could participate in teaching the graduate programs envisioned here.

4. The Basic Core: Implementation

The Panel recommends that all life science majors be required to complete two semesters of calculus and one semester each of linear algebra and probability (including some statistics). These courses, Mathematics 1 and 2, Mathematics 3, and Mathematics 2P, are discussed in detail in Commentary on A General Curriculum in Mathematics for Colleges, page 33. In many undergraduate curricula these courses must serve many needs: prospective biology majors find themselves in the same classes with students from a wide variety of disciplines (such as engineering, economics, business administration, one of the physical sciences, or even mathematics). When this is the case, it is unlikely that special emphasis on biological applications will be featured in any part of this four-course program. A few institutions, however, can afford to present all, or some part, of this core program exclusively for students whose main interests lie in the life sciences. We do not address ourselves to the task of making detailed recommendations to this group since we feel that an institution offering a special mathematics core for life scientists will wish to take advantage of local features and design a hand-tailored program. Between these extremes, we find institutions able to give varying amounts of special attention to life science orientation in the mathematics core. Our suggestions below are directed primarily to this group. We expect that there will be considerable latitude in the extent and manner that these recommendations are utilized.

The life science major should be given more consideration than has been the custom in the past, even by the first group of institutions that cannot afford to provide special courses or sections of courses. Traditionally, the applications given in calculus, for example, are almost exclusively chosen from the physical sciences. With the rapid growth of the life sciences, it is only reasonable that increased emphasis be given to illustrative examples from this field, even in a calculus course in which the interests of most of the students lie elsewhere.

We now proceed to our comments on the modifications necessary to make the core courses more suitable to the needs of the life science students.

[Editor's note: In the case of Mathematics 1 and 2, the Panel's suggestions relative to the original course outlines in the 1965 General Curriculum in Mathematics for Colleges have been incorporated into the revised outlines in Commentary on A General Curriculum in Mathematics for Colleges (1972). Thus, we refer the reader to the new outlines.]

Mathematics 1. Calculus I.

See Commentary on A General Curriculum in Mathematics for Colleges, page 44.

Mathematics 2. Calculus II.

See Commentary on A General Curriculum in Mathematics for Colleges, page 51.

Mathematics 2P. Probability.

Relative to this course, an outline of which is given in Commentary on A General Curriculum in Mathematics for Colleges, page 76 , we make the following comments:

a. With respect to (1) of the GCMC description, the introduction of the probability axioms should be properly motivated by the frequency interpretation (see Hodges, J. L. and Lehmann, E. L. Basic Concepts of Probability and Statistics. San Francisco, California, Holden-Day, Inc., 1964) in order to connect these concepts with the empirical traditions of the life sciences.

b. With respect to sections (2) and (3), some time could be saved by merging the sections so that the Poisson and normal distributions would be introduced as limits of the binomial distribution. For the normal approximation, we feel that the most efficient presentation--in consideration of both time spent and student understanding--would be to discover numerically that a sequence of binomial cumulative distribution functions, after the usual normalization, tends to the normal distribution (see Mosteller, F. R., et al. Probability with Statistical Applications, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970).

c. We feel that two-state Markov chains, as a generalization of sequences of Bernoulli trials, should be included in the course. The presentation of these could begin in the discussion of conditional probability in section (1). A solution for the limiting distribution of the process and a numerical demonstration of convergence to this limit should then follow the other limit theorem discussions, taking up one or two lessons.

d. If Mathematics 3 is included as a prerequisite for Mathematics 2P, topics such as the Markov process in (c) can be presented more efficiently and in greater depth in matrix form.

A modified course description of 2P appropriate for students in the life sciences can then be given as follows:

1. Probability as a mathematical system. (11 lessons) Variability of experimental results, sample spaces, events as subsets, probability axioms and immediate consequences, finite sample spaces and equiprobable measure as a special case, random variables (discrete and continuous), conditional probability and stochastic independence, Bayes' formula. Sequences of independent Bernoulli trials, two-state Markov chains.

2. Probability distributions. (15 lessons) Characterization of probability distributions by density and distribution functions, illustrated by the binomial and uniform distributions. Expected values, mean and variance. Chebychev inequality, Poisson distribution introduced as approximation to the binomial, normal approximation to the binomial, Central Limit Theorem, stationary distribution of a simple Markov chain, Law of Large Numbers, discussion of special distributions motivated by relevant problems in the life sciences.

3. Statistical inference. (13 lessons) Concept of random sample, point and interval estimates, hypothesis-testing, power of a test, regression, examples of nonparametric methods, illustrations of correct and of incorrect statistical inference.

Mathematics 3. Elementary Linear Algebra.

[Editor's note: Here the Panel on Mathematics for the Life Sciences referred the reader to Mathematics 3 (Linear Algebra) in the 1965 General Curriculum in Mathematics for Colleges and made several suggestions for modifying this course in order that it be more appropriate for students in the life sciences. Some of the Panel's suggestions were incorporated into the revised version of Mathematics 3 (Elementary Linear Algebra) which appears in Commentary on A General Curriculum in Mathematics for Colleges, page 55. Another appropriate course, featuring many of the Panel's suggested modifications, is Mathematics L (Linear Algebra) of A Transfer Curriculum in Mathematics for Two-Year Colleges, page 231.

5. Recommendations for Computing

Automatic Computing

We recommend that every undergraduate in the life sciences have some contact with an automatic digital computer, and that this contact begin as early as possible in his program of study. Among the many bases for this recommendation are: that many mathematical models in the life sciences, as witnessed by the current technical literature, are procedural in nature and are best studied with the computer; that many analytic techniques of experimental biology are of practical value only when applied with an automatic computer; and that the automatic computer could play an important role in undergraduate biology lectures and laboratory if the students were prepared to make use of it.

This recommendation is stated separately from our recommendation of a CORE mathematics program for the life sciences student because we feel that experience in automatic computing will become part of a general liberal arts requirement rather than part of a major in either biology or mathematics. In many colleges a first course in computing is not the responsibility of the mathematics department but of a computer science department or a department in which computer applications are already numerous. As applications in subject-area courses increase, the need for an introduction to computing separate from the courses in biology, chemistry, mathematics, and physics may disappear.

CUPM has established a panel to consider instruction in computer science and the use of computers for instruction in mathematics. Our recommendations may be used to select options from their recommendations when they become available and to set an amount of experience appropriate for the biology major. [See Recommendations for an Undergraduate Program in Computational Mathematics (1971) and Recommendations on Undergraduate Mathematics Courses Involving Computing (1972).]

It should be noted that our basic recommendation in automatic computing is minimal. For example, the recommended experience does not include the introduction to analog or to hybrid analog-digital computing, and it includes only the briefest view of the complex problems of numerical analysis. We hope that some analog experience could be gained in advanced biology laboratory work. We also urge that the student be cautioned against the misuse of computing techniques, to avoid any tendency toward confusing the mastery of a programming language with an adequate knowledge of mathematics.

We suggest two alternatives for a one-semester course by which this computing experience can be gained. These are described in detail in Section 9. [See also the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563.] The first alternative is an informal program of weekly lectures and discussions of one hour extending through the freshman year, supplemented by a large number of assigned programming exercises to be developed and run at the student's convenience. It would amount to about one half a semester course for which credit might or might not be given. The second is a formal 3-semester-hour course with five or six assigned programming exercises, to be taken normally in the sophomore year. These suggestions will be stated more completely, but first it seems proper to point out the advantages and problems of the two approaches.

Basic computer programming skills have commonly been acquired through programs of self-instruction. Many computer scientists feel that it is best to provide the student with a computing facility, some reference manuals as to its use, an introductory lesson or two, and then to stay out of his way as he practices by developing programs which are of particular interest to him or are relevant to his other studies. They would not give formal academic credit for this work. We would temper this plan by continuing the lessons or discussion periods beyond the most basic introduction and assigning some specific programs to insure that the student is exposed to various classical and valuable computing techniques. Even with this modification, the plan has the obvious advantage of not making significant demands on either faculty or student time, an important factor considering the already heavy required curriculum in the life sciences. It reasonably could, in fact, be carried out without academic credit. Unfortunately, the student freedom which permits this implies a lack of control over facility usage, so this alternative can prove expensive in machine charges.

The second alternative, a formal semester course introducing the student to computing, has the disadvantage of adding to an already exacting schedule. A most troublesome additional problem is that we do not feel that the introductory course in many computer science programs is appropriate to students in the life sciences. In that they are planned to set the foundation for further work in mathematics or computer science, they often do not cover computing applications adequately. The course we propose may, therefore, add an additional load on the mathematics faculty, particularly distasteful because of its partial redundancy. A formal course has two powerful advantages, however. The students will be brought to a higher level of competency and machine charges per student can be kept relatively low. This applied computing course could be relevant to fields beyond the life sciences, of course, and could be planned to serve all undergraduates not intending to specialize in computer science.

Continuing the Computing Experience

Given the contact with computers which we recommend, the student will be able to use the computer to extend his studies both in mathematics and in the life sciences. Experience has shown that he will, in fact, do so. It is important, therefore, that facilities be available to support this use. While accurate estimates of potential use are impossible, many students will continue computing at about the rate begun in the introductory course if given a chance.

Of importance in taking proper advantage of the student's computing experience is the use of computing exercises and demonstrations whenever relevant in the regular biology curriculum. We point out that the relevance is striking in many areas. For example, a course in population genetics could use a computing facility as a regular laboratory instrument, and some topics such as genetic drift are difficult to present without the computer. Too often, the student will recognize this value before the instructor. It is essential that every effort be made to introduce the potential of applied computing, as well as all other mathematical techniques, to the life science faculty. Among the possible means for this faculty education are: the involvement of the life science faculty in the program of computer instruction, the preparation and distribution of materials and understandable manuals on local computing facilities, the preparation and distribution of computer demonstration and laboratory materials for specific courses, and, most important, a demonstration of interest by the mathematics or computing faculty in biological research along with patient collaborative effort with life scientists.

6. Undergraduate Preparation for Biomathematics

The present state of biomathematics is such that one cannot expect to study this subject as an undergraduate. The best that can be expected of an undergraduate curriculum is to provide the student with a strong background in mathematics, physics, chemistry, and biology as preparation for graduate study. Indeed, because of its dependence upon the other sciences, biology may be emphasized the least in the undergraduate program and then, presumably, the most in the graduate program. In most colleges the undergraduate who is enrolled in such a program will be regarded as a major in mathematics.

Before going into details concerning the mathematics component, we consider some general principles on which a biomathematics program should be based.

(1) About one third of the student's undergraduate curriculum will be devoted to the mathematical sciences, including statistics and computing. A second third will be devoted to physics, chemistry, and biology, and the remainder to the humanities and social sciences to fulfill degree requirements. Since the student will normally be a major in mathematics, it is important that departments of mathematics allow their majors to choose electives freely in the biological sciences.

(2) Many institutions give several different versions of basic courses in the sciences. The crucial difference is usually the extent to which mathematics is used. It is vital, therefore, that the student plan his program so as to take the most sophisticated version of each course that is available. This injunction applies especially to courses in physics and in physical chemistry.

(3) Very few universities have a department of biomathematics. Most graduate students who study this subject will be enrolled in some life science department. It is essential, therefore, that the undergraduate program of such a student include enough courses in biology for him to gain admission to a graduate program in a life science area. This need not imply that the undergraduate program must contain very many courses in biology. Many leading life science departments will admit a person with as strong a background in mathematics and chemistry as is contemplated here if he has had as few as four semester courses of undergraduate biology and some may even require no undergraduate biology.

(4) It is neither practical nor desirable for a student to make an irrevocable commitment to a particular specialty early in his college career. As was stated in the preceding paragraph, a student who elects the program that is being presented here should be qualified for admission to a graduate life science program. He may then choose to specialize in some area of biology other than

biomathematics. On the other hand, he may decide to do graduate work in mathematics only. By adding a substantial course in abstract algebra to the program described below, he should become eligible for admission to most graduate departments of mathematics.

(5) Of the natural sciences, chemistry will receive the greatest emphasis. Courses in organic chemistry and the strongest possible course in physical chemistry will certainly be included in the program; biochemistry may also be included, although some schools prefer to introduce this topic at the graduate level.

With these considerations in mind, we now turn to the mathematics in this program. The computer experience and the core of four mathematics courses discussed in Sections 4 and 5, as well as in the GCMC report, form the foundation of this preparation. To this we add semester courses in Calculus, Advanced Multivariable Calculus, Statistics and Probability, and a two-semester sequence of Real Variable Theory, as described in Mathematics 4, 5, 7, 11, and 12 in Commentary on A General Curriculum in Mathematics for Colleges, page 33 The GCMC Mathematics 10, preferably in the version described in Section 7 below, and a Numerical Analysis course (see Mathematics 8, page 83) should be included.

A biomathematician will need to know more mathematics than is presented in this program. For example, he will have only a touch of differential equations in Mathematics 2 and 4, and will ordinarily need considerably more probability and statistics than is covered in Mathematics 2P and 7. Thus, his graduate program will include additional work in mathematics, although it will consist predominantly of biology. A biomathematics student (as well as other biology graduate students) may wish to follow a plan that is currently used in many other graduate fields: electing one mathematics course each term until the Master's degree requirements in mathematics are met. A few biomathematicians may wish to include course preparation for the Ph.D. degree both in mathematics and in the life sciences.

7. A Course in Applications of Mathematics in the Life Sciences

A course in applied mathematics (Mathematics 10) is briefly described in Commentary on a General Curriculum in Mathematics for Colleges. The essential feature of this course is "model building and analysis" coupled with appropriate interpretation and theoretical prediction. The philosophy of this approach to applied mathematics is well stated on page 92 , and we recommend the reader's careful attention to that material in order to establish the necessary point of view for consideration of a course entitled Introduction to Applied Mathematics: Life Sciences Option. [See also the 1972 report Applied Mathematics in the Undergraduate Curriculum, page 705.

Three versions of a model-building course are developed in detail in Applied Mathematics in the Undergraduate Curriculum, page 705. We consider here another version designed for students with a particular interest in the life sciences.

Applications of mathematics in the life sciences may be classified basically into two broad categories, deterministic and stochastic. Moreover, a third category should also be added, that of mixed models, wherein the particular phenomenon under consideration may be modeled in either deterministic or stochastic fashion.

Specific prerequisites for a life sciences version of Mathematics 10 will vary according to which topics are studied, but in any case they include the basic core, supplemented suitably--usually with additional work in calculus and differential equations (Mathematics 4 and 5) and perhaps with additional work in probability and statistics (Mathematics 7).

One feature of life science models is that the mathematics used tends to be either almost trivial or relatively advanced; good "junior-level" models seem hard to find. Thus, for the present, successful offering of a life science version of Mathematics 10 would seem to call for an instructor who is well qualified both in mathematics and in the life sciences. Moreover, this instructor should be broadly interested and knowledgeable in applied mathematics and, in particular, in model building. Earlier CUPM reports have recommended that, in the absence of such a member of the mathematics faculty, Mathematics 10 should not be offered. It has been found in a number of institutions, however, that a viable alternative may be obtained through a joint effort of an interested member of the mathematics faculty and specialists in various other disciplines. Both mathematics and biology can thus be adequately represented and the essential feature of strong motivation is present. An undergraduate seminar led jointly by such a faculty team, with models being proposed by the members of the class, has been found to work well in practice. A format suitable for this purpose has been described by S. A. Altman ("A Graduate Seminar on Mathematics in Biology." CUEBS News, Vol. V, No. 1, October, 1968, pp. 9-10).

An institution with a strong, modern biological sciences department should be able to offer a course such as that suggested above. This is especially true if the members of the life sciences faculty are interested in bringing in mathematical ideas and there is also present at the institution a mathematics cadre interested in the applied mathematical sciences. Several members of the Panel have had some experience in offering courses based on model building in both physical and life sciences. Such an approach revolves around an artful use of the case study method, with the class thereafter pursuing the mathematical structure, detective story fashion, wherever it may lead. Usually the mathematical structure itself is developed en route only to the extent that is demanded by the model, although appropriate avenues are of course indicated to the students for following up any particular portions of the mathematics that may especially interest them.

We conclude this section with a few general comments concerning mathematical models in the life sciences.

Model construction consists, for the mathematician at least, of laying down an appropriate axiom system, either as a formal set of axioms or by means of a system of defining equations. Equations of motion in physiology and biophysics, linear algebra formulations of protein sequences or of population state vectors, dynamical systems describing population interactions, combinatorial models of genetic phenomena or of macromolecule configurations are all instances of such axiom systems. Once an appropriate mathematical structure (i.e., a set or sets with operations) has been specified, the further analysis proceeds within the mathematical structure, emerging at certain strategic times with interpretations or theoretical predictions drawn from the mathematical deductions themselves. To the extent that these conclusions are in accord with those aspects of the actual phenomenon that are regarded as significant in that context, so, too, may the original mathematical model be regarded as a good one. One of the virtues of such a procedure, as noted in the GCMC report, is that "the attempt to build a satisfactory mathematical model (often) forces the right question about the original situation to come to the surface." It is clear, therefore, that the modeling process is often one of successive approximations, hopefully convergent to a sound theory at some stage. An essential part of the instructor's responsibility would seem to be conveying to the life sciences student the realization that once an appropriate mathematical structure has been determined via axiomatization, he can work strictly within this mathematical structure, to come back in the end with certain interpretations and theoretical predictions relevant to the particular life science phenomenon under consideration. All too often there seems to be what amounts to almost a mental block in many biologists' thinking that precludes their leaving the realm of empirical laws and statistical description (mathematics as curve fitting) to work within the mathematical structure itself. The great significance of this latter mode of procedure for the astonishing growth of the physical sciences during the past half century has been well described by Mostow, Sampson, and Meyer (Fundamental Structures of Algebra. New York, McGraw-Hill Book Company, 1963. Preface) in the following terms:

> "The great evolution of the physical, engineering and social sciences during the past half century has cast mathematics in a role quite different from its familiar one of a powerful but essentially passive instrument for computing answers. In fact that view of mathematics was never a correct one... . Its inadequacy is becoming increasingly apparent with the growing recognition that mathematics is at the very heart of many modern scientific theories--not merely as a calculating device, but much more fundamentally as the sole language in which the theories can be expressed. Thus mathematics plays an organic and creative part in science, as a limitless source of concepts which provide fruitful new ways of representing natural phenomena."

The objective of the proposed course in applications of mathematics in the life sciences is to develop in students the capability to utilize these powerful mathematical methods in the fashion indicated above.

III. APPENDICES

8. Course Outlines for Mathematics 0, 1, 2, 3, 2P

See Commentary on A General Curriculum in Mathematics for Colleges, page 33.

9. Course Outlines for an Introduction to Computing

Following are outlines of programs for the two alternatives suggested in Section 5. [See also the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563.]

Introduction to Computing: Alternative 1

The course is comprised of weekly or biweekly one-hour lecture and discussion meetings and ten or more student programming exercises. It is set primarily for freshmen, although, if done with informality, it could involve the entire life science community, including the faculty. The prime goal is the development of basic applied programming skills in an algebraic language. There is little concern for the logical organization of the machine or detailed representation of information in the computer.

Materials

1. An introduction to a common algebraic language, such as FORTRAN, PL/1, ALGOL, or the conversational languages BASIC, CAL, JOSS.

2. A reference for elementary numerical methods.

3. A reference for statistical methods.

4. A reference for simulation and other general problem-solving techniques.

Facilities

Computing facilities will be required to handle the submittal of approximately 60 batch process jobs of very short duration per student over the period of the course. If a time-shared facility is available, about 20 to 25 console hours will be required equivalently. The conversational use of a time-shared facility is to be preferred from the point of view of efficient use of student time.

Faculty

In order to assure relevance of the exercises to the life sciences, it would be desirable for an instructor from the life sciences faculty to handle the lectures and discussions for the life science students in this course, initially, with the advice and assistance of a member of the mathematics or computer science faculty. This would also serve as a logical entree to the education of the life science faculty to the potential of automatic computers and related models for their fields. We feel that many biologists will accept the challenge posed in this context, when assured adequate guidance.

There are a number of possible logistic problems related to running the programming exercises that will usually make teaching assistants at a very junior level valuable to this course.

Content

Topics	Suggested Problems
(Approximate number of lecture hours in parentheses)	(These may be presented in the context of a biological problem)
1. Algorithms, flowchart representations. (1)	
2. First principles of an algebraic language. (6)	2a. Mean and standard deviation of a sample.
BASIC, FORTRAN, CAL, PL/I, or ALGOL. Organized so that students may begin programming as soon as possible.	2b. Selection sort.
	2c. Table look-up.
	2d. Linear interpolation in a table.

3. Very simple introduction to numerical calculus. (This <u>can</u> be carried out before the students have had any appreciable instruction in calculus.) (4)	3a. Area under a curve by Simpson's rule.
	3b. Euler's method (point-slope).
Cautionary discussion of error in calculation, with examples.	3c. Root finding by method of false position.
4. Pseudo-random numbers. Simulation. (2)	4. Simulation of the rolling of a die.
5. Introduction to the literature of computer programs. (2)	5. Use of a standard data analysis package such as the BIMD statistical programs.
	Additional program or programs on topics of special interest to the student.

Introduction to Computing: Alternative 2

This course is a one-semester 3-credit-hour introduction to applied computing. While the concern for representation of algorithms and data overlaps that of a first course in computer science, our suggestion differs in that the accent is always on application, on problem solving with a digital computer. There are three or four suggested programming exercises in an algebraic language and one or two in special-purpose languages suited for biological problems. The course is set primarily for sophomores.

Materials

1. An introductory text on computing.
2. References for elementary numerical methods and statistical methods.

Faculty

The course must be handled by a specialist in computing, as opposed to Alternative 1, although the elective programming exercise could be directed by a teaching assistant from the life sciences.

Facilities

Computing facilities will be required to handle the submittal of approximately 30 batch process jobs of short duration per student.

About 15 console hours on a time-shared remote access computer would be required equivalently.

Content

Topics	Suggested Problems
(Approximate number of lecture hours in parentheses)	Problems listed under alternative 1 and others, such as
1. The concept of an algorithm: discussion of its connotations. (2)	
2. Representation of algorithms: natural language, flowchart, algebraic language. (2)	
3. Principles of an algebraic language: FORTRAN, PL/I, ALGOL, or a conversational language: BASIC, CAL, etc., as available. Illustrations from simple numerical and statistical methods. (10)	
4. The evaluation of algorithms, logical organization of a computing machine. (5)	4. Test for well formation of string of parentheses.
5. A sampling of computer applications and methods. Simple symbol manipulation, list structures, simulation examples, pseudo-random number generation, a simulation language such as SIMSCRIPT, and specific applications from the life sciences. (12)	5a. Generation of Markov chain from transition matrix (presented in behavioral terms) in algebraic language, or a flow simulation in SIMSCRIPT. 5b. Elective problem on a topic from the life sciences such as an epidemic simulation, or the analysis of data from an actual experiment. 5c. Numerical integration, or linear regression with printer plot of graphical output.
6. Discussion or demonstration of special computing equipment involving graphical displays or real-time control of experiments. (3)	

APPLIED MATHEMATICS
IN THE
UNDERGRADUATE CURRICULUM

A Report of

The Panel on Applied Mathematics

January 1972

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I. INTRODUCTION AND STATEMENT OF RECOMMENDATIONS

Traditionally, attempts to solve problems in the physical sciences have stimulated and, in turn, extensively utilized basic developments in mathematics. This essential interaction between mathematics and the sciences is experiencing new vigor and growth. Recently, mathematical methods have been introduced into the social and life sciences, and even into some areas of the humanities. This has led to the development of new mathematical ideas and to new ways of using mathematics. The Committee on the Undergraduate Program in Mathematics (CUPM) appointed a Panel on Applied Mathematics to consider the implications for the undergraduate curriculum of this new growth of the uses of mathematics.

Instead of training students to handle all of the steps involved in solving a realistic problem, typical courses in applied mathematics generally confine themselves to a treatment of various mathematical techniques; in particular, mathematical model building is neglected. While courses in mathematical techniques are necessary, they do not provide a sufficiently broad training for students interested in applied mathematics.

The Panel therefore makes the following recommendations:

1. Every mathematics department should offer one or two courses in applied mathematics which seriously and comprehensively treat realistic problems and which emphasize model building.

2. Mathematics courses in the first two years of college should contain many realistic applications.

3. Every student taking a substantial number of courses in mathematics should include at least one course in applied mathematics.

4. A concentration in applied mathematics should be made available if the resources of the college permit.

The Panel is aware that the fourth recommendation is the most difficult to implement, especially in smaller departments. However, we feel strongly that most college departments can begin to implement the first three recommendations without undue difficulty or delay. For instance, having one instructor offer a course emphasizing model building could be an initial step toward implementing the first recommendation. Although the course may not have all of the desired characteristics the first time it is taught, the instructor's experience, along with ideas from this report, should enable him to come closer to meeting the objectives described here when he teaches the course again. Instructors in calculus, for example, can help to implement the second recommendation by introducing in their courses some applications different from the usual ones. In any case, the first of these recommendations can be effected by instituting one or two new courses at the upper-division level, and the second by incorporating applications in the lower-division courses.

II. DISCUSSION

Pure mathematics has undergone tremendous development during the past 25 years. Consequently, the recent generation of mathematicians is concerned primarily with pure mathematics, not only in research but also in educational activities. This is evidenced by the abstractness of some high school mathematics courses and the early introduction of axiomatic courses in colleges.

While the Panel applauds the advances in pure mathematics, it feels that it is unfortunate that education in applied mathematics has not received the same attention as that in pure mathematics. As a result, many other departments offer courses having substantial mathematical content, and mathematics faculties have tended to be unaware of the mathematization of many areas. It is encouraging, however, that there seems to be a recognition of this tendency and that a sympathetic interest in applications of mathematics is spreading. There is much more emphasis now than there was ten years ago on areas which directly attack problems of contemporary society such as ecological studies, city planning, water and atmosphere restoration, etc. This interest manifests a return to an attitude held in earlier times when mathematics was viewed as closely related to other areas such as the physical sciences and engineering. The unique way in which mathematics can contribute to an understanding of important problems in modern society is acknowledged, and many mathematicians have been attracted to the new ideas involved in recent applications because they are eager to have their teaching and research contribute to solutions of problems which are practical and contemporary.

These recent applications have contributed to changes in applied mathematics, both in its nature and in its methods. Applied mathematics may once have been identified exclusively with areas of analysis which had particular bearing on physics and engineering. But because mathematics is used in the social, life, and managerial sciences, and even in the humanities, applied mathematics must now include topics such as linear programming, graph theory, optimization theory, combinatorics, game theory, and linear algebra, in addition to those which have been traditionally associated with it. Similarly, methods of applied mathematics may have been thought of as involving complicated calculations with numbers or analytic expressions. While techniques for calculation are important, they are only part of the professional resources of an applied mathematician. Theory construction and model building are now essential for him. In studying the role of applied mathematics in the undergraduate curriculum, the Panel has taken into account these new topics and methods.

Having considered all of these points, we conclude that undergraduate instruction in applied mathematics must have a strong component specifically devoted to model building, and that undergraduates generally should be more aware of the many uses of mathematics in other areas.

III. NEW COURSES IN APPLIED MATHEMATICS

In our considerations we have been guided by the steps a working applied mathematician follows in studying a given situation. This process has been described in many ways by various authors. We use a description which is reminiscent of the one given by Murray Klamkin in the American Mathematical Monthly, 78 (1971) pp. 55-56 (ascribed to Henry O. Pollak):

1. Recognition of the nonmathematical problem.

2. Formulation of the mathematical model.

3. Solution of the mathematical problem.

4. Relevant computations.

5. Explanation of results in the context of the original problem.

Courses in mathematical topics give training in the solutions of mathematical problems (step 3), and courses in computer science and numerical analysis explain computational and approximative techniques (step 4), but very few courses adequately treat the processes involved in recognition, formulation, and explanation (steps 1, 2, and 5). While the student must, of course, have sufficient mathematical and computational techniques at his command to solve the mathematical problems he confronts and to obtain the numerical results which are needed, we are convinced that the training of a student of applied mathematics must be more comprehensive. He must be thoroughly grounded in the techniques of mathematical model building, and he must have ample practice in interpreting the results of his mathematical solution in the original setting.

The first recommendation of the Panel is that each department should offer a course or two in applied mathematics which treat some realistic situations completely, beginning with a careful analysis of the nonmathematical origin of the problem; giving extremely careful consideration to formulation of a mathematical model, solution of the mathematical problem, and relevant computations; and presenting thoughtful interpretations of the theoretical results to the original problem. In other words, there should be a few courses which give the students the experience of grappling with an entire problem from beginning to end.

To aid colleges in implementing the first recommendation, the Panel has constructed outlines of courses which emphasize model building. These courses are not intended to replace courses stressing mathematical techniques which are offered for students majoring in other areas, nor should they replace those standard mathematical offerings in which applications play a useful motivational role. Service courses are valuable and should continue to be offered by the mathematics department; indeed, they should be designed in active collaboration with members of concerned departments. Courses in mathematical topics which have their origins in applications are also important. However, the courses we recommend here provide a complementary training by giving students active experience in mathematical model building.

The Panel has given at the end of this report three course outlines which illustrate how a course stressing model building can be designed. These outlines are centered around the topics of optimization, graph theory and combinatorics, and fluid mechanics. The optimization course is intended as an example of a sophomore-junior course, the course on graph theory and combinatorics is appropriate at the junior level, and a course along the lines of the fluid mechanics option can be taught at the senior level. The optimization course and the course in graph theory and combinatorics can be offered at various levels by changing the level of rigor, varying the pace, concentrating longer on problems from a specific area, etc. These particular topics were chosen as unifying themes because of the experiences, interests, and competencies of individual Panel members, and because the courses on optimization and on graph theory and combinatorics illustrate the use of topics not traditionally viewed as being part of applied mathematics. In choosing these topics, the Panel does not mean to exclude other topics which might be used as the unifying element of an applied mathematics course. On the contrary, we hope that these outlines will stimulate instructors to construct similar courses around other topics. In fact, within reasonable limits, the particular topics chosen are not nearly so crucial as the emphasis on the model building process.

IV. GUIDELINES FOR TEACHING THE NEW COURSES

In planning or in teaching courses which emphasize model building, the instructor should keep in mind certain points which are essential for proper implementation of our recommendations.

First, the role of model construction must be made clear and amply illustrated throughout the course. The student must have as much experience as possible in constructing models. Real-life situations are often so complex that it is impossible to formulate a satisfactory model immediately; quite often it is necessary to construct a succession of models in an effort to find a satisfactory one. The student should have experience with this process. Furthermore, he should be aware that there may be several approaches which lead to essentially different mathematical models for the same problem. Therefore, a critical evaluation of the steps in constructing a model is essential in order that the student know what kind of information he can expect or cannot expect from a model and that he be able to choose the model which is most effective for his purpose.

Constructing a model for a given situation requires originality and a thorough understanding of the original nonmathematical situation. To appreciate what is involved, students must be active in formulating models. This aspect of the training is so important that the instructor should be willing to sacrifice some topics to insure the student's thorough grounding in model building. If the instructor conducts his class in the traditional lecture fashion, then he should prepare homework projects which require his students to formulate and to refine models for various situations. However, the Panel explicitly calls attention to the possibility of conducting these courses as seminars in which students and faculty members work cooperatively. Such a seminar could be organized around various problems, or it could develop a model for a complicated system which can be subdivided into smaller units. A benefit of the latter format is the experience of teamwork. Another possibility is for students to choose projects which they pursue independently. These projects could range from original investigations to reports based on the literature. In this case, students should periodically report their progress to the other participants in the seminar.

It is important to realize that model building has many forms. The activity which is most usually associated with the term modeling and which is actually always present in some form consists of formulating in explicit terms the dependence of the phenomenon under investigation upon the relevant factors. A classic example is the construction of a model for the motion of a vibrating string leading to a linear partial differential equation. In this case the factors which are to be neglected as well as those which have considerable effect on the motion can be identified, and the sort of physical assumptions which simplify the model are relatively clear. With appropriate assumptions, an analysis of the physical laws governing the motion of a particle lead to a mathematical model for the motion of the string consisting of a partial differential equation and suitable boundary conditions. The solution of this mathematical problem aids in the description of the motion of the string. The degree to which the solution of the mathematical problem contributes to an understanding of the physical one depends upon the degree to which the assumptions fit the real situation.

The model for motion of a vibrating string is a deterministic one; that is, it is based on the assumption that the physical laws and the initial conditions determine the response of the system exactly. Such models are not always appropriate, and there are instances in which uncertainty in the real situation should be reflected in the model, as, for example, in stochastic models. As an illustration, consider the construction of a model for the spread of a disease. The number of people who become ill during an epidemic depends on a number of factors associated with the disease--its virulence and period of contagion for example--and also on the random contacts between infected and susceptible individuals. In some instances the results obtained by ignoring the probabilistic features of the situation may be adequate, while in others inclusion of the probabilistic features may be required in order to obtain a satisfactory fit between the predictions of the model and the results of observations.

Alternatively, it may be that any model which accounts for what appear to be the essential features and which is formulated for mathematical analysis will lead to mathematical problems which are either totally intractable or beyond the scope of investigation. In such cases a computer simulation may be useful. Simulations may be performed on both deterministic and stochastic models, and they may provide much of the same type of information that is obtained from a mathematical analysis when such analysis is feasible.

The point is that there are many kinds of models, and the student of applied mathematics should be aware of them. Consequently, the topics for investigation must be chosen carefully so that different types of models will be illustrated.

Second, the problems chosen for investigation must be realistic. In this report, when we use the term "realistic" referring to problems or situations, we have in mind those which arise directly from nature or from social behavior and which have some current significance. We label as "artificial" those problems which seem to be designed purely to illustrate some mathematical point. While some artificial problems have undeniable pedagogical value, relying almost exclusively on such problems will not instill the attitude of mind which should characterize the modern applied mathematician. In a contrived situation it is difficult to create and maintain interest in the multitude of concerns which arise in problems occurring in the real world. Since it is the Panel's intention that the student recognize the complexities of the real world and that he come to terms with these complexities in his model building process, the student must face real problems. In the course outlines we have given references to assist those who wish to acquaint themselves with significant problems in other fields.

Third, the original nonmathematical situation should not be forgotten once a mathematical formulation has been achieved. The results of the mathematical study need to be interpreted in the original setting. Stopping short of this gives the impression that manipulation of symbols or that techniques of computation or approximation are the important points, whereas they are only intermediate steps, although absolutely essential, in studying a realistic nonmathematical situation. For this reason we urge that in these courses the situations should not only be realistic but that they should be treated as completely as possible.

Fourth, the mathematical topics treated should be worthwhile and have applicability beyond the specific problem being discussed. They should be appropriate to the level at which the course is offered; problems and examples should be chosen to illustrate more than just elegant or ingeneous applications of mathematically trivial ideas. It is impossible for a single course to contain all of the mathematical techniques which all students may need; nevertheless, it is possible to select as illustrative techniques those which will be valuable to a large portion of the students.

Finally, an instructor of applied mathematics should not view his work as being confined to one academic department or, for that matter, restricted to his college or university. Applied mathematics affords unique opportunities for cooperative projects with other members of the college community and with people outside the college whose professional work is related to mathematics. We encourage instructors to invite active participation by students and faculty members from other departments in planning and conducting courses or seminars. In some instances it may be valuable to include nonacademic professional people having interests and competencies related to the area being studied; their experience and point of view may add a new dimension to the investigations. It is our view that instructors in applied mathematics are in a particularly good position to initiate cooperative ventures of this type.

V. USE OF COMPUTERS IN APPLIED MATHEMATICS

Mathematics education has been influenced in several ways by the recent trend toward the widespread use of computers. This is particularly true of instruction involving applications. The role of computing in the mathematics curriculum is being studied in detail by the CUPM Panel on the Impact of Computing on Mathematics Courses [see Recommendations on Undergraduate Mathematics Courses Involving Computing, page 571], and comments on computing as a part of a concentration in applied mathematics can be found in Section VIII of this report. The purpose of this section is to draw attention to the ways in which machine experience can reinforce ideas and techniques which the student is learning and thereby contribute to the teaching of applications.

The use of computers makes it possible to consider situations having a much greater complexity than would be possible if the associated numerical work were to be carried out by hand or with the assistance of a desk calculator. This is true not only in courses specifically oriented toward applications but also in the standard undergraduate courses. As an illustration of the sort of activity which illustrates the process of applied mathematics and which becomes feasible through the use of computers, consider the example of determining as a function of time the position and velocity of a rocket traveling to the moon. The depth of the study obviously depends heavily on the audience, but certain versions are appropriate for students in courses in elementary calculus or ordinary differential equations. A sample discussion in the spirit of this report would include the following features.

1. Newton's laws of motion and gravitation and a mathematical model for the system. A careful discussion of the idealizations and approximations made in constructing the model.

2. Derivation of the differential equations governing the motion of a rocket in one dimension between the earth and moon.

3. Discussion of the qualitative features of the solution.

4. Selection of a numerical method.

5. Preparation and testing of a computer program for the integration of a system of first-order ordinary differential equations.

6. Use of the program to obtain quantitative information on the motion of the rocket. Determine the escape velocity of the earth-moon system and compare it with that of the earth alone.

7. Comparison of results predicted by the 1-dimensional model with observed phenomena and a discussion of the inadequacies of such a model.

8. Derivation of the differential equations describing the motion of a rocket in two dimensions.

9. Numerical solution of these equations in two dimensions [repeat steps 3, 4, and 5 in this case]. Use a plotter to graph the trajectories as functions of initial velocity and firing angle.

10. Comparison of these results with observations. Discussion of discrepancies.

In addition to its use in the activities described above, the computer can also be used to obtain the best values of parameters occurring in the model and to test the validity of the model. The latter usually involves comparing predictions based on the model with experimental data by using statistical techniques. Finally, both analog and digital computers are useful tools for simulation when the situation cannot be modeled in a form susceptible to mathematical analysis.

VI. RECOMMENDATIONS CONCERNING MATHEMATICS COURSES IN THE FIRST TWO YEARS

The Panel believes that many students lose their enthusiasm for mathematics even as a tool because their mathematics courses seem unrelated to their own discipline. A large segment of students in lower-division mathematics courses is primarily interested in fields outside mathematics. These students want to use the ideas and techniques of mathematics in their fields of interest; they are not interested in majoring or minoring in mathematics. We feel that the best way to demonstrate the power and utility of mathematical ideas to these students and thereby to sustain their interest is to introduce applications to other fields in the early mathematics courses. Therefore, <u>the second recommendation of the Panel is that a greater number of realistic applications from a greater variety of fields be introduced into the mathematics courses of the first two years</u>.

The suggestions made earlier about the choice of problems and examples apply here too. Instructors should strive to avoid artificial or contrived examples and applications. It is especially important to formulate the problem clearly and to mention explicitly the assumptions, approximations, and idealizations used to obtain a reasonable mathematical model. If simplifications are needed to make the mathematical problem workable, then they should be clearly stated and discussed. In other words, the applications should be significant and their treatment should be as complete and intellectually honest as the level of the course will allow.

The applications should be chosen from various fields in order to illustrate the use of a mathematical model or idea in different settings. If the course and the background of the students permit, some problems should be treated which require one to construct a succession of mathematical models in an effort to conform better to experimental data. Numerical methods might be included.

As we have already mentioned, some students who are not mathematics majors lose enthusiasm for mathematics because their courses do not contain applications. However, the Panel is also concerned that the mathematics major have an appreciation for the importance of mathematics in other areas. Even if he becomes a research mathematician, he is very likely to teach some undergraduate mathematics courses. His effectiveness in these courses can be greatly increased by a grasp of the relations among different branches of mathematics and the relations between mathematics and other disciplines. Therefore, we feel that he should see many significant applications in his elementary mathematics courses.

Unfortunately, very little literature on applications of elementary mathematics exists at the present time. One source is the Proceedings of the Summer Conference for College Teachers on Applied Mathematics held at the University of Missouri--Rolla with the support of the National Science Foundation, published by CUPM. These proceedings contain applications of elementary calculus, linear algebra, elementary differential equations, and probability and statistics.

Textbooks for most undergraduate mathematics courses vary considerably in their emphasis on applications, and instructors should consult various books so that they can provide their classes with a variety of interesting applications. For example, in differential equations there are many modern texts which contain discussions of genuine applications. Two books which contain a variety of applications not duplicated in many other places are:

Bellman, R. and Cooke, K. L. Modern Elementary Differential Equations, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1971.

Carrier, G. F. and Pearson, C. E. Ordinary Differential Equations. Waltham, Massachusetts, Blaisdell Publishing Company, 1968.

Also, modern texts in general physics and mechanics usually have examples suitable for discussion in a course on differential equations.

Another standard undergraduate course--linear algebra--has many applications to both physical problems and linear programming. In addition to the references listed in connection with the optimization outline given later in this report, the following text deserves mention:

Noble, B. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

The following book is a collection of realistic problems suitable for undergraduate mathematics courses. The problems are catelogued according to the mathematical tools used in their solution. Every teacher of freshman and sophomore mathematics should be aware of this source of applications.

Noble, B. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967.

As an example of the way in which a specific subject matter area may be used to provide applications for elementary courses, consider the biological sciences. Population growth, for example, can serve as a motivation for the introduction of elementary differential equations. Also, population growth problems can be considered from a probabilistic point of view; indeed, many problems in the biological and social sciences admit both deterministic and stochastic models, so it may be wise to introduce probability along with calculus in order to be able to study both kinds of models. Books are now available which take this approach; for example,

Chover, J. The Green Book of Calculus. Menlo Park, California, W. A. Benjamin, Inc., 1971.

Stein, Sherman K. Calculus in the First Three Dimensions. New York, McGraw-Hill Book Company, 1967.

The instructor who wishes to include applications to the biological sciences will find the following references useful. Although some of this material can be treated with little modification in lower-division classes, these sources are more suitable for the instructor than for the student.

Gerstenhaber, M. et al. AMS Lectures on Mathematics in the Life Sciences: Some Mathematical Problems in Biology, vol. I. Providence, Rhode Island, American Mathematical Society, 1968. (See particularly the first paper.)

Pielou, E. C. An Introduction to Mathematical Ecology. New York, John Wiley and Sons, Inc., 1969.

Smith, J. M. Mathematical Ideas in Biology. New York, Cambridge University Press, 1968.

VII. COMMENTS ON SECONDARY SCHOOL TEACHER TRAINING

Our third recommendation is that every student whose degree program includes a substantial number of courses in mathematics should take at least one course in applied mathematics. This recommendation clearly should apply to mathematics majors, but the Panel wishes to emphasize that every prospective secondary school teacher of mathematics should also have at least one course in applied mathematics. The role of applied mathematics in the training of teachers of secondary school mathematics has been underscored by the American Association for the Advancement of Science* and by other CUPM panels. [See Recommendations on Course Content for the Training of Teachers

* Guidelines and Standards for the Education of Secondary School Teachers of Science and Mathematics. Washington, D. C., American Association for the Advancement of Science, 1971.

of Mathematics, page 158.] The AAAS recommendations state that "an undergraduate program for secondary school mathematics teachers should ... provide substantial experiences with mathematical model building so that future teachers will be able to recognize and construct models illustrating applications of mathematics." The CUPM Panel on Teacher Training recommends that prospective teachers should complete a major in mathematics and that the courses in the program should include not only a mixture of motivation, theory, and application but also an introduction to model building. Indeed, that Panel recommends that a course in applied mathematics is particularly desirable as an upper-division option for the mathematics major.

The Panel on Applied Mathematics strongly supports these recommendations and emphasizes the following reasons for a secondary school teacher of mathematics to have a knowledge of applications:

1. Appropriate applications provide excellent motivational material.

2. The teacher should be aware that most of the mathematics encountered in the secondary school has its origins in problems in the real world, and he should know what these origins are.

3. The teacher should be aware of the applications of mathematics in the social and life sciences as well as in the physical sciences. Since mathematical notions are occurring with increasing frequency in elementary texts in the social and life sciences, and since it is unlikely that most teachers of these subjects have adequate mathematical training to appreciate this material, the mathematics teacher may well be called upon to serve as a resource person for other teachers.

4. Carefully selected applications may aid significantly in developing the student's ability to recognize familiar processes which occur in complex situations.

Further discussion of these and other ideas can be found in references [E] and [P] at the end of this section.

We make the following recommendations:

1. In those courses of the basic curriculum which are taken by substantial numbers of prospective secondary school teachers (viz., Mathematics 1, 2, 3, 4 and 2P of Commentary on A General Curriculum in Mathematics for Colleges (CGCMC)), applications of the subject to problems arising outside mathematics should receive more attention than is generally given now.

2. Each prospective teacher should be strongly encouraged to take one of the courses proposed in Section III of this report or a course in applied mathematics designed especially for secondary school teachers. Sample materials appropriate for an applications-oriented course for teachers include [B], [Po], and [S].

References

[B] Bell, M. S., ed. Some Uses of Mathematics: A Source for Teachers and Students of School Mathematics. (Studies in Mathematics, vol. XVI, SMSG) Pasadena, California, A. C. Vroman, Inc., 1967.

[E] Engle, A. "The relevance of modern fields of applied mathematics for mathematics education." Educational Studies in Mathematics, 2 (1969), pp. 257-269.

[P] Pollak, H. O. "How can we teach applications of mathematics?" Educational Studies in Mathematics, 2 (1969), pp. 393-404.

[Po] Pólya, G. Mathematical Methods in Science. (Studies in Mathematics, vol. XI, SMSG) Pasadena, California, A. C. Vroman, Inc., 1963.

[S] Schiffer, M. M. Applied Mathematics in the High School. (Studies in Mathematics, vol. X, SMSG) Pasadena, California, A. C. Vroman, Inc., 1968.

VIII. RECOMMENDATIONS CONCERNING A CONCENTRATION IN APPLIED MATHEMATICS

The fourth recommendation of the Panel is that an undergraduate concentration in applied mathematics should be offered if the resources of the college permit. In many institutions there are students who desire such a program. These students should take some courses in model building such as those described in Section III, and they should be trained in mathematical topics useful in applications. We are concerned both that the training of the students properly reflect the changes taking place in applied mathematics and that a department of mathematics be able to begin implementation of our recommendations immediately with a relatively small change in course offerings. For these reasons our recommendations center around courses of the type we have already described and courses in various mathematical techniques which are common in many colleges.

A student interested in a concentration in applied mathematics should take three courses in calculus (Mathematics 1, 2, 4 of CGCMC) and a course in linear algebra (Mathematics 3 of CGCMC). (For those who notice the omission of differential equations, we point out that Mathematics 2 of CGCMC contains an introduction to differential equations.) To insure training and practice in modeling, he should take at least one and preferably two of the new courses described in this report. A student who has a particular area to which he wishes to apply his mathematics should select courses in mathematical topics which are useful in that area as well as courses in the field of application which utilize significant mathematics. The topics suggested below can be organized into courses in various ways. However, we do recommend that applications be introduced in these courses, and we feel that the comments made in Section VI on applications in the freshman and sophomore courses are particularly appropriate here.

A student who is interested in applications to the physical sciences or in some areas of life sciences (e.g., ecology) should take a physical science version of an applied mathematics course such as the one in fluid mechanics outlined in this report. His further mathematics courses should include as many of the following topics as possible: probability theory; elementary partial differential equations (some of this is already contained in the fluid mechanics course); topics in ordinary differential equations such as asymptotic solutions, stability, and periodic solutions; boundary value problems (including Fourier series); computer-oriented topics from numerical analysis such as those which emphasize numerical solutions of ordinary differential equations, numerical linear algebra, solution of nonlinear equations, or numerical quadrature.

A student interested in applications to business and social sciences should take courses such as the optimization course and the graph theory course outlined in this report. His further mathematics course work should include as many topics as possible from the following: probability theory and applications as described in the report of the CUPM Panel on Statistics, Preparation for Graduate Work in Statistics; statistics as described in the same document; computational linear algebra.

Furthermore, because much work in applied mathematics involves computations, approximations, and estimates, it is clear that students concentrating in applied mathematics should have training in the use of computers. Beyond increasing computational power, a knowledge of the uses of computers can provide a new perspective for formulating and analyzing problems of applied mathematics. Consequently, the Panel strongly recommends that the following phases of computer experience be included in the program of every student of applied mathematics:

1. Computer programming. The student should have sufficient familiarity with a programming language to be able to use computer facilities in ways that are appropriate for his mathematical course work.

2. Computational mathematics. The approximations, estimations, algorithms, and programming necessary to derive numerical solutions of mathematical questions should be presented.

3. Training and experience in the use of a computer at the various stages of solving a problem in applied mathematics. The

student should have experience in using the computer to organize large quantities of numerical data and to simulate complicated processes.

IX. COURSE OUTLINES

To exemplify the kinds of courses recommended in Section III, the Panel has constructed three course outlines. These courses do not deal merely with mathematical topics; they are courses in which the momentum comes from real situations. In particular, stress should be placed on model building and on interpretation of mathematical results in the original nonmathematical situation.

These outlines are not offered as perfect models of the kinds of courses we recommend. Rather, they represent our present best efforts to construct courses with these new emphases. We hope that they will produce a thoughtful response in the form of even better outlines for applied mathematics courses.

It is essential that these outlines be read with the recommendations of Section IV in mind. Also, the reader should have in hand one or two of the primary references in order to find examples of the kind of treatment we are suggesting.

In reading these outlines, in teaching these courses, or in constructing other courses along the lines of our recommendations, instructors should strive to stay well between the extremes of: (a) a course about mathematical methods whose reference to science consists mainly of assigning appropriate names to problems already completely formulated mathematically, and (b) a kind of survey of mathematical models in which only trivial mathematical development of the models is carried out.

The course in optimization was planned as a one-quarter course, with additional material in the sections marked * bringing the total to a one-semester course. The courses in graph theory and combinatorics and in fluid mechanics were designed as one-semester courses.

The number of lectures specified indicates the relative emphasis we have in mind for the various topics and serves as an actual time estimate for a well-prepared class. The Panel appreciates the fact that some instructors will find these time estimates somewhat unsuitable (for instance, they do not take into account the pursuit of finer points or the review of prerequisite material) and will find it necessary to make modifications in the courses for their classes. The Panel was tempted to construct less ambitious outlines but decided against this, because it felt that a prospective instructor would be helped by having more examples of the treatment we recommend rather than fewer. Nonetheless, a valuable course can be constructed by choosing a few of the topics listed and treating them carefully and thoroughly. Furthermore, if the students become actively engaged in the model building activity, then the time estimates given are not appropriate. In any case, we encourage instructors to engage in open-ended discussions with class participation in the modeling aspect of the course and, if necessary, to restrict the subject matter content of the course in order to accommodate this.

IX.1. OPTIMIZATION OPTION

This course was designed to provide an introduction to the applications of mathematics in the social and management sciences. The goals of this course, as stated in Sections III and IV, are a study of the role of mathematics as a modeling tool and a study of some mathematical notions of proven usefulness in problems arising in the social and management sciences. The mathematical content consists of programming and game theory. This selection is a considered choice, although it is recognized that several other alternatives could serve as well.

The proposed course can be taught at several levels to fit the competencies and interests of the class. In particular, one version might be appropriate for freshmen whereas another might be appropriate for upper-class students in the management and social sciences. The course outlined here is intended for an average junior-level class. The students should have completed the equivalent of two semesters of calculus and should have some familiarity with elementary probability theory. Linear algebra is not included as a prerequisite, as the necessary background is developed in the course. No specific knowledge of any other discipline is assumed.

A bibliography and an appendix, important adjuncts to the course coutline, are found after the outline. References to the bibliography are enclosed in square brackets [ ], and references to the appendix are enclosed in braces { }. The bibliography contains a selection of books and other references which have proved useful in courses of this sort. Certain references have been designated as primary references, and comments have been provided which indicate those features of particular interest for an instructor. Most of the citations in the course outline are to the primary references. The instructor should have at least one of the primary references at hand while reading the outline. The appendix contains examples of the types of problems which can be studied using the ideas and methods of this course.

COURSE OUTLINE

1. Mathematical foundations of model building (4 lectures)

The real world and abstractions to mathematical systems; axiom systems as used in model building.

The ideas of a mathematical model and model building are introduced by using several examples which can be developed quickly and which illustrate applications in several different fields. Typical examples might be drawn from business (programming models for resource allocation), ecology (linear programming models of pollution control), psychology (2- or 3-state Markov chain models for learning), and sociology (game theory models for conflict). Assumptions made in the construction of these models should be carefully identified. The status of empirical "laws" should be discussed: law of gravity, law of reflection, law of supply and demand. It should be pointed out that all model building requires some essentially arbitrary decisions on the part of the person who is constructing the model. For example, whether to select a deterministic or a stochastic model is ultimately a decision of the investigator. In most instances there is no single best model. A model which was constructed to account for observed phenomena of one type may not give predictions which agree with other observations. The role of approximation and idealization in model building is fundamental. Approximations which are made and justified for real-world reasons should be distinguished from those whose basis is mathematical. Students need practice in making connections between assumptions about the real world and axioms in a mathematical system. Some of the examples should bring out the fact that an important (and frequently difficult) part of model building is asking the right question and viewing the real world problem from the right perspective. Some attention should be given to the practical problems of critically evaluating models and estimating parameters.

Most of the references contain some comments on model building. The initial chapters of primary references [D], [KS], and [Sa] have more comprehensive discussions. The books [ABC] and [LR] discuss modeling from the point of view of the social scientists.

2. Linear programming models (18 lectures)

a) Construction of linear programming models. (1 lecture) A detailed discussion of a real-world situation which can be reasonably modeled in terms of a linear program.

Examples similar to {1} or {2} might be used. Assumptions which lead to the axioms of linearity should be explicitly noted and adequately justified. It may be that the linear model is meant to serve only as a first approximation to a more complicated situation. Also, a linear model is frequently realistic only for restricted values of some variables. Such questions need to be considered. It is desirable to introduce both deterministic and stochastic models and later to compare two models of the same situation. The history of the development of linear programming during and after World War II is interesting. The book [D] is a useful reference for this material.

b) The basic problem. (6 lectures) The algebra and geometry of systems of linear inequalities in R^n. Matrix and vector notation and elementary linear algebra. Systems of linear equations and their application to systems of linear inequalities (e.g., if A is an $m \times n$ matrix and $\underline{b} \in R^m$ ($\underline{b} \neq 0$), then there exists $\underline{x} \in R^n$ satisfying $A\underline{x} = \underline{b}$ or there exists $\underline{y} \in R^m$ satisfying $A^t\underline{y} = 0$, $\underline{b}\cdot\underline{y} \neq 0$).

The notion of duality and the fundamental theorem should be introduced and illustrated. Consider complementary slackness and its economic interpretation. Selections from the primary references [D], [Ga], [SpT], and [W] provide appropriate sources.

* Proof of the fundamental duality theorem.

c) Algorithms: the simplex method. (6 lectures)

Much of the usefulness of linear programming models rests on the fact that the resulting mathematical problems can be efficiently solved. Accordingly, it is important to give some attention to computation, although only a bare introduction is proposed here. The method can be introduced as a sequence of replacement operations similar to a method for solving systems of linear equations. Algebraic and economic considerations can be used to describe and motivate the method. The concept of degeneracy arises naturally, but a complete discussion of this idea is beyond the scope of the course. Larger and more realistic problems should be solved, and students with computer competence should be encouraged to use it. The references are the same as those cited in b).

* Further remarks on degeneracy.

* A proof of the convergence of the simplex algorithm.

d) Refined models: linear programming and uncertainty. (5 lectures)

These models should be introduced by discussing the inadequacy of deterministic models for certain problems. One example is the allocation of aircraft to routes (this is discussed in Chapter 28 of [D]). There is no single formulation for stochastic models, as for deterministic ones, and there is little general theory. However, this is an important modeling technique which serves to demonstrate how models can be refined to take account of additional information. Examples can be given which show that one is not usually justified in simply substituting expected values for coefficients which are actually random variables. The basic problem is to formulate the stochastic model in such a way that relevant information can be obtained by studying an ordinary deterministic model. Chance constrained programming provides an interesting special case. Primary references [D] and [W] contain this material.

* Multistage models and dynamic programming.

* Geometry of the simplex method.

* Linear models of exchange and production.

3. <u>Game-theoretic models</u> (10 lectures)

a) Games and decision-making with uncertainty models for systems involving opposing interests. (3 lectures) The role of games as a modeling technique in the social sciences. The basic assumption of rational behavior and its validity.

Introduce utility theory, in both its qualitative and quantitative aspects. Consider individual decision-making under uncertainty and compare this to games. Discuss examples and, in particular, the relevance of a mathematical theory of games for the real world. The basic reference [LR] is useful here. Both [LR] and [BN] discuss game theory from the social scientist's point of view.

b) Games with two sets of opposing interests. (3 lectures) Two-person zero-sum matrix games and the connection between such games and linear programming.

Although such games are of limited use in applications, they provide a convenient vehicle for introducing basic notions of strategy and payoff. The fundamental (minimax) theorem of two-person zero-sum games. In primary references [D] and [Ga] this material is closely connected with linear programming. The discussion in [LR] is more comprehensive, and the notions of extensive and normal forms for games are introduced.

c) Nonzero-sum games. (3 lectures)

Games of the "prisoner's dilemma" type are of particular interest to the social scientist and can be used to illustrate the difficulties which arise in more complex models. The theory for such games is not nearly so well developed as for the games of b), but the study should bring out many of the questions that arise in mathematical work in the social sciences. Primary reference [LR] contains some of this material; more detailed expositions can be found in [BN] and, among the additional references, in [R].

d) n-person games. (1 lecture)

There is a qualitative difference between two-person situations and those involving three or more independent interests. Thus, there are new difficulties which arise in modeling three-interest conflict situations. The notion of a "solution" to such games requires careful analysis. The role of bargaining and coalitions is important in such models. See primary reference [LR].

* Games of timing. Reference [Dr] is especially complete on this topic.

* Two-person cooperative games.

References

A bibliography consisting of several hundred items on the topics listed in the course outline could easily be compiled. Thus, with some exceptions, the list of references is restricted to those sources specifically cited in the course outline. Several of the books listed here contain extensive bibliographies. The books given extended annotation are, with one exception, examples of writing which reflect the spirit of the course. The exception [0] is a mathematics textbook which presents some of these notions from a purely mathematical point of view. Critical reviews are indicated according to the following scheme: AMM, American Mathematical Monthly; MR, Mathematical Reviews; OR, Operations Research; and SR, SIAM Review. Also, each reference has been broadly classified according to whether it is primarily concerned with the mathematical content (M) or applications (A), and whether it is most useful for the student (S) or instructor (I). Several of the other references have been given a one-line annotation where useful.

Primary References

[D] Dantzig, George B. Linear Programming and Extensions. Princeton, New Jersey, Princton University Press, 1963, 625 p. AMM 72, p. 332; MR 34 #1073; OR 14, p. 734. (M and A, S) A textbook on mathematical programming written by one of the founders of the field. It includes chapters on the history of the subject and on model formulation. Chapter 3 contains five detailed examples. Standard topics in linear programming, extensions to integer, stochastic, and nonlinear programming, and many applications. Connection between programming and matrix games is included. Basic linear algebra is covered rapidly, and some probability is needed for the chapters on stochastic programming and games. No other prerequisites. Last two chapters contain detailed examples of formulation and study of models for nutrition and transportation. Extensive bibliography, many examples, and exercises.

[Dr] Dresher, Melvin. Games of Strategy: Theory and Applications. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961, 184 p. AMM 69, p. 243; MR 22 #13310; OR 10, p. 272. (M and A, S) The basic theory of two-person zero-sum games is developed in the first three chapters independent of its connection with linear programming. The remaining chapters include methods of solution, extensions to games with an infinite number of strategies, and games of timing. Basic calculus and probability are required. Several thoroughly discussed examples and applications, particularly to military problems. No exercises.

[Ga] Gale, David. The Theory of Linear Economic Models. New York, McGraw-Hill Book Company, 1960, 330 p. MR 22 #6599. (M and A, S) Theoretically oriented presentation of linear programming and game theory with emphasis on the use of these concepts in developing linear economic models. An advanced undergraduate textbook with no specific mathematics prerequisites, but almost everything is proved and several proofs require considerable sophistication. Chapter 2 provides a self-contained study of linear algebra including linear inequalities. The approach is both algebraic and geometrical throughout. Validity of the models is not discussed. Exercises are a definite asset; they vary from routine to nontrivial extensions of the theory.

Kaufmann, A. and Faure, R. Introduction to Operations Research. New York, Academic Press, Inc., 1968. (A, S) Collection of 18 chapters, each a completely worked-out independent example, generally written in anecdotal form. Few specific mathematics prerequisites; calculus and finite mathematics certainly sufficient. Basic ideas thoroughly explained but involved mathematical arguments are avoided. Applications are mostly to business situations and problems have an aura of reality.

[KS] Kemeny, J. C. and Snell, J. L. Mathematical Models in the Social Sciences. Waltham, Massachusetts, Blaisdell Publishing Company, 1962, 145 p. AMM 71, p. 576; MR 25 #3797. (A, S) Collection of eight independent examples of the construction and study of mathematical models drawn from several scientific disciplines. Stated mathematics prerequisites are one year of calculus and a good course in finite mathematics, but most students will require more background. No specific social science knowledge is assumed. There is an introductory chapter on the methodology of mathematical model building. Exercises and projects at the end of each chapter.

[LR] Luce, R. D. and Raiffa, H. Games and Decisions. New York, John Wiley and Sons, Inc., 1957, 509 p. MR 19, p. 373. (A, S) This is more a book about the concepts and results of game theory than a mathematics textbook; there are almost no proofs. Modest prerequisites: some knowledge of finite mathematics plus a bent for mathematical thinking. Thoroughly motivated discussions of the heuristic considerations which precede the mathematical formulation of the problems. These discussions are colored by a social science point of view. The introductory chapters consider the role of game theory in the social sciences and give a relatively complete discussion of utility theory including an axiomatic treatment. Extensive bibliography. No exercises.

[O] Owen, Guillermo. Game Theory. Philadelphia, Pennsylvania, W. B. Saunders Company, 1968, 228 p. MR 36 # 7420. (M, I) Mathematics textbook on two-person (chapters 1-5) and n-person (chapters 6-10) game theory, including those aspects of linear programming which are important for the study. Assumes basic calculus and probability. Convexity used but developed in an appendix. Chapters on infinite games and utility theory. Written in Definition-Theorem-Proof style. Many exercises of varying difficulty. Few applications.

[Sa] Saaty, Thomas L. Mathematical Methods of Operations Research. New York, McGraw-Hill Book Company, 1959, 421 p. AMM 68, p. 188; MR 21 #1223. (M and A, I) A textbook on operations research consisting of three major units. Part I contains chapters on the scientific method, mathematical existence and proofs, and some methods of model formation. The first chapter is particularly relevant for this course. Part II includes classical optimization techniques as well as linear programming and game theory. Part III is devoted to probability theory and its applications, particularly to queueing. There are many examples with convenient references to the literature, and a large bibliography accompanies each chapter. Assumes basic calculus and matrix theory. Some sections require multidimensional calculus. No exercises.

Simonnard, Michele. Linear Programming (translated by William S. Jewell). Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966, 430 p. MR 34 #1079, original French edition MR 25 #1952; SR 9, p. 608. (M, I) A textbook on linear programming covering the general theory (chapters 1-7), integer programming (chapters 8-9), and the transportation problem (chapters 11-15). The connection of linear programming with game theory and extensions to stochastic and dynamic programming are omitted. The book is oriented in a practical direction, and emphasis is on effective methods. It contains references to the literature, and there is an extensive bibliography but no exercises. Appendices on linear algebra, convex polyhedra, and graphs. No specific prerequisites.

[SpT] Spivey, W. A. and Thrall, R. M. Linear Optimization. New York, Holt, Rinehart and Winston, Inc., 1970, 530 p. (M and A, S) A mathematics textbook on linear programming with emphasis on the development of the simplex algorithm. The approach is a spiral one, and most topics are developed at several levels of difficulty. Chapter 2 discusses modeling and presents several examples. There is a chapter on game theory. The necessary background material on foundations, sets, functions, and linear algebra is given in appendices. Many exercises. Suitable as a text for students with limited backgrounds.

[W] Wagner, H. M. Principles of Management Science with Applications to Executive Decisions. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1970, 562 p. (A, S) A textbook of mathematical model building and optimization in a business setting. Prerequisites are a year of calculus and finite mathematics. No knowledge of business administration or economics is assumed. Emphasis on linear, dynamic, and stochastic programming with chapters on waiting-line models and computer simulation. Broad selection of exercises ranging from computational to "Form a mathematical

model for ... ." Some proofs, but many results are provided only heuristic justification.

Additional References

Ackoff, Russell L., with the collaboration of S. K. Gupta and J. S. Minas. Scientific Method: Optimizing Applied Research Decisions. New York, John Wiley and Sons, Inc., 1962. AMM 72, p. 216; OR 11, p. 157. The philosophy and formulation of mathematical models, including utility theory.

[ABC] Atkinson, R. C.; Bower, G. H.; Crothers, E. J. An Introduction to Mathematical Learning Theory. New York, John Wiley and Sons, Inc., 1965.

Bellman, Richard, ed. Mathematical Optimization Techniques. Berkeley, California, University of California Press, 1963.

Blalock, H. M. Theory Construction, from Verbal to Mathematical Formulations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969. AMM 77, p. 216.

[BN] Buchler, I. R. and Nutini, H. G., eds. Game Theory in the Behavioral Sciences. Pittsburgh, Pennsylvania, University of Pittsburgh Press, 1969.

Charnes, Abraham and Cooper, W. W. Management Models and Industrial Applications of Linear Programming, vol. I, II. New York, John Wiley and Sons, Inc., 1961. MR 28 #1003a,b.

Dantzig, George B. and Veinott, A. F., eds. Mathematics in the Decision Sciences, Parts 1, 2. (Lectures in Applied Mathematics, vol. XI, XII) Providence, Rhode Island, American Mathematical Society, 1968.

Duffin, R. J.; Peterson, E. L.; Zener, C. Geometric Programming. New York, John Wiley and Sons, Inc., 1967. AMM 77, p. 1024; SR 10, p. 235.

Freudenthal, H., ed. The Concept and the Role of the Model in Mathematics and Natural and Social Sciences. New York, Gordon and Breach, Science Publishers, Inc., 1961.

Gale, D.; Kuhn, H. W.; Tucker, A. W. "Linear programming and the theory of games." Cowles Commission for Research in Economics, Monograph #13. New York, John Wiley and Sons, Inc., 1951.

Goldman, A. J. and Tucker, A. W. "Theory of linear programming." Annals of Mathematics Studies No. 38. Princeton, New Jersey, Princeton University Press, 1956, pp. 53-98.

Hadley, G. Linear Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962. AMM 71, p. 815; SR 6, p. 191.

Rapoport, Anatol. Fights, Games, and Debates. Ann Arbor, Michigan, University of Michigan Press, 1960.

Rapoport, Anatol. Two-Person Game Theory: The Essential Ideas. Ann Arbor, Michigan, University of Michigan Press, 1966. MR 35 #1356. Nonmathematical exposition of basic ideas of two-person game theory.

[R] Rapoport, Anatol and Chammah, A. M. Prisoner's Dilemma; a Study in Conflict and Cooperation. Ann Arbor, Michigan, University of Michigan Press, 1965.

Shubik, Martin, ed. Game Theory and Related Approaches to Social Behavior. New York, John Wiley and Sons, Inc., 1964. OR 12, p. 637. Collection of articles on the applications of game theory in the social sciences.

Spivey, W. A. Linear Programming: An Introduction. New York, The Macmillan Company, 1963.

Vajda, S. Mathematical Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

Ventzel, E. S. Lectures on Game Theory. New York, Gordon and Breach, Science Publishers, Inc., 1961.

Wilder, R. L. Introduction to the Foundations of Mathematics, 2nd ed. New York, John Wiley and Sons, Inc., 1965.

Williams, J. D. The Compleat Strategyst, rev. ed. New York, McGraw-Hill Book Company, 1966. Intended for readers with no background in mathematics, this book develops the basic ideas of game theory through a variety of clever examples.

Appendix

The problems given here are indicative of the sorts of questions that can be studied using the techniques and ideas of this course. Problems similar to these should be approached in the spirit of Section 1 of the outline, where the question is phrased in real-world terms and a mathematical model is constructed. In such a discussion, close attention should be paid to assumptions, both explicit and tacit. The student should be made aware of the strengths and shortcomings of the resulting models.

1. Linear programming

Here are two linear optimization problems, one concerning diet and another concerning transportation, posed in a business context. The first is given in considerable detail, while the second is merely sketched. Possible extensions are indicated.

1.1 This problem, the determination of an adequate diet of minimum cost, was one of the first studied using a linear programming model. Detailed comments on the formulation of a mathematical model may be found in [D] and in the original paper of G. J. Stigler ("The cost of subsistence," J. of Farm Econ., 27 (1945), pp. 303-314). The following is a linear programming model.

Consider n different types of foods (apples, cheese, onions, peanut butter, etc.) and m nutrients (proteins, iron, vitamin A, ascorbic acid, etc.). In the original problem of Stigler, $n = 77$ and $m = 9$. Suppose that one can determine the daily allowance of each nutrient required by an individual and the nutrient values of the foods per dollar of expenditure. (These assumptions are at best approximations and should be presented as such.) Let

a_{ij} = amount of nutrient i obtained from an expenditure of one dollar on food j,

b_i = daily requirement of nutrient i,

x_j = number of dollars spent on food j.

With these definitions the condition that the diet provide at least the daily requirement of each nutrient becomes

$$\sum_{j=1}^{n} a_{ij}x_j \geqq b_i, \quad i = 1, 2, \ldots, m.$$

The problem of finding an adequate diet of least cost is then the problem of minimizing $\sum_{j=1}^{n} x_j$ subject to the above inequalities.

1.2 Suppose an oil company has m producing wells, n refineries, and pipelines connecting certain pairs of wells and refineries. Given the output at each well, the demand at each refinery, and the cost of transporting one barrel of oil through each pipeline, determine how the production of the wells should be allocated among refineries in order to minimize transportation costs.

1.3 In 1.2 consider the case that only allocations in whole barrels are permitted. Also consider the case where supply, demand, or other parameters are not known exactly, but instead some random behavior for each is assumed.

2. Game theory

Here are two examples involving decision making under uncertainty. The first example can be completely analyzed in terms of the elementary theory; the second cannot, but it illustrates a game that occurs frequently in the social sciences.

2.1 Two political parties compete for public favor by stating their views in n different media, labeled $1, \ldots, n$. Each party has finite resources and must distribute its expenditures among the various media without knowing the intentions of the opposing party. The payoff (a numerical measurement of the gain of one party or, equivalently, the loss of the other) resulting from use of medium i is given by a function $p(i,x,y)$ depending only on the medium and the resources x and y committed to that medium by the opposing parties. The payoff for the entire game is the sum of the payoffs in individual media. Given a knowledge of the resources and payoffs, how should each political party allocate its expenditures?

The following is a very simple model of a social situation involving conflicting interests. Models of this sort and their refinements are currently being studied by mathematically oriented social scientists. Although these models are only rough approximations to very complex situations, the results obtained from them are far from completely understood from a psychological and sociological point of view.

2.2 In an isolated and self-contained environment two retail stores compete for the local soft drink market. Each retailer handles only one brand of soda pop, different from the brand handled by the other retailer, and the two brands are identical in quality. In ordinary circumstances each retailer pays 70¢ for a carton of pop which he sells for $1. However, the soft drink distributors realize that from time to time price competition will develop, and they agree to sell their products to the retailer at 60¢ per carton provided that it is offered at retail for 80¢ per carton. Every Saturday each retailer must decide independently what his price for soda pop will be for the following week. Each has available the following information concerning demands: At the usual price they will each sell 1,000 cartons per week. If one retailer discounts while the other does not, then the discount store will sell 2,000 cartons while the store maintaining the usual price will sell only 200 cartons. If both stores sell at the discount price, then the total demand will be for 2,300 cartons and each will sell half that amount. Supposing that this decision must be made each week, how should the store managers proceed?

IX.2 GRAPH THEORY AND COMBINATORICS OPTION

This is an outline for a one-semester course designed to acquaint students with some fundamental concepts, results, and applications of graph theory and combinatorial mathematics. Only high school mathematics is required, but the student needs to be thoroughly familiar with this material. It should be kept in mind that this course represents just one of a number of (essentially equivalent) possible courses and is intended to offer the student not only specific facts and applications but also a feeling for the underlying philosophy of combinatorial mathematics.

A bibliography and an appendix follow the course outline. References to the bibliography are in brackets [ ], and references to the appendix are in braces { }. The bibliography contains references to books and other sources, together with comments about the primary references. The appendix contains examples of problems which can be treated using the ideas and methods of this course.

COURSE OUTLINE

1. Mathematical foundations of model building (4 lectures)

Real models, mathematical models, axiom systems as used in model building. (For discussion, see Section 1 of the course outline for the Optimization Option.)

2. Graph theory (18-20 lectures)

a) Basic concepts: relations, isomorphism, adjacency matrix, connectedness, trees, directed graphs, Euler and Hamiltonian circuits. (3 lectures)

In this section the student is introduced to a number of elementary (but fundamental) ideas of graph theory. He should be given as soon as possible the opportunity to formulate and discuss various models of real situations in these terms. [BS] is an especially good source of appropriate, relatively simple examples.

This material is available from numerous sources. The presentation in [L] is suitable here; more technical treatments are given in [Har1] and [O1], while that of [O2] is probably too elementary. Other sources are [Bel] and {1, 2}.

b) Circuits, cutsets, spanning trees, incidence matrices, vector spaces associated with a graph, independent circuits and cutsets, orthogonality of circuit and cutset subspaces. (5-7 lectures)

The linear algebra required for this section is minimal and, if necessary, could be developed in several hours. The concepts covered here lead directly to one of the more important applications of graph theory, namely, electrical network analysis. This material is covered rather briefly in [L] with no applications, very compactly in [Bec], more completely in [BS], and comprehensively in [SR] (on which an entire course could easily be based). [SR] is also an excellent source of applications of these topics.

c) Flows in networks, max-flow min-cut theorem, Ford-Fulkerson algorithm, integrity theorem, applications (e.g., linear programming, König-Egerváry theorem, multicommodity flows, marriage theorem). (4 lectures)

An appropriate discussion of this material occurs in [L] and in selected passages of [BS]. An exhaustive treatment occurs in [FF], which is also a good source of examples and applications ({3} is typical).

This section allows for a wide selection of applications for which these techniques are appropriate. Examples of multicommodity flow problems might be given here in order to illustrate the difficulties often encountered in more complex models.

d) Planarity, Kuratowski's theorem, duality, chromatic graphs, matching theory. (6 lectures)

The concepts presented in this section allow the student to become familiar with some slightly more advanced material in graph theory. These can be used to model more complex situations, e.g., {4} and {5} (cf. [Si], [Ben]).

This material may be found in nearly all standard graph theory texts (e.g., [O1], [Be1], [Har1], or briefly in [L]). Typical applications occur in [BS]. Example {4} gives a nice application of some of these subjects (cf. [Si]). These topics are perhaps not so fundamental as the preceding and may be omitted if time pressure is a problem.

3. Combinatorial mathematics (19-22 lectures)

a) Basic tools: permutations, combinations, generating functions, partitions, binomial coefficients, recurrence relations, difference equations, inclusion-exclusion. (10-12 lectures)

The concepts introduced in this section are fundamental and should be part of every applied mathematician's stock in trade. Typical applications of this material are literally too numerous to be singled out. See, e.g., [F], [Rio], [L], [Bec], [Sa], [Kn], [Pe].

Two standard sources are the initial chapters of [F] and [Rio], but these might tax some students a bit. [L] is easier to read but says less. Crisp discussions of most of the material are given in [Ry].

b) Somewhat more advanced material. Systems of distinct representatives, Möbius inversion, theorems of Ramsey type, block designs, Hadamard matrices. (3-4 lectures)

It is important for the student to see models which use somewhat more sophisticated concepts from combinatorial mathematics. Good examples of this are the studies of the dimer problem and the Ising model presented in [Pe] and the analysis of telephone switching networks in [Ben]. The topics listed in this section serve to introduce the student to more advanced ideas. (Of course, other similar topics listed in the available references may be substituted at the discretion of the instructor.) These subjects are covered adequately (although perhaps somewhat disjointly) in [Hal]. The treatment in [Ry] would be suitable for the better students. The relevant sections of [Hal] are suitable if more emphasis on block designs is desired. Historically, block designs arose primarily in the design of statistical experiments. Recently, these concepts have been useful in a variety of fields, e.g., coding theory [Ber1], spectroscopy [SFP], and data compression. (Also see {5}.)

c) Pólya counting theory: equivalence classes, (permutation) groups, cycle structure, Burnsides' theorem, Pólya's theorem, generalizations. (6 lectures)

Historically, this subject arose from Pólya's work on enumerating chemical isomers ([Po]; see also {6}). Typical applications include enumeration of Boolean functions [Sle] and enumeration of random walks on lattices [Pe]. Other examples are also available in [L], [Be2], [Rio].

[L] is appropriate here if only minimal depth is required. [Bec, Ch.5] gives a more detailed picture. The presentation of [Rio] has a reputation of being somewhat hard to read. Pólya counting theory offers students an opportunity to apply some elementary concepts from group theory to their models. Of course, several additional lectures may be needed to prepare students who have had no exposure to the concept of an equivalence relation or a group. Numerous examples and applications of this material are available, e.g., [Sle], [L], [Be2], [Rio].

It should be kept in mind that the particular choice of models and results presented is not critical. The underlying object here is to develop in the student a feeling for the formulation and analysis of various models using the ideas of combinatorial mathematics.

Many of the topics covered involve techniques for which efficient algorithms are known (e.g., network flows, matching, connectivity, and planarity). It would be quite appropriate for students to implement these algorithms on computers if facilities are available. This very effectively illustrates the savings in time and money achieved by using an efficient algorithm rather than, for example, an enumerative search.

References

In the list of references below, there is no attempt to be exhaustive. Each primary reference is accompanied by a short description and a suggestion whether it is of interest mainly to the instructor (I) or to a student in the k^{th} year of college (S-k). References to <u>Mathematical Reviews</u> (MR) are given.

Primary References

[Bel] Berge, Claude. The Theory of Graphs and its Applications. New York, John Wiley and Sons, Inc., 1962, 247 p. MR21 #1608. (I; S-3,4) This is one of the original standard texts on graph theory. In addition to the standard topics, e.g., chromatic numbers, connectivity, planarity, Hamilton paths, and transport networks, this volume contains several nice chapters dealing with games on graphs and Sprague-Grundy functions. There are no exercises and a moderate selection of examples.

[BS] Busacker, Robert G. and Saaty, Thomas L. Finite Graphs and Networks: An Introduction with Applications. New York, McGraw-Hill Book Company, 1965, 294 p. MR 35 #79. (S-2) This is a nice introduction to the basic topics of graph theory, slanted somewhat toward applications to network theory. A major feature of this book is the 140-page section on applications. They are varied and interesting and include applications to economics and operations research (linear programming and PERT), combinatorial problems, games, communication networks, statistical mechanics, chemistry, genetics, human sciences, group theory, and a number of other subjects. Exercises are included.

[FF] Ford, L. R., Jr. and Fulkerson, D. R. Flows in Networks. Princeton, New Jersey, Princeton University Press, 1962, 194 p. MR 28 #2916. (I) This book, written by two of the principal developers of the field, contains the most complete treatment of network flows. A sampling of the contents includes the max-flow min-cut theorems, the König-Egerváry theorem, sets of distinct representatives, linear programming and duality, Dilworth's theorem, minimal cost flow problems, and 0-1 matrices. There are some examples but no exercises. It is probably more useful as a reference than as a text.

[Hal] Hall, Marshall, Jr. Combinatorial Theory. Waltham, Massachusetts, Blaisdell Publishing Company, Inc., 1967, 310 p. MR 37 #80. (S-3,4) In addition to most of the standard topics, some less common subjects such as Möbius inversion and finite geometries are touched upon. By far the chief emphasis of the book is on block designs, a topic on which the author is well qualified to write. Some exercises and a few examples are contained in the book.

[L] Liu, C. L. Introduction to Combinatorial Mathematics. New York, McGraw-Hill Book Company, 1968, 393 p. MR 38 #3154. (S-1,2) This is a very well-rounded presentation of most of the basic concepts of both graph theory and combinatorial mathematics mentioned in the course outline. Numerous exercises and examples are contained in the book, although it is not particularly strong in mentioning

applications to other fields. However, this defect is offset by an extensive bibliography. The book is on the whole quite readable and could easily serve as a textbook for a freshman-sophomore course.

[O1] Ore, Oystein. Theory of Graphs. Providence, Rhode Island, American Mathematical Society, 1962, 270 p. MR 27 #740. (I; S-4+) This is currently the most mathematical treatment of graph theory available. The subject material ranges widely and includes, e.g., product graphs, Euler paths in infinite graphs, homomorphic images of graphs, the axiom of choice, partial orders, and groups and their graphs. Unfortunately, the density of definitions is rather high (especially at the beginning), and this may discourage many readers. The patient reader will be well-rewarded for his perseverance, however. While numerous examples are included, the primary orientation of the book is toward basic concepts rather than applications of graph theory.

[Rio] Riordan, John. An Introduction to Combinatorial Analysis. New York, John Wiley and Sons, Inc., 1958, 244 p. MR 20 #3077. (I; S-3,4) This is the standard modern text on combinatorial analysis. It contains complete discussions of permutations and combinations, generating functions, inclusion and exclusion, occupancy problems, permutations with restricted positions, and, above all, enumeration (including Pólya theory). There are many exercises and examples at a variety of levels with which the reader may test his skill. An extensive list of references is also included. This is suitable as a text for upper-division and good lower-division students or as a reference for an instructor.

[Ry] Ryser, H. J. Combinatorial Mathematics, MAA Carus Monograph 14. New York, John Wiley and Sons, Inc., 1963, 154 p. MR 27 #51. (I; S) This short book, already considered by many to be a classic in its field, provides the reader with an accurate (although necessarily abbreviated) introduction to some fundamental ideas in combinatorics. The subjects include permutations and combinations, inclusion and exclusion, recurrences, Ramsey's theorem and applications, systems of distinct representatives, 0-1 matrices, orthogonal Latin squares and the Bruck-Ryser theorem, block designs, and perfect difference sets. Each chapter concludes with a number of references. There are few examples and no exercises. The initial sections of the book could be read by freshmen; the latter material would be more suitable for a good junior or senior.

Additional References

Avondo-Bodino, G. Economic Applications of the Theory of Graphs. New York, Gordon and Breach, Science Publishers, Inc., 1962.

[Bec] Beckenbach, E. F., ed. Applied Combinatorial Mathematics. New York, John Wiley and Sons, Inc., 1964. (I; S-3,4) A formidable collection of chapters written by a number of experts in combinatorial fields. Especially good is the section by deBruijn on counting theory.

[Ben] Beneš, V. E. Mathematical Theory of Connecting Networks and Telephone Traffic. New York, Academic Press, Inc., 1965.

[Be2] Berge, Claude. Principles of Combinatorics. New York, Academic Press, Inc., 1971.

Berge, Claude and Chouila-Houri, A. Programming, Games, and Transportation Networks. New York, John Wiley and Sons, Inc., 1965.

[Berl] Berlekamp, E. R. Algebraic Coding Theory. New York, McGraw-Hill Book Company, 1968.

Combinatorial Structures and Their Applications. Proceedings of Calgary International Conference on Combinatorial Structures and Their Applications. New York, Gordon and Breach, Science Publishers, Inc., 1970.

[ER] Erdös, P. and Rényi, A. "On random graphs I." Publicationes Mathematicae, 6 (1959), pp. 290-297.

[F] Feller, W. An Introduction to Probability Theory and Its Applications, vol. 1, 3rd ed. New York, John Wiley and Sons, Inc., 1968.

[GP] Gilbert, E. N. and Pollak, H. O. "Steiner minimal trees." SIAM Journal, 16 (1968), pp. 1-29.

Guillemin, E. A. Introductory Circuit Theory. New York, John Wiley and Sons, Inc., 1953.

[Harl] Harary, F. Graph Theory. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969. A coherent presentation of the basic results in what could be called "exact" graph theory (as opposed to the asymptotic graph-theoretic results of Erdös and others, cf. [ER]).

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Harary, F. Proof Techniques in Graph Theory. New York, Academic Press, Inc., 1969.

Harary, F.; Norman, R. Z.; Cartwright, D. Structural Models: An Introduction to the Theory of Directed Graphs. New York, John Wiley and Sons, Inc., 1965.

Harris, B. ed. Graph Theory and Its Applications. New York, Academic Press, Inc., 1970.

Harrison, M. A. Introduction to Switching and Automata Theory. New York, McGraw-Hill Book Company, 1965.

Kattsoff, L. O. and Simone, A. J. Finite Mathematics with Applications in the Social and Management Sciences. New York, McGraw-Hill Book Company, 1965.

Kemeny, J. G.; Snell, J. L.; Thompson, G. L. Introduction to Finite Mathematics, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

[KL] Kernighan, B. and Lin, S. "A heuristic algorithm for the traveling salesman problem." Operations Research, to appear.

[Kn] Knuth, D. E. The Art of Computer Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., Vol. I (1968), Vol. II (1969), Vol. III (1971). The first two volumes of this (intended) seven-volume series are literally overflowing with a wealth of results, applications, references, and problems having to do with computational (and combinatorial) algorithms (among other things).

König, D. Theorie der endlichen und unendlichen Graphen. New York, Chelsea Publishing Company, 1950.

[Kr] Kruskal, J. B., Jr. "On the shortest spanning subtree of a graph and the traveling salesman problem." Proceedings of the American Mathematical Society, 7 (1956), pp. 48-50.

[Li] Lin, S. "Computer solutions of the traveling salesman problem." Bell System Technical Journal, 44 (1965), pp. 2245-2269.

Proceedings of the Louisiana Conference on Combinatorics, Graph Theory, and Computing. Baton Rouge, Louisiana, Louisiana State University, March, 1970.

MacMahon, P. A. Combinatorial Analysis, vols. I and II reprinted in one volume (originally published in two volumes in 1915 and 1916). New York, Chelsea Publishing Company, 1960. This is the old classical treatise on combinatorial analysis.

Marcus M. A Survey of Finite Mathematics. Boston, Massachusetts, Houghton Mifflin Company, 1969.

Milne-Thomson, L. M. The Calculus of Finite Differences. New York, St. Martin's Press, Inc., 1951.

Minty, G. J. "On the axiomatic foundations of the theories of directed linear graphs, electrical networks, and network-programming." J. Math. and Mech., 15 (1966), pp. 485-520.

Moon, J. Topics on Tournaments. New York, Holt, Rinehart and Winston, Inc., 1968.

Noble, B., ed. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967 (especially Ch. 2 and Sect. 4.6).

International Conference on Combinatorial Mathematics, proceedings. New York Academy of Sciences, July 31, 1970.

Recent Trends in Graph Theory. Proceedings of the First New York City Graph Theory Conference. Jamaica, New York, St. John's University, June, 1970.

[O2] Ore, Oystein. Graphs and Their Uses. New York, Random House, Inc., 1963. A good reference for graph theory as applied to electrical network theory.

[Pe] Percus, J. K. Combinatorial Methods. New York University, Courant Institute of Mathematical Sciences, 1969. An excellent reference for the application of combinatorial techniques to some of the classical lattice statistics problems of physics, e.g., the dimer problem and the Ising model.

[Po] Pólya, G. "Sur le nombre des isomètres de certains composes chimique." C. R. Acad. Sciences, 202 (1936), p. 1554.

Richardson, W. H. Finite Mathematics. New York, Harper and Row, Publishers, 1968.

Rota, G.-C. "On the foundations of combinatorial theory, I." Theory of Möbius Functions. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 2 (1964), pp. 340-368.

[Sa] Saaty, T. L. Optimization in Integers and Related Extremal Problems. New York, McGraw-Hill Book Company, 1970. A recent book which contains a wide variety of interesting results and examples from optimization theory, many of which are well in the spirit of this course.

[SR] Seshu, S. and Reed, M. B. Linear Graphs and Electrical Networks. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

[Si] Sinden, F. W. "Topology of thin film RC circuits." Bell Systems Technical Journal, 45 (1966), pp. 1639-1662.

[Sle] Slepian, D. "On the number of symmetry types of Boolean functions of n variables." Can. J. Math., 5 (1953), pp. 185-193.

[SFP] Sloane, N. J. A.; Fine, Terrence; Phillips, P. G. "New methods for grating spectrometers." Optical Spectra, 4 (1970), pp. 50-53.

Whitworth, W. A. Choice and Chance, reprint of the 5th ed. (1901). New York, Hafner Publishing Company, Inc., 1965.

Appendix

1. Organization X has offices located in a number of cities. It wishes to establish a communication network among all its locations so that any two offices may communicate with one another, possibly by going through some of the other locations. Furthermore, it is desired to minimize the total length (cost) of the network. How should the cities be connected? If one is allowed to locate switching junctions arbitrarily rather than just at the office locations, then how can a minimal network be obtained? (See [Kr] and [GP].)

2. A (traveling) salesman has a fixed set of locations (farm houses) that he is required to visit. He leaves from his home office, travels to each location once in some order, and then returns. In what order should he visit the locations in order to minimize his total distance, cost, time (energy)? (See [Li] and [KL].)

3. An oil company has a number of oil wells (sources) and a number of refineries, customers, etc. (sinks), all connected by some intricate network of pipelines. The portions of pipeline between various points of the network have different (known) capacities. How can one route the oil through the system in order to maximize the flow of oil to the sinks? What if the direction of flow in certain pipelines is restricted? What if there are several grades of oil available in varying amounts from the sources and it is desired to maximize the value of the mixture received at the sinks? (See [FF].)

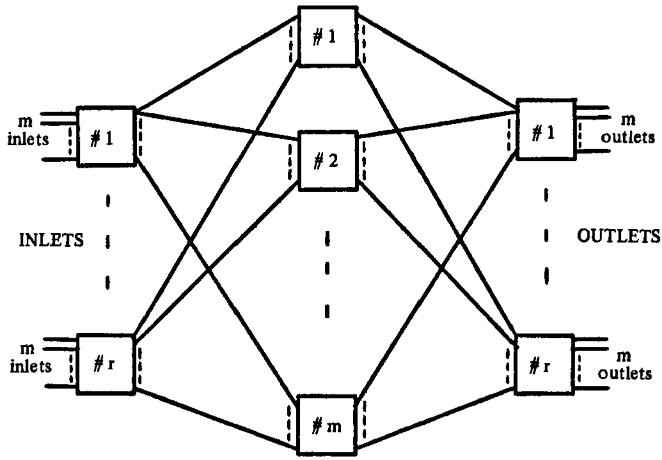
4. Certain integrated circuits can be made by depositing very thin metallic and dielectric films in suitable patterns on an insulating substrate. Ordinarily printed circuits are strictly planar; crossovers are made only by leading one of the conductors entirely out of the plane of the circuit. In the thin film technique, however, conductors can be separated by thin insulating layers within the plane of the circuit, causing a nonzero capacitance between the crossing conductors. Thus, crossovers can be permitted, provided this nonzero capacitance between the crossing conductors is permitted. The general problem is to determine which circuits can be realized by some suitable thin film circuit. This leads to a number of interesting questions in graph theory, one of which is the following: Given a set $S = \{s_1, \ldots, s_n\}$ of arcs or "strings," what are necessary and sufficient conditions on a set P of pairs $\{s_i, s_j\}$ so that there is some configuration of the s_k <u>in the plane</u> for which s_i and s_j intersect if and only if $\{s_i, s_j\}$ belongs to P? (See [Si].)

5. The Hall theorem on systems of distinct representatives occurs in a variety of applications. Several of these are:

a) In a certain company, n employees are available to fill n positions, each employee being qualified to fill one or more of these jobs. When can each employee be assigned to a job for which he is qualified? (See [Bel].)

b. An $m \times n$ chessboard has a certain subset of its squares cut out. When is it possible to place a collection of 2×1 "dominoes" on this board so that each of its squares is covered exactly once? (See [Pe].)

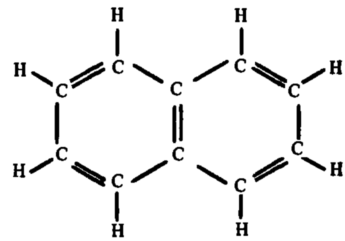
c) A telephone switching network connecting mr inlets with mr outlets is made up of three stages as indicated in the figure. (See [Ben].)



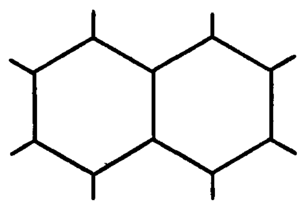
Each square box represents a switching unit for which any of the possible permutations of connecting its local inlets to its local outlets is possible. The problem is to show that this network is rearrangeable, i.e., given any set of calls in progress and any pair of idle terminals, the existing calls can be reassigned new routes (if necessary) so as to make it possible to connect the idle pair. How is the reassignment made so as to change the minimum number of existing calls? (cf. [Ben].)

d) If there are as many r-element subsets of an n-element set as there are k-element subsets, then it is possible to associate with each k-element subset a distinct r-element subset which contains it. How?

6. A naphthalene molecule $C_{10}H_8$ (See figure on next page.)



contains 8 hydrogen atoms which are available for substitution. The symmetry group G of the underlying figure



has order 4 and consists of the identity and three rotations about axes which are horizontal, vertical, or coming out of the page. Think of this group as just permuting the eight hydrogen atoms. The identity fixes them all and has cycle index S_1^8; each of the other three permutations moves them in four pairs of two each and contributes to the cycle index S_2^4. The cycle index of the group (considered as acting only on the hydrogen atoms) is thus

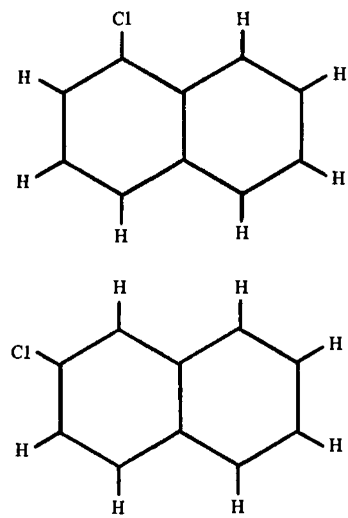
$$P_G(S_1,S_2) = \frac{1}{4}\,(S_1^8 + 3S_2^4).$$

Now suppose we replace k of the hydrogen atoms by chlorine atoms and r of the hydrogen atoms by bromine atoms. How many different molecules can be formed? This is exactly the kind of question that Pólya's theorem answers.

Answer: In $P_G(S_1,S_2)$ replace S_1 by $1 + x + y$ and S_2 by $1 + x^2 + y^2$. Then the coefficient of $x^k y^r$ is the desired number. In fact, after making the substitution, we have

$$\frac{1}{4}[(1 + x + y)^8 + 3(1 + x^2 + y^2)^4] = 1 + 2x + 2y + 10x^2 + 14xy + 10y^2 + \ldots .$$

Each term in this series can be interpreted: 1 corresponds to the original molecule (no substitution); 2x corresponds to



(substituting one Cl for an H), etc. (For the notation used here, see [L] and [Bec]; problems of this type occur in [Pe].)

IX.3 FLUID MECHANICS OPTION

The course described below is based on a view of applied mathematics as a natural science distinguished from other natural sciences by a mathematical content that is significant in its own right. Fluid mechanics was chosen because it exemplifies applied mathematics in this sense: it is important historically, it encompasses many interesting physical problems, and it can be taught in the spirit of this report. However, to teach such a course at the undergraduate level requires special care in order to avoid the two possible extremes of, on the one hand, pursuing mathematical topics for their own sake and, on the other hand, studying physical models which involve only trivial mathematical ideas.

The approach to the subject proposed here has been selected with the audience and our objectives in view. Although this material can be taught from a more modern perspective, it would then require more sophisticated mathematical techniques and would be feasible only with very well prepared undergraduates. Our approach was selected because we feel that it is accessible to a wide audience and because it effectively attains our goals.

The course is intended for seniors. Prerequisites are elementary courses in calculus, differential equations, linear algebra, and physics. A course in advanced calculus or analysis is desirable. The student should be familiar, for instance, with the mathematical issues involved in the termwise integration of infinite series. This course should be valuable in solidifying and extending the student's grasp of areas of analysis and differential equations. The course outlined here does <u>not</u> assume prior knowledge of complex analysis, partial differential equations, or fluid mechanics.

A potential instructor of this course is faced with issues not present in the preceding outlines. It requires more specialized knowledge and would most easily be offered by someone with a background in applied mathematics. Nevertheless, we feel that the present outline is sufficiently detailed so that it can serve as a guide to instructors and so that it can encourage teachers to experiment with courses in this area. The main point which the instructor must keep in mind is that this is to be a course about applied mathematics using fluid mechanics as its representative element; it is not a course on fluid mechanics alone.

The main needs of the instructor, in addition to mathematics, are a basic knowledge of classical physics, a willingness to read, and perhaps above all an interest in nature. Those who are not specialists in fluid mechanics will find it particularly important to read this outline with one of the references at hand. While there are many books on fluid mechanics, there are very few which emphasize the point of view which the Panel has taken here. A list of books which may be helpful to the teacher is given in section 5 below, with brief comments. References to specific sections of some of these books are given for each of the topics in the detailed outline in section 4. Unfortunately, there seems to be no book which would be completely satisfactory as a text for this kind of a course; the book by Prandtl, which includes most of the topics in the outline discussed clearly from the physical point of view, is perhaps the most appropriate. But the mathematical side of many of these discussions will require appreciable expansion for the purposes of this course; for digressions of a more purely mathematical nature which will from time to time be appropriate, one can perhaps rely on the general mathematical background of the instructor.

COURSE OUTLINE

This course has two main parts, the first of a fairly general nature concerned with the mathematical formulation of continuum models for fluids and the second dealing with more specific problems illustrative of the more important simplified models. In the outline each part is broken down into several areas, for each of which some remarks in the style of a "catalog description," with some suggestions on the general approach, are given. These remarks are followed by a list of topics for each of several lectures on this area, with attention drawn to specific sections of the references in which a treatment of these topics is given. For definiteness, these specific references have been restricted mainly to the books of Prandtl and Yih. The format of the references is indicated by this example: [P] II:1.1 means section 1.1 in the second chapter of Prandtl's book.

1. Continuum models for fluids

This part of the course concerns primarily the formulation and basic properties of the principal mathematical models used in fluid mechanics. Here one can well emphasize the central role of model building in applied mathematics and the importance of models which are both mathematically self-consistent and capable of being critically compared with the experimental or observational facts which they are supposed to describe. Fluid mechanics is a particularly good example to illustrate that a mathematical model can be very helpful even though it is in a sense definitely incorrect (e.g., the molecular structure of matter is completely missing from continuum models) and that in reality all theoretical science is done in terms of models, none of which should be assigned any absolute validity.

a) The concept of continuous matter as a useful macroscopic model of real matter. (2 lectures) Mass and density. Kinematics: velocity field and the idea of a "fluid particle" as a theoretical concept in the continuum model, not the same thing at all as a molecule. "Eulerian" and "Lagrangian" variables and the mathematical form of the continuum model. The continuity equation.

Mathematical ideas: flow as a continuous mapping, Jacobians in the transformation of multiple integrals, the divergence theorem. One might well emphasize here the reverse of the familiar physical "proof" of the divergence theorem--the mathematical theorem shows that the continuum model is in accord with our intuitive ideas about the continuity of matter.

References. Continuum models for fluids; mass, density, "fluid particle," velocity, acceleration: [Y] I:1,2,3,4 and [P] II:1.1. Continuity equation, divergence theorem, and Jacobians: [Y] I:5,6 and [P] II:1.2. More kinematics: [Y] I:7,8,9.

b) Dynamics. (4 lectures) Introduction to the basic ideas from particle mechanics (momentum, force, kinetic energy) into the continuum model. Pressure and stress. Stress tensor and the momentum equation. Mechanical energy equation. Angular momentum and symmetry of the stress tensor in the absence of body torques and "torque-stresses."

Mathematical ideas: divergence theorem again, with more vector calculus. Tensors as geometric objects. Components of a symmetric second-order tensor form a symmetric matrix, hence have real eigenvalues and an orthonormal basis of eigenvectors (principal stresses).

References. Pressure and stress: [P] I:1,2,3. Concept of a tensor as a geometric object and its representation by components; stress tensor of a continuum: [Y] I:10. Yih's rather classical and formalistic view of tensors might well be given a more geometrical and contemporary flavor. Symmetrical second-order tensors, in particular stress and deformation tensors, relationship between a second-order tensor and the matrix of its components; application of linear algebra; principal stresses and eigenvalues, principal directions, etc.: [Y] I:11,12. Navier-Stokes form of the stress tensor and the corresponding fluid equations of motion: [Y] I:13 and [Y] II:1,2. Conservative body forces and the mechanical energy equation, vorticity equation: [Y] II:3,4.

c) Thermodynamics. (3 lectures) The equation of state. Internal energy, heat, and entropy. Heat conduction and the total energy equation.

In the absence of sufficient background in physics, this part may have to be limited mainly to equations of state in the simplest cases: incompressible fluids and the isothermal and adiabatic ideal gas. However, thermodynamics, where accessible, provides a good source of exercises in changing variables, Jacobians, etc., and also often illustrates rather well the advantages of a careful mathematical formulation over a loose intuitive description.

At this point various examples of hydrostatics problems can conveniently be introduced. Two important points to be emphasized here (and throughout the course in other contexts) are: i) Hydrostatics is a "simplified model," relevant not only when there is strictly no motion but also a good approximation in appropriate circumstances (vertical accelerations small compared with that of gravity). One can introduce here the idea of simplifying the model on the basis of the smallness of certain dimensionless parameters characteristic of the particular case in hand. ii) By discussing some problems related to familiar situations, one can help the student to form the habit of using mathematics to enhance his perception of nature. For example, the hydrostatics of the isothermal and adiabatic atmospheres can answer questions like: Is it plausible that oxygen should be needed when climbing Mt. Everest? or How much colder is it likely to be on the top of some local peak than it is at ground level?

Mathematical ideas: in addition to Jacobians, etc., some simple ordinary differential equations.

References. Review of thermodynamics: [Y] appendix I, possibly truncated and treated in a mathematically more sophisticated manner. Ideal gases: [Y] appendix I and [P] I:5. Heat or energy equation: [Y] II:8. Applications from hydrostatics: [P] I:6,7,10.

2. The more simplified models

Geometrical or physical parameters needed to specify a problem completely lead to characteristic dimensionless parameters (e.g., Mach number, Reynolds number) whose smallness or largeness in particular cases indicate the usefulness of simplified models (e.g., incompressible or inviscid flow). In the discussion of simplified models, emphasis should be shared between their general properties (e.g., Kelvin's circulation theorem) and careful consideration of the extent to which the simplified model is in fact relevant. In particular, the prevalence of nonuniform convergence in going over to the simplified model and the kinds of additional considerations required in the regions of nonuniformity ("boundary layers") should be brought out, at least qualitatively. In assessing relevance, it is probably best to include with the general discussion a number of applications of the basic models to concrete situations. Simple and familiar cases which emphasize the two points mentioned under 1c) in connection with hydrostatics should be considered where possible.

a) Ideal irrotational flow and surface waves on water. (5 lectures)

Here there are a number of opportunities for introducing important mathematical ideas and techniques, for instance: i) some general properties of harmonic functions; ii) solution of boundary value problems for Laplace's equation by superposition of wave solutions (i.e., "separation of variables" or use of Fourier representations); iii) free waves--phase and group velocity; iv) forced waves, e.g., the linear wave-maker problem (radiation condition at infinity, Sturm-Liouville equations, and eigenfunction expansions for boundary value problems).

If the students have not seen a proof of the Fourier series theorem, the instructor might like to insert a lecture on this topic, proving the theorem for piecewise continuously differentiable periodic functions.

References. General properties of the inviscid flow model, dimensionless form of general equations, and inviscid flow as idealization for large Reynolds number: [Y] II:5,6. Kelvin circulation theorem: [Y] III:1,2 and [P] II:2.8. Bernoulli theorems: [Y] III:8,9 and [P] II:2.3,4. Ideal irrotational flow: [Y] IV:1,2,3,4 and [P] II:2.9. Surface waves: [P] II:2.15 and [Y] V:1,2.1,2.2,2.3. Standing waves and group velocity: [Y] V:2.4,2.5,2.6,2.11. The wave-maker problem: [Y] V:2.13.

b) Linear shallow-water theory. (4 lectures)

This provides another simplified model and gives opportunity for further discussion of Sturm-Liouville eigenvalue problems. The relationship with variational techniques can be brought out here in the estimation with trial functions or comparison theorems of the resonant frequencies of soup bowls, swimming pools, harbors, and lakes.

Some properties of the wave equation, for instance the significance of characteristics, can also be included. (Nonlinear shallow-water theory, its analogy with compressible flow and shock waves might be discussed, but probably there will not be sufficient time for this.)

References. Linear shallow-water theory: [L] pp. 169-72, 189. Forced motion and normal modes: [L] pp. 176-79. Tides: [L] pp. 180-82, 198-200. Variable depth or section: [L] pp. 185-86, 191-93.

c) Ideal flow past bodies. (5 lectures)

Flow past circles and spheres gives simple problems in potential theory which can be tied in with Fourier series and spherical harmonics, notably by considering flow past near-circles or near-spheres; ideas of regular perturbation theory enter here as well.

D'Alembert's "paradox" provides a striking example of the failure of a simplified model when interpreted too literally, combined with its rescue and continued usefulness when the main source of the difficulty (flow separation) is identified and appropriately modeled. The elementary theory of airfoils and drag estimates via dynamic pressure arguments could be discussed with questions like: Why do sailplanes have very long slender wings? How big should a parachute be? How much air resistance is a car subject to?

References. Examples: [Y] IV:7.4, [P] II:2.9, and [Y] IV:18. Flow past a near-sphere: [Y] IV:13, possibly generalized and with further discussion of spherical harmonics. (Yih's discussion is perhaps too brief and formalistic, and the fact that surface harmonics are to spheres what sines and cosines are to circles is rather obscured.) Perhaps another mathematical digression could be added here: students are too often so put off by excessive emphasis on associated Legendre functions that they never seem to realize that the rotation group is behind it all. Two-dimensional flows with circulation: [P] II:10. Blasius and Kutta-Joukowski theorems: [Y] IV:22,22.1. Airfoils: [P] III:16,17.

d) Inviscid flow with vorticity. (3 lectures)

Some interesting phenomena of this sort can be studied without too much complication by considering linearized flow in rotating systems. Unfortunately, a more complete picture of the applications of hydrodynamic theory in meteorology and oceanography probably involves too many other considerations to be feasible in this course.

References. Geostropic flow and the Taylor-Proudman theorem: [G] I and [Y] III:12,12.1,12.2. Effect of the earth's rotation on atmospheric and oceanic flows: [P] V:8. Motion of parallel rectilinear vortex filaments: [L] pp. 154-55.

e) Viscosity. (4 lectures)

References. Couette and Poiseuille flow: [Y] VII:2.1,2.2,2.3, 2.10,2.11 and [P] III:1.9b. Ekman and Stokes-Rayleigh boundary layers: [Y] VII:2.12,3.4; [P] V:9; and [G] 2.3. Secondary flow: [P] III:8, [G] 2.4, and [P] V:9. The boundary layer: [P] III:3 and [Y] VII:5,6.

f) Instabilities. (2 lectures)

(Why does water run out of an inverted glass even though the atmospheric pressure can support the weight of a 30-foot column of water--and why does it not similarly run out of a narrow tube?) Kelvin-Helmholtz instability, although an over-simplified model, can be related to wave generation by wind.

References. Gravitational instability: [Y] IX:2.1,2.3 and [P] V:16. Kelvin-Helmholtz instability: [Y] IX:8.1, [L] p. 232, [Y] IX:6.1.

References

The books referred to in the outline are:

[G] Greenspan, H. P. The Theory of Rotating Fluids. New York, Cambridge University Press, 1968. A rather advanced text or research monograph.

[L] Lamb, H. Hydrodynamics. New York, Dover Publications, Inc., 1945 (reprint). The classic work in the field. Its rather old-fashioned mathematical style and extensive character,

combined with a certain tendency to present important results without adequate identification, make it sometimes rather difficult for the novice. As with some other classics, results given almost parenthetically in Lamb continue occasionally to be rediscovered (and published!).

[P] Prandtl, L. Essentials of Fluid Dynamics. New York, Hafner Publishing Company, 1952. An excellent and interesting book from the physical point of view, with clear discussions of many scientific and engineering applications. Most of the less elementary mathematical aspects, however, have (intentionally) been left aside.

[Y] Yih, Chia-Shun. Fluid Mechanics. New York, McGraw-Hill Book Company, 1969. A good graduate-level textbook, emphasizing the theoretical side. Quite a few exercises.

Some other books which the instructor may find useful to have on hand are listed below.

Additional References

Batchelor, G. K. An Introduction to Fluid Mechanics. New York, Cambridge University Press, 1967.

Landau, L. D. and Lifschitz, E. Fluid Mechanics. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1959.

Rouse, H. Fluid Mechanics for Hydraulic Engineers. New York, McGraw-Hill Book Company, 1938.

Von Arx, W. S. Introduction to Physical Oceanography. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

The books by Batchelor and by Landau and Lifschitz are both good; Landau and Lifschitz is written perhaps more from the physicist's point of view, Batchelor from the applied mathematician's.

Also, a good engineering text such as the book by Rouse and, in connection with 2d), the book by Von Arx, may be found helpful.

There are a number of interesting 8mm. film strips on topics in fluid mechanics, as well as some longer films, prepared by the National Committee for Fluid Mechanics Films and available from Encyclopaedia Brittanica Films. They do not on the whole contribute much on the mathematical side but may well add interest and appreciation for the physics. Some which might be found useful in connection with the course outlined above are:

FM-3: Shear Deformation of Viscous Fluids [continuity equation]

FM-14A and B: Visualization of Vorticity with Vorticity Meter [continuity equation, conservative body forces and the mechanical energy equation, vorticity equation]

FM-13: The Bathtub Vortex [general properties of the inviscid flow model, two-dimensional flows with circulation, effect of the earth's rotation on atmospheric and oceanic flows]

FM-10: Generation of Circulation and Lift for an Airfoil [airfoils]

FM-11: The Magnus Effect [secondary flow]

FM-6: Boundary Layer Formation [the boundary layer, Ekman and Stokes-Rayleigh boundary layers]

FM-31: Instabilities in Circular Couette Flow [instabilities, Couette and Poiseuille flows]

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