MATHEMATICAL ASSOCIATION OF AMERICA

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

A COMPENDIUM OF CUPM RECOMMENDATIONS

VOLUME I

A COMPENDIUM OF CUPM RECOMMENDATIONS

STUDIES

DISCUSSIONS

and

RECOMMENDATIONS

by the

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

of the

MATHEMATICAL ASSOCIATION OF AMERICA

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PREFACE

The Committee on the Undergraduate Program in Mathematics (CUPM) was established as a standing committee of the Mathematical Association of America (MAA) in 1959.* With financial assistance from the National Science Foundation, CUPM in 1960 began to engage in several projects and activities related to improvement in the undergraduate curriculum. These projects often involved the publication of reports, which were widely disseminated throughout the mathematical community and were available from the CUPM Central Office upon request. Since a change in the funding policy of the United States government makes the continuing production and free distribution of such reports extremely unlikely, the MAA has decided to publish in permanent form the most recent versions of many of the CUPM recommendations so that these reports may continue to be readily available to the mathematical community and may conveniently be kept on the reference shelves of mathematics libraries.

This COMPENDIUM is published in two volumes, each of which has been divided into sections according to the category of reports contained therein. These CUPM documents were produced by the cooperative efforts of literally several hundred mathematicians in the United States and Canada. The reports are reprinted here in essentially their original form; there are a few editorial comments which serve to update or cross-reference some of the materials.

The editorial work for the COMPENDIUM was started by William E. Mastrocola during his term as Director of CUPM and completed after his return to Colgate University. He was assisted in the early stages by Andrew Sterrett and Paul Knopp, Executive Directors of CUPM during 1972 and 1973. Preparation of the final manuscript for the printer was the joint work of William E. Mastrocola and Katherine B. Magann. The considerable efforts of these individuals is deserving of special recognition.

* A detailed history of CUPM can be found in an article by W. L. Duren which appeared in the American Mathematical Monthly, vol. 74, pp. 23-37.

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January 1965

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INTRODUCTION

One of the many channels by which the Mathematical Association of America offers advice and guidance to colleges is the Committee on the Undergraduate Program in Mathematics. A project of this Committee has been an attempt to define a minimal college mathematics library. Preliminary versions of the accompanying list have been used to improve mathematics libraries.

This list of some 300 books, from which approximately 170 are to be chosen to form a basic library in undergraduate mathematics, is intended to do the following:

1. Provide the student with introductory material in various fields of mathematics which he may not previously have encountered

2. Provide the student, whose interest has been aroused by his teachers, with reading material collateral to his course work

3. Provide the student with reading at a level beyond that ordinarily encountered in his undergraduate curriculum

4. Provide the faculty with reference material

5. Provide the general reader with elementary material in the field of mathematics

The list is minimal and is not intended to provide anyone with the grounds of an argument that a particular library is complete, and hence cannot be improved. On the contrary, the list is _basic_ in that it provides a nucleus for a library whose further acquisitions should be dictated by student and faculty interests. There has been a concerted effort to keep the list small, in the exercise of which many books of merit have had to be excluded; several equally attractive areas sometimes have been combined into one group from which one book is to be selected. In many cases similar books are suggested as alternate choices so that a library may exploit its present holdings.

The Advisory Group on Communications of CUPM has prepared this list over a period ending in 1964; hence, recently published books do not appear on the list.

BASIC LIBRARY LIST

I. Background and Orientation

The volumes listed here offer a variety of topics which must have representation in any basic library. Of the three books on the history of mathematics, _Men of Mathematics_ can be read with enjoyment by students at any level. Equally readable are What is Mathematics?, Number, the Language of Science, and The Enjoyment of Mathematics. Symmetry, An Introduction to Mathematics, and Mathematical Snapshots are well-known classics, while the books on finite mathematics (1.10) bring numerous modern topics to the freshman level.

1.1 Bell, Eric T. Development of Mathematics, 2nd ed. New York, McGraw-Hill Book Company, 1945.

1.2 Bell, Eric T. Men of Mathematics. New York, Simon and Schuster, Inc., 1937.

1.3 Courant, R. and Robbins, H. What is Mathematics? New York, Oxford University Press, Inc., 1941.

1.4 Dantzig, Tobias. Number, The Language of Science, 4th rev. and augm. ed. New York, The Macmillan Company, 1954; New York, Doubleday and Company, 1956.

1.5 Rademacher, Hans and Toeplitz, Otto. The Enjoyment of Mathematics. (translated by H. Zuckerman) Princeton, New Jersey, Princeton University Press, 1957.

1.6 Steinhaus, H. Mathematical Snapshots, 2nd ed., rev. and enl. New York, Oxford University Press, Inc., 1960.

1.7 Struik, Dirk Jan. A Concise History of Mathematics, 2nd rev. ed. New York, Dover Publications, Inc., 1948.

1.8 Weyl, Hermann. Symmetry. Princeton, New Jersey, Princeton University Press, 1952.

1.9 Whitehead, Alfred North. An Introduction to Mathematics, rev. ed. New York, Oxford University Press, Inc., 1959.

1.10 At least one of the following: (a-c)

1.10a Kemeny, John G.; Snell, J. Laurie; Thompson, Gerald L. Introduction to Finite Mathematics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1957.

1.10b Kemeny, John G.; Mirkil, H.; Snell, J. Laurie; Thompson, Gerald L. Finite Mathematical Structures. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1959.

1.10c Kemeny, John G.; Snell, J. Laurie; Thompson, Gerald L.; Schleifer, Arthur. Finite Mathematics with Business Applications. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

1.11 At least one of the following: (a-b)

1.11a James, Glenn and Robert C., eds. Mathematical Dictionary. New York, Van Nostrand Reinhold Company, 1959.

1.11b Karush, William. The Crescent Dictionary of Mathematics. New York, The Macmillan Company, 1962.

II. Algebra

For reference and for systematic study, a basic library should contain general treatments of abstract algebra at successive levels (2.15, 2.7, 2.2, 2.4, 2.9). Because of the tremendous importance of the basic structures, models, and tools of linear algebra, there should be introductions emphasizing linear transformations (2.11) and also emphasizing matrices (2.10). For the casual reader there should be attractive elementary approaches to modern algebra via special topics such as groups (2.16), rings (2.6), and other subjects (2.5). For the serious student there should be more advanced works in a few key special fields, e.g., group theory (2.17), linear algebra (2.12, 2.13), fields and Galois theory (2.1). The uniquely useful book 2.3 provides for a transition from linear algebra towards the theory of Hilbert space. Connections between linear algebra and geometry deserve attention (2.14).

2.1 Artin, Emil. Galois Theory, 2nd rev. ed. (edited by Arthur Milgram) Notre Dame, Indiana, University of Notre Dame Press, 1946.

2.2 Birkhoff, Garrett and MacLane, Saunders. A Survey of Modern Algebra, rev. ed. New York, The Macmillan Company, 1965.

2.3 Halmos, Paul R. Finite-dimensional Vector Spaces, 2nd ed. New York, Van Nostrand Reinhold Company, 1958.

2.4 Herstein, I. N. Topics in Algebra. New York, Blaisdell Publishing Company, 1963.

2.5 MAA Studies in Mathematics, vol. II. Studies in Algebra. (edited by A. A. Albert) Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1963.

2.6 McCoy, Neal H. Rings and Ideals (Carus Monograph No. 8). Chicago, Illinois, The Open Court Publishing Company, 1948.

2.7 Mostow, George D.; Sampson, J. H.; Meyer, J. P. Fundamental Structures of Algebra. New York, McGraw-Hill Book Company, 1963.

2.8 Uspensky, J. V. Theory of Equations. New York, McGraw-Hill Book Company, 1948.

2.9 At least one of the following: (a-b)

2.9a Jacobson, Nathan. Lectures in Abstract Algebra, vols. I, II, III. New York, Van Nostrand Reinhold

Company. Vol. I, Basic Concepts, 1951; Vol. II, Linear Algebra, 1953; Vol. III, Theory of Fields and Galois Theory, 1964.

2.9b van der Waerden, Bartel L. Modern Algebra, vols. I, II. (translated by Fred Blum) New York, Frederick Ungar Publishing Company. Vol. I, rev. ed., 1953; Vol. II, 1950.

2.10 At least one of the following: (a-e)

2.10a Aitken, Alexander C. Determinants and Matrices, 8th ed. New York, Interscience, 1956.

2.10b Hohn, Franz Edward. Elementary Matrix Algebra, 2nd ed. New York, The Macmillan Company, 1964.

2.10c MacDuffee, Cyrus C. Vectors and Matrices (Carus Monograph No. 7). Chicago, Illinois, The Open Court Publishing Company, 1943.

2.10d Murdoch, D. C. Linear Algebra for Undergraduates. New York, John Wiley and Sons, Inc., 1957.

2.10e Perlis, Sam. Theory of Matrices. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1952.

2.11 At least one of the following: (a-e)

2.11a Curtis, C. Linear Algebra: An Introductory Approach. Boston, Massachusetts, Allyn and Bacon, Inc., 1963.

2.11b Finkbeiner, Daniel T. Introduction to Matrices and Linear Transformations. San Francisco, California, W. H. Freeman and Company, 1960.

2.11c Shields, Paul C. Linear Algebra. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1964.

2.11d Paige, Lowell J. and Swift, J. Dean. Elements of Linear Algebra. New York, Blaisdell Publishing Company, 1961.

2.11e Stewart, Frank Moore. Introduction to Linear Algebra. New York, Van Nostrand Reinhold Company, 1963.

2.12 At least one of the following: (a-d)

2.12a Hoffman, Kenneth and Kunze, Ray. Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961.

2.12b Nering, Evar Dave. Linear Algebra and Matrix Theory. New York, Interscience, 1963.

2.12c Stoll, Robert Roth. Linear Algebra and Matrix Theory. McGraw-Hill Book Company, 1952.

2.12d Thrall, Robert McDowell and Tornheim, L. Vector Spaces and Matrices. New York, John Wiley and Sons, Inc., 1957.

2.13 At least one of the following: (a-c)

2.13a Gantmakher, Feliks R. Theory of Matrices, vols. I, II. New York, Chelsea Publishing Company, Inc., 1959.

2.13b Mal'cev, A. I. Foundations of Linear Algebra. (translated from the Russian by T. C. Brown, edited by J. B. Roberts) San Francisco, California, W. H. Freeman and Company, 1963.

2.13c Varga, Richard S. Matrix Iterative Analysis. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

2.14 At least one of the following: (a-c)

2.14a Jaeger, Arno. Introduction to Analytic Geometry and Linear Algebra. New York, Holt, Rinehart and Winston, Inc., 1960.

2.14b Kuiper, N. H. Linear Algebra and Geometry. (translated from the Dutch edition) New York, Interscience, 1962.

2.14c Rosenbaum, R. A. Introduction to Projective Geometry and Modern Algebra. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

2.15 At least one of the following: (a-c)

2.15a Johnson, Richard Edward. First Course in Abstract Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1953.

2.15b McCoy, Neal H. Introduction to Modern Algebra. Boston, Massachusetts, Allyn and Bacon, Inc., 1960.

2.15c Weiss, Marie J. Higher Algebra for the Undergraduate, 2nd ed. (revised by Roy Dubisch) New York, John Wiley and Sons, Inc., 1962.

2.16 At least one of the following: (a-b)

2.16a Alexandroff, P. S. An Introduction to the Theory of Groups. (translated by Hazel Perfect and J. M. Petersen) New York, Hafner Publishing Company, 1959.

2.16b Ledermann, Walter. Introduction to the Theory of Finite Groups. New York, Interscience, 1953.

2.17 At least one of the following: (a-c)

2.17a Hall, Marshall, Jr. The Theory of Groups. New York, The Macmillan Company, 1961.

2.17b Kurosh, A. G. The Theory of Groups, vols. I, II, 2nd ed. (translated from the Russian and edited by K. A. Hirsch) New York, Chelsea Publishing Company, Inc., 1960.

2.17c Zassenhaus, Hans J. The Theory of Groups, 2nd ed. (translated by S. Kravetz) New York, Chelsea Publishing Company, Inc., 1958.

III. Analysis

Analysis covers a broad spectrum of mathematical disciplines. This section contains a selection of books which may serve to introduce the mathematics undergraduate to many of these disciplines.

In those areas in which undergraduate courses are usually offered, books of mathematical depth and sophistication are recommended. Thus, for advanced calculus, or what is rapidly being renamed real analysis, we list 3.25, 3,26, and 3.27; the last all contain elements of Lebesgue integration. In addition, we recommend the now classic 3.4, 3.6. Interesting and unusual presentations of material in this general area occur in 3.11 and 3.15a.

The elements of ordinary differential equations appear in 3.20. More advanced treatments are contained in 3.21 and 3.22; the former have excellent material on boundary value problems while the latter stress the geometrical and qualitative aspects of differential equations. An excellent problem source is 3.3.

Presentations of the theory of functions of a complex variable are to be found in 3.13, 3.23, and 3.24. Introductions to topics in the theory of linear spaces and functional analysis are contained in 3.10, 3.15b, 3.16, among others. In 3.17 two distinct elementary treatments of generalized functions are listed. Finally, attention is called to the note on calculus books which is at the end of this section.

3.1 Bliss, Gilbert A. Calculus of Variations (Carus Monograph No. 1). Chicago, Illinois, The Open Court Publishing Company, 1925.

3.2 Boas, Ralph P., Jr. A Primer of Real Functions (Carus Monograph No. 13). New York, John Wiley and Sons, Inc., 1960.

3.3 Brenner, Joel Lee. Problems in Differential Equations. San Francisco, California, W. H. Freeman and Company, 1963.

3.4 Courant, R. Differential and Integral Calculus, vols. I, II. (translated by E. J. McShane) New York, Interscience. Vol. I, 2nd ed. rev., 1937; Vol. II, 1st ed., 1936.

3.5 Flanders, Harley. Differential Forms, with Applications to the Physical Sciences. New York, Academic Press, Inc., 1963.

3.6 Hardy, Godfrey H. Pure Mathematics. New York, Cambridge University Press, 1959.

3.7 Knopp, Konrad. Elements of the General Theory of Analytic Functions, 1st American ed. (translated by F. Bagemihl) New York, Dover Publications, Inc., 1952.

3.8 Knopp, Konrad. Problem Book in the Theory of Functions, vols. I, II. New York, Dover Publications, Inc. Vol. I, Problems in the Elementary Theory of Functions, 1948; Vol. II, Problems in the Advanced Theory of Functions, 1952.

3.9 Knopp, Konrad. Theory and Application of Infinite Series. (translated from the 2nd German edition) New York, Hafner Publishing Company, 1948.

3.10 MAA Studies in Mathematics, vol. I. Studies in Modern Analysis (edited by R. C. Buck) Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

3.11 Nickerson, H. K.; Spencer, D. C.; Steenrod, N. E. Advanced Calculus. New York, Van Nostrand Reinhold Company, 1959.

3.12 Rogosinski, Werner. Fourier Series, 2nd ed. New York, Chelsea Publishing Company, Inc., 1959.

3.13 Titchmarsh, Edward C. Theory of Functions, 2nd ed. New York, Oxford University Press, Inc., 1939.

3.14 Williamson, John Hunter. Lebesgue Integration. New York, Holt, Rinehart and Winston, Inc., 1962.

3.15 At least one of the following: (a-b)

3.15a Dieudonné, Jean. Foundations of Modern Analysis. New York, Academic Press, Inc., 1960.

3.15b Simmons, George F. Introduction to Topology and Modern Analysis. New York, McGraw-Hill Book Company, 1963.

3.16 At least one of the following: (a-b)

3.16a Kolmogorov, Andree N. and Fomin, S. V. Elements of the Theory of Functions and Functional Analysis, vols. I, II (translated from the 1st Russian edition) Baltimore, Maryland, Graylock Press. Vol. I, Metric and Normed Spaces, 1957; Vol. II, Measure, the Lebesgue Integral, Hilbert Space, 1961.

3.16b Lorch, Edgar Raymond. Spectral Theory. New York, Oxford University Press, Inc., 1962.

3.17 At least one of the following: (a-b)

3.17a Erdélyi, Arthur. Operational Calculus and Generalized Functions. New York, Holt, Rinehart and Winston, Inc., 1962.

3.17b Lighthill, Michael James. Introduction to Fourier Analysis and Generalized Functions. New York, Cambridge University Press, 1958.

3.18 At least one of the following: (a-b)

3.18a Akhiezer, Naum I. Calculus of Variations. (translated by Aline H. Frink) New York, Blaisdell Publishing Company, 1962.

3.18b Gelfand, I. M. and Fomin, S. V. Calculus of Variations. (translated by R. A. Silverman) Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1963.

3.19 At least one of the following: (a-c)

3.19a Beckenbach, E. F. and Bellman, R. Introduction to Inequalities. New York, Random House, Inc., 1961.

3.19b Kazarinoff, N. D. Geometric Inequalities. New York, Random House, Inc., 1961.

3.19c Korovkin, Pavel P. Inequalities. (translated from the Russian by Halina Moss, edited by Ian N. Sneddon) New York, Blaisdell Publishing Company, 1962.

3.20 At least one of the following: (a-f)

3.20a Agnew, Ralph Palmer. Differential Equations, 2nd ed. New York, McGraw-Hill Book Company, 1960.

3.20b Coddington, Earl A. An Introduction to Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961.

3.20c Ford, Lester R. Differential Equations, 2nd ed. New York, McGraw-Hill Book Company, 1955.

3.20d Golomb, Michael and Shanks, Merrill. Elements of Ordinary Differential Equations, 2nd rev. ed. New York, McGraw-Hill Book Company, 1965.

3.20e Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations. New York, Harper and Row, Publishers, 1963.

3.20f Pontryagin, Lev S. Ordinary Differential Equations. (translated by L. Kocinskas and W. Counts) Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

3.21 At least one of the following: (a-b)

3.21a Birkhoff, Garrett and Rota, Gian-Carlo. Ordinary Differential Equations. New York, Blaisdell Publishing Company, 1962.

3.21b Coddington, Earl A. and Levinson, Norman. Theory of Ordinary Differential Equations. New York, McGraw-Hill Book Company, 1955.

3.22 At least one of the following: (a-c)

3.22a Hurewicz, Witold. Lectures on Ordinary Differential Equations. Cambridge, Massachusetts, MIT Press, 1958.

3.22b Lefschetz, Solomon. Differential Equations: Geometric Theory, 2nd ed. New York, Interscience, 1963.

3.22c Tricomi, F. G. Differential Equations. New York, Hafner Publishing Company, 1961.

3.23 At least one of the following: (a-c)

3.23a Ahlfors, Lars V. Complex Analysis. New York, McGraw-Hill Book Company, 1953.

3.23b Knopp, Konrad. Theory of Functions, parts I, II. New York, Dover Publications, Inc. Part I, Elements of the General Theory of Analytic Functions, 1945; Part II, Applications and Continuations of the General Theory, 1947.

3.23c Nehari, Zeev. Introduction to Complex Analysis. Boston, Massachusetts, Allyn and Bacon, Inc., 1961.

3.24 At least one of the following: (a-d)

3.24a Carathéodory, C. Theory of Functions of a Complex Variable, vols. I, II, 2nd ed. (translated by F. Steinhardt) New York, Chelsea Publishing Company, Inc. Vol. I, 1958; Vol. II, 1960.

3.24b Fuchs, B. A. and Shabat, B. V. Functions of a Complex Variable and Some of Their Applications. (translated by J. Berry, edited by T. Kovari) Reading, Massachusetts, Addison-Wesley Publishing Company, Inc. Vol. I, rev. and expanded by J. W. Reed, 1964; Vol. II, 1962.

3.24c Hille, Einar. Analytic Function Theory, vols. I, II. New York, Blaisdell Publishing Company. Vol. I, 1959; Vol. II, 1962.

3.24d Saks, S. and Zygmund, A. Analytic Functions. (translated by E. J. Scott) Warsaw, Poland, Nakładem Polskiego Towarziptwa Matematycznego, 1952 (not in print in U. S.). Rev. ed., New York, Dover Publications, Inc., 1964.

3.25 At least one of the following: (a-f)

3.25a Bartle, Robert G. The Elements of Real Analysis. New York, John Wiley and Sons, Inc., 1964.

3.25b Franklin, Philip. Treatise on Advanced Calculus. New York, John Wiley and Sons, Inc., 1940.

3.25c Kaplan, Wilfred. Advanced Calculus. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1952.

3.25d Olmsted, J. M. H. Advanced Calculus. New York, Appleton-Century-Crofts, 1961.

3.25e Taylor, Angus Ellis. Advanced Calculus. New York, Blaisdell Publishing Company, 1955.

3.25f Widder, David Vernon. Advanced Calculus, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961.

3.26 At least one of the following: (a-d)

3.26a Apostol, Tom M. Mathematical Analysis. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1957.

3.26b Buck, R. C. Advanced Calculus, 2nd ed. New York, McGraw-Hill Book Company, 1964.

3.26c Maak, Wilhelm. An Introduction to Modern Calculus. (translated by G. Strike) New York, Holt, Rinehart and Winston, Inc., 1963.

3.26d Rudin, Walter. Principles of Mathematical Analysis, 2nd ed. New York, McGraw-Hill Book Company, 1964.

3.27 At least one of the following: (a-e)

3.27a Goffman, Casper. Real Functions. New York, Holt, Rinehart and Winston, Inc., 1953.

3.27b Graves, Lawrence M. Theory of Functions of Real Variables, 2nd ed. New York, McGraw-Hill Book Company, 1956.

3.27c McShane, Edward J. and Botts, Truman. Real Analysis. New York, Van Nostrand Reinhold Company, 1959.

3.27d Royden, H. L. Real Analysis. New York, The Macmillan Company, 1963.

3.27e Thielman, Henry P. Theory of Functions of Real Variables. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1953.

3.28 At least one of the following: (a-d)

3.28a Green, J. A. Sequences and Series. Glencoe, Illinois, Free Press of Glencoe, 1958.

3.28b Hirschman, Isidore I., Jr. Infinite Series. New York, Holt, Rinehart and Winston, Inc., 1962.

3.28c Hyslop, James Morton. Infinite Series, 4th rev. ed. New York, Interscience, 1954.

3.28d Knopp, Konrad. Infinite Sequences and Series. (translated by F. Bagemihl) New York, Dover Publications, Inc., 1956.

3.29 At least one of the following: (a-b)

3.29a Epstein, Bernard. Partial Differential Equations, An Introduction. New York, McGraw-Hill Book Company, 1962.

3.29b Garabedian, P. R. Partial Differential Equations. New York, John Wiley and Sons, Inc., 1964.

3.30 At least one of the following: (a-b)

3.30a Halmos, Paul R. Measure Theory. New York, Van Nostrand Reinhold Company, 1950.

3.30b Munroe, Marshall Evans. Introduction to Measure and Integration. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1953.

Two books on mathematical tables: one numerical, such as 3.31, and one functional, such as 3.32.

3.31 Cogan, Edward J. and Norman, R. Z. Handbook of Calculus, Difference and Differential Equations, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1963.

3.32 At least one of the following: (a-b)

3.32a Jahnke, E. and Emde, F. Tables of Functions with Formulas and Curves, 6th ed. New York, McGraw-Hill Book Company, 1960.

3.32b National Bureau of Standards, U. S. Department of Commerce, Applied Mathematics, Series 55. Handbook of Mathematical Functions. (edited by M. Abramowitz and I. A. Stegun) Superintendent of Documents, U. S. Government Printing Office, Washington, D. C.

The Library should also contain a selection of several calculus books to which students may refer for supplementary reading. These books should be chosen so as to describe a variety of approaches and motivations. It is felt that there should be at least one careful, detailed development such as is contained in any of the following (or similar works):

Apostol, Tom M. Calculus, vols. I, II. New York, Blaisdell Publishing Company. Vol. I, Introduction with Vectors and Analytic Geometry, 1961; Vol. II, Calculus of Several Variables with Applications to Probability and Vector Analysis, 1962.

Begle, Edward G. Introductory Calculus with Analytic Geometry. New York, Holt, Rinehart and Winston, Inc., 1954.

Kuratowski, K. C. Introduction to Calculus. (translated from the Polish by J. Musielak) Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

Landau, Edmund G. H. Differential and Integral Calculus. (translated by M. Hausner and M. Davis) New York, Chelsea Publishing Company, Inc., 1960.

IV. Applied Mathematics

Because of the increasing interaction between mathematics and the natural and social sciences, it is virtually impossible to list a definitive collection of library books in this area. We urge the student and the teacher, intent on following this interaction, to make use of materials already available in libraries under the science, social science, and engineering listings. Nevertheless, we do recommend that the libraries contain certain books on the mathematical aspects of physical science and engineering. These are 4.5, 4.6, 4.7, 4.12, 4.15, and 4.18. Recent developments in applied mathematics which bear a close relationship to the developments in social sciences are 4.9, 4.23, 4.24, 4.27, 4.28, and 4.29.

Since mathematical methods form part of applied mathematics, we recommend a few of the many compilations of mathematical analysis methods such as those listed in 4.20 and 4.21. We note that 4.1 consists of a definitive study of problems of partial differential equations occurring in many applications of mathematics. Introductions to functional analytical methods useful in applied mathematics are listed in 4.14.

In the past decade or so, with the advent of highspeed computing machines, numerical analysis and some brances of algebra and logic have become an important area of applied mathematics. Numerical analysis books are listed in 4.2, 4.26, 4.18. The last (4.18) stresses algebraic aspects. Incidentally, the books on linear algebra contained in the algebra section of this report furnish material indispensable in the area of numerical analysis. Selection 4.17 contains introductions to computing machines--their modes of operation, programming techniques, computer logic, and the use of algorithms.

4.1 Courant, R. and Hilbert D. Methods of Mathematical Physics, 1st English ed. (translated from the German) New York, John Wiley and Sons, Inc., Vol. I, 1953.

4.2 Henrici, Peter. Discrete Variable Methods in Ordinary Differential Equations. New York, John Wiley and Sons, Inc., 1962.

4.3 Hopf, L. Introduction to Differential Equations of Physics. (translated by Walter Nef) New York, Dover Publications, Inc., 1948.

4.4 Kemeny, John G. and Snell, J. Laurie. Mathematical Models in the Social Sciences. New York, Blaisdell Publishing Company, 1962.

4.5 Khinchin, A. I. Mathematical Foundations of Statistical Mechanics. (translated by G. Gamow) New York, Dover Publications, Inc., 1949.

4.6 Lamb, Sir Horace. Hydrodynamics, 6th rev. ed. New York, Dover Publications, Inc., 1956.

4.7 Landau, Lev D. and Lifshitz, E. M. The Classical Theory of Fields, 2nd ed. (translated from the Russian by M. Hamermesh) Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

4.8 Love, A. E. H. Treatise on the Mathematical Theory of Elasticity, 4th rev. ed. New York, Dover Publications, Inc., 1956.

4.9 Luce, Robert Duncan and Raiffa, Howard. Games and Decisions. New York, John Wiley and Sons, Inc., 1957.

4.10 National Physical Laboratory, Teddington, England. Modern Computing Methods, 2nd ed. Notes on Applied Science #16. London, Her Majesty's Stationery Office, 1962. (In U. S. Philosophical Library)

4.11 Parzen, Emanuel. Stochastic Processes with Applications to Science and Engineering. San Francisco, California, Holden-Day, Inc., 1962.

4.12 Rayleigh, John W. S. Theory of Sound, 2nd rev. ed. New York, Dover Publications, Inc., 1955. 2 vols.

4.13 Stiefel, E. L. An Introduction to Numerical Mathematics. (translated by W. C. and C. J. Rheinboldt) New York, Academic Press, Inc., 1963.

4.14 At least one of the following: (a-b)

4.14a Friedman, Bernard. Principles and Techniques of Applied Mathematics. New York, John Wiley and Sons, Inc., 1956.

4.14b Vulikh, Boris Z. Introduction to Functional Analysis for Scientists and Technologists. (translated by Ian N. Sneddon) Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

4.15 At least one of the following: (a-b)

4.15a Lichnerowicz, André. Elements of Tensor Calculus. (translated by J. W. Leech and D. J. Newman) New York, John Wiley and Sons, Inc., 1962.

4.15b Synge, John L. and Schild, A. Tensor Calculus. Toronto, Ontario, University of Toronto Press, 1949.

4.16 At least one of the following: (a-c)

4.16a Fano, Robert M. Transmission of Information. Cambridge, Massachusetts, MIT Press, 1961.

4.16b Reza, F. M. An Introduction to Information Theory. New York, McGraw-Hill Book Company, 1961.

4.16c Shannon, Claude E. and Weaver, W. The Mathematical Theory of Communication. Urbana, Illinois, University of Illinois Press, 1949.

4.17 At least one of the following: (a-c)

4.17a Arden, B. W. An Introduction to Digital Computers. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

4.17b Galler, Bernard A. The Langauage of Computers. New York, McGraw-Hill Book Company, 1962.

4.17c Leeds, Herbert D. and Weinberg, Gerald M. Computer Programming Fundamentals. New York, McGraw-Hill Book Company, 1961.

4.18 At least one of the following: (a-d)

4.18a Faddeev, D. K. and Faddeeva, V. N. Computational Methods in Linear Algebra. (translated by Robert C. Williams) San Francisco, California, W. H. Freeman and Company, 1963; Authorized translation by Curtis Benster. New York, Dover Publications, Inc., 1959.

4.18b Fox, Leslie. An Introduction to Numerical Linear Algebra. Fair Lawn, New Jersey, Clarendon Press, 1964.

4.18c Frazer, Robert A.; Duncan, W. J.; Collar, A. R. Elementary Matrices. New York, Cambridge University Press, 1938.

4.18d Householder, Alston Scott. The Theory of Matrices in Linear Algebra. New York, Blaisdell Publishing Company, 1964.

4.19 At least one of the following: (a-b)

4.19a Goldstein, Herbert. Classical Mechanics. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1950.

4.19b Synge, John L. and Griffith, B. A. Principles of Mechanics, 3rd ed. New York, McGraw-Hill Book Company, 1959.

4.20 At least one of the following: (a-c)

4.20a Jeffreys, Sir Harold and Jeffreys, Bertha Swirles. Methods of Mathematical Physics, 3rd ed. New York, Cambridge University Press, 1956.

4.20b Morse, Philip M. and Feshbach, H. Methods of Theoretical Physics, parts I, II. New York, McGraw-Hill Book Company, 1953.

4.20c Whittaker, Edmund T. and Watson, G. N. A Course of Modern Analysis, 4th ed. New York, Cambridge University Press, 1958.

4.21 At least one of the following: (a-c)

4.21a Kreyszig, Erwin. Advanced Engineering Mathematics. New York, John Wiley and Sons, Inc., 1962.

4.21b Tychonov, A. N. and Samarski, A. A. Partial Differential Equations in Mathematical Physics, vol. I. (translated by S. Radding) San Francisco, California, Holden-Day, Inc., 1964.

4.21c von Kármán, Theodore and Biot, M. A. Mathematical Methods in Engineering. New York, McGraw-Hill Book Company, 1940.

4.22 At least one of the following: (a-b)

4.22a Riordan, John. An Introduction to Combinatorial Analysis. New York, John Wiley and Sons, Inc., 1958.

4.22b Ryser, Herbert John. Combinatorial Mathematics (Carus Monograph #14). New York, John Wiley and Sons, Inc., 1963.

4.23 At least one of the following: (a-c)

4.23a Aris, Rutherford. Discrete Dynamic Programming. New York, John Wiley and Sons, Inc., 1963.

4.23b Bellman, Richard E. and Dreyfus, Stuart E. Applied Dynamic Programming. Princeton, New Jersey, Princeton University Press, 1962.

4.23c Howard, Ronald A. Dynamic Programming and Markov Processes. Cambridge, Massachusetts, MIT Press, 1960.

4.24 At least one of the following: (a-d)

4.24a Dantzig, George B. Linear Programming and Extensions. Princeton, New Jersey, Princeton University Press, 1962.

4.24b Gass, Saul I. Linear Programming, 2nd ed. New York, McGraw-Hill Book Company, 1964.

4.24c Hadley, George. Linear Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

4.24d Vajda, S. Theory of Games and Linear Programming. New York, John Wiley and Sons, Inc., 1956.

4.25 At least one of the following: (a-b)

4.25a Hohn, Franz E. Applied Boolean Algebra. New York, The Macmillan Company, 1960.

4.25b Whitesitt, John Elden. Boolean Algebra and Its Applications. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

4.26 At least one of the following: (a-c)

4.26a Hildebrand, Francis B. Introduction to Numerical Analysis. New York, McGraw-Hill Book Company, 1956.

4.26b Householder, Alston Scott. Principles of Numerical Analysis. New York, McGraw-Hill Book Company, 1953.

4.26c Lanczos, Cornelius. Applied Analysis. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1956.

4.27 At least one of the following: (a-c)

4.27a Cox, D. R. and Smith, W. L. Queues. New York, John Wiley and Sons, Inc., 1961.

4.27b Riordan, John. Stochastic Service Systems. New York, John Wiley and Sons, Inc., 1962.

4.27c Takács, Lajos. Introduction to the Theory of Queues. New York, Oxford University Press, 1962.

4.28 At least one of the following: (a-b)

4.28a Gale, David. The Theory of Linear Economic Models. New York, McGraw-Hill Book Company, 1960.

4.28b Dorfman, Robert; Samuelson, Paul A.; Solow, Robert M. Linear Programming and Economic Analysis. New York, McGraw-Hill Book Company, 1958.

4.29 At least one of the following: (a-c)

4.29a Berge, Claude. The Theory of Graphs and Its Applications. (translated by Alison Doig) New York, John Wiley and Sons, Inc., 1962.

4.29b Ford, L. R., Jr., and Fulkerson, D. R. Flows in Networks. Princeton, New Jersey, Princeton University Press, 1962.

4.29c Ore, Oystein. Theory of Graphs. Providence, Rhode Island, American Mathematical Society, 1962. (American Mathematical Society Colloquium Publications, Vol. 38)

V. Geometry-Topology

The following 38 books, of which a minimum of 15 are to be selected, are intended to cover topics in geometry and topology. Besides general reading and introductory material on geometry as found in 5.3 and 5.5, various other topics such as projective geometry (5.4, 5.8), algebraic geometry (5.7), non-Euclidean geometry (5.10), and differential geometry (5.11) are represented. In addition to general and introductory material on topology (5.1, 5.3), increasing levels of sophistication in general topology (5.12, 5.13, 5.14) are mentioned, as is algebraic topology (5.9).

5.1 Arnold, Bradford Henry. Intuitive Concepts in Elementary Topology. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

5.2 Artin, Emil. Geometric Algebra. New York, Interscience, 1957.

5.3 Hilbert, David and Cohn-Vossen, S. Geometry and the Imagination. (translated by P. Nemenyi) New York, Chelsea Publishing Company, Inc., 1952.

5.4 Young, J. W. Projective Geometry (Carus Monograph No. 4). Chicago, Illinois, The Open Court Publishing Company, 1930.

5.5 At least of the following: (a-b)

- 5.5a Coxeter, H. S. M. Introduction to Geometry. New York, John Wiley and Sons, Inc., 1961.
- 5.5b Eves, Howard. A Survey of Geometry, vol. I. Boston, Massachusetts, Allyn and Bacon, Inc., 1963.

5.6 At least one of the following: (a-c)

- 5.6a Eggleston, Harold G. Problems of Euclidean Space: Applications of Convexity. Elmsford, New York, Pergamon Press, Inc., 1957.
- 5.6b Hadwiger, Hugo and Debrunner, Hans. Combinatorial Geometry in the Plane. (translated by Victor Klee) New York, Holt, Rinehart and Winston, Inc., 1964.
- 5.6c Yaglom, Isaak M. and Boltyanskiĭ, B. G. Convex Figures. (translated by P. J. Kelly and L. F. Walton) New York, Holt, Rinehart and Winston, Inc., 1961.

5.7 At least one of the following: (a-b)

- 5.7a Jenner, William E. Rudiments of Algebraic Geometry. New York, Oxford University Press, Inc., 1963.
- 5.7b Walker, Robert John. Algebraic Curves. New York, Dover Publications, Inc., 1962.

5.8 At least one of the following: (a-c)

5.8a Baer, Reinhold. Linear Algebra and Projective Geometry. New York, Academic Press, Inc., 1952.

5.8b Busemann, Herbert and Kelly, Paul J. Projective Geometry and Projective Metrics. New York, Academic Press, Inc., 1953.

5.8c Seidenberg, A. Lectures in Projective Geometry. New York, Van Nostrand Reinhold Company, 1962.

5.9 At least one of the following: (a-d)

5.9a Aleksandrov, P. S. Combinatorial Topology, 3 vols. Baltimore, Maryland, Graylock Press. Vol. I., Introduction, Complexes, Coverings, Dimension, 1956; Vol. II, Betti Groups, 1957; Vol. III, Homological Manifolds, Duality, Classification, and Fixed Point Theorems, 1960.

5.9b Lefschetz, Solomon. Introduction to Topology. Princeton, New Jersey, Princeton University Press, 1949.

5.9c Pontryagin, Lev S. Foundations of Combinatorial Topology. (translated by Bagemihl, Kohm, and Seidu) Baltimore, Maryland, Graylock Press, 1952.

5.9d Wallace, Andrew Hugh. Introduction to Algebraic Topology. Elmsford, New York, Pergamon Press, Inc., 1957.

5.10 At least one of the following: (a-b)

5.10a Coxeter, H. S. M. Non-Euclidean Geometry, 4th rev. ed. Toronto, Ontario, University of Toronto Press, 1957.

5.10b Wolfe, Harold E. Introduction to Non-Euclidean Geometry. New York, Holt, Rinehart and Winston, Inc., 1945.

5.11 At least one of the following: (a-d)

5.11a Guggenhein, Heinrich W. Differential Geometry. New York, McGraw-Hill Book Company, 1963.

5.11b Kreyszig, Erwin. Differential Geometry, 2nd ed. Toronto, Ontario, University of Toronto Press, 1963.

5.11c Struik, Dirk Jan. Differential Geometry, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

5.11d Willmore, Thomas James. Introduction to Differential Geometry. New York, Oxford University Press, Inc., 1959.

5.12 At least one of the following: (a-f)

5.12a Baum, John D. Elements of Point Set Topology. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964.

5.12b Bushaw, Donald Wayne. Elements of General Topology. New York, John Wiley and Sons, Inc., 1963.

5.12c Hu, Sze-Tsen. Elements of General Topology. San Francisco, California, Holden-Day, Inc., 1964.

5.12d Kuratowski, Kazimierz. Introduction to Set Theory and Topology. (translated from the revised Polish edition by L. Boron) Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

5.12e Mendelson, Bert. Introduction to Topology. Boston, Massachusetts, Allyn and Bacon, Inc., 1962.

5.12f Pervin, William J. Foundations of General Topology. New York, Academic Press, Inc., 1964.

5.13 At least one of the following: (a-b)

5.13a Hall, Dick Wick and Spencer, G. L. Elementary Topology. New York, John Wiley and Sons, Inc., 1955.

5.13b Newman, M. H. A. Topology of Plane Sets of Points. New York, Cambridge University Press, 1951.

5.14 At least one of the following: (a-b)

5.14a Hocking, John and Young, Gail. Topology. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

5.14b Kelley, John L. General Topology. New York, Van Nostrand Reinhold Company, 1955.

5.15 At least one of the following: (a-d)

5.15a Crowell, Richard Henry and Fox, Ralph H. Introduction to Knot Theory. New York, Blaisdell Publishing Company, 1963.

5.15b Hurewicz, Witold and Wallman, Henry. Dimension Theory. Princeton, New Jersey, Princeton University Press, 1941.

5.15c Pontryagin, Lev S. Topological Groups. (translated by Emma Lehmer) Princeton, New Jersey, Princeton University Press, 1958.

5.15d Springer, George. Introduction to Riemann Surfaces. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956.

VI. Logic, Foundations, and Set Theory

Of the following 23 books on logic, foundations, and set theory, at least 13 are to be selected. Besides historical and introductory material on set theory (6.1, 6.4, 6.8), this field is covered in increasingly sophisticated fashion in 6.8, 6.2, and 6.11. Foundational material is to be found in 6.5, 6.9, and 6.10, while logic is covered in increasing levels of sophistication in 6.6, 6.8, 6.7, 6.3, 6.12, and 6.13.

6.1 Cantor, George. Contributions to the Founding of the Theory of Transfinite Numbers. (translated by P. E. B. Jourdain) Chicago, Illinois, The Open Court Publishing Company, 1961; New York, Dover Publications, Inc.

6.2 Halmos, Paul R. Naive Set Theory. New York, Van Nostrand Reinhold Company, 1960.

6.3 Hilbert, David and Ackerman, W. Principles of Mathematical Logic. (translated from the 2nd German edition) New York, Chelsea Publishing Company, Inc., 1950.

6.4 Kamke, Erich. Theory of Sets. (translated by F. Bagemihl) New York, Dover Publications, Inc., 1950.

6.5 Landau, Edmund G. H. The Foundations of Analysis. (translated by E. Steinhardt) New York, Chelsea Publishing Company, Inc., 1951.

6.6 Nagel, Ernest and Newman, James R. Gödel's Proof. New York, New York University Press, 1958.

6.7 Rosenbloom, Paul Charles. The Elements of Mathematical Logic. New York, Dover Publications, Inc., 1951.

6.8 Stoll, Robert Roth. Sets, Logic and Axiomatic Theories. San Francisco, California, W. H. Freeman and Company, 1961.

6.9 Wilder, Raymond L. Introduction to the Foundations of Mathematics. New York, John Wiley and Sons, Inc., 1952.

6.10 At least one of the following: (a-e)

6.10a Cohen, Leon W. and Ehrlich, G. The Structure of the Real Number System. New York, Van Nostrand Reinhold Company, 1963.

6.10b Feferman, Solomon. The Number Systems: Foundations of Algebra and Analysis. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1964.

6.10c Henkin, Leon A.; Smith, Norman; Varineau, V. J.; Walsh, Michael J. Retracing Elementary Mathematics. New York, The Macmillan Company, 1962.

6.10d Kershner, Richard B. and Wilcox, L. R. Anatomy of Mathematics. New York, Ronald Press Company, 1950.

6.10e Landin, Joseph and Hamilton, N. T. Set Theory: The Structure of Arithmetic. Boston, Massachusetts, Allyn and Bacon, Inc., 1961.

6.11 At least one of the following: (a-b)

6.11a Quine, Willard von Orman. Set Theory and Its Logic. Cambridge, Massachusetts, Harvard University Press, 1963.

6.11b Suppes, Patrick C. Axiomatic Set Theory. New York, Van Nostrand Reinhold Company, 1960.

6.12 At least one of the following: (a-e)

6.12a Copi, Irving Marmer. Symbolic Logic. New York, The Macmillan Company, 1954.

6.12b Kalish, Donald and Montague, Richard. Logic: Techniques of Formal Reasoning. New York, Harcourt Brace Jovanovitch, 1964.

6.12c Quine, Willard von Orman. Mathematical Logic, rev. ed. Cambridge, Massachusetts, Harvard University Press, 1951.

6.12d Suppes, Patrick C. Introduction to Logic. New York, Van Nostrand Reinhold Company, 1958.

6.12e Tarski, Alfred. Introduction to Logic and to the Methodology of Deductive Sciences, 2nd ed. rev. New York, Oxford University Press, Inc., 1946.

6.13 At least one of the following: (a-b)

6.13a Church, Alonzo. Introduction to Mathematical Logic, vol. 1. Princeton, New Jersey, Princeton University Press, 1956.

6.13b Kleene, Stephen C. Introduction to Metamathematics. New York, Van Nostrand Reinhold Company, 1952.

VII. Probability-Statistics

The first five books listed are authoritative reference books in this rapidly growing field. The remainder of the list consists of pairings of books, one book from each pair being sufficient in a minimum library. Probability is treated in increasing levels of sophistication in 7.6, 7.7, 7.2, 7.4, and 7.3, and statistics in the order 7.8, 7.9, 7.10, 7.5, and 7.1. Items 7.6 and 7.8 do not assume a knowledge of the calculus.

7.1 Cramér, Harald. Mathematical Methods of Statistics. Princeton, New Jersey, Princeton University Press, 1946.

7.2 Feller, William. An Introduction to Probability Theory and Its Applications, vol. I, 2nd ed. New York, John Wiley and Sons, Inc., 1957.

7.3 Loève, Michel Moise. Probability Theory, 3rd ed. New York, Van Nostrand Reinhold Company, 1963.

7.4 Parzen, Emanuel. Modern Probability Theory and Its Applications. New York, John Wiley and Sons, Inc., 1960.

7.5 Wilks, Samuel S. Mathematical Statistics, 2nd ed. New York, John Wiley and Sons, Inc., 1962.

7.6 At least one of the following: (a-b)

7.6a Gnedenko, Boris V. and Khinchin, A. I. An Elementary Introduction to the Theory of Probability. (translated from the Russian by W. R. Stahl, edited by J. B. Roberts) San Francisco, California, W. H. Freeman and Company, 1961; New York, Dover Publishing Company.

7.6b Goldberg, Samuel. Probability: An Introduction. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1960.

7.7 At least one of the following: (a-b)

7.7a Cramér, Harald. The Elements of Probability Theory and Some of Its Applications. New York, John Wiley and Sons, Inc., 1955.

7.7b Gnedenko, Boris V. Theory of Probability. (translated by E. D. Seckler) New York, Chelsea Publishing Company, Inc., 1962.

7.8 At least one of the following: (a-d)

7.8a Hodges, J. L. and Lehmann, E. L. Basic Concepts of Probability and Statistics. San Francisco, California, Holden-Day, Inc., 1964.

7.8b Mosteller, Frederick; Rourke, R. E. K.; Thomas, G. B. Probability with Statistical Applications. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

7.8c Neyman, Jerzy. First Course in Probability and Statistics. New York, Holt, Rinehart and Winston, Inc., 1950.

7.8d Wolf, Frank Louis. Elements of Probability and Statistics. New York, McGraw-Hill Book Company, 1962.

7.9 At least one of the following: (a-b)

7.9a Hogg, Robert V. and Craig, A. T. Introduction to Mathematical Statistics. New York, The Macmillan Company, 1959.

7.9b Lindgren, Bernard William. Statistical Theory. New York, The Macmillan Company, 1962.

7.10 At least one of the following: (a-b)

7.10a Brunk, Hugh Daniel. Introduction to Mathematical Statistics, 2nd ed. New York, Blaisdell Publishing Company, 1964.

7.10b Mood, Alexander M. and Graybill, F. A. Introduction to the Theory of Statistics, 2nd ed. New York, McGraw-Hill Book Company, 1963.

VIII. Number Theory

The theory of numbers has a perennial appeal for amateurs as well as for specialists. Both for browsers and for serious students, a basic library should contain some of the lore of number theory as well as systematic works.

8.1 Dickson, Leonard E. History of the Theory of Numbers, vols. I, II, III. New York, Chelsea Publishing Company, Inc., 1952.

8.2 Hardy, Godfrey H. and Wright, E. M. An Introduction to the Theory of Numbers, 4th ed. New York, Oxford University Press, Inc., 1960.

8.3 Niven, Ivan. Irrational Numbers (Carus Monograph No. 11). New York, John Wiley and Sons, Inc., 1956.

8.4 Ore, Oystein. Number Theory and Its History. New York, McGraw-Hill Book Company, 1948.

8.5 Pollard, Harry S. The Theory of Algebraic Numbers (Carus Monograph No. 9). New York, John Wiley and Sons, Inc., 1950.

8.6 At least one of the following: (a-d)

8.6a Jones, Burton W. The Theory of Numbers. New York, Holt, Rinehart and Winston, Inc., 1955.

8.6b LeVeque, William Judson. Elementary Theory of Numbers. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

8.6c Stewart, Bonnie Madison. Theory of Numbers, 2nd ed. New York, The Macmillan Company, 1964.

8.6d Wright, Harry Nable. First Course in the Theory of Numbers. New York, John Wiley and Sons, Inc., 1939.

8.7 At least two of the following: (a-g)

8.7a Landau, Edmund G. H. Elementary Number Theory. (translated by Jacob E. Goodman) New York, Chelsea Publishing Company, Inc., 1958.

8.7b LeVeque, William Judson. Topics in Number Theory, vols. I, II. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956.

8.7c Nagell, Trygve. Introduction to Number Theory, reprint, 2nd ed. New York, Chelsea Publishing Company, Inc., 1964.

8.7d Niven, Ivan and Zuckerman, H. S. An Introduction to the Theory of Numbers. New York, John Wiley and Sons, Inc., 1960.

8.7e Rademacher, Hans A. Lectures on Elementary Number Theory. New York, Blaisdell Publishing Company, 1964.

8.7f Uspensky, James V. and Heaslet, M. A. Elementary Number Theory. New York, McGraw-Hill Book Company, 1939.

8.7g Vinogradov, Ivan M. Elements of Number Theory. (translated from the 5th revised edition by Saul Kravetz) New York, Dover Publications, Inc., 1954; 6th edition translated by H. Popova, Elmsford, New York, Pergamon Press, Inc., 1955.

IX. Miscellaneous

Inevitably there are some books which a library needs, not because they neatly fit a category, but because they themselves have unique appeal or utility. The titles under Miscellaneous resist omission for miscellaneous reasons. A mathematics library is made more useful by the inclusion of collections of problems, more diverting because of the less technical or even whimsical insights of capable mathematicians, and better suited for browsing if it is stocked with collections of mathematical fragments or synopses. The following two dozen volumes are an especially good investment because they are likely to wear out first!

9.1 Beaumont, Ross A. and Pierce, Richard S. Algebraic Foundations of Mathematics. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

9.2 Blumenthal, Leonard M. A Modern View of Geometry. San Francisco, California, W. H. Freeman and Company, 1961.

9.3 Burkill, J. C. and Cundy, H. M. Mathematical Scholarship Problems. New York, Cambridge University Press, 1961.

9.4 Eves, Howard and Newsom, C. V. Introduction to the Foundations and Fundamental Concepts of Mathematics, rev. ed. New York, Holt, Rinehart and Winston, Inc., 1964.

9.5 Hadamard, Jacques. Psychology of Invention in the Mathematical Field. New York, Dover Publications, Inc., 1954.

9.6 Hall, Henry S. and Knight, S. R. Higher Algebra, 4th ed. New York, St. Martin's Press, Inc., 1932.

9.7 Hardy, Godfrey Harold. A Mathematician's Apology, rev. ed. New York, Cambridge University Press.

9.8 Jones, Burton W. Elementary Concepts of Mathematics, 2nd ed. New York, The Macmillan Company, 1963.

9.9 Kac, Mark. Statistical Independence in Probability, Analysis and Number Theory (Carus Monograph No. 12). New York, John Wiley and Sons, Inc., 1959.

9.10 Klein, Felix. Elementary Mathematics from an Advanced Standpoint, vols. I, II. (translated from the 3rd German edition) New York, Dover Publications, Inc., 1961. Vol. I, Arithmetic, Algebra, Analysis, 1924; Vol. II, Geometry, 1939.

9.11 National Council of Teachers of Mathematics. Insights into Modern Mathematics (23rd Yearbook). Washington, D. C., National Council of Teachers of Mathematics, 1957.

9.12 Newman, James R. The World of Mathematics, 4 vols. New York, Simon and Schuster, Inc., 1962. Vol. I, Men and Numbers; Vol. II, World of Laws and the World of Chance; Vol. III, Mathematical Way of Thinking; Vol. IV, Machines, Music and Puzzles.

9.13 Pólya, Gyorgy. How to Solve It, 2nd ed. New York, Doubleday and Company, 1957.

9.14 Saaty, Thomas L. Lectures on Modern Mathematics, 3 vols. New York, John Wiley and Sons, Inc. Vol. I, 1963; Vol. II, 1964; Vol. III, 1965.

9.15 Stein, Sherman K. Mathematics: The Man-made Universe. San Francisco, California, W. H. Freeman and Company, 1963.

9.16 Steinhaus, H. One Hundred Problems in Elementary Mathematics. New York, Basic Books, Inc., 1964.

9.17 Toeplitz, Otto. The Calculus: A Genetic Approach. (translated by Luise Lange) Chicago, Illinois, University of Chicago Press, 1963.

9.18 Ulam, Stanislaw. A Collection of Mathematical Problems. New York, Interscience, 1960.

9.19 van der Waerden, Bartel L. Science Awakening. (translated by Arnold Dresden) New York, Oxford University Press, Inc., 1961.

9.20 Weyl, Hermann. Philosophy of Mathematics and Natural Science, rev. and augm. English ed. based on trans. by Olaf Helmar. Princeton, New Jersey, Princeton University Press, 1949; New York, Atheneum Publishers, 1953; Gloucester, Massachusetts, Peter Smith.

9.21 Williams, John Davis. The Compleat Strategyst. New York, McGraw-Hill Book Company, 1954.

9.22 At least one of the following: (a-c)

9.22a Ball, Walter W. R. Mathematical Recreations and Essays. New York, The Macmillan Company, 1939.

9.22b Gardner, Martin, ed. Scientific American Book of Mathematical Puzzles and Diversions, 2 vols. New York, Simon and Schuster. Vol. I, 1964; Vol. II, 1961.

9.22c Kraitchik, Maurice. Mathematical Recreations, 2nd ed. New York, Dover Publications, Inc., 1942.

9.23 At least one of the following: (a-b)

9.23a Pólya, Gyorgy. Mathematics and Plausible Reasoning, 2 vols. Princeton, New Jersey, Princeton University Press, 1954. Vol. I, Induction and Analogy in Mathematics; Vol. II, Patterns of Plausible Inference.

9.23b Pólya, Gyorgy. Mathematical Discovery. New York, John Wiley and Sons, Inc., 1962.

9.24 Shklarsky, D. O.; Chentzov, N. N.; Yaglom, I. M. The USSR Olympiad Problem Book. (translated by J. Maykovitch, edited by I. Sussman) San Francisco, California, W. H. Freeman and Company, 1962.

FURTHER MATHEMATICAL MATERIALS

The value of a mathematical library is considerably enhanced by the inclusion of materials beyond those in the preceding basic list. Much of mathematical value can be found in general reference works, such as encyclopedias. In addition, it is recommended that the basic library be supplemented by items under the following headings.

Journals

The American Mathematical Monthly. Mathematical Association of America, Inc. 1225 Connecticut Avenue, N.W., Washington D. C. 20036 Ten issues per year.

The Mathematical Gazette. G. Bell and Sons, Ltd., Portugal Street, London, W.C. 2, England. Five issues per year.

Mathematics Magazine. Mathematical Association of America, Inc., 1225 Connecticut Avenue, N.W., Washington, D.C. 20036 Five issues per year.

Scripta Mathematica. Yeshiva University, New York, New York 10033 Quarterly.

SIAM Review. Society for Industrial and Applied Mathematics, 33 South 17th Street, Philadelphia, Pennsylvania 19103 Quarterly.

The Mathematics Teacher. National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D. C. 20036 Eight issues per year.

Series

There exist series of excellent inexpensive books whose inclusion in a library for undergraduates is suggested. Individual volumes in some of the following series are included in the basic list. In general, the following series are recommended, although, of course, individual volumes vary in quality and no endorsement of future volumes in any series is implied.

The Athena Series (Selected Topics in Mathematics). Holt, Rinehart and Winston, Inc., New York. This is a series of small books that form excellent supplements to standard junior- and senior-level courses.

Blaisdell Scientific Paperbacks. Blaisdell Publishing Company, New York. This is a series of small pamphlets that are translations of the Russian series "Popular Lectures in Mathematics."

The Carus Mathematical Monographs. The Mathematical Association of America, Inc., Washington, D. C. There are now 16 volumes in this series.

Library of Mathematics. Routledge and Kegan Paul, London. Available from the Free Press, New York. These are small paperback books covering a wide variety of topics at quite elementary levels.

The MAA Studies in Mathematics. The Mathematical Association of America, Inc., Washington, D. C.

School Mathematics Study Group New Mathematical Library. Random House, Inc., New York. This is a series of monographs.

University Mathematical Texts. Interscience, New York. This is a series of small books at the advanced undergraduate level.

Topics in Mathematics. D. C. Heath and Company, Boston, Massachusetts. This is a series of booklets translated and adapted from the Russian series "Popular Lectures in Mathematics." These American editions have been prepared by the Survey of Recent East European Mathematical Literature at the University of Chicago under a grant from the National Science Foundation. These booklets provide students of mathematics at various levels, as well as other interested readers, with valuable supplementary material to further their mathematical knowledge and development.

The Slaught Memorial Papers. The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. When they are issued, copies are sent free of charge to all members of the Association and subscribers to the Monthly. Additional copies may be purchased from the Mathematical Association of America.

Books in Foreign Languages

We recommend that some books in foreign languages--especially French, German, and Russian--be included in the collection. The principal purpose of these books would be to provide an opportunity for the student to learn to read mathematics in the language rather than to provide additions to the mathematical content of the list. Thus, in some cases it is suggested that, where available, both the English translation and the foreign language original be provided (good examples are van der Waerden's Modern Algebra, and the Heath Series Topics in Mathematics, in the preceding list).

There also should be included some books which do not exist in translation, such as Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, or de la Vallée Poussin, Cours d'Analyse.

COMMENTARY ON

"A GENERAL CURRICULUM IN MATHEMATICS

FOR COLLEGES"

A report of

The _ad hoc_ Committee on the Revision of GCMC

January 1972

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I. PREAMBLE--THE NEED FOR REAPPRAISAL

In 1965 the Committee on the Undergraduate Program in Mathematics (CUPM) published a report entitled A General Curriculum in Mathematics for Colleges (GCMC); this report has had an extensive influence on undergraduate mathematics programs in U. S. colleges and universities. Earlier CUPM reports had recommended specific undergraduate programs in mathematics for a variety of careers (teaching; mathematical research; physics and engineering; biological, management, and social sciences; and computer science). In contrast, the GCMC report undertook to identify a central curriculum beginning with calculus that could be taught by as few as four qualified teachers of mathematics (or four full-time equivalents) and that would serve the basic needs of the more specialized programs as well as possible. The extent to which the GCMC report achieved its purpose is indicated by the large number of colleges that have revised their course offerings in directions indicated by that report. Indeed, its influence has been widespread in spite of its stringent, self-imposed restrictions.

Many departments offer courses in addition to those mentioned in the GCMC report, such as a mathematics appreciation course for students in the arts and humanities, courses for prospective elementary school teachers, courses for students whose high school preparation is seriously deficient in mathematics, and specialized courses for most of the careers mentioned above. Thus the four-man "department" of the GCMC report often consists of four full-time equivalents within a much larger department having 10, 15, or even more members.

Numerous conferences of collegiate mathematicians have been held, both by the Sections of the Mathematical Association of America (MAA) and by CUPM, to discuss the GCMC report and to identify difficulties in following its suggestions. Although the response has been generally favorable, two criticisms have been made repeatedly: (a) The pace of some course outlines is unrealistically fast and in particular leaves no time for applications. (b) Many of the colleges for which the GCMC report was intended have substantial commitments to programs that are not discussed in the GCMC report, and they would welcome assistance with their problems.

For these reasons CUPM felt that the GCMC report should be reviewed. Such a re-evaluation of the entire program has been in progress for two years, and this commentary is the result of these deliberations.

During this review of the GCMC report, many problems have been considered by CUPM, of which three central ones are briefly mentioned below. Aspects of the first two are subjects of other CUPM studies (see Section II). We hope that these problems will be considered by individual departments of mathematics in the light of their local conditions.

1) The Evolving Nature of Mathematics Curricula. During the recent past, mathematics has been growing at a phenomenal rate, both internally and in its interconnections with other human activities. The subject continues to grow, and its influence continues to broaden beyond the traditional boundaries of pure mathematics and classical applied mathematics to include statistics, computer science, operations research, mathematical economics, mathematical biology, etc. When thinking about undergraduate education, therefore, is it not now more appropriate to speak of the mathematical sciences in a braod sense rather than simply mathematics in the traditional sense? Although large universities may have separate departments for the various aspects of the mathematical sciences, this alternative is not feasible at most colleges. Even in institutions where separate departments exist, how can one coordinate the various course offerings to take advantage of the impact that each branch of the mathematical sciences has upon the others and on related disciplines?

A closely related question is whether the "core" of pure mathematics that all departments should offer is now the same as it was presumed to be a few years ago. As new fields develop, some older fields seem less relevant, and today some mathematicians even question the assumption that calculus is the basic component of all college mathematics.

However, we wish to emphasize that no matter what changes occur in the undergraduate mathematics curriculum, one of the desirable alternatives will surely include basic calculus and algebra courses closely akin to Mathematics 1, 2, 3, 4, and 6 of the GCMC report.

2) The Service Functions of Mathematics. Mathematically educated people are needed in many kinds of work. It is therefore pertinent to ask whether the present undergraduate curriculum is sufficiently broad, especially in the freshman and sophomore years, to meet the mathematical needs of students interested in preparing for a variety of careers.

The traditional mathematics curriculum was heavily weighted toward analysis and its applications to physical sciences. One of the major innovations of the program in the original GCMC report was the introduction of linear algebra in the sophomore year and probability in the freshman year, thus exposing a large number of undergraduates to a wider range of mathematical topics. But because of its limited scope, the 1965 GCMC program is necessarily a single-track system, or essentially so. Should a college de-emphasize calculus and offer a variety of entrances and exits in its lower-level mathematics program, assuming that it has adequate staff? If so, what options should be available, and what advanced work should follow these courses? What service courses should be given? How should courses be taught in the light of the availability of computers? How should students be introduced to the mathematics needed for modern applications in the behavioral, biological, and engineering sciences?

3) The Initial Placement of Students. Although increasing numbers of college freshmen arrive with mathematical preparation that qualifies them for advanced placement, there is a simultaneous need for a greater variety of precalculus courses; the latter problem is especially critical at colleges having a policy of open admission. Does the mathematics curriculum provide suitable points of entry and exit for all students? Are placement procedures and policies in mathematics sufficiently flexible?

Thus, for a variety of reasons, it is no longer clear that there should be a single general curriculum in mathematics. Several alternative curricula in mathematics are emerging, and colleges with limited resources will soon have to make difficult choices from among these alternatives.

II. THE NATURE OF THIS STUDY

The intention of CUPM in establishing a committee to review the GCMC report was to publish a new version, incorporating changes as needed to correct deficiencies in the original study and modifying the curriculum in accordance with new conditions in mathematics and mathematics education. Some of the technical shortcomings of the original course outlines (pace and content) proved to be manageable and are taken up below, whereas other problems mentioned in Section I are more difficult, both intrinsically and in their effect on the whole concept of a compact general curriculum. Several of these problems have been considered by other CUPM panels. They include:

(a) Basic mathematics. See A Course in Basic Mathematics for Colleges (1971) and A Transfer Curriculum in Mathematics for Two-Year Colleges (1969).

(b) The training of elementary and secondary school teachers. See Recommendations on Course Content for the Training of Teachers of Mathematics (1971).

(c) A program in computational mathematics. See Recommendations for an Undergraduate Program in Computational Mathematics (1971).

(d) The impact of the computer on the content and organization of introductory courses in mathematics. See Recommendations on Undergraduate Mathematics Courses Involving Computing (1972).

(e) Upper-division courses in probability and statistics. See Preparation for Graduate Work in Statistics (1971).

(f) Lower-division courses in statistics. See Introductory Statistics Without Calculus (1972).

(g) Courses in the applications of mathematics. See Applied Mathematics in the Undergraduate Curriculum (1972).

(h) New teaching techniques and unusual curricula. See Newsletter #7, "New Methods for Teaching Elementary Courses and for the Orientation of Teaching Assistants" (not included in this COMPENDIUM).

Clearly, a definitive restatement of the GCMC report, if possible at all, would have to take into account not only these reports but others that will yet emerge from further study. However, suggestions for improvements in the recommendations of the GCMC report have been developed, and there is no need to defer their publication until a comprehensive reformulation is completed. Accordingly, the present pamphlet gives the current suggestions of CUPM for the half-dozen courses that include a substantial part of the mathematics enrollment in almost all colleges, namely first- and second-year calculus, linear algebra, and the elements of modern algebra. In the next section we shall discuss our proposed changes and our reasons for proposing them. It is entirely possible that when the questions raised in Section I are answered, the needs of large numbers of students will be met more adequately by some completely new selections of courses, rather than by the traditional ones. However, as we stated in Section I, basic calculus and algebra courses like Mathematics 1, 2, 3, 4, and 6 of the GCMC report will surely continue to be taught. Thus, those departments that have made or are making efforts to implement the recommendations of the 1965 GCMC report should continue to do so, with attention to the changes of detail proposed in Section III, changes that do no violence to the basic content of the core program originally proposed.

III. NEW DESCRIPTIONS OF THE BASIC CALCULUS AND ALGEBRA COURSES

As CUPM did in 1965, we use two devices to obtain enough flexibility to accommodate the diversity of achievement and ability of college freshmen. We describe a basic set of semester courses rather than year courses; this arrangement makes it easier for students to take advantage of advanced placement or to leave the mathematics program at a variety of levels. We also suggest that, wherever possible, a college should offer the basic courses Mathematics 1 through 4 every semester. This allows advanced placement students to continue a normal program in mathematics without interruptions. Moreover, students who need to begin with precalculus mathematics can follow it immediately with a calculus sequence.

The following list of basic courses is deliberately given with bare "college catalogue" descriptions, for we do not wish to seem overly prescriptive. In Section IV of this report, however, we include detailed course outlines and commentaries which are meant to identify those topics that we feel are most significant and to convey the spirit in which we recommend that these basic courses be taught.

Mathematics 1. Calculus I. Differential and integral calculus of the elementary functions with associated analytic geometry. [Prerequisite: Mathematics 0 or its equivalent. A description of Mathematics 0 is given in Section VI.]

Mathematics 2. Calculus II. Techniques of integration, introduction to multivariable calculus, elements of differential equations. [Prerequisite: Mathematics 1]

Mathematics 3. Elementary Linear Algebra. An introduction to the algebra and geometry of 3-dimensional Euclidean space and its extension to n-space. [Prerequisite: Mathematics 2 or, in exceptional cases, Mathematics 0]

Mathematics 4. Multivariable Calculus I. Curves, surfaces, series, partial differentiation, multiple integrals. [Prerequisites: Mathematics 2 and 3]

Mathematics 6L. Linear Algebra. Fields, vector spaces over fields, triangular and Jordan forms of matrices, dual spaces and tensor products, bilinear forms, inner product spaces. [Prerequisite: Mathematics 3]

Mathematics 6M. Introductory Modern Algebra. The basic notions of algebra in modern terminology. Groups, rings, fields, unique factorization, categories. [Prerequisite: Mathematics 3]

(More upper-division courses are described in Section V, and outlines for them can be found in Section VI.)

A reader who is familiar with the 1965 GCMC report will notice at once that some significant changes are being proposed here.

In the first place, that document sketched only the broad outlines of a curriculum, giving for each course a (rather ample) college catalogue description. Those who accepted the broad outlines immediately had to face the specific details of implementation: What is a reasonable rate at which to cover new material for the average student? What specific topics can be included if this rate is to be achieved?

CUPM has now attempted to answer these questions by means of commentaries on the course outlines. We have tried to develop a sense of what is meant by "the average student," taking account of the changing capabilities and preparation of the students in most undergraduate courses. Because of the frequent objection that the rate apparently suggested by the 1965 GCMC report was unreasonably fast, we have made a special effort to be realistic about the material that can be covered and to offer suggestions about the pace and style of its presentation. The course outlines are intended as existence proofs rather than as prescriptive recommendations; they represent solutions that CUPM feels are feasible, but we are aware that these are not the only possible solutions. In fact, we encourage others to devise different and more effective ways of achieving the same ends.

The commentaries accompanying the course outlines attempt to convey some specific ideas about the manner of presentation that CUPM feels is appropriate. The suggested pace has been indicated by assigning a number of hours to each group of topics and, in many cases, by more detailed suggestions of what to omit, what to mention only briefly, what to stress. Since a standard semester contains 42 to 48 class meetings, we arbitrarily allowed approximately 36 hours for each one-semester course, representing class time mainly devoted to the discussion and illustration of new material; thus the assignment of, say, six hours to a topic is a guide to the relative proportion of time to be spent on the topic. CUPM hopes that the commentaries are sufficiently detailed to show that the suggested material, in the recommended spirit, can actually be covered in 36 hours. The slack time that we have left provides for tests, review, etc. CUPM feels that a department that wishes to cover additional topics, or to provide deeper penetration of the topics listed, should not attempt to crowd such material into the course as outlined, but rather should either move to courses of four semester-hours or lengthen the program.

<u>The structure of the calculus sequence</u>. The 1965 GCMC report envisioned a program extending over four semesters to cover the traditional subject matter of calculus courses augmented by elementary linear algebra. The present study, on the other hand, seeks to return to the tradition of a basic two-semester calculus course serving both as an introduction to further work in calculus and as a unit for students who will end their study at this point. What is not traditional is that this course (Mathematics 1 and 2) should be a self-contained introduction to the essential ideas of calculus of both one and several variables, including the first ideas of differential equations. Students who stop at the end of a year generally need calculus as a tool rather than as an end in itself or as preparation for a heavily mathematical subject like physics, and they ought to encounter all the main topics, at least in embryo. The present arrangement was suggested in 1965 only as an alternative to a more conventional arrangement. The arguments given above for the present arrangement seem so compelling that now CUPM does not wish to suggest any alternative for the first year of calculus.

We have, however, preserved the feature of GCMC which makes the first semester (Mathematics 1) a meaningful introduction to the major ideas of calculus (limit, derivative, integral, Fundamental Theorem) in a single-variable setting.

To achieve the aims both of Mathematics 1 in this spirit and of Mathematics 1 and 2 as set forth above, a very intuitive treatment is necessary. The course should raise questions in the minds of students rather than rush to answer questions they have not asked. We consider such a treatment to be the right one in any case. It serves the needs of the many students who are taking calculus for its applications in other fields. It is also appropriate for mathematics majors.

Although the recommended treatment is intuitive, it is not intended to be careless. Theorems and definitions should be stated with care. Proofs should be given whenever they constitute part of the natural line of reasoning to a conclusion but are not technically complicated. Those proofs that require detailed epsilon-delta arguments, digressions, or the use of special tricks or techniques should be consciously avoided. Every theorem should be made plausible and be supported by pictures when appropriate and by examples exhibiting the need for the hypotheses. It is often the case that such preparation for a theorem falls short of a proof by only a little. In such cases the proof should be completed. However, stress should always be placed on the meaning and use of the theorem. The following examples should clarify these ideas.

(1) A student may get along, at least for a while, without the formal definition of a limit. But limits, and all other concepts of calculus, should be taught as <u>concepts</u> in some form at every stage. For example, the Fundamental Theorem of Calculus involves <u>two</u> concepts: the "limit" of a sum and the antiderivative. The theorem states that if f is continuous and if

$$\int_a^b f(x)\,dx$$

has been defined by approximating sums, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F' = f$. There is, to begin with, no obvious relation between the two sides of this equation, and an effort is required to make it credible. One natural approach depends on proving that if

$$G(x) = \int_a^x f(t)\,dt,$$

then $G' = F' = f$, whence G and F differ by a constant which can only be $F(a)$. Thus a simple test to determine whether a student understands the Fundamental Theorem is to ask him to differentiate

$$G(x) = \int_0^x \sqrt{1 + t^8}\,dt.$$

If he does not know how, he does not understand the theorem. It is dishonest to conceal the connection between the two concepts by conditioning the student to accept the formalism without his being aware that the concepts are there. On the other hand, to give the student only the concepts without making him fully aware of the formalism is to lose sight of the aspect of calculus that makes it such a powerful tool in applications as well as in pure mathematics.

(2) A "cookbook" course might teach the students to find the maximum of a function by setting its derivative equal to zero, solving the equation, and perhaps checking the sign of the second derivative; it might not discuss other kinds of critical points. A thoroughly rigorous course, on the other hand, might demand careful proofs of the existence of a maximum of a continuous function, Rolle's theorem, and so on. What we suggest for the first calculus course is a clear statement of the problem of maximizing a function on its domain, a precise statement of such pertinent properties as the existence of the maximum, and examples to indicate that the maximum, if it exists, may occur either at endpoints, points where the derivative equals zero, or points where the derivative does not exist.

The commentaries on Mathematics 1 through 4, given in Section IV, may also be consulted for a more detailed presentation of what we have in mind.

The computational aspects of calculus should be the center of attention in Mathematics 1 and 2. This means both the techniques of differentiation and integration and the numerical-computational methods that go along with them. Many people believe that a computer should be used, if possible, to supplement the formal procedures and reinforce their teaching. Guidelines on the use of computers in calculus courses appear in the report of the Panel on the Impact of Computing in Mathematics Courses (Recommendations on Undergraduate Mathematics Courses Involving Computing).

Finally, Mathematics 1 and 2, and indeed all the courses discussed here, should include examples of applications to other fields--the more concrete, the better.

The introduction of Mathematics 3 (Elementary Linear Algebra) was suggested in the 1965 GCMC report for the following reasons:

> Our arguments for placing a formal course in linear algebra in the first semester of the second year are more concerned with the values of the subject itself and its usefulness in other sciences than with linear algebra as a prerequisite for later semesters of calculus. Let us first consider prospective mathematics majors. Their official commitment to major in mathematics is usually made before the junior year of college. It is desirable that this decision be based on mathematical experience which includes college courses other than analysis. For these students linear algebra is a useful subject which

involves a different and more abstract style of reasoning and proof. The same contrasts could be obtained from other algebraic or geometric subjects but hardly with the same usefulness that linear algebra offers.

The usefulness of linear algebra at about the stage of Mathematics 3 is becoming more and more apparent in physics and engineering. In physics it is virtually essential for quantum mechanics, which is now being studied as early as possible in the undergraduate curriculum, especially in crystal structures where matrix formulation is most appropriate. In engineering, matrix methods are increasingly wanted in the second year or earlier for computation, for network analysis, and for linear operator ideas. The basic ideas and techniques of linear algebra are also essential in the social sciences and in business management. Students in these specialties are best served by an early introduction to the material in Mathematics 3.

We think, however, that Mathematics 3 is about the earliest stage at which the subject can profitably be taught to undergraduates generally. It can be taught to selected students in high school, though the high school version of the subject tends to be somewhat lacking in substance. High school students do not have a sufficiently broad scientific or mathematical background to motivate it and have not yet reached the stage of their curriculum when they can use it outside the mathematics classroom.

These reasons seem equally cogent today. However, CUPM is now more persuaded than in 1965 that it is important to have the terminology and elementary results of linear algebra available for the study of the calculus of several variables, and we propose a version of Mathematics 4 that takes as much advantage as possible of what the student has learned in Mathematics 3. How this can be done is explained in some detail in the commentary on Mathematics 4.

The present version of Mathematics 3 is a less demanding course than the Mathematics 3 described in the 1965 GCMC report, which indeed has frequently been criticized as containing too much material. Students who need more linear algebra than can reasonably be included in Mathematics 3 should also take Mathematics 6L.

In 1965 the GCMC report presented a calculus sequence that culminated in Mathematics 5, a course in vector calculus and Fourier methods. This has long been the accepted culmination of the calculus sequence. CUPM no longer feels that this material is to be regarded as basic in the same sense as the material of Mathematics 1 through 4. It is needed for graduate study of mathematics and for physics, but not for many other purposes. In fact, we do not suggest any single sequel to Mathematics 4 as part of the basic program but mention several possible courses at this level, recommending that each college choose one or more of these, or a course of its own design, according to its capabilities and the needs of its students.

Mathematics 6M (Introductory Modern Algebra) introduces the student to the basic notions of algebra as they are used in modern mathematics. We regard this course, or one of similar content, as an essential course that should be available in every college. We also recommend that every college that can do so offer a semester course containing further topics in linear algebra (Mathematics 6L; this is independent of Mathematics 6M). The rationale behind these recommendations is contained in the course descriptions.

IV. NEW OUTLINES FOR THE BASIC CALCULUS AND ALGEBRA COURSES

The following course outlines are intended in part as extended expositions of the ideas that we have in mind, in part as feasibility studies or existence proofs, and in part as proposals for the design of courses and textbooks. They are intended only to suggest content, not to prescribe it; they do, however, convey the spirit in which we believe the lower-division courses should be presented.

Mathematics 1. Calculus I.

[Prerequisite: Mathematics 0] Mathematics 1 is a one-semester intuitive treatment of the major concepts and techniques of single-variable calculus, with careful statements but few proofs; in particular, we think that epsilon-delta proofs are inappropriate at this level. We give a brief outline suggesting the amount of time for each topic; a more detailed commentary follows the outline.

COURSE OUTLINE

1. Introduction. (4 hours) Review of the ideas of function, graph, slope of a line, etc.

2. Limits, continuity. (3 hours) Limit and approximation defined intuitively. Derivatives as examples. Definition of continuity, types of discontinuity, Intermediate Value Theorem.

3. Differentiation of rational functions; maxima and minima. (5 hours)

4. Chain rule. (3 hours) Include derivatives of functions defined implicitly, inverse function and its derivative.

5. Differentiation of trigonometric functions. Higher derivatives. (3 hours)

6. Applications of differentiation. (3 hours) Tangent as "best" linear approximation. Differential, approximations using differentials. Extrema, curve sketching.

7. Intuitive introduction to area. (2 hours)

8. Definite integral. (3 hours)

9. Indefinite integrals, Fundamental Theorem. (4 hours)

10. Logarithmic and exponential functions. (3 hours)

11. Applications of integration. (3 hours)

COMMENTARY ON MATHEMATICS 1

The idea of this course is to provide the student with some understanding of the important ideas of calculus as well as a fair selection of techniques that will be useful whether or not he continues his study of calculus. If all this is to be done, formal proofs must necessarily be slighted. The following comments attempt to bring out the spirit that we have in mind.

1. Introduction. The basic ideas of slope of a straight line and of functions and their graphs can be reviewed in the context of an applied problem leading to the search for an extreme value of a quadratic or cubic polynomial. The ideas of increasing and decreasing functions and of maxima and minima should appear early. The direction of a graph at a point can be introduced as the limiting slope of chords. No formal definition of a limit need be given here: the derivative can be understood as a slope-function, and the vanishing of the derivative can be explored. Alternative interpretations are useful: derivative as velocity, as rate of change in general, and abstractly as $\lim_{x \to a} \frac{f(x)-f(a)}{x - a}$ using the intuitive idea of a limit. Derivatives of the functions $x \to x^2$, the general quadratic function, $x \to 1/x$, $x \to \sqrt{x}$ can be determined. The need for a deeper study of limits can be shown by the attempted computation of $f'(0)$ for $f: x \to \sin x$. Students can use tables or a computer to obtain values of $\frac{\sin x}{x}$ for x near 0.

2. Limits, continuity. We do not intend that this should be a rigorous treatment with ϵ-δ proofs. Rather, the presentation of continuity and the Intermediate Value Theorem should strive to make the definitions and the theorems (and the need for their hypotheses) clear by pictorial means. Limit theorems for sums, products, and quotients should be mentioned and various types of discontinuity illustrated by examples. A discontinuity not of jump type can be illustrated by sketching $\sin(1/x)$ near $x = 0$. The students should be convinced that rational functions are continuous (except at zeros of the denominator).

3. Differentiation of rational functions. The definition of derivative can be repeated with alternative notations:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} .$$

(It is desirable for students to be aware of all the notations that they are likely to meet in other subjects.) Application of limit theorems will yield differentiation formulas for integral powers, sums, products, polynomials; products and quotients; higher derivatives. Calculation of maxima and minima furnishes an immediate application. The distinction between local and global extrema needs to be made here. For curve-sketching, one can make good use of the proposition that a continuous function is monotone between successive local extrema. This is intuitively clear from a diagram and is easily proved.

4. Chain rule. Composite functions can be thought of as compositions of mappings from a line to a line. If the derivative is thought of geometrically as a local magnification, the chain rule then expresses the result of two successive magnifications.

It is worth exploring the geometrical interpretation of the derivative for the inverse function in terms of reflection in the line $y = x$.

5. Differentiation of trigonometric functions. An appropriate argument for demonstrating the value of $\lim_{x\to 0} \frac{\sin x}{x}$ is the geometrical argument using areas (which can be more readily justified than the one using lengths).

6. Applications of differentiation. The Mean Value Theorem is needed here. When this theorem is discussed, the student should see its pictorial representation and should understand that the conditions placed on the function (continuity on the closed interval, differentiability on the open interval) are no more, and no less, than is necessary.

The phrase "tangent as 'best' linear approximation" is intended to suggest the geometric meaning of the formula

$$f(x) = f(a) + f'(a)(x-a) + E,$$

where $E/(x-a) \to 0$ as $x \to a$.

7. Intuitive introduction to area. What is intended is a presentation along such lines as the following: Properties of area (e.g., $A(S) \geq 0$, $S \cap T = \emptyset \Rightarrow A(S \cup T) = A(S) + A(T)$). Area of rectangle accepted from geometry. Area within closed curve expressed in terms of areas under the graphs of functions. Approximation from above and below by sums of areas of rectangles. Idea of area as a limit by squeeze between upper and lower estimates. Error estimates (pictorially obtained) for monotone functions.

8. Definite integral. No formal proofs of the existence and properties of the integral are expected. A possible outline is as follows: Integral as a formal generalization of the idea of area--a number approximated by upper and lower sums formed for any function regardless of sign. Integral as limit of Riemann sums. Interpretation of integral as signed area. Integral of af and of $f + g$. Reversal of order of limits of integration. $\int_a^b f + \int_b^c f = \int_a^c f$. If $f(x) \leq g(x)$, then, for $a < b$, $\int_a^b f \leq \int_a^b g$.

Improper integrals: a 15-minute introduction to the idea, with some simple illustrative examples.

9. Indefinite integrals, Fundamental Theorem. Integral as a function of the upper endpoint, $F(x) = \int_a^x f(t)\,dt$. Intuitive discussion of the derivative of this function for continuous f; one can geometrically motivate the inequalities $\min_{[x,x+h]} f(t) \leq \frac{F(x+h) - F(x)}{h} \leq \max_{[x,x+h]} f(t)$ and then apply the squeezing or

pinching principle. The student should have some practice in the use of simple substitutions to evaluate integrals by the use of the Fundamental Theorem, including integrals of trigonometric functions. A brief introduction to tables of integrals is desirable at this point, to be continued in the next section when more functions are available.

10. Logarithmic and exponential functions. The definition of the logarithm as an integral is recommended.

One can give a heuristic argument for the formula for differentiating the logarithmic function: from assumed differentiability of the exponential function $f: x \to a^x$ $(a > 0)$, obtain $f'(x) = f'(0)a^x = Cf(x)$; hence, for the inverse function $g: x \to \log_a x$, note that $g'(x) = 1/(Cx)$. This is one way of suggesting the definition of the logarithmic function as an integral.

The Fundamental Theorem can be used to derive some basic rules for logarithms. For example, using $D(\log ax) = \frac{1}{x} = D(\log x)$ and integrating from 1 to b, one obtains $\log(ab) - \log(a) = \log(b) - \log(1)$ or $\log(ab) = \log(a) + \log(b)$.

Integration exercises requiring simple substitutions and the use of integral tables may be continued with special emphasis on integrands involving logarithmic and exponential functions.

The discussion of the differential equation $y' = ky$ provides an alternate approach to the definition of the exponential function. One starts with the solution $y = y_o e^{kx}$ for the differential equation with initial condition $y'(0) = y_o$. To show that this initial value problem defines the exponential function, we must prove that the problem has a unique solution. To do this, suppose z is any solution. Let $u = ze^{-kx}$. Then $z = ue^{kx}$ and, since $z' = kz$, it follows that $u' = 0$. Hence u = constant and the initial condition requires $u = y_o$. Hence $z = y$ and the solution is unique. The discussion of the equation $y' = ky$ also leads naturally to a discussion of growth and decay models as in the next section.

Students may be reminded at this point of the basic rules for operations with exponents, and these rules may be justified.

With the derivatives of logarithmic and exponential functions available, it is now possible to justify the expected rule for differentiating general powers and hence to provide more diversified drill problems on differentiation of elementary functions.

Further use of tables of integrals is now possible and is recommended in place of integration by ingenious devices. Of course, students must be able to make simple substitutions in order to use integral tables effectively.

11. Applications of integration. It is very desirable for the students to see applications of integration to as many fields as possible besides geometry and physics. Since such applications do not yet appear in many textbooks, we have included some specific suggestions with references to places where more information can be found.

It is particularly desirable to have some applications of the integral as a limit of Riemann sums, not merely as an antiderivative. Examples like the following can be used: defining volume of a solid by the parallel slice procedure; defining work done by a variable force applied over an interval as an integral over that interval suggested by Riemann sums; defining the capital value of an income stream obtained over time at a given rate and with interest compounded continuously as the limit of a Riemann sum (see Allen, Roy G. Mathematical Analysis for Economists. New York, St. Martin's Press, Inc., 1962).

An intuitive understanding of probability density (perhaps using the analogy with mass density for a continuous distribution of mass on a line) can also supply sufficient background for interesting applications of definite integrals, since if f is the probability density function (pdf) of a random variable X, then $\Pr(a < X < b) = \int_a^b f(x)\,dx$. Such important practical pdf's as the exponential and normal can be introduced, as well as the uniform, triangular, and other pdf's defined on a finite interval, e.g., $f(x) = 3(1 - x)^2$ if $0 \leqq x \leqq 1$, $f(x) = 0$ elsewhere. The normal pdf offers an opportunity to point out a function that cannot be integrated in elementary form and for which tables are available.

At the conclusion of this semester course, one is able to discuss the growth of a population governed by a differential equation of the form $N'(t) = (a - bN)N$. Here $N(t)$ is the size of the population at time t. If $b = 0$, then we have exponential growth with growth coefficient a. If, however, the growing population encounters environmental resistance (due to limited food or space, say), then $b > 0$ and the differential equation model involves a growth coefficient $(a - bN)$ that diminishes with increasing population size. This leads, when the differential equation is solved, to the logistic curve.

This differential equation and the corresponding logistic curve arise in many different contexts: (i) in the study of the phenomenon of diffusion through some population of a piece of information, of an innovative medical procedure, of a belief, or of a new fashion in clothes (see Coleman, James S. Introduction to Mathematical Sociology. New York, Free Press, 1964); (ii) in epidemiology where one studies the spread of a communicable disease (see Bailey, N. T. The Mathematical Theory of Epidemics. New York, Hafner Publishing Company, 1957); (iii) in biological studies of the size of populations of fruit flies as well as in demographic models of the U. S. population (see references in Keyfitz, Nathan. Introduction to the Mathematics of Population. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968); (iv) in the analysis of autocatalytic reactions in chemistry (see Frost, Arthur A. and Pearson, R. G. Kinetics and Mechanism; A Study of Homogeneous Chemical Reactions, 2nd ed. New York, John Wiley and Sons, Inc., 1961); (v) in studies of individual response and learning functions in psychology and in operations research models of advertising-sales relationships (see references in Rao, A. G. Quantitative Theories in Advertising. New York, John Wiley and Sons, Inc., 1970).

An hour or two spent on this differential equation offers an opportunity for students to review many parts of the course (inverse functions, the Fundamental Theorem, integration of a rational function, relationships between logarithms and exponentials, sketching the graph of a function with special attention to the asymptotic limiting population size $t \to \infty$). But the logistic example enables the instructor also to make other useful points: that mathematics is widely applied in not only the physical sciences and engineering, but also in the biological, management, and social sciences, and that the same piece of mathematics (the logistic differential equation) often makes an appearance in many different disguises and contexts. Finally, one can point out the progression from simple to more complex models (from pure exponential population growth to logistic growth) as one strives to develop mathematical models that better describe real-world phenomena and data, and one may conclude by pointing out that the logistic itself can be significantly improved by generalizations that take account of the age structure of the population and of stochastic and other complicating features of the growth process.

Mathematics 2. Calculus II.

[Prerequisite: Mathematics 1] Mathematics 2 develops the techniques of single-variable calculus begun in Mathematics 1 and extends the concepts of function, limit, derivative, and integral to functions of more than one variable. The treatment is intended to be intuitive, as in Mathematics 1.

COURSE OUTLINE

1. Techniques of integration. (9 hours) Integration by trigonometric substitutions and by parts; inverse trigonometric functions; use of tables and numerical methods; improper integrals; volumes of solids of revolution.

2. Elementary differential equations. (7 hours)

3. Analytic geometry. (10 hours) Vectors; lines and planes in space; polar coordinates; parametric equations.

4. Partial derivatives. (5 hours)

5. Multiple integrals. (5 hours)

COMMENTARY ON MATHEMATICS 2

1. Techniques of integration. The development of formal integration has been kept to the minimum necessary for intelligent use of

tables.

At the beginning of the course the instructor should review briefly the concepts of derivative, antiderivative, and definite integral, and should emphasize the relationships which hold among them (Fundamental Theorem). The importance of the antiderivative as a tool for obtaining values of definite integrals makes it desirable to have a sizable list of functions with their derivatives. This should motivate the study of the inverse trigonometric functions and the further development of integration methods through trigonometric substitutions and integration by parts. We recommend the use of the latter technique to obtain some of the reduction formulas commonly appearing in integral tables.

The instructor should point out that not all elementary functions have elementary antiderivatives and should use this fact to motivate the study of numerical methods for approximating definite integrals (trapezoidal rule, Simpson's rule). If students have access to a computer, they should be required to evaluate at least one integral numerically with programs they have written.

The improper integral with infinite interval of integration should be introduced. Comparison theorems should be discussed informally as there is not enough time for an excursion into theory. If additional time can be spared, the improper integral for a function with an infinite discontinuity in the interval of integration may be considered.

The method of "volumes by parallel slices" from Mathematics 1 should be applied here to find the volume of solids of revolution by the disk method.

2. Elementary differential equations. Solution of differential equations is a natural topic to follow a unit on formal integration, because it extends the ideas developed there and gives many opportunities to practice integration techniques. The coverage recommended below provides only a brief introduction to the subject, and it is intended that examples be simple and straightforward with time allowed for a variety of applications.

a. First-order equations. The notion of tangent field, solution curve. Separable equations. Linear homogeneous equations of first order. Applications: orthogonal trajectories, decay and mixing problems, falling bodies.

b. Second-order linear equations with constant coefficients. Homogeneous case, case of simple forcing or damping function; initial conditions. Applications: harmonic motion, electric circuits. These topics will require a brief discussion of the complex exponential function and DeMoivre's theorem.

3. Analytic geometry. Vectors and vector operations (sums, multiples, inner products; $\vec{i}$, $\vec{j}$, $\vec{k}$) should be introduced at the beginning of this unit because they greatly simplify the analytic geometry of lines and planes in 3-dimensional space. It is desirable to discuss the algebraic laws for vector operations, but proofs should be kept informal. The efficiency of vector notation can be illustrated by proving one or two theorems from elementary geometry by vector methods, e.g., that the three medians of a triangle intersect in a point.

Equations of lines and planes in 3-dimensional space should first be obtained in vector form and then translated into scalar equations. The students should be able to solve problems involving parallelism, orthogonality, and intersections; they should be familiar with the derivation (by vector methods) of the formula for the distance from a point to a plane.

A very brief introduction to polar coordinates is suggested. Students should learn how to draw simple polar graphs and to convert from x,y to r,θ and vice versa; they should be able to compute areas using polar coordinates.

The brief unit on parametric equations should include parametric representation of curves, motion along curves, velocity, acceleration, and arc length.

4. Partial derivatives. This section is intended to provide a basic acquaintance with functions of two or three variables and with the concept of and notation for partial derivatives.

Examples of functions of two or three variables should be given, and methods of representing such functions as surfaces by means of level curves or level surfaces should be shown. The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ should be defined and explained geometrically as slopes of appropriate curves in the planes $y = b$ and $x = a$, respectively. The concept of a tangent plane to a surface at a point should be introduced. In particular, the tangent plane, if it exists, is generated by the tangent lines in the x- and y-directions. Let these be, respectively,

$$z = c + \alpha(x - a), \quad y = b$$

and

$$z = c + \beta(y - b), \quad x = a,$$

where $c = f(a,b)$, $\alpha = f_x(a,b)$, $\beta = f_y(a,b)$. The normal $\vec{N}$ to the plane must therefore be perpendicular to the directions $\vec{i} + \alpha\vec{k}$ and $\vec{j} + \beta\vec{k}$; hence $\vec{N} = -\alpha\vec{i} - \beta\vec{j} + \vec{k}$, and the tangent plane has the equation

$$z = \alpha(x - a) + \beta(y - b) + c.$$

Extremum problems may be treated briefly as follows: At a point (a,b,c) where $z = f(x,y)$ has a maximum or minimum value, the tangent plane, if it exists, must be parallel to the xy-plane. This gives the necessary conditions that $f_x(a,b)$ and $f_y(a,b)$ both vanish at an interior extremum. Examples should be given to show that this condition is not sufficient. The second derivative test for extrema may be stated and illustrated by examples. Applications should be considered, including the method of least squares.

Topics such as the general concept of differentiability, the chain rule, and implicit functions are not included. (If it is possible to spend another hour or two on this section, it would be worthwhile to invest the time in studying the directional derivative for $z = f(x,y)$, noting that the directions of greatest increase of the function are orthogonal to level curves.)

5. Multiple integrals. The notions of double and triple integrals should be introduced through consideration of areas, volumes, or moments. Evaluation of double integrals by means of iterated integrals can be made plausible by calculating the volume of a solid by integrating the cross-sectional areas. Computations in both rectangular and polar coordinates should be included.

Mathematics 3. Elementary Linear Algebra.

This course is an introduction to the algebra and geometry of R^3 and its extension to R^n. Most students electing Mathematics 3 will have studied some calculus, but Mathematics 2 need not be considered a prerequisite.

Since the content and methods of linear algebra are new to most students, this course should begin by emphasizing computation and geometrical interpretation in R^3, to allow the student time to absorb unfamiliar concepts. In the outline below, the first 18 hours are devoted to this phase of the course. During the second half of the course, many of the same ideas are re-examined and extended in R^n, so that theorem-proving techniques can be developed gradually. Classroom experience has shown that the two outlines given for Mathematics 3 in the original GCMC report are too extensive, so the content of this outline has been reduced. Students who need to go further in linear algebra should resume their study of this subject in Mathematics 6L.

In selecting topics for this first course in linear algebra we confirm the judgments of the 1965 GCMC report: (1) the course content should be as geometrical as possible to offset its natural abstractness; (2) the treatment of determinants should be very brief; (3) the next topics to abbreviate under pressure of limited time are abstract vector spaces and linear transformations.

To prepare students adequately for Mathematics 4, this course must provide a knowledge of vectors in R^n, geometry in R^n, linear mappings from R^n into R^m and their matrix representations, matrix algebra, and determinants of small order. These topics, coupled with the solution of systems of linear equations, also provide a very useful course for students in the social and life sciences, and applications to those subjects serve to enliven the course. This much can be accomplished in one semester, but careful planning is required, and the degree of generality attempted in this first course must be controlled. For most classes it will be necessary to defer to Mathematics 6L consideration of such topics as $n \times n$ determinants, eigenvalues and eigenvectors, canonical forms, quadratic forms, orthogonal mappings, and the spectral theorem.

The instructor is expected to use judgment in adjusting the level of this course to the ability of his class by deciding upon a proper balance between concreteness and generality. Not all theorems have to be proved, but all should be motivated convincingly and illustrated amply. Coordinate-free methods should be used for efficiency and generality in definitions, proofs, and derivations, but students should also be required to perform computations with n-tuples. The examples developed early in the course for R^2 and R^3 should be carried along as illustrations in R^n.

COURSE OUTLINE

1. Vector algebra and geometry of R^3. (7 hours) Vector sum and scalar multiple, with geometric interpretations. Basic properties of vector algebra, summarized in coordinate-free form. Linear combinations of vectors; subspaces of R^3. Points, lines, and planes as translated subspaces. Vector and cartesian equations of lines and planes in R^3. Dot product in R^3; Euclidean length, angle, orthogonality, direction cosines. Projection of a vector on a subspace; the Gram-Schmidt process; vector proofs of familiar geometric theorems. Cross product in R^3, interpreted geometrically; the triple scalar product and its interpretation as the volume of the associated parallelepiped.

2. Systems of linear equations. (4 hours) Geometric interpretation of one linear equation in three variables and of a system of m linear equations; geometric description of possible solutions. Systems of m linear equations in n variables; solution by Gaussian elimination. Matrix representation of a linear system. Analysis of Gaussian elimination as the process of reducing the matrix to echelon form by three basic row operations (transposition of two rows, addition of one row to another, multiplication of a row by a nonzero scalar), followed by backward substitution. The consistency condition; use of an echelon form of the matrix of the system to obtain information about the existence, uniqueness, and form of the solution.

3. Linear transformations on R^3. (7 hours) Linear dependence and independence; the use of Gaussian elimination to test for linear independence. Bases of R^3; representation of a vector relative to a chosen basis; change of basis. Linear transformations on R^2 and R^3; matrix representation relative to a chosen basis. Magnification of area by a linear transformation on R^2; 2×2 determinants. Magnification of volume by a linear transformation on R^3; 3×3 determinant expressed as a triple scalar product and as a trilinear alternating form. The algebra of 3×1 and 3×3 matrices, developed as a representation of the algebra of vectors and linear transformations. Extension to $m \times n$ matrices; sum, scalar multiple, and product of matrices.

4. Real vector spaces. (8 hours) R^n as a vector space; subspaces of R^n. Linear independence, bases, standard basis of R^n. Representation of a linear mapping from R^n to R^m by an $m \times n$ matrix relative to standard bases. Range space and null space of a linear mapping from R^n to R^m; vector space interpretation of the solution of a system of linear equations in n variables, homogeneous and nonhomogeneous. Axiomatic definition of a vector space over R. A variety of examples in addition to R^n, such as polynomial spaces, function spaces, the space of $m \times n$ matrices, solutions of a homogeneous s stem of linear equations, solutions of a linear homogeneous differential equation with constant coefficients. Subspaces; linear combinations; sum and intersection of subspaces. Linear dependence, independence; extension of a linearly independent set of vectors to a basis. Basis and dimension; relation of bases to coordinate systems.

5. Linear mappings. (6 hours) Linear mappings of one real vector space into another. Images and preimages of subspaces; numerous examples to illustrate the algebra of mappings. Range space and null space of a mapping and their dimensions. Nonsingularity. Matrix representations of a linear mapping relative to chosen bases; review of matrix algebra and its relation to the algebra of mappings. Important types of square matrices, including the identity matrix, nonsingular matrices, elementary matrices, diagonal matrices. The relation of elementary matrices to Gaussian elimination, row operations, and nonsingular matrices. Rank of a matrix; determination of rank and computation of the inverse of a nonsingular matrix by elementary row operations.

6. Euclidean spaces. (4 hours) Real inner products introduced axiomatically; examples. Schwarz inequality; metric concepts and their geometric meaning in R^n. Orthogonality, projections, the Gram-Schmidt process, orthogonal bases. Proofs of geometric theorems in R^n.

7. Determinants. (optional) If time is available, the properties and geometric meaning of 2×2 and 3×3 determinants may be used to motivate a brief treatment of $n \times n$ determinants. Emphasis should be given to properties of determinants that are useful in

matrix computations.

COMMENTARY ON MATHEMATICS 3

1. Vector algebra and geometry of R^3. The primary objectives of this first section are to develop geometric insight into R^3 and to gain experience in the methods of vector algebra. Vectors should be introduced both as ordered triples and as translations, the latter leading naturally to a coordinate-free interpretation. Algebraic properties of vectors should be stated in coordinate-free form; later in the course they can be taken as axioms for an abstract vector space. The geometry of lines and planes should be stressed, as should the geometric meanings of the dot and cross product. The triple scalar product should be shown to be an alternating trilinear form, later to be called a 3×3 determinant.

2. Systems of linear equations. The problem of determining the subspace spanned by a given set of vectors in R^3 leads directly to a system of m linear equations in three variables. The solutions of such a system can first be interpreted geometrically as intersections of translated subspaces to provide insight for the consideration of $m \times n$ systems. To solve a system of m linear equations in n variables, Gaussian elimination provides an effective algorithm that should be stressed as a unifying computational method of linear algebra. The system $AX = Y$ can be represented by the augmented matrix $(A|Y)$. A succession of elementary row operations can be used to replace the matrix $(A|Y)$ by $(E|Z)$, where E is in row echelon form. The solutions of $AX = Y$ coincide with those of $EX = Z$ and are easily obtained by backward substitution since E is in row echelon form. At this stage the major emphasis should be concrete and computational. Formal representation of elementary row operations by elementary matrices and the concept of row equivalence are considered in Section 5. For some classes it may be appropriate to suggest that the operations discussed above can be carried out with complex numbers as well as with real numbers.

3. Linear transformations on R^3. Linear independence, basis, linear transformations, and matrix representations are introduced

concretely here and then are repeated in the next section for R^n and for the general vector spaces to provide a gradual, spiral development of these important concepts.

Determinants are introduced geometrically for the 2 × 2 and 3 × 3 cases. Properties of these determinants should be observed in a way that facilitates generalization to n × n determinants, perhaps in a later course.

Matrix algebra arises naturally as a representation of the algebra of vectors and linear transformations on R^3 and then is easily generalized to matrices of arbitrary size.

4. Real vector spaces. Consideration of R^n can be motivated by a geometric interpretation of the algebra of m × n matrices. The basic concepts of linear algebra in R^n should be studied briefly as natural extensions of the same concepts in R^3. The stage is then set for a general study of real vector spaces in coordinate-free form, illustrated amply by a wide variety of familiar examples. Theorems of various degrees of difficulty can now be proved for any finite-dimensional vector space, and students can be expected to prove some of them.

The concepts of linear independence, basis, and dimension need to be illustrated with many examples. The student should understand that questions about linear independence reduce to questions about the solution of a system of linear equations to which Gaussian elimination provides an answer. The same method can be used to express a given vector in terms of a given basis.

A brief mention of complex vector spaces is appropriate for some classes.

5. Linear mappings. Properties of linear mappings, including rank and nullity and their relation to the dimension of the domain space, should now be treated generally. Prove that if R and T are nonsingular, then the rank of RST equals the rank of S. The isomorphism of matrix algebra with the algebra of linear transformations should be exploited. Elementary matrices, one for each of the three types of elementary row operations, can be used to effect row operations on matrices. A matrix is nonsingular if and only if it is the product of elementary matrices. For some nonsingular P, PA

is in echelon form. By observing that the column rank of a matrix in echelon form is the number of nonzero rows, one can show that the row rank and the column rank of any matrix are equal. Elementary row operations should be used to develop a constructive method for computing the inverse of a nonsingular matrix.

6. Euclidean spaces. The coordinate-free formulation of a real inner product as a bilinear, symmetric, positive-definite function from $V \times V$ to R, where V is a vector space over R, can be viewed as a natural abstraction of the dot product in R^3. Its role as a source of all metric concepts should be emphasized. The Schwarz inequality should be derived in coordinate-free from and then interpreted concretely in various inner product spaces to obtain the classical inequalities. The flavor of this section should be strongly geometric.

Mathematics 4. Multivariable Calculus I.

[Prerequisites: Mathematics 2 and 3] This course completes a four-semester introductory sequence of calculus and linear algebra, building on the intuitive notions of multivariable calculus from Mathematics 2 and the linear algebra of Mathematics 3. The four semesters contain all the topics that seem to us to be essential for every student who has only this much time to spend on calculus; subsequently, students with various interest will need different courses.

A considerable advance in conceptual depth should be possible in Mathematics 4, but there is not enough time for full formal proofs of the theorems; these proofs are not needed except by students who are going at least as far as Mathematics 12, and their omission makes it possible to cover more topics here. Since maximum use should be made of Mathematics 3 and since some of the material suggested here is not yet standard, we give a fairly extensive commentary on the outline.

COURSE OUTLINE

1. Curves and particle kinematics. (5 hours)
2. Surfaces; functions from R^m to R^1. (7 hours)
3. Taylor's theorem for $f: R^m \to R^1$. (5 hours)

4. Sequences, series, power series. (6 hours)
5. Functions from R^m to R^n $(m, n \leqq 3)$. (2 hours)
6. Chain rule. (5 hours)
7. Iterated and multiple integrals. (6 hours)

COMMENTARY ON MATHEMATICS 4

1. Curves. A (parametrically represented) curve in R^n is thought of here as the range of a function $f: R^1 \to R^n$ (with principal emphasis on $n = 2, 3$). Set $\vec{x} = (x_1,\ldots,x_n) = f(t)$. The idea of $\lim_{t\to\alpha} f(t) = \vec{a}$ can be introduced through $\lim_{t\to\alpha} |f(t) - \vec{a}| = 0$; this limit is the same as the component-by-component limit. Continuity can be defined via $\lim_{t\to\alpha} f(t) = f(\alpha)$. The derivative of f is associated with the tangent vector. A curve in R^2 or R^3 can be thought of as the path of a particle; the first and second derivatives with respect to time are then interpreted as velocity and acceleration. At this point plane curves should be reviewed with attention to curve tracing and convexity. The present point of view makes it easy to derive the reflection properties of the conic sections: for example, if $\vec{a}$ and $\vec{b}$ are the foci of an ellipse and $\vec{x}$ is a point on the ellipse, then

$$|\vec{x} - \vec{a}| + |\vec{x} - \vec{b}| = k.$$

Differentiate with respect to the parameter t, using

$$\frac{d|z|}{dt} = \frac{1}{|\vec{z}|}\left(\vec{z} \cdot \frac{d\vec{z}}{dt}\right),$$

to obtain

$$\vec{v} \cdot \frac{\vec{x} - \vec{a}}{|\vec{x} - \vec{a}|} = - \vec{v} \cdot \frac{\vec{x} - \vec{b}}{|\vec{x} - \vec{b}|},$$

where $\vec{v}$ = unit tangent vector. This implies that the rays to the foci from a point on the ellipse make equal angles with the tangent at that point.

2. Surfaces. Consider functions $f: R^m \to R^1$ with emphasis on the case $m = 2$, interpreting the graph of such a function as a surface in R^3. The Euclidean norm $|\ldots|$ in R^m is the most useful, but

it is sometimes also useful to have the maximum norm $\|\ldots\|$ and the inequality $\|\vec{x}\| \leq |\vec{x}|$. The limit of $f\colon R^m \to R^1$ at $\vec{a}$ should be defined, and continuity should be defined by $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$. The derivative J of f at $\vec{a}$ can be introduced as the linear transformation from R^m to R^1 satisfying

$$f(\vec{x}) = f(\vec{a}) + J(\vec{x} - \vec{a}) + o(|\vec{x} - \vec{a}|)$$

(but the o-notation itself should not be introduced unless there is time to get the students thoroughly used to it). Thus with R^m as a space of column vectors, $f'(\vec{a})$ is a $1 \times m$ matrix (or row vector), also called the gradient. This should be illustrated especially for $m = 2$ and compared with the treatment of the tangent plane in Mathematics 2. Here $J = \text{grad } f|_{\vec{x} = \vec{a}} = (f_1(\vec{a}), f_2(\vec{a}))$, where $f_i(\vec{a}) = \frac{\partial f}{\partial x_i}|_{\vec{x} = \vec{a}}$. The directional derivative is the rate of increase of $f(\vec{x})$ in the direction of a given unit vector $\vec{t}$, namely $\vec{t}\cdot\text{grad } f$. The notation of differentials should be at least mentioned since books on other subjects will presumably continue to use it. From the present point of view, $df = \vec{v}\cdot\text{grad } f$, where $\vec{v}$ is an arbitrary vector, conventionally denoted by $\vec{i}\,dx + \vec{j}\,dy$. The gradient is a vector in the direction of maximal rate of increase and is orthogonal to level lines.

In general, J is the $1 \times m$ matrix (row vector) with components $\frac{\partial f}{\partial x_i}|_{\vec{x} = \vec{a}}$, $i = 1, \ldots, m$, and the idea of the directional derivative and of the gradient extend to the general case.

It is desirable to use the linear approximation also for non-geometric applications, in particular to estimate the effect on the computed value of a function resulting from small errors in the variables (conventionally done in differential notation).

The Implicit Function Theorem for $f(x,y) = 0$ should be treated geometrically. If J is not the zero vector, the level line $z = 0$ of the surface $z = f(x,y)$ defines a function $y = g(x)$ locally so that $f(x,g(x)) = 0$ (this should be treated with a picture, not a proof). The equation $\frac{dy}{dx} = -\frac{\partial f}{\partial x}/\frac{\partial f}{\partial y}$ follows.

3. Taylor's theorem. Begin with $f: R^1 \to R^1$. An easy approach to the the theorem assumes $|f^{(n+1)}(t)| < M$ for $|t - a| < |x - a|$; repeated integration on (a,x) yields

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + R_n(x),$$

where

$$|R_n(x)| < \frac{M|x - a|^{n+1}}{(n+1)!}.$$

Typical examples: binomial, sine, cosine, exponential, logarithm, arctangent. Such examples lead naturally to the idea of convergence of an infinite series.

As an application one can expand $f: R^2 \to R^1$ to second-degree terms, first with respect to x and then with respect to y, and in reverse sequence; assuming continuous third derivatives one then shows that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Taylor's theorem can now be derived for $f: R^2 \to R^1$ and applied to extreme value problems. (Extreme value problems for $f: R^m \to R^1$, $m > 2$, should be treated lightly, if at all.)

4. Sequences, series, power series. It is appropriate to introduce the epsilon and neighborhood definitions of limit of sequences and series of constants, but little attention need be paid to conditional convergence; in the context of this course, absolute convergence is the significant idea. The comparison test, ratio test, and integral test can be treated.

For power series it is important to know that there are an interval and a radius of convergence; a useful formula for the radius is $\lim |a_n/a_{n+1}|$, provided the limit exists. The students should know that the differentiated and integrated series have the same interval of convergence as the original series; proofs can be omitted unless there is ample time. Applications; for example, approximate computation of $\int_0^x e^{-t^2} dt$ for small x.

5. <u>Functions from R^m to R^n</u>. Interpret $f: R^m \to R^n$ in various ways, e.g., the graph as a subset of R^{n+m}; representation of the range of f as a hypersurface. Interpretation by vector fields, e.g., stationary field of force or stationary flow ($R^3 \to R^3$); parametric representation of a surface ($R^2 \to R^3$); unsteady plane flow ($R^3 \to R^2$). Limit and continuity of f: $\vec{L} = \lim_{\vec{x}\to\vec{a}} f(\vec{x})$ means

$\lim_{|\vec{x}-\vec{a}|\to 0} |f(\vec{x}) - \vec{L}| = 0$. Note that this is equivalent to taking the limit component by component: setting $\vec{z} = f(\vec{x}) = (z_1, \ldots, z_n)$, f can be considered as an ordered set of n mappings $\varphi_k: \vec{x} \to z_k$ from R^m to R^1, and $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{b}$ if and only if $\lim_{\vec{x}\to\vec{a}} \varphi_k(\vec{x}) = b_k$, $k = 1, \ldots, n$.

The derivative J of f at $\vec{a}$ is defined as the linear transformation satisfying

$$f(\vec{x}) = f(\vec{a}) + J(\vec{x} - \vec{a}) + o(|\vec{x} - \vec{a}|).$$

As a linear mapping from R^m to R^n, J may be represented as an $n \times m$ matrix, the Jacobian matrix, with elements

$$J_{ki} = \frac{\partial \varphi_k}{\partial x_i} \Big|_{\vec{x} = \vec{a}}.$$

6. <u>Chain rule</u>. Composition of functions $f: R^m \to R^n$, $g: R^n \to R^p$; emphasis on application to change in parametic equations of a surface under a coordinate transformation (either of domain or range space). Lemma (continuity of linear transformation): For each linear transformation L there is a constant K such that $|L\vec{x}| \leq K|\vec{x}|$ for all $\vec{x}$. Proof: Let $\vec{e}_1, \ldots, \vec{e}_m$ be unit coordinate vectors. $L\vec{x} = L(\Sigma\, x_i\vec{e}_i) = \Sigma\, x_i L\vec{e}_i$, whence

$|L\vec{x}| \leq \Sigma|x_i L\vec{e}_i| \leq \max|x_i| \cdot \Sigma|L\vec{e}_i| = K \max|x_i| \leq K|\vec{x}|$.

Theorem: If J_f, J_g, J_{gf} are the derivatives (Jacobian matrices) of f, g, and gf, then $J_{gf} = J_g J_f$.

Proof: Set $f(\vec{x}) = f(\vec{a}) + J_f(\vec{x} - \vec{a}) + o(\vec{x} - \vec{a})$,

$g(\vec{z}) = g(\vec{b}) + J_g(\vec{z} - \vec{b}) + o(\vec{z} - \vec{b})$,

$\vec{z} = f(\vec{x})$, $\vec{b} = f(\vec{a})$, and apply the lemma above.

Special cases: $R^1 \to R^1 \to R^1$, etc. Applications in spaces of dimension at most 3, particularly to polar and cylindrical coordinates.

Coordinate transformations; interpretation of the Jacobian determinant det J as a local scale factor for "volume."

7. Iterated and multiple integrals. A more careful and more general treatment than in Mathematics 2. Iterated integrals of functions on R^2 and R^3 (partly review from Mathematics 2). Multiple integrals as limits of sums; evaluation by iterated integration.

Additivity, linearity, positivity of integrals. Application to volumes, etc.

Change of variables of integration; geometrical interpretation as coordinate transformation. Special attention to polar and cylindrical coordinates. Further applications.

Mathematics 6L. Linear Algebra.

[Prerequisite: Mathematics 3] This course is the first course in linear algebra proper, although it assumes the material on that subject taught in Mathematics 3. It contains the usual basic material of linear algebra needed for further study in mathematics except that the rational canonical form is omitted and the Jordan form is given only brief treatment.

We point out that Mathematics 6L and 6M together do not include the following topics in the outline of the course Abstract Algebra given in the 1965 CUPM report Preparation for Graduate Study in Mathematics [page 453]: Jordan-Hölder theorem, Sylow theorems, exterior algebra, modules over Euclidean rings, canonical forms of matrices, elementary theory of algebraic extensions of fields.

COURSE OUTLINE

1. Fields. (4 hours) Definition. Examples: Q, R, C, Q(x), R(x), C(x), $Q(\sqrt{2})$. The fields of 2, 3, 4 elements explicitly constructed by means of addition and multiplication tables. Characteristic of a field.

2. Vector spaces over fields. (9 hours) Definition. Point out that the material of Mathematics 3 and 4 on vector subspaces of

R^n and their linear transformations carries over verbatim to vector spaces over arbitrary fields. Linear dependence. Bases, dimension, subspaces, direct sums. Linear transformations and matrices. Rank, image and kernel. The preceding material is to be thought of as review of the corresponding material in Mathematics 3 and 4. Matrix representation of linear transformations. Change of basis. A transformation is represented by two matrices A and B if and only if there exist nonsingular matrices P and Q so that $A = PBQ$, i.e., if and only if A and B are equivalent. Systems of linear equations. Relation to linear transformations. Existence and uniqueness of solutions in both the homogeneous and nonhomogeneous cases. Two systems have the same solution if their matrices are row equivalent. Equivalence under elementary row operations of equations and matrix, row echelon form, explicit method for calculating solutions.

3. Triangular and Jordan forms. (6 hours) State without proof that C is algebraically closed. Any linear transformation (matrix) over C has a triangular matrix with respect to some basis (is similar to a triangular matrix). Nilpotent matrices and transformations and their similarity invariants, i.e., such a transformation is completely determined by vectors v_i on which it is nilpotent of index q_i, $i = 1, \ldots, r$. Definition of eigenvalue. Jordan form over C via the theorem: If T is a linear transformation on a vector space V over C with $\dim V < \infty$ and if T has eigenvalues λ_i with multiplicities m_i, $i = 1, \ldots, r$, then $V = \bigoplus_1^r V_i$ with $T(V_i) \subset V_i$, $\dim V_i = m_i$, and $T - \lambda_i$ is nilpotent on V_i. Elementary divisors and minimum polynomial. The Cayley-Hamilton theorem.

4. Dual spaces and tensor products. (6 hours) The dual space of a vector space. Adjoints of linear transformations and transposes of matrices. Finite-dimensional vector spaces are reflexive. Tensor products of vector spaces as the solution of a universal problem. Behavior of tensor product with regard to direct sums, basis of a tensor product, change of base fields by means of tensor products.

5. Forms. (5 hours) Definition of bilinear and quadratic forms. Matrix of a form with respect to different bases. A form yields a linear transformation of the vector space into its dual. General theory of symmetric and skew-symmetric forms, forms over

fields where $2 \neq 0$. Diagonalization and the canonical forms, both a form and a matrix approach. The case of the real and complex fields, Sylvester's theorem.

6. Inner product spaces. (6 hours) Definition over R and C. Orthogonal bases, Gram-Schmidt process, Schwarz inequality for the general case. Review of the treatment of Euclidean space in Mathematics 3. Self-adjoint and hermitian linear transformations and their matrices with respect to an orthonormal basis. Eigenvalues and eigenvectors. All eigenvalues of self-adjoint linear transformations are real. The spectral theorem in several equivalent forms both for transformations and for matrices. Applications to classification of quadrics. Relations between quadratic forms and inner products. Positive-definite forms.

COMMENTARY ON MATHEMATICS 6L

At all times a computational aspect must be preserved. The students should be made aware of the constant interplay between linear transformations and matrices. Thus they should be required to solve several systems of linear equations; find the Jordan form, invariant factors, and elementary divisors of numerical matrices; diagonalize symmetric matrices and find the matrix P such that PAP^{tr} is diagonal; and also diagonalize symmetric and hermitian matrices by means of orthogonal and unitary similarity. In Section 6 the concept of tensor product should be exploited in complexifying a real space in order to prove that eigenvalues of self-adjoint transformations are real.

A treatment of the Jordan form along the lines of Section 3 can be found in Halmos, Paul R. Finite-Dimensional Vector Spaces, 2nd ed. New York, Van Nostrand Reinhold Company, 1958.

In addition to the definitive treatment of tensor products to be found in Book I, Chapter II of Bourbaki's treatise Algèbre Linéaire (Bourbaki, N. Éléments de Mathématiques, Livre I, Chapitre II (Algèbre Linéaire), 3ème éd. Paris, Hermann et Cie., 1962) or in MacLane and Birkhoff's Algebra (MacLane, Saunders and Birkhoff, Garrett. Algebra. New York, The Macmillan Company, 1967), briefer and perhaps more accessible treatments may also be found in Goldhaber and Ehrlich (Goldhaber, Jacob K. and Ehrlich, Gertrude. Algebra. New York, The Macmillan Company, 1970) and in Sah (Sah, Chih-Han. Abstract Algebra. New York, Academic Press, Inc., 1966).

All theorems dealing with linear transformations should be accompanied by parallel statements about matrices. Thus, for example, the spectral theorem should be stated in the following three forms for real vector spaces:

I(a). Let V be a real finite-dimensional inner product space and let T be a symmetric linear transformation on V. Then V has an orthonormal basis of eigenvectors of T.

I(b). With the same hypotheses as I(a), there exists a set of orthogonal projections $E_1, \ldots, E_r$ of V such that $T = \sum_1^r \lambda_i E_i$ where λ_i are the distinct eigenvalues of T.

II. Let A be a symmetric real matrix. Then there exists an orthogonal matrix P such that $PAP^{-1} = PAP^{tr}$ is diagonal.

The student should understand that these are equivalent theorems and, given T or A, should be able to compute $\lambda_i E_i$ and P explicitly in low-dimensional cases.

In dealing with positive-definite forms, one should point out that these are equivalent to inner products and that yet another form of the spectral theorem asserts:

Let A, B be symmetric real matrices with A positive-definite. Then there is a matrix P such that $PAP^{-1} = I$ and PBP^{-1} is diagonal.

Mathematics 6M. Introductory Modern Algebra.

[Prerequisite: Mathematics 3] This course introduces the student to the basic notions of algebra as they are used in modern mathematics. It covers the notions of group, ring, and field and also deals extensively with unique factorization. The language of categories is to be used from the beginning of the course, but the formal introduction of categories is deferred to the end of the term. In order to make the material meaningful to the student, the instructor must devise concrete examples that will relate to the student's earlier experiences.

We again point out that Mathematics 6L and 6M together do not include the following topics in the outline of the course Abstract Algebra given in the 1965 CUPM report Preparation for Graduate Study in Mathematics [page 453]: Jordan-Hölder theorem, Sylow theorems, exterior algebra, modules over Euclidean rings, canonical forms of matrices, elementary theory of algebraic extensions of fields.

COURSE OUTLINE

1. Groups. (10 hours) Definition. Examples, vector subgroups of R^n, linear groups, additive group of reals, permutation and transformation groups, cyclic groups, groups of symmetries of geometric figures. Subgroups. Order of an element. Theorem: Every subgroup of a cyclic group is cyclic. Coset decomposition. Lagrange's theorem. Normal subgroups. Homomorphisms of groups. The first two homomorphism theorems.

2. Rings and fields. (9 hours) Definitions. Examples: integers, integers modulo m, polynomials over the reals, the rationals, the Gaussian integers, all linear transformations on a vector subspace of R^n, rings of functions. Zero divisors and inverses. Division rings and fields. Domains, quotient fields as solution to a universal problem. Homomorphisms, isomorphisms, monomorphisms. Ideals. Congruences in Z. Tests for divisibility by 3, 11, 9, etc. Fermat's little theorem: $a^{p-1} \equiv 1 \pmod{p}$, using group theory. Residue class rings.

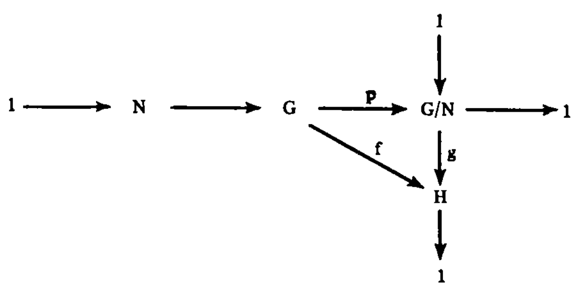
3. Unique factorization domains. (11 hours) Prime elements in a commutative ring. Reminder of unique factorization in Z. Examples where unique factorization fails, say in $Z[\sqrt{-5}]$. Definition of Euclidean ring, regarded as a device to unify the discussion for Z and $F[x]$, F a field. Division algorithm and Euclidean algorithm in a Euclidean ring; greatest common divisor; Theorem: If a prime divides a product, then it divides at least one factor. Unique factorization in a Euclidean ring. GCD and LCM. Theorem: A principal ideal domain is a unique factorization domain. Gauss' lemma. Theorem: If D is a unique factorization domain, then $D[x]$ is also a unique factorization domain.

4. Categories of sets. (6 hours) The notion of a category

of sets. The categories of sets, groups, abelian groups, rings, fields, vector spaces over the reals. Epimorphisms, monomorphisms, isomorphisms, surjections, injections. Examples to show that epimorphisms may not always be surjections, etc. Exact sequences. Functors and natural transformations. The homomorphism theorems of group and ring theory in categorical language, monomorphisms and epimorphisms in the categories of groups, rings, and fields.

COMMENTARY ON MATHEMATICS 6M

From the beginning of the course the language of category theory should be used. Thus arrows, diagrams (commutative and otherwise), and exact sequences should be defined and used as soon as possible. For example, the first homomorphism theorem for groups should be stated as follows: Let G be a group, f a surjective homomorphism of G onto H, and $N = \ker f$. If $p: G \to G/N$ is the natural projection, then there exists a unique homomorphism g which makes the diagram



commutative with exact row and column.

In addition to extensive treatments of categories in such treatises as Mitchell's (Mitchell, Barry. Theory of Categories. New York, Academic Press, Inc., 1965) and MacLane and Birkhoff's (MacLane, Saunders and Birkhoff, Garrett. Algebra. New York, The Macmillan Company, 1967), a brief treatment of this subject can also be found in Goldhaber and Ehrlich (Goldhaber, Jacob K. and Ehrlich, Gertrude. Algebra. New York, The Macmillan Company, 1970).

In the section on groups, the general linear and orthogonal groups should be introduced and based on the material of Mathematics 3. The affine group and its relationship to the general linear group should be discussed. The students should do a considerable number of concrete computations involving groups and counting problems.

In Section 2 there is an opportunity to introduce some elementary number theory: the Euler phi-function counts the number of units of the ring $Z/(n)$; $a^{\varphi(n)} \equiv a$ is a theorem that can be demonstrated by these methods; the divisibility of $2^{32}+1$ by 641 can easily be asserted using congruences; calendar and time problems can also be introduced to illustrate the notions of congruence and ideals. Again, the homework should include many problems of this kind so that the student gains some familiarity with the notions introduced here. Fields of 2, 4, 3, 9, and p^n elements, p a prime, should be introduced, at least in the exercises.

In Section 3 the Euclidean algorithm should be introduced and used to calculate the greatest common divisor of large integers and of polynomials having degree higher than three. If time permits, Euclidean rings different from Z and $F[x]$ should be introduced in the homework. The integers of certain quadratic number fields are especially suitable for this.

In Section 4 the material of the first three sections must be used to illustrate the definitions at each step; when natural transformations are discussed, the "naturality" of the homomorphism theorems should be underlined and many examples given. The language of categories should be familiar to all students who pursue mathematics beyond this level. This language reveals how much various mathematical disciplines have in common and how different disciplines may be related to each other. By virtue of its generality, category theory is a very valuable source of meaningful conjectures and an effort should be made, even at this level, to emphasize this.

V. A FOUR-YEAR CURRICULUM

In the 1965 GCMC report, CUPM presented a curriculum for four years of college mathematics. It devoted a considerable amount of attention to both upper- and lower-division courses other than basic calculus and algebra, indicating their relationships and their significance for various kinds of students. The 1965 GCMC report is now out of print, but many of its suggestions are still relevant, at least to one very common kind of mathematics curriculum. Consequently, CUPM feels that it will be useful to repeat some of its suggestions of 1965 with modifications prompted by recent developments and to reprint some of the course outlines even though experience has shown they are open to objections such as excessive length.

We have not described a special one-year course in mathematics appreciation for students in liberal arts colleges because we think that it is better for the student to take Mathematics 1 and 2, 1 and 2P, or 1 and 3. (A description of the probability course Mathematics 2P is given in Section VI.) These ways of satisfying a liberal arts requirement open more doors for the student than any form of appreciation course, and they are consistent with our view that mathematics is best appreciated through a serious effort to acquire some of its content and methodology and to examine some of its applications.

A student who has successfully completed Mathematics 1 may select Mathematics 2, 2P, or 3 according to his interests. In particular, many students who are interested in the social sciences will choose Mathematics 2P or 3 in preference to Mathematics 2.

For those students for whom a sequence beginning with Mathematics 1 is not possible or not appropriate, there are several possibilities. In the first place, Mathematics 0 and 1 forms a reasonable year sequence for students whose preparation will not permit them to start with Mathematics 1. In many colleges students have been taking and will continue to take a full year course like Mathematics 0. (A description of Mathematics 0 is given in Section VI.)

Among the students for whom neither Mathematics 0 nor Mathematics 1 is appropriate we recognize a sizable number who are preparing to become elementary school teachers. Their needs should be met by special courses described in the CUPM publication Recommendations on Course Content for the Training of Teachers of Mathematics (1971).

Finally, there is a rather large number of students who need further study of mathematics in order to function effectively in the modern world. Some have never had the usual mathematics courses in high school, whereas others have not achieved any mastery of the topics they studied. These students are older and more mature than high school students, and so they need a fresh approach to the necessary topics, if possible one involving obviously significant applications to the real world. One suggestion is the course Mathematics A, "Elementary functions and coordinate geometry," from the CUPM report A Transfer Curriculum in Mathematics for Two-Year Colleges (1969). For students who are not ready even for Mathematics A, we suggest the less conventional course Mathematics E described in considerable detail in the CUPM report A Course in Basic Mathematics for Colleges (1971).

1. Lower-division courses.

By lower-division courses we mean Mathematics 1, 2, 2P, 3, 4, Mathematics 0, and any other basic precalculus courses that are offered. Mathematics 1, 2, 3, and 4 have already been described in detail. Outlines of Mathematics 0 and of Mathematics 2P appear in Section VI reproduced from the 1965 GCMC report.

2. Upper-division courses.

The following list of typical courses might be offered once a year or, in some cases, in alternate years, to meet the needs of students requiring advanced work in mathematics. At many colleges some of these upper-division courses are combined into year courses. Which of them are offered will depend on the needs of the students and special qualifications of the staff. The order is a rough indication of the level. The course outlines for Mathematics 6L and 6M appear in Section IV and the outlines for the remaining courses appear in Section VI.

Although we describe the upper-division work in terms of semester courses, these advanced subjects may also be treated by independent or directed study, tutorials, or seminars. This is especially appropriate in a small college where it may not be possible to organize classes in every subject.

Mathematics 5. Multivariable Calculus II. This is a calculus course to follow Mathematics 4. Two possibilities are (1) a course in vector calculus and (2) a course consisting of selected topics in analysis. Two examples of the first possibility are quoted from the 1965 GCMC report in Section VI. An example of the second, appropriate not only for statisticians but also for physical scientists and mathematics majors, is quoted from the 1971 CUPM report Preparation for Graduate Work in Statistics.

Mathematics 6L and 6M. Linear Algebra and Introductory Modern Algebra. Mathematics 6M is essential for all mathematics majors including prospective high school teachers. Both courses are essential for students preparing for graduate work in mathematics and are useful for computer science students as well. Many physical science students are now finding both courses important, and social science students often require the material of Mathematics 6L.

Mathematics 7. Probability and Statistics. In place of a one-semester course recommended in the 1965 GCMC report we now recommend the two-semester course in probability and statistics suggested in Preparation for Graduate Work in Statistics (1971) and reproduced in Section VI. This course is essential for students preparing for graduate work in statistics. It is desirable for mathematics majors, for mathematically oriented biology or social science students, for engineering students, particularly in communication fields or industrial engineering, and for theoretical physicists and chemists.

Mathematics 8. Introduction to Numerical Analysis. This course is desirable not only for mathematics majors but also for students majoring in a science that makes extensive use of mathematics. In place of the course outlined in the 1965 GCMC report we now suggest the course outlined in Section VI.

Mathematics 9. Geometry. This course should cover a single concentrated geometric theory from a modern axiomatic viewpoint; it is not intended to be a descriptive or survey course in "college geometry." If the college undertakes the training of prospective secondary school teachers, the essential content of this course is Euclidean geometry. A more widely ranging full-year course in the same spirit is desirable if it is possible. Other subjects which provide the appropriate depth include topology, convexity, projective geometry, and differential geometry. A serious introduction to geometric ideas and geometric proof is valuable for all undergraduates majoring in mathematics.

In Section VI two geometry courses of general appeal are quoted from the CUPM report Recommendations on Course Content for the Training of Teachers of Mathematics (1971).

Mathematics 10. Applied Mathematics. Although this course is not yet a standard part of the curriculum, it is desirable for mathematics majors to become aware of the ways in which their subject is applied. Several versions of such a course--optimization theory, graph theory and combinatorial analysis, and fluid mechanics--are described in the CUPM report Applied Mathematics in the Undergraduate Curriculum (1972) [page 705].

Mathematics 11-12. Introductory Real Variable Theory. Preferably this is a one-year course, but if necessary it may be offered in a one-semester version or combined with complex analysis in a one-year course. The student should learn to prove the basic propositions of real variable theory.

At least one semester is desirable for any mathematics major. Mathematics 11-12 is essential for students preparing for graduate work in mathematics. On completion of Mathematics 12 a student should be ready to begin a graduate course in measure and integration theory or in functional analysis. The topics and skills are basic in such fields of analysis as differential equations, calculus of variations, harmonic analysis, complex variables, probability theory, and many others. We feel an extensive coverage of subject matter, especially in the directions of abstract topologies and functional analysis, should be sacrificed in favor of active practice by the student in proving theorems. For an outline of Mathematics 11-12, see Section VI.

Mathematics 13. Complex Analysis. This course contains standard material in the elementary theory of analytic functions of a single complex variable.

Many prefer to have this course precede Mathematics 11-12. It is important for mathematics majors, engineering students, applied mathematicians, and theory-oriented students of physics and chemistry. For an outline of Mathematics 13 see Section VI.

VI. ADDITIONAL COURSE OUTLINES

Mathematics 0. Elementary Functions and Coordinate Geometry. (3 or 4 semester hours) (Reprinted from the 1965 GCMC report)

1. Definition of function and algebra of functions. (5 lessons) Various ways of describing functions, examples from previous mathematics and from outside mathematics, graphs of functions, algebraic operations on functions, composition, inverse functions.

2. Polynomial and rational functions. (10 lessons) Definitions, graphs of quadratic and power functions, zeros of polynomial functions, remainder and factor theorems, complex roots, rational functions and their graphs.

3. Exponential functions. (6 lessons) Review of integral and rational exponents, real exponents, graphs, applications, exponential growth.

4. Logarithmic functions. (4 lessons) Logarithmic function as inverse of exponential, graphs, applications.

5. Trigonometric functions. (10 lessons) Review of numerical trigonometry and trigonometric functions of angles, trigonometric functions defined on the unit circle, trigonometric functions defined on the real line, graphs, periodicity, periodic motion, inverse trigonometric functions, graphs.

6. Functions of two variables. (4 lessons) Three-dimensional rectangular coordinate system, sketching graphs of $z = f(x,y)$ by plane slices.

Mathematics 2P. Probability. (3 semester hours) (Reprinted from the 1965 GCMC report) [Prerequisite: Mathematics 1]

1. Probability as a mathematical system. (9 lessons) Sample spaces, events as subsets, probability axioms, simple theorems, finite sample spaces and equiprobable measure as special case, binomial coefficients and counting techniques applied to probability problems, conditional probability, independent events, Bayes' formula.

2. Random variables and their distributions. (13 lessons) Random variables (discrete and continuous), probability functions, density and distribution functions, special distributions (binomial, hypergeometric, Poisson, uniform, exponential, normal, etc.), mean and variance, Chebychev inequality, independent random variables, functions of random variables and their distributions.

3. Limit theorems. (4 lessons) Poisson and normal approximation to the binomial, Central Limit Theorem, Law of Large Numbers, some statistical applications.

4. Topics in statistical inference. (7-13 lessons) Estimation and sampling, point and interval estimates, hypothesis-testing, power of a test, regression, a few examples of nonparametric methods.

Remarks:

For students with only the minimum prerequisite training in calculus (Mathematics 1), about six lessons will have to be devoted to additional calculus topics needed in Mathematics 2P: improper integrals, integration by substitution, infinite series, power series, Taylor's expansion. For such students there will remain only about seven lessons in statistical inference. Students electing Mathematics 2P after Mathematics 4 will be able to complete the entire course as outlined above.

Mathematics 5. Multivariable Calculus II. (3 semester hours) (Conventional version of Advanced Multivariable Calculus as printed in the 1965 GCMC report) [Prerequisites: Mathematics 1, 2, 3, 4]

The differential and integral calculus of Euclidean 3-space, using vector notation, leading up to the formulation and solution (in simple cases) of the partial differential equations of mathematical physics. Considerable use can and should be made of the students' preparation in linear algebra.

1. Vector algebra. (4 lessons) Dot and cross product, identities. Geometric interpretation and applications. Invariance under change of orthogonal bases.

2. Differential vector calculus. (8 lessons) Functions from V_m to V_n, continuity. Functions from V_1 to V_3, differential geometry of curves. Functions from V_3 to V_1, scalar fields, directional derivative, gradient. Functions from V_3 to V_3, vector fields, divergence, curl. The differential operator ∇, identities. Expression in general orthogonal coordinates.

3. Integral vector calculus. (15 lessons) Line, surface, and volume integrals. Change of variables. Green's, divergence, and Stokes' theorems. Invariant definitions of gradient, divergence, and curl. Integrals independent of path, potentials. Derivation of the Laplace, heat, and wave equations.

4. Fourier series. (6 lessons) The vector space of square-integrable functions, orthogonal sets, approximation by finite sums, notion of complete orthogonal set, general Fourier series. Trigonometric functions as a special case, proof of completeness.

5. Boundary value problems. (6 lessons) Separation of variables. Use of Fourier series to satisfy boundary conditions. Numerical methods.

Mathematics 5. Multivariable Calculus II. (3 semester hours) (Alternate version of Advanced Multivariable Calculus employing differential forms as printed in the 1965 GCMC report) [Prerequisites: Mathematics 1, 2, 3, 4]

A study of the properties of continuous mappings from E_n to E_m, making use of the linear algebra in Mathematics 3, and an introduction to differential forms and vector calculus based upon line integrals, surface integrals, and the general Stokes theorem. Application should be made to field theory, elementary hydrodynamics, or other similar topics so that some intuitive understanding can be gained.

1. Transformations. (15 lessons) Functions (mappings) from E_n to E_m, for $n, m = 1, 2, 3, 4$. Continuity and implications of continuity; differentiation and the differential of a mapping as a matrix-valued function. The role of the Jacobian as the determinant of the differential; local and global inverses of mappings and the Implicit Function Theorem. Review of the chain rule for differentiation and reduction to matrix multiplication. Application to change of variable in multiple integrals and to the area of surfaces.

2. Differential forms. (6 lessons) Integrals along curves. Introduction of differential forms; algebraic operations; differentiation rules. Application to the change of variable in multiple integrals. Surface integrals; the meaning of a general k-form.

3. Vector analysis. (4 lessons) Reinterpretation in terms of vectors; vector function as mapping into E_3; vector field as mapping from E_3 into E_3. Formulation of line and surface integrals (1-forms and 2-forms) in terms of vectors. The operations Div, Grad, Curl, and their corresponding translations into differential forms.

4. Vector calculus. (8 lessons) The theorems of Gauss, Green, Stokes, stated for differential forms and translated into vector equivalents. Invariant definitions of Div and Curl. Exact differential forms and independence of path for line integrals. Application to a topic in hydrodynamics, or to Maxwell's equations, or to the derivation of Green's identities and their specializations for harmonic functions.

5. Fourier methods. (6 lessons) The continuous functions as a vector (linear) space; inner products and orthogonality; geometric concepts and analogy with E_n. Best L^2 approximation; notion of an orthogonal basis and of completeness. The Schwarz and Bessel inequalities. Generalized Fourier series with respect to an orthonormal basis. Treatment of the case $\{e^{inx}\}$ and the standard trigonometric case. Application to the solution of one standard boundary value problem.

Mathematics 5. Multivariable Calculus II. (3 semester hours)
(Reprint of Selected Topics in Analysis from the 1971 report
Preparation for Graduate Work in Statistics)

The Panel on Statistics feels that the course Mathematics 5 presented in the 1965 GCMC report is not particularly appropriate for statistics students, and it has recommended that a course including the special topics listed below be offered in place of Mathematics 5 for students preparing for graduate work in statistics.

The course it recommends gives the student additional analytic skills more advanced than those acquired in the beginning analysis sequence. Topics to be included are multiple integration in n dimensions, Jacobians and change of variables in multiple integrals, improper integrals, special functions (beta, gamma), Stirling's formula, Lagrange multipliers, generating functions and Laplace transforms, difference equations, additional work on ordinary differential equations, and an introduction to partial differential equations.

It is possible that the suggested topics can be studied in a unified course devoted to optimization problems. Such a course, at a level which presupposes only the beginning analysis and linear algebra courses and which may be taken concurrently with a course in probability theory, would be a valuable addition to the undergraduate curriculum, not only for students preparing for graduate work in statistics but also for students in economics, business administration, operations research, engineering, etc. Experimentation by teachers in the preparation of written materials and textbooks for such a course would be useful and is worthy of encouragement.

Mathematics 7. Probability and Statistics. (6 semester hours)
(Reprinted from the 1971 report Preparation for Graduate Work in Statistics)

This key course is a one-year combination of probability and statistics. On the semester system, a complete course in probability should be followed by a course in statistics. If the course is given on a quarter system, it may be possible to have a quarter of probability, followed by two quarters of statistics or by a second quarter of statistics and a third quarter of topics in probability and/or statistics. In any case, these courses should be taught as one sequence.

Prerequisites for this one-year course are Mathematics 1, 2, and 4 (Calculus). Students should also be encouraged to have taken Mathematics 3 (Elementary Linear Algebra). [For detailed course descriptions see Section IV.] All students in this course, whether they be prospective graduate students of statistics, other mathematics majors, or students from other disciplines, should be encouraged to take the full year rather than only the first-semester probability course. Almost all students will have studied the calculus sequence and perhaps linear algebra without interruption during their first two years in college. Although our recommended probability course and Mathematics 2P differ only little in content, our course assumes the additional maturity and ability of students who have successfully completed the three or four semesters of the core curriculum described above.

The probability course should include the following topics:

> Sample spaces, axioms and elementary theorems of probability, combinatorics, independence, conditional probability, Bayes' theorem.
>
> Random variables, probability distributions, expectation, mean, variance, moment-generating functions.
>
> Special distributions, multivariate distributions, transformations of random variables, conditional and marginal distributions.
>
> Chebychev's inequality, limit theorems (Law of Large Numbers, Central Limit Theorem).
>
> Examples of stochastic processes such as random walks and Markov chains.

The course in probability should provide a wide variety of examples of problems which arise in the study of random phenomena. With this aim in mind, we recommend that this course be taught so as to maintain a proper balance between theory and its applications.

The time allotted to the probability course will not permit detailed treatment of all topics listed above. We recommend that such topics as the Central Limit Theorem and the use of Jacobians in transformations of random variables be presented without proof. Also, discussion of multivariate distributions should include only a brief description of the multivariate normal distribution. Random walks and Markov chains may serve as useful topics for two or three lectures to illustrate interesting applications of probability theory. Even though the topics of this paragraph are not treated in depth mathematically, we recommend their inclusion to enrich the student's comprehension of the scope of probability theory.

The statistics course can be implemented in a variety of ways, giving different emphases to topics and, indeed, including different topics. Widely divergent approaches are acceptable as preparation for graduate work and are illustrated in the statistics books listed below, selected from many appropriate texts for this course:

Brunk, H. D. Introduction to Mathematical Statistics, 2nd ed. New York, Blaisdell Publishing Company, 1965.

Freeman, H. A. Introduction to Statistical Inference. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

Freund, John E. Mathematical Statistics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

Hadley, G. Introduction to Probability and Statistical Decision Theory. San Francisco, California, Holden-Day, Inc., 1967.

Hoel, Paul G.; Port, Sidney C.; Stone, Charles J. Introduction to Statistical Theory. Boston, Massachusetts, Houghton Mifflin Company, 1971.

Hogg, Robert V. and Craig, A. T. Introduction to Mathematical Statistics, 3rd ed. New York, The Macmillan Company, 1970.

Lindgren, B. W. Statistical Theory, 2nd ed. New York, The Macmillan Company, 1968.

Mood, Alexander M. and Graybill, F. A. Introduction to the Theory of Statistics, 2nd ed. New York, McGraw-Hill Book Company, 1963.

Despite the diversity of possible approaches, most will include the following topics:

Estimation: consistency, unbiasedness, maximum likelihood, confidence intervals.

Testing hypotheses: power functions, Type I and II errors, Neyman-Pearson lemma, likelihood ratio tests, tests for means and variances.

Regression and correlation.

Chi-square tests.

Other topics to be included in the statistics course will depend on the available time and method of approach. Possible topics include:

Estimation: efficiency, sufficiency, Cramer-Rao theorem, Rao-Blackwell theorem.

Linear models.

Nonparametric statistics.

Sequential analysis.

Design of experiments.

Decision theory, utility theory, Bayesian analysis.

Robustness.

The above list of additional topics for the key course in statistics is much too large to be adequately covered in its entirety. The fact that many topics will have to be omitted or treated superficially gives the statistics course much more flexibility in approach and coverage than is possible in the probability course. The instructor's choice of topics may be influenced by the following factors. Decision theory, Bayesian analysis, and sequential analysis dealing with foundations of inference will appeal to the philosophically inclined students. The Cramer-Rao theorem and the Rao-Blackwell theorem appeal to mathematically oriented students and illustrate statistical theory. In design of experiments and estimation, one has an opportunity to apply techniques of optimization. Nonparametric techniques utilize combinatorial probability and illustrate the high efficiency that can be attained from simple methods. Analysis of variance provides an application of linear algebra and matrix methods and should interest students who have taken Mathematics 3.

Detailed outlines for the probability and statistics courses have not been presented on the assumption that the choice of texts, which is difficult to anticipate, will tend to determine the order of presentation and the emphasis in a satisfactory fashion. It may be remarked that most statistics texts at this level begin with a portion which can be used for the probability course.

To avoid a formal, dull statistics course and to provide sufficient insight into practice, we recommend that meaningful cross-reference between theoretical models and real-world problems be made throughout the course. Use of the computer will help to accomplish this goal. Three reports that are valuable in appraising the potential role of computers in statistics courses are:

Development of Materials and Techniques for the Instructional Use of Computers in Statistics Courses. University of North Carolina, Chapel Hill, North Carolina, 1971.

Proceedings of a Conference on Computers in the Undergraduate Curricula. The University of Iowa, Iowa City, Iowa, 1970.

Proceedings of the Second Annual Conference on Computers in the Undergraduate Curricula. Dartmouth College, Hanover, New Hampshire, 1971.

Mathematics 8. Introduction to Numerical Analysis. (3 semester hours) [Prerequisites: Mathematics 1, 2, 3, 4]

1. Introduction. (1 hour) Number representation on a computer, discussion of the various types of errors in numerical processes, the idea of stability in numerical processes.

2. Solution of a single nonlinear equation. (7 hours) Existence of a fixed point; contraction theorem and some consequences; Ostrowski's point-of-attraction theorem; the rate of convergence for successive approximations; Newton's method: local convergence and rate of convergence, convergence theorem in the convex case; secant methods, including regula falsi; roots of polynomials: Newton-Raphson method, Sturm sequences, discussion of ill-conditioning.

3. Linear systems of equations. (7 hours) Gaussian elimination with pivoting, the factorization into upper and lower triangular matrices, inversion of matrices, discussion of ill-conditioning, vector and matrix norms, condition numbers, discussion of error bounds, iterative improvement, Gaussian elimination for symmetric positive-definitive matrices.

4. Interpolation and approximation. (6 hours) Lagrange interpolating polynomial; Newton interpolating polynomial; error formula for the interpolating polynomial; Chebychev polynomial approximation; least squares approximation: numerical problem associated with the normal equations, the use of orthogonal polynomials.

5. Numerical integration and differentiation. (6 hours) Quadrature based on interpolatory polynomials, error in approximate integration, integration over large intervals, Romberg integration including development of the even-powered error expansion, error in differentiating the interpolating polynomial, differentiation by extrapolation to the limit.

6. Initial value problems in ordinary differential equations. (9 hours) Taylor's series expansion technique; Euler's method with convergence theorem; Runge-Kutta methods; predictor-corrector methods: convergence of the corrector as an iteration, local error bound for predictor-corrector of same order; general discussion of stability using the model problem $y' = Ay$, consistency and convergence; reduction of higher-order problems to a system of first-order problems.

Mathematics 9. Geometry. (3 semester hours) (Reprint of Foundations of Euclidean Geometry from the 1971 report Recommendations on Course Content for the Training of Teachers of Mathematics)

The purpose of this course is two-fold. On the one hand it presents an adequate axiomatic basis for Euclidean geometry, including the one commonly taught in secondary schools, while on the other hand it provides insight into the interdependence of the various theorems and axioms. It is this latter aspect that is of the greater importance for it shows the prospective teacher that there is no one royal road to the classical theorems. This deeper appreciation of geometry will better prepare the teacher to assess the virtues of alternative approaches and to be receptive to the changes in the secondary school geometry program that loom on the horizon.

Courses similar to this have now become commonplace. As a consequence, no great detail should be necessary in this guide. There is a greater abundance of appropriate topics than can be covered in one course, so some selection will always need to be made.

Although enough consideration should be given to 3-space to build spatial intuition, the major emphasis should be on the plane, since it is in 2-space that the serious and subtle difficulties first become apparent. The principal defects in Euclid's Elements relate to the order and separation properties and to the completeness of the line. Emphasis should be directed to clarifying these subtle matters with an indication of some of the ways by which they can be circumvented. The prospective teacher must be aware of these matters and have enough mathematical sophistication to proceed to new topics with only an indication of how they are resolved.

The course consists of six parts, after a brief historical introduction and a critique of Euclid's Elements. The allotment of times that have been assigned for these parts are but suggestions to be used as a guide, because emphasis will vary with the background of the students, the text used, and the tastes of the instructor. Prerequisites for the course are a modest familiarity with rigorous deduction from axioms, for example as encountered in algebra, and the completeness of the real number system.

1. Incidence and order properties. (8 lessons) In this part of the course, after a brief treatment of incidence properties, the inherent difficulties of betweenness and separation are discussed. The easiest, and suggested, way to proceed is in terms of distance. The popular method today is to use the Birkhoff axioms or a modification such as given by the School Mathematics Study Group. In addition, one should give some indication of a synthetic foundation for betweenness such as that of Hilbert. A brief experience with a synthetic treatment of betweenness is enough to convince the student

of the power of the metric apparatus.

Alternatively, one can begin with a synthetic treatment of betweenness and then introduce the metric apparatus. With this approach, metric betweenness is a welcome simplification.

2. Congruence of triangles and inequalities in triangles. (8 lessons) It is recommended that angle congruence be based on angle measure (the Birkhoff axioms). Yet here too some remarks on a synthetic approach are desirable.

The order of presentation of the congruence theorems can depend on the underlying axiom system used. What is perhaps more important is to observe their interrelations. At this point a global view of transformations of the plane should receive attention. Ruler and compass constructions should be deferred, as the treatment is simpler and more elegant after the parallel axiom has been introduced. The triangle inequality and the exterior angle theorem occur here.

3. Absolute and non-Euclidean geometry. (6 lessons) Up to this point there has been no mention of the parallel postulate. It is desirable to explore some of the attempts to prove it. One should prove a few theorems in absolute geometry, in particular ones about Saccheri quadrilaterals. Then some theorems in hyperbolic geometry can be given, among which the angle-sum theorem for angles in a triangle is most important. A model, without proof, for hyperbolic geometry is natural here.

This part of the course can also be taught after Part 4 when Euclid's parallel axiom and consequences of it have been covered.

4. The parallel postulate. (8 lessons) There are many topics, of central importance in high school, that need to be discussed in this part of the course. It is desirable to give here, as well as in Part 3, considerable attention to the history of the parallel axiom. Due to time limitations, it will probably be necessary to omit some topics. Nevertheless, some attention should be given to: parallelograms, existence of rectangles, Pythagorean theorem, angle-sum theorem for triangles, similarity, ruler and compass construction, and an introduction to the notion of area.

5. The real numbers and geometry. (8 lessons) This part is

devoted to matters in which the completeness of the real number system plays a role. Some attention must be given to the completeness of the line and the consequences thereof. Archimedes' axiom arises naturally here. Important topics are: similarity of triangles for the incommensurable case; circumference; area in general and, in particular, area of circles; and, finally, a coordinate model of Euclidean geometry. It is possible to give a coordinate model of a non-Archimedean geometry at this time.

6. Recapitulation. (3 lessons) This part is intended to give perspective on the preceding sections. It should have a strong historical flavor and might well include lectures with outside reading or a short essay.

Mathematics 9a. Geometry. (Reprint of Vector Geometry from the 1971 report Recommendations on Course Content for the Training of Teachers of Mathematics)

There are approaches to geometry other than the classical synthetic Euclidean approach, and several of these are being suggested for use in both the high school and college curricula. Moreover, exposure to different foundations for geometry yields deeper insights into geometry and can serve to relate Euclidean geometry to the mainstream of current mathematical interest. It is this latter reason which underlies much of the discussion about geometry that is now prevalent. There are at least three approaches that merit consideration.

I. The classical approach of Felix Klein, wherein one begins with projective spaces and, by considering successively smaller subgroups of the group acting on the space, one eventually arrives at Euclidean geometry. A course of this nature might be called projective geometry, but it should proceed as rapidly as possible to Euclidean geometry. Besides books on projective geometry, other references are:

Artin, Emil. Geometric Algebra. New York, John Wiley and Sons, Inc., 1957.

Gans, David. Transformations and Geometries. New York, Appleton-Century-Crofts, 1968.

Klein, Felix. Vorlesungen über Nicht-Euklidische Geometrie. New York, Chelsea Publishing Company, Inc., 1959.

Schreier, Otto and Sperner, Emanuel. Projective Geometry of n Dimensions. New York, Chelsea Publishing Company, Inc., 1961.

(Throughout this outline, references are given because of their content, with no implication that the level of presentation is appropriate. Indeed, adjustments will normally be necessary.)

II. The transformation approach, which in some ways is a variant of Klein's, uses the Euclidean group to define congruence and other familiar concepts. As a further variant of this, one finds books which begin with synthetic Euclidean geometry and proceed to the Euclidean group. References are:

Bachmann, F. Aufbau der Geometrie aus dem Spiegelungsbegriff. Berlin, Springer-Verlag, 1959.

Choquet, Gustave. Geometry in a Modern Setting. Boston, Massachusetts, Houghton Mifflin Company, 1969.

Coxford, A. F. and Usiskin, Z. P. Geometry, A Transformation Approach, vol. I, II. River Forest, Illinois, Laidlaw Brothers, 1970.

Eccles, Frank. An Introduction to Transformational Geometry. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1971.

III. The vector space approach, the one suggested for this course, uses vector spaces as an axiomatic foundation for the investigation of affine and Euclidean geometry. Through the use of vector spaces, classical geometry is brought within the scope of the central topics of modern mathematics and, at the same time, is illuminated by fresh views of familiar theorems. Some of the references below contain isolated chapters which are relevant to this approach; in such cases these chapters are indicated.

Artin, Emil. Geometric Algebra. New York, John Wiley and Sons, Inc., 1957.

Artzy, Rafael. Linear Geometry. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1965.

Dieudonné, Jean. Linear Algebra and Geometry. Boston, Massachusetts, Houghton Mifflin Company, 1969.

Gruenberg, K. W. and Weir, A. J. Linear Geometry. New York, Van Nostrand Reinhold Company, 1967.

MacLane, Saunders and Birkhoff, Garrett. Algebra. New York, The Macmillan Company, 1967. (Chapters VII, XI, XII)

Mostow, George; Sampson, Joseph; Meyer, Jean-Pierre. Fundamental Structures of Algebra. New York, McGraw-Hill Book Company, 1963. (Chapters 8, 9, 14)

Murtha, J. A. and Willard, E. R. Linear Algebra and Geometry. New York, Holt, Rinehart and Winston, Inc., 1969.

Snapper, Ernst and Troyer, Robert. Metric Affine Geometry. New York, Academic Press, Inc., 1971.

The course outlined below has as prerequisite an elementary course in linear algebra (Mathematics 3). The main topics are:

1. Affine geometry and affine transformations

2. Euclidean geometry and Euclidean transformations

3. Non-Euclidean geometries

Because of the relative unfamiliarity of this approach to geometry, more details such as definitions and typical results will be included. Also, a brief justification is given.

In Euclidean geometry one considers the notion of a translation of the space into itself. These translations form a real vector space under the operation (addition) of function composition and multiplication by a real number. Thus the "vector space of translations" acts on the set of points of Euclidean space and satisfies the following two properties:

A. If (x,y) is an ordered pair of points, there is a translation T such that $T(x) = y$. Moreover, this translation is unique.

B. If T_1 and T_2 are translations and x is a point, then the definition of "vector addition" as function composition is indicated by the formula

$$(T_1 + T_2)(x) = T_1(T_2(x)).$$

With this intuitive background, the details of the course outline are now given. The definitions and propositions are stated for dimension n since this causes no complication, but the emphasis will be on dimensions 2 and 3.

1. Affine geometry and affine transformations. One defines real n-dimensional affine space as the triple (V,X,μ) where V is a real vector space of dimension n (the vector space of the translations), X is the set of points of the geometry, and $\mu: V \times X \to X$

defined by $\mu(T,x) = T(x)$ is the action of V on X which satisfies properties A and B above. For convenience, the affine space (V,X,μ) is usually denoted simply by X.

Affine subspaces of X are defined as follows. Let $x \in X$ and let U be a linear subspace of V (a subspace of translations). The affine subspace determined by x and U is denoted by $S(U,x)$ and consists of the set of points

$$\{T(x):T \in U\},$$

i.e., $S(U,x)$ consists of all translates of x by a translation belonging to U. The dimension of $S(U,x)$ is defined to be the dimension of U. Then 1-dimensional affine subspaces are called lines, 2-dimensional affine subspaces are called planes, and (n-1)-dimensional affine subspaces are called hyperplanes (n = dimension of V).

Two affine subspaces S and S' are called parallel $(S \parallel S')$ if there exists a translation T such that $T(S) \subset S'$ or $T(S') \subset S$. Parallelism and incidence are investigated, with special emphasis on dimensions two and three. Results such as the following are obtained.

a. Lines ℓ and m in the plane are parallel if and only if $\ell = m$ or $\ell \cap m = \emptyset$.

b. A line ℓ and a plane π in 3-space are parallel if and only if $\ell \subset \pi$ or $\ell \cap \pi = \emptyset$. If $\ell \nparallel \pi$, then $\ell \cap \pi$ is a point.

c. There exist skew lines in 3-space.

d. Planes π and π' in 3-space are parallel if and only if $\pi = \pi'$ or $\pi \cap \pi' = \emptyset$. If $\pi \nparallel \pi'$, then $\pi \cap \pi'$ is a line.

A coordinate system for the affine space X consists of a point $c \in X$ and an ordered basis for V. A point $x \in X$ is assigned the coordinates $(x_1,\ldots,x_n)$ if T is the unique translation such that $T(c) = x$ and T has coordinates $(x_1,\ldots,x_n)$ with respect to the given ordered basis for V. Using these notions, one can study analytic geometry, e.g., the parametric equations for lines, the linear equations for hyperplanes, the relationship between the linear equations of parallel hyperplanes, incidence in terms of coordinate representations, etc.

For each point $c \in X$, there is a natural way to make X into a vector space which is isomorphic to V. Namely, if r is a real number, $x, y \in X$, and T_1, T_2 are the unique translations satisfying $T_1(c) = x$ and $T_2(c) = y$, then one defines

$$x + y = T_2(T_1(c)) \quad \text{and} \quad rx = (rT_1)(x).$$

The vector space X_c with origin c is the tangent space of classical differential geometry. (Affine space is often defined as the vector space V itself; this approach to affine geometry is based on the isomorphism between X_c and V.)

An affine transformation is a function $f: X \to X$ with the following properties:

a. f is one-to-one and onto.

b. If ℓ and ℓ' are parallel lines, then $f(\ell)$ and $f(\ell')$ are parallel lines.

The affine transformations form a group called the affine group which contains the translation group as a commutative subgroup. For each point $c \in X$, the set of affine transformations which leave c fixed form a subgroup of the affine group; moreover, this subgroup is the general linear group of the vector space X_c and is therefore isomorphic to the general linear group of V. Finally, properties of affine transformations are investigated.

Other topics of affine geometry which are studied include orientation, betweenness, independence of points, affine subspace spanned by points, and simplexes.

2. Euclidean geometry and Euclidean transformations. Euclidean space is defined as the affine space (V,X,μ), where V has been given the additional structure of a positive-definite inner product. Thus for each $T \in V$, T^2 is a nonnegative real number. A distance function is introduced on X by defining the distance between an ordered pair (x,y) of points of X to be $\sqrt{T^2}$, where T is the unique translation such that $T(x) = y$. A Euclidean transformation (rigid motion, isometry) of X is a mapping of X which preserves distance.

The Euclidean transformations form a subgroup of the affine group. For each $c \in X$, the Euclidean transformations which leave c fixed form a subgroup of the Euclidean group. In fact, this is the orthogonal group of the vector space X_c (with the inner product induced on it from V through the given isomorphism) and therefore is isomorphic to the orthogonal group of V.

Rotations and reflections are first defined for the Euclidean plane and then for n-dimensional space. The Cartan-Dieudonné theorem becomes an important tool in the investigation of the Euclidean group. It states that every Euclidean transformation of n-space is the product of at most $n + 1$ reflections in hyperplanes. It follows immediately that there are four kinds of Euclidean transformations of the Euclidean plane: translations, rotations, reflections, and glide reflections.

Rotations and reflections of Euclidean 3-space are investigated. From the Cartan-Dieudonné theorem it follows that every rotation of 3-space has a line of fixed points (the axis of rotation). The set of all rotations with a given line ℓ as axis is a subgroup of the rotation group of 3-space. Moreover, this rotation group with axis ℓ is isomorphic to the rotation group of the Euclidean plane, thus giving the classical result that every rotation of 3-space is determined by an axis and a given "angle of rotation."

One now defines a <u>figure</u> to be a subset of X and calls two figures congruent if there is a Euclidean transformation which maps one figure onto the other. Using these concepts, one proceeds to proofs of the classical congruence theorems of plane geometry (S.S.S., S.A.S., A.S.A, H.S.).

Finally, orthogonality and similarity are investigated.

3. <u>Non-Euclidean geometries</u>. The classical method of obtaining a non-Euclidean plane geometry is to replace the parallel postulate by another postulate on parallel lines and thus obtain hyperbolic geometry. Here the approach is different. The positive-definite inner product is replaced by other (nonsingular) inner products. The geometry obtained is non-Euclidean, but the parallel postulate is still valid! This startling result is true because the underlying space is the affine plane (in which the parallel

postulate is valid) and the change of inner product does not disturb the affine structure.

Actually, the investigation of non-Euclidean geometries can be made concurrently with that of Euclidean geometry. For example, the Lorentz plane and the negative Euclidean plane can be defined and investigated at the same time as the Euclidean plane. "Circles" in the Lorentz plane are related to hyperbolas of the Euclidean plane, etc.

One of the major results is Sylvester's theorem, from which one concludes that there are precisely $n + 1$ distinct nonsingular geometries which can be placed on n-dimensional affine space.

Mathematics 10. <u>Applied Mathematics</u>.

Applied mathematics is a mathematical science distinguished from other branches of mathematics in that it actively employs the scientific method. A working applied mathematician is usually confronted with a real situation whose mathematical aspects are not clearly defined. He must identify specific questions whose answers will shed light on the situation, and he must construct a mathematical model which will aid in his study of these questions. Using the model he translates the questions from the original terms into mathematical terms. He then uses mathematical ideas and techniques to study the problem. He must decide upon methods of approximation and computation which will enable him to determine relevant numbers. Finally, he must interpret the results of his mathematical work in the setting of the original situation.

Mathematics 10 was designed to introduce the student to applied mathematics and, in particular, to model building. Courses concentrating primarily on mathematical techniques which are useful in applications do not satisfy the goals set here for Mathematics 10. Rather, it is intended that the student participate in the total experience of applied mathematics from formulating precise questions to interpreting the results of the mathematical analysis in terms of the original situation, and that particular emphasis be given to model building. A number of courses involving different mathematical topics can be constructed which fulfill these goals. In constructing such a course the instructor should have the following recommendations in mind.

First, the role of model building must be made clear and should be amply illustrated. The student should have considerable experience in building models, in noting their strengths and weaknesses, and in modifying them to fit the situation more accurately. Also, he must realize that often there is more than one approach to a situation and that different approaches may lead to different models. He should be trained to be critical of the models he constructs so that he will know what kind of information to expect from the model and what kind not to expect.

Second, the situations investigated must be realistic. Throughout the course the student should be working on significant problems which are interesting and real to him.

Third, the mathematical topics which arise in the course should be worthwhile and should have applicability beyond the specific problem being discussed. The mathematical topics and the depth of treatment should be appropriate for the level at which the course is offered.

Fourth, the mathematical results should always be interpreted in the original setting. Stopping short of this gives the impression that the manipulation of symbols, methods of approximation, techniques of computation, or other mathematical points are the primary concerns of the course, whereas they are only intermediate steps, essential though they are, in the study of a real situation.

Finally, the course should avoid the extremes of (1) a course about mathematical methods whose reference to the real world consists mainly of assigning appropriate names to problems already completely formulated in mathematical terms and (2) a kind of survey of mathematical models in which only trivial mathematical development of the models is carried out.

The 1972 report of the Panel on Applied Mathematics, Applied Mathematics in the Undergraduate Curriculum, offers three outlines as aids to constructing courses of the type recommended here. [See page 705.]

Mathematics 11-12. Introductory Real Variable Theory. (6 semester hours) (Reprinted from the 1965 GCMC report)

FIRST SEMESTER - 39 lessons

1. Real numbers. (6 lessons) The integers; induction. The rational numbers; order structure, Dedekind cuts. The reals defined as a Dedekind-complete field. Outline of the Dedekind construction. Least upper bound property. Nested interval property. Denseness of the rationals. Archimedean property. Inequalities. The extended real number system.

2. Complex numbers. (3 lessons) The complex numbers introduced as ordered pairs of reals; their arithmetic and geometry. Statement of algebraic completeness. Schwarz inequality.

3. Set theory. (4 lessons) Basic notation and terminology: membership, inclusion, union and intersection, cartesian product, relation, function, sequence, equivalence relation, etc.; arbitrary unions and intersections. Countability of the rationals; uncountability of the reals.

4. Metric spaces. (6 lessons) Basic definitions: metric, ball, boundedness, neighborhood, open set, closed set, interior, boundary, accumulation point, etc. Unions and intersections of open or closed sets. Subspaces. Compactness. Connectedness. Convergent sequence, subsequences, uniqueness of limit. A point of accumulation of a set is a limit of a sequence of points of the set. Cauchy sequence. Completeness.

5. Euclidean spaces. (6 lessons) R^n as a normed vector space over R. Completeness. Countable base for the topology. Bolzano-Weierstrass and Heine-Borel-Lebesgue theorems. Topology of the line. The open sets; the connected sets. The Cantor set. Outline of the Cauchy construction of R. Infinite decimals.

6. Continuity. (8 lessons) (Functions into a metric space:) Limit at a point, continuity at a point. Continuity; inverses of open sets, inverses of closed sets. Continuous images of compact sets are compact. Continuous images of connected sets are connected. Uniform continuity; a continuous function on a compact set is uniformly continuous. (Functions into R:) Algebra of continuous functions. A continuous function on a compact set attains its maximum. Intermediate Value Theorem. Kinds of discontinuities.

7. Differentiation. (6 lessons) (Functions into R:) The derivative. Algebra of differentiable functions. Chain rule. Sign of the derivative. Mean Value Theorems. The Intermediate Value Theorem for derivatives. L'Hôpital's rule. Taylor's theorem with remainder. One-sided derivatives; infinite derivatives. (This material will be relatively familiar to the student from his calculus course, so it can be covered rather quickly.)

SECOND SEMESTER - 39 lessons

8. The Riemann-Stieltjes integral. (11 lessons) [Alternative: the Riemann integral] Upper and lower Riemann integrals. [Existence of the Riemann integral: for f continuous, for f monotonic.] Monotonic functions and functions of bounded variation. Riemann-Stieltjes integrals. Existence of $\int_a^b f \, d\alpha$ for f continuous and α of bounded variation. Reduction to the Riemann integral in case α has a continuous derivative. Linearity of the integral. The integral as a limit of sums. Integration by parts. Change of variable. Mean Value Theorems. The integral as a function of its upper limit. The Fundamental Theorem of Calculus. Improper integrals. The gamma function.

9. Series of numbers. (11 lessons) (Complex:) Convergent series. Tests for convergence (root, ratio, integral, Dirichlet, Abel). Absolute and conditional convergence. Multiplication of series. (Real:) Monotone sequences; lim sup and lim inf of a sequence. Series of positive terms; the number e. Stirling's formula, Euler's constant.

10. Series of functions. (7 lessons) (Complex:) Uniform convergence; continuity of uniform limit of continuous functions. Equicontinuity; equicontinuity on compact sets. (Real:) Integration term-by-term. Differentiation term-by-term. Weierstrass approximation theorem. Nowhere-differentiable continuous functions.

11. Series expansions. (10 lessons) Power series, interval of convergence, real analytic functions, Taylor's theorem. Taylor expansions for exponential, logarithmic, and trigonometric functions. Fourier series: orthonormal systems, mean square approximation, Bessel's inequality, Dirichlet kernel, Fejér kernel, localization theorem, Fejér's theorem. Parseval's theorem.

Mathematics 11. Introductory Real Variable Theory. (3 semester hours) (One-semester version) (Reprinted from the 1965 GCMC report)

1. Real numbers. (3 lessons) Describe various ways of constructing the real numbers but omit details. Least upper bound property, nested interval property, denseness of the rationals.

2. Set theory. (4 lessons) Basic notation and terminology: membership, inclusion, union and intersection, cartesian product, relation, function, sequence, equivalence relation, etc; arbitrary unions and intersections. Countability of the rationals; uncountability of the reals.

3. Metric spaces. (4 lessons) Material of topic 4 in Mathematics 11-12, condensed.

4. Euclidean spaces. (4 lessons) R^n as a normed vector space over R. Completeness. Bolzano-Weierstrass and Heine-Borel-Lebesgue theorems. Topology of the line. Outline of the Cauchy construction of R. Infinite decimals.

5. Continuity. (5 lessons) (Functions into a metric space:) Limit at a point, continuity at a point, inverses of open or closed sets. Uniform continuity. (Functions into R:) A continuous function on a compact set attains its maximum. Intermediate Value Theorem.

6. Differentiation. (3 lessons) Review of previous information, including sign of the derivative, Mean Value Theorem, L'Hôpital's rule, Taylor's theorem with remainder.

7. Riemann-Stieltjes or Riemann integration. (5 lessons) Functions of bounded variation (if the Riemann-Stieltjes integral is covered), basic properties of the integral, the Fundamental Theorem of Calculus.

8. Series of numbers. (8 lessons) Tests for convergence, absolute and conditional convergence. Monotone sequences, lim sup, series of positive terms.

9. Series of functions. (3 lessons) Uniform convergence, continuity of uniform limit of continuous functions, integration and differentiation term-by-term.

Mathematics 13. Complex Analysis. (3 semester hours) (Reprinted from the 1965 GCMC report)

This course is suitable for students who have completed work at the level of vector analysis and ordinary differential equations. The development of skills in this area is very important in the sciences, and the course must exhibit many examples which illustrate the influence of singularities and which require varieties of techniques for finding conformal maps, for evaluating contour integrals (especially those with multivalued integrands), and for using integral transforms.

1. Introduction. (4 lessons) The algebra and geometry of complex numbers. Definitions and properties of elementary functions, e.g., e^z, $\sin z$, $\log z$.

2. Analytic functions. (2 lessons) Limits, derivatives, Cauchy-Riemann equations.

3. Integration. (6 lessons) Integrals, functions defined by integrals. Cauchy's theorem and formula, integral representation of derivatives of all orders. Maximum modulus, Liouville's theorem, Fundamental Theorem of Algebra.

4. Series. (5 lessons) Taylor and Laurent series. Uniform convergence, term-by-term differentiation, uniform convergence in general. Domain of convergence and classification of singularities.

5. Contour integration. (3 lessons) The residue theorem. Evaluation of integrals involving single-valued functions.

6. Analytic continuation and multivalued functions. (6 lessons) Analytic continuation, multivalued functions, and branch points. Technique for contour integrals involving multivalued functions.

7. Conformal mapping. (6 lessons) Conformal mapping. Bilinear and Schwarz-Christoffel transformations, use of mapping in contour integral evaluation. Some mention should be made of the general Riemann mapping theorem.

8. Boundary value problems. (3 lessons) Laplace's equation in two dimensions and the solution of some of its boundary value problems, using conformal mappings.

9. Integral transforms. (4 lessons) The Fourier and Laplace transforms, their inversion identities, and their use in boundary value problems.

THE TRAINING OF TEACHERS OF MATHEMATICS

CUPM's interest in the training of mathematics teachers has pervaded its activities throughout the Committee's existence.

The Panel on Teacher Training, one of the original four panels, began its work at a time when mathematics instruction in elementary and secondary schools was undergoing significant changes. Throughout the years since its original report was issued, the Panel's recommendations and ongoing activities have had a profound influence on the education of elementary and secondary school teachers.

The 1961 Recommendations for the Training of Teachers of Mathematics* identified five levels of mathematics teachers:

I. Teachers of elementary school mathematics--grades K through 6
II. Teachers of the elements of algebra and geometry
III. Teachers of high school mathematics
IV. Teachers of the elements of calculus, linear algebra, probability, etc.
V. Teachers of college mathematics

To complement the 1961 recommendations, CUPM also published Course Guides for the Training of Teachers of Elementary School Mathematics* and Course Guides for the Training of Teachers of Junior High School and High School Mathematics.* When it was proposed, the Level I curriculum received widespread attention and approval. It was approved formally by the Mathematical Association of America and it was endorsed by three conferences held by the National Association of State Directors of Teacher Education and Certification (NASDTEC) and the American Association for the Advancement of Science (AAAS). It formed a part of the Guidelines for Science and Mathematics in the Preparation Program of Elementary School Teachers, published by NASDTEC-AAAS in 1963.

In the years 1962-66 CUPM made an intensive effort to explain its proposed Level I program to that part of the educational community especially concerned with the mathematics preparation of elementary school teachers. Forty-one conferences were held for this purpose. Participants in these conferences, who came from all fifty states, represented college mathematics departments and education departments, state departments of education, and the school systems. The details of CUPM proposals were discussed and an effort was made to identify the realistic problems of implementation of the recommendations. As a result of these conferences and of other forces for change, there was a marked increase in the level of mathematics training required for the elementary teacher.

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* Not included in this COMPENDIUM.

Level II and III conferences similar to those held for Level I were deemed unnecessary because the Level II and III guidelines had apparently been accepted by the teaching community through distribution of the recommendations and course guides. One indication of this acceptance has been the publication of numerous textbooks whose prefaces claim adherence to the CUPM guidelines.

Throughout the decade of the 1960's, CUPM continued to expend considerable effort on the problems associated with the preparation of teachers. Minor revisions of the original recommendations were produced in 1966, and the course guides for Level I were similarly revised in 1968.

In 1965 CUPM published A General Curriculum in Mathematics for Colleges* (GCMC) as a model for a mathematics curriculum in a small college. GCMC became a standard reference in other CUPM documents. The shortage of mathematicians, already severe by the late 1950's, had seriously impaired the ability of many colleges to implement CUPM recommendations, including GCMC. Qualified new faculty members were extremely difficult to obtain, and many established teachers were so overloaded with teaching responsibilities that they could not keep abreast of developments in their field. By 1965 the time was obviously ripe for CUPM to see what could be done to alleviate this problem. An *ad hoc* Committee on the Qualifications of College Teachers of Mathematics was appointed to study and report on the proper academic qualifications for teaching the GCMC courses. Simultaneously, CUPM established a Panel on College Teacher Preparation and instructed it to study a number of related topics: existing programs for the preservice and inservice training of college teachers, opportunities for support of college teacher programs by government and foundations, the supervision and training of teaching assistants, supply and demand data, etc.

In 1967 the Qualifications Committee issued its report, Qualifications for a College Faculty in Mathematics. The report identifies four possible components in the formal education of college teachers and describes teaching duties suitable for individuals with academic attainment equivalent to a given component. It also makes suggestions concerning the composition of a small undergraduate department.

Immediately upon publication of the qualifications report, the Panel on College Teacher Preparation fell heir to several tasks. One of these was the responsibility for a series of regional conferences designed to bring together mathematicians and college administrators to discuss some of the issues raised by the report. Another was the task of preparing a detailed description of a graduate program modeled after the "first graduate component" defined in the qualifications

* Not included in this COMPENDIUM. However, the 1972 Commentary on A General Curriculum in Mathematics for Colleges appears on page 33.

report. This latter project was undertaken by the Graduate Task Force, a group with membership drawn from the Panel and from CUPM. Its report, A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, was issued in 1969. Meanwhile, other members of the Panel conducted a study on the supervision and training of teaching assistants in mathematics. Their findings were reported in a newsletter published in 1968.

The need for a "companion volume" for the qualifications report was established when the Panel on Mathematics in Two-Year Colleges issued its 1969 report A Transfer Curriculum in Mathematics for Two-Year Colleges. CUPM felt it was necessary to describe the qualifications for persons to teach the courses in the Transfer Curriculum, and for this purpose it appointed an ad hoc Committee on Qualifications for a Two-Year College Faculty in Mathematics. This group's recommendations are given in the document Qualifications for Teaching University-Parallel Mathematics Courses in Two-Year Colleges, published in 1969.

Publication of the several reports mentioned in the preceding paragraphs completed CUPM's original plan of providing course guides for each of the five teaching levels defined in 1961. By 1967, however, the pressure for further change was already beginning to be felt. A minor revision (1968) of the Level I course guides contained the statement, "The five years that have elapsed since the preparation of the Course Guides have seen widespread adoption of the ideas of the new elementary school curricula, not only of the work of such experimental or quasi-experimental groups as the School Mathematics Study Group (SMSG) or the University of Illinois Curriculum Study in Mathematics (UICSM), but also of many new commercial textbook series which incorporate such ideas. In addition, there have been attempts to influence the future direction of elementary school mathematics by such groups as the Cambridge Conference. In the near future, the Panel believes, it will be necessary to examine our courses to take account of these developments. We hope in the next couple of years to begin the sort of detailed, intellectual study of current trends in the curriculum and of predictions of the future which will be necessary in order to prepare teachers for the school mathematics of the next twenty years."

During the years 1968-72 the Panel on Teacher Training continued this promised study. It sought to understand current trends and future possibilities through a variety of means: in the spring of 1968 it sponsored a conference, "New Directions in Mathematics," to obtain the views and advice of a large number of mathematicians and educators; it followed the deliberations of the CUPM Panel on Computing; it followed with interest, and contributed to, continuing discussion on pedagogy, the changing attitudes toward experimentation in mathematics education, and the role of mathematics in society today; and, finally, the Panel met with representatives of the National Council of Teachers of Mathematics, the American Association for the Advancement of Science, and the National Association of State Directors of Teacher Education and Certification, and maintained contact with national curriculum planning groups. The Panel concluded from this study that a revision of the CUPM recommendations and course guides for Levels I, II, and III was indeed required. Its 1971 report, Recommendations on Course Content for the Training of Teachers of Mathematics, was a result of that decision.

During the early seventies the Panel on College Teacher Preparation continued its interest in the role and preparation of teaching assistants. A 1972 newsletter, "New Methods for Teaching Elementary Courses and for the Orientation of Teaching Assistants,"* contains a statement by the Panel on teaching experience as part of Ph.D. programs. In 1972 the Panel also issued a booklet entitled Suggestions on the Teaching of College Mathematics,* whose purpose was to disseminate some ideas about practices that are believed to have contributed to successful teaching of mathematics in colleges and universities.

* Not included in this COMPENDIUM.

QUALIFICATIONS FOR A COLLEGE FACULTY IN MATHEMATICS

A report of

The *ad hoc* Committee on the Qualifications of College Teachers of Mathematics

January 1967

BACKGROUND

[Editor's note: Although statements in this report which refer to the shortage of mathematics teachers are no longer valid, the main theme of these recommendations is still cogent.]

The curriculum reforms in mathematics at the elementary and secondary school levels during the past decade have necessitated immediate programs of support to provide the quality of teaching needed in those schools. Today we are beginning to notice many changes in college mathematics courses stemming from previous CUPM recommendations. Hence, it is now time to focus attention on the training and qualifications of teachers needed in our colleges and universities in order to effect the required changes in the undergraduate curriculum.

The recent dramatic growth of mathematical research activity, combined with the growing demands of industry and government for people with mathematical training, has created a severe shortage of mathematics teachers who have doctoral degrees. The rapidly increasing mathematics enrollments within a growing college population and the expansion of areas of application of mathematics have left many college mathematics departments seriously understaffed, greatly overworked, and quite unprepared to initiate urgently required modifications of their course offerings. It is imperative that decisions for curriculum changes, as well as for the other critical problems facing mathematics departments, be made and carried out by people with the highest possible mathematical qualifications.

The simple traditional requirement of many colleges, and of some junior colleges, that new appointments to the mathematics faculty be awarded only to people with a Ph.D. degree is, at the present time, quite unrealistic. Recipients of new Ph.D.s in mathematics are simply not available in the required numbers. For example, in 1964-1965 barely more than one quarter of the new full-time mathematics teachers employed by four-year colleges had Ph.D.s. The shortage is likely to continue, and junior colleges and four-year colleges will, of necessity, continue to use teachers whose academic preparation is intermediate between the bachelor's degree and the doctor's degree.

Our principal goal in this report is to set forth appropriate qualifications for teaching the courses recommended by CUPM in its report A General Curriculum in Mathematics for Colleges* (GCMC) in terms of a teacher's own academic background. As a further task, we consider the distribution of training within a mathematics

* The original GCMC report is not included in this COMPENDIUM. However, the 1972 Commentary on A General Curriculum in Mathematics for Colleges appears on page 33.

faculty today which makes it possible for the department to teach effectively the program recommended in the GCMC report.

It should be understood that no academic program or degree in itself qualifies an individual to teach effectively at any level unless this preparation is accompanied by a genuine interest in teaching and by professional activities reflecting continuing mathematical growth. These activities may assume the form of several of the following:

(a) taking additional course work

(b) reading and studying to keep aware of new developments and to explore new fields

(c) engaging in research for new mathematical results (even if unpublished)

(d) developing new courses and new ways of teaching

(e) publishing expository or research articles

(f) participating in the activities of professional mathematical organizations

The preceding list reflects our conviction that an effective teacher must maintain an active interest in the communication of ideas and have a dedication to studying, learning, and understanding mathematics at levels significantly beyond those at which he is teaching.

A college mathematics department, whose staff members are engaged in activies such as those described above and have the academic qualifications to be described below, should have confidence in its ability to provide the quality of teaching required of it.

THE TRADITIONAL ROLE OF THE Ph.D DEGREE

Colleges and universities have come to place considerable emphasis on the doctor's degree as a necessary requirement for college teaching. This emphasis is quite understandable, since the Ph.D. is the most advanced degree offered by American universities and is therefore a symbol of maximal academic achievement. Unfortunately, the relevance of the doctoral degree in the qualification of a college teacher is often misunderstood, and the resulting confusion has, in many cases, led to serious abuses. We have in mind such abuses as the preferential treatment frequently assured the holder of a doctoral degree over an otherwise well-qualified teacher who lacks a Ph.D.; or unrealistic emphasis at some institutions on the number of doctoral degrees, regardless of origin, held by members of the faculty. We shall examine the requirements for a Ph.D. in mathematics in order to analyze the relevance of each of them in evaluating the qualifications of a college teacher.

The Ph.D. in mathematics is by long tradition a research degree; research in mathematics has meant the creation of new mathematics and not, as in many fields, the scholarly analysis or synthesis of previous work. A mathematics student working toward a Ph.D. is expected to spend a considerable portion of his time, in the later undergraduate and early graduate years, acquiring a broad general background in mathematics. The breadth of his knowledge is usually tested by special examinations after one, two, or more years of graduate study. After these examinations the student's work becomes highly specialized with seminars, independent study, and thesis work penetrating in depth some area of particular interest.

The earlier years of graduate study provide a breadth of knowledge essential to a college teacher. The subsequent, very specialized, graduate study is equally essential for research work.

An institution that has, or that aspires to have, a legitimate graduate program must necessarily have a substantial number of research mathematicians on its faculty, to implement the program and to provide the necessary leadership. This is a condition which obviously should not be changed. Even in an undergraduate college which does not offer a graduate program in mathematics, there are good reasons for wanting faculty members to have the kind of preparation required for the Ph.D. Both the nature and the content of undergraduate courses in mathematics must undergo frequent revision to reflect the rapid developments in mathematics and in related fields. The competent teacher of undergraduate mathematics must be able to master new material independently and to prepare new courses involving material which he never studied in his own course work; frequently he must guide independent study by gifted undergraduates. These challenges call for a degree of mathematical maturity which comes only with extended effort. A confident approach to new material is made possible not alone by the amount of knowledge a teacher may have; it requires in addition a broad understanding and a deep appreciation of the nature of mathematics. A significant research experience such as that demanded for the Ph.D. dissertation is perhaps the best guarantee that a person actually has the kind of maturity we have in mind. The research work itself may not provide the prospective teacher with the necessary breadth of knowledge, but it provides him with maturity which should enable him to continue his mathematical education independently and indefinitely.

THE FORMAL EDUCATION OF COLLEGE TEACHERS OF MATHEMATICS

There are a number of levels of mathematical preparation which are appropriate for teaching the various courses described in A General Curriculum in Mathematics for Colleges. In discussing these levels we shall begin with a prospective college teacher's undergraduate program and then indicate the teaching responsibilities compatible with successive components of his additional mathematical education.

A. Strong Undergraduate Mathematics Major Program

Mathematics major programs differ widely from one institution to another. For present purposes we shall refer to a major program based on courses described in the GCMC report.* This report suggests that a mathematics major program for students preparing for graduate work in mathematics should include the lower-division analysis courses 1, 2, 4, 5, the lower-division probability course 2P, the lower-division linear algebra course 3, and the upper-division courses in algebra (6M and 6L), analysis (11, 12, 13), and applied mathematics (10). The report adds that, where possible, a stronger major is desirable, with options to be selected from the courses in probability and statistics (7), numerical analysis (8), and geometry (9). The CUPM Panel on Pregraduate Training, in reviewing these recommendations in their report Preparation for Graduate Study in Mathematics [page 447], observed that most graduate departments desire an incoming student to be especially well-grounded in algebra and analysis. Consequently they recommended, and described in outline, a year course in algebra to replace the GCMC course, as well as the content for the year course in real analysis (11-12) which they considered essential to preparation for graduate study.

We do not favor special undergraduate curricula for prospective college teachers. Instead, we recommend a strong mathematics major program which begins with the mathematics major as described in GCMC and includes the analysis courses 11-12 outlined by the Pregraduate Panel, the additional work in algebra recommended by the Pregraduate Panel, and two additional courses selected from probability and statistics (7), numerical analysis (8), and geometry (9). We firmly believe that applications should be presented in all mathematics courses and that, where possible, students also should have some courses in fields where mathematics is applied (for example, theoretical physics or mathematical economics).

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* These courses will be cited below using the numbers given them in the report A General Curriculum in Mathematics for Colleges; all of them are semester courses.

While the strong mathematics major program which we have described is certainly desirable for a college teacher, one must expect and encourage wide variation in the undergraduate programs which students actually encounter. Indeed, it is to be expected that strongly motivated research-oriented students will be advised to proceed to graduate work without some of the undergraduate courses we have listed. There are institutions where this strong major will be completed by many students at the time they receive the bachelor's degree. On the other hand, some students, including many in other disciplines or in training programs for secondary school teachers, will not encounter some of the more advanced upper-division courses until they reach graduate school.

Graduate students who have completed a strong undergraduate mathematics major program with distinction and who have a definite interest in teaching are qualified to assist more mature teachers in teaching elementary courses at the college level. Completion of this strong mathematics major should not be considered permanent qualification for a teacher of even the most elementary college courses. As we pointed out, continued intellectual growth is an essential qualification for sustained competence as a teacher. (In junior colleges, or at other institutions where remedial mathematics courses are offered, there could be some justification for having outstanding teachers with training equivalent to that of a strong mathematics major as members of the faculty, responsible for these courses.)

B. First Graduate Component

In this section we describe the additional graduate work which a prospective college teacher, who has completed the strong major program, will need in order to acquire the mathematical background necessary to teach the lower-division curriculum of GCMC (and hence the mathematics courses for junior college students who plan to transfer to a college or university). Those who complete both the strong major program and this first graduate component will also have the technical qualifications needed to teach some of the upper-division courses of the GCMC program.

We must emphasize that the courses to be described are not meant to be minimal introductions to their subject matter. The courses demand a serious involvement with graduate mathematics. Where questions of substance arise, mathematics departments should tend in the direction of the recommendations of the CUPM Pregraduate Panel's report Pregraduate Preparation of Research Mathematicians [page 369].

The time required to complete the first graduate component will vary considerably; obviously, a student who achieves only minimal success in his course work or whose undergraduate training has fallen short of the strong mathematics major will require more than the usual amount of time to reach the necessary level of mathematical maturity. We have found, incidentally, that the programs of many

Academic Year Institutes bring the student to a level only slightly beyond that of a strong mathematics major.

Hopefully, a student who completes the first graduate component will have developed a mathematical maturity that will enable him to bring to his classes an awareness of the fact that the mathematics taught in lower-division courses is a part of the basic fabric of applied mathematics. He should be able to present illustrations from outside of mathematics including both the physical and the behavioral sciences, where appropriate. It would be desirable, but it is not necessary, that he have made a serious study of some field of application (as represented, for example, by a year's course work), but it would also be possible for him to broaden his appreciation for the applications of mathematics by supplementary reading outside of his regular course assignments.

The first graduate component, which is an essential part of the preparation of a college mathematics teacher, and for which a master's degree would be suitable recognition, includes:

1. The completion of the strong mathematics major, if it has not been completed by the time the student begins graduate work.

2. At least two of the following three items:

   a. A substantial year's work in modern algebraic theory building on the earlier courses which presented the fundamental concepts of algebra.

   b. A year's work in analysis designed to follow the undergraduate analysis courses 11, 12, 13 of GCMC.

   c. A full year of "geometry" from a topological point of view following an undergraduate geometry course such as 9 of GCMC. This should include a semester of general topology and at least an introduction to algebraic topology.

3. At least one semester, preferably two, of teaching a class of undergraduate mathematics under the close supervision of an experienced teacher. Serious special attention should be devoted to the pedagogical problems involved in developing mathematical material for an immature audience. If possible, this teaching experience should also be accompanied by a proseminar designed to give students experience in articulating mathematical concepts before a critical audience.

We have repeatedly stated that a college teacher must continue his mathematical growth throughout his career. While the early graduate years are themselves a period of growth, it is also desirable that the student review college mathematics from the more advanced point of view of his graduate courses. There are many books by distinguished mathematicians which can help in this review and which provide a wealth of illustrations to enrich his teaching.

C. Advanced Graduate Component

In this section we describe a program of study which, when offered in a graduate department having an established Ph.D. program in mathematics, should provide the prospective college mathematics teacher with the mathematical background and with the maturity he will need to be prepared to teach all of the courses in the four-year GCMC program. Successful completion of both the first graduate component and the advanced graduate component should also provide a sound basis for the continued professional and intellectual growth which a college teacher requires in order to qualify, in due course, for promotion, tenure, and administrative responsibility in his department--whether or not he subsequently earns an advanced degree. Some of the work which we include in the advanced graduate component is intended specifically for prospective college teachers and to this extent it complements regular graduate programs designed to prepare research mathematicians.

The work of the advanced component builds on that of the first component and consists of the satisfactory completion of the following:

1. A year course in any of the three fields--algebra, analysis, topology-geometry--not included in satisfying Recommendation 2 of the first graduate component.

2. A second year of graduate study in at least one of the three fields mentioned above, as well as additional graduate courses in mathematics representing areas of special interest to the faculty.

3. A graduate research seminar designed to bring the student into active contact with the creative efforts of a member of the research faculty.

4. A seminar or reading course designed to provide a critical review of the relationship of the student's graduate courses to the undergraduate courses he might be called upon to teach: briefly, a form of "Elementary Mathematics from an Advanced Viewpoint."

5. A general examination designed to test the breadth of knowledge essential to a college mathematics teacher. It would cover each of the major areas of mathematics in which the student has taken courses at the graduate level.

6. A lecture project designed to test the student's ability to prepare and deliver a seminar talk and to provide him

an opportunity to develop his expository ability. We suggest that the topic assigned for the lecture be one outside of the student's field of specialization, in order that he may also demonstrate his competence to pursue mathematics on his own initiative.

While the time and the course work required to complete the advanced graduate component will vary a great deal among individuals and institutions, it should be clear that a candidate who reaches this level must have a strong personal commitment to mathematics and that he will have successfully completed at least two or three years of serious full-time graduate study beyond the strong major program.

The depth of understanding, the breadth of knowledge, and the mathematical maturity attested to by the successful achievement of the advanced graduate component are essential for the effective teaching of the various courses in mathematics offered at the college level. We believe that such achievement should be recognized by appropriate certification. Recent action of the faculties at Michigan, at Yale, and on the Berkeley Campus of the University of California seems to indicate a growing sentiment in favor of some such formal recognition.

D. The Doctorate

Although we have asserted that a Ph.D. degree in mathematics should not be regarded as an absolute necessity for the academic qualification of a college teacher of mathematics, we certainly would not suggest that the work and the study required to earn a Ph.D. are not important or that they would not enhance the effectiveness of any college teacher. The significant difference between the advanced graduate component and the Ph.D. degree consists of research seminars and independent reading in the candidate's field of specialization, leading to an original contribution to mathematical knowledge reported in the thesis. Making an original contribution to mathematical knowledge is extremely valuable for the college teacher, for by engaging in research he becomes a participating member of the mathematical profession and thus is able to transmit to his students, both in the classroom and outside of it, the knowledge and the stimulation that comes from the experience of creating new ideas.

It is our intention that the successful completion of the advanced graduate component when followed by an appropriate thesis should be worthy of a doctor's degree. Thus we believe that it should be offered only in those departments which already have established Ph.D. programs in mathematics; only in the vital research atmosphere of such a department can the required quality be attained. We also believe that graduate schools should be encouraged to seek ways of increasing the opportunities for qualified college teachers of mathematics to earn the Ph.D. after some years of teaching.

THE COMPOSITION OF AN UNDERGRADUATE DEPARTMENT

It is clear from the preceding discussion that we consider it neither necessary nor desirable to specify a single standard to be applied to all college teachers of mathematics. A very effective department can be composed of staff members with different levels of preparation and experience. Of course, there is no such thing as being too highly qualified to teach any course: higher qualifications can always be translated into more effective teaching, the design of an improved course, the preparation of better materials, and so on. However, the critical shortage of mathematics teachers requires that the available staff be used as effectively as possible, both in the individual college and in the country as a whole.

Let us consider reasonable academic qualifications for the mathematics faculty of a small college, one with a mathematics staff of six. We assume that the college has no graduate students and hence no graduate teaching assistants. At least two thirds of the teaching load is likely to be in lower-division courses. We believe that if three or four of the six staff members are at or near the level of the advanced graduate component or have Ph.D.s in mathematics, the department will have the technical qualifications needed to do an excellent job. Care must be exercised in the selection of staff members to assure that advanced study is not concentrated in only one area of mathematics. For example, the GCMC courses in applied mathematics, numerical analysis, and probability and statistics require special attention; there should be members of the staff who have graduate work in these areas.

We do not suggest that all the lower-division courses ought to be taught by teachers in the first group and all the advanced courses by the others. On the contrary, we consider it essential that some of the most highly qualified teachers be involved in the elementary courses, just as we believe that many of the less well-prepared teachers can be expected to do excellent work in some of the more advanced courses. Indeed, one very effective way for any teacher to increase his knowledge is for him to give an advanced course in which he may learn as he teaches. The level of qualification of any staff member cannot be regarded as permanent or fixed. Since continued intellectual growth is required for good teaching, every staff member at whatever level must be considered as on his way to higher qualifications. This applies just as much to a man with a Ph.D. in mathematics as it does to any other teacher in the department.

Finally, we do not wish to imply that rank or salary should depend entirely on the levels of academic preparation we have described. In general, rank should correspond to professional competence and achievement, as indicated by all professional activities and by teaching effectiveness, as well as by earned degrees.

Our suggestions are, of course, subject to modification to fit the needs of individual institutions. We predict, for example, that for the foreseeable future the first graduate component should represent adequate preparation for teaching transfer students in junior colleges, provided the teacher continues to remain "intellectually alive." At universities, and at colleges near universities, it is certainly appropriate to make use of teaching assistants who have reached only the level of a strong mathematics major, or who have not yet completed the first graduate component, provided that the teaching is adequately supervised and that there is clear evidence of progress toward the next level.

FINAL REMARKS

We have repeatedly stated our conviction that continued intellectual and professional growth is essential to continued competence as a teacher. One needs to move forward in order not to fall behind. A significant reason for recognizing the Ph.D. as a meaningful and desirable level of qualification for college teachers is that it is both evidence of an individual's ability to continue his mathematical growth by himself and an indication of momentum in that direction. However, for reasons of isolation or inadequate training, many college teachers are unable to provide for their own professional growth. For them, and for college teachers who do not have even minimal academic qualifications for the responsibilities they are asked to assume, there is an urgent need for expanded programs of external stimuli for improvement: guidance, financial assistance, and easily accessible and attractive study programs. Institutes, internships, and new forms of retraining need to be explored and developed. We must recognize, however, that the intellectual growth of college teachers depends primarily not on opportunities of this kind but on the conditions of their daily work. If their teaching and administrative duties leave them no time or energy for study and reflection, then it cannot be expected that their scientific qualifications will improve from year to year, or even that they will be maintained.

We have no definite advice to offer for solving these problems. We can only call attention to them and suggest that the difficulties involved in upgrading many of our present teachers and in stimulating continued growth in others provide some of the most important and pressing problems faced by the mathematical community.

A BEGINNING GRADUATE PROGRAM

IN MATHEMATICS

FOR PROSPECTIVE TEACHERS OF UNDERGRADUATES

A Report of

The Graduate Task Force

February 1969

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INTRODUCTION

More than half of the college and university mathematics teachers in the United States do not hold a Ph.D. in one of the mathematical sciences. Thus, the bulk of undergraduate teaching of mathematics in the country is being done by men and women whose graduate training, for one reason or another, has broken off short of the doctorate. There is no convincing evidence that this situation will soon change: the rising output of Ph.D.s in mathematics is probably more than offset by rapidly rising college enrollments and by increasing demands on mathematics as a service discipline.*

Those concerned with the preparation of college teachers are therefore faced with a basic problem: What is the best way to arrange the early part of the graduate program in mathematics to provide background for effective college teaching? This problem is further complicated because the research potential of most students is still untested when they begin graduate work; thus, it is not possible to separate those who will complete a Ph.D. from those who will not. Accordingly, the choice of topics for the first year or two of graduate study must permit students to progress unretarded toward the Ph.D. This booklet explores one solution to this problem.

We contend that all graduate students of mathematics should be treated as future teachers--first, because most of them do in fact go into teaching, and second, because virtually all professional mathematicians are engaged to some extent in the communication of mathematics. Hence, graduate programs aimed at producing better teachers may be expected to benefit everyone.

The CUPM *ad hoc* Committee on the Qualifications of College Teachers of Mathematics, in its report *Qualifications for a College Faculty in Mathematics* [page 102], outlined a graduate program ("first graduate component") which provides both a reasonable first segment of a Ph.D. program and adequate background for teaching the lower-division courses described in *Commentary on A General Curriculum in Mathematics for Colleges* (GCMC) [page 33]. In 1967 the Graduate Task Force, a group with membership drawn from CUPM and its Panel on College Teacher Preparation, was given the assignment of preparing a more detailed description of the first graduate component. A description appears in the pages that follow.

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* The situation has, in fact, changed dramatically since this report was written. According to the 1972 document *Undergraduate Education in the Mathematical Sciences, 1970-71* (Report of the Survey Committee of the Conference Board of the Mathematical Sciences), university departments of mathematical sciences had, in the fall of 1970, 6,304 doctorates in mathematical sciences, 348 other doctorates, and 971 nondoctorates. In four-year college departments there were 3,508 mathematical science doctorates, 758 other doctorates, and 5,158 nondoctorates.

Clearly, this set of recommendations is not the only possible solution to the problem stated above, but we believe that it forms a sound basic program which each university can adapt to local conditions. The time for its completion will depend upon the student's ability and preparation, but in most cases one to two years beyond the bachelor's degree should be adequate. Satisfactory completion will insure that the student has sound academic qualifications for teaching lower-division courses in calculus, linear algebra, probability, and advanced multivariable calculus.

Besides serving the purposes already described, appropriate parts of the course of study we recommend would constitute an excellent sabbatical program for established teachers who wish to improve their acquaintance with modern approaches to mathematics.

The recommended program is discussed in detail in the following section. Here we mention some of its characteristic features and reasons for them. Two considerations figure prominently in the selection of topics for courses: first, the relative importance of the topic in all of mathematics, and second, its relevance to teaching the lower-division courses described in the GCMC report.

In analysis, to follow a year of undergraduate real analysis and a semester of undergraduate complex analysis (like the GCMC courses 11, 12, 13), the program includes a semester of measure and integration followed by a semester of functional analysis. The course in measure and integration is obviously relevant to the GCMC courses in calculus and probability. We believe that the course in functional analysis is more important at this stage than a second course in complex analysis, since functional analysis will further develop the methods of linear algebra, the concept of uniform convergence, and various other topics in analysis. Moreover, functional analysis provides an immediate application of the course in measure and integration.

In topology we recommend a sequence which, in addition to the usual material in basic topology, includes an introduction to manifold theory and differential forms, to provide the prospective teacher with a deeper understanding of multivariable calculus.

Since lower-division mathematics needs to be illustrated liberally with uses of the subject, college teachers must command a broad knowledge of the applications of mathematics. Also, many will be called upon to teach elementary probability and statistics. Hence, we recommend that a course of study for college teachers include two or three semesters of work at the advanced undergraduate or beginning graduate level chosen from courses in probability, statistics, differential equations, numerical analysis, or subjects in applied mathematics. Moreover, all courses in the program should give attention to the relevance of their subjects to undergraduate mathematics and related disciplines.

In algebra we believe that a year-long advanced undergraduate course such as that described in the CUPM publication Preparation for Graduate Study in Mathematics [page 453] is essential, but that further study of algebra is less important in the preparation of teachers of lower-division mathematics than the suggested work in analysis, topology, and applied mathematics; therefore, graduate-level algebra has been treated as an elective. Likewise, we do not advocate a special requirement in geometry, partly because a considerable amount of geometry in various forms is distributed throughout other recommended courses, and partly because advanced training in geometry does not seem essential either as background for lower-division teaching or as *general* preparation for further graduate study. Nevertheless, geometrical points of view should be stressed in courses whenever they are appropriate.

It is certain that many teachers of lower-division mathematics will, in the very near future, be called upon to use the computer to some extent in their courses. However, in view of the rapid developments in computer science and the many nonmathematical factors involved, any explicit recommendations on the role of computing in this program must be regarded as tentative at this time. We do expect that students completing this program will have acquired at least a basic knowledge of computers [for example, the content of the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563].

Apart from formal course work, we feel that a meaningful apprenticeship in teaching is an essential aspect of the student's preparation. Activities to provide such an apprenticeship should form an integral part of beginning graduate work.

A master's degree would suitably recognize completion of the first graduate component. However, in place of a master's thesis we strongly recommend the substitution of a comprehensive examination. Foreign language requirements are not discussed here because we believe that they are irrelevant for a student whose graduate training stops at the first graduate component; however, the student who hopes to earn a Ph.D. should be advised that a reading knowledge of foreign languages is likely to be essential in his subsequent work.

It must be understood that the student who has completed this or any other program will not be, by that reason alone, a complete teacher or mathematician for the rest of his career; sustained intellectual and professional growth is essential to continued competence as a teacher and as a mathematician. For this reason, we urge a graduate faculty to make vigorous efforts to involve the students seriously, as participants rather than observers, in the mathematics they are studying. It is important for the student who stops short of the Ph.D., even more than for the one who will complete it, that course work of the first two years of graduate study emphasize fundamentals and basic understanding. This applies especially to the prospective college teacher who must be able to relate his graduate work to the material he will be teaching later. Courses which involve the student in *doing* mathematics as distinct from hearing about mathematics would seem to be particularly valuable. Depth of understanding on the part of the student is to be preferred to superficial exposure to mathematical terms. Our course outlines should be understood in this context.

Finally, we emphasize that it has not been our objective to design a separate track in graduate mathematics. This program is intended to prepare the student as an effective and well-informed teacher of lower-division mathematics; but, at the same time, we believe it moves him toward the Ph.D. at a satisfactory rate.

PROGRAM DESCRIPTION

The "first graduate component," as described in Qualifications for a College Faculty in Mathematics [page 107], is a program of graduate study built upon strong undergraduate preparation in mathematics. Because the undergraduate preparation of graduate students varies widely, it is useful to describe the first graduate component in terms of the combined undergraduate and graduate preparation of the candidate. It is likely that many students will have to complete in graduate school some undergraduate-level work; for such students, up to two years of post-baccalaureate study may be required to complete this program.

We assume that every student has already completed lower-division courses equivalent to the GCMC courses Mathematics 1, 2, 2P, 3, 4, 5, including a basic course in computer science, like C1 [page 563].

In addition, he will have studied some, but probably not all, of the following upper-division courses: Mathematics 7 (Probability and Statistics)*, 8 (Introduction to Numerical Analysis), 9 (Geometry), 10 (Applied Mathematics) [see page 79].

The following five courses form the core of preparation for graduate study: Mathematics 11-12 (Introductory Real Variable Theory), 13 (Complex Analysis), D-E (Abstract Algebra). For outlines of Mathematics 11-12 and 13, see page 93. For outlines of D-E, see page 453.

Graduate courses which are especially appropriate for the first graduate component, and for which suggested course descriptions are given starting on page 121 of this COMPENDIUM, are:

* We use 7A and 7B to refer to the probability component and statistics component, respectively, of Mathematics 7.

P Measure and Integration

Q Functional Analysis

R Complex Analysis

S Topology

T Homology and Multivariable Integration

U Topology and Geometry of Manifolds

V Galois and Field Theory

W Ring Theory and Multilinear Algebra

X Advanced Ordinary Differential Equations with Applications

Y Problem-oriented Numerical Analysis

Z Seminar in Applications

Courses 11-12, 13, D-E, P, Q, S, and T should be in every student's program. Because of the great importance of applied mathematics, every program of study should include at least one year of applied work, of which the following four sequences are examples: 7A-7B, X-10, X-Y, X-Z.

Each student should, if possible, include a third course from among the courses 7A, 7B, 10, X, Y, Z in his program. Students who plan to continue into the advanced graduate component and to specialize in some area of pure mathematics are advised to take as many as possible of the courses R, U, V, W. Other students may substitute electives in geometry, logic, foundations, number theory, or other subjects.

For the sake of convenience, we have stated our recommendations in terms of semester courses. However, we believe that courses at the graduate level are best thought of as year courses. The material outlined for pairs of related courses can, of course, be rearranged within the year to suit local conditions.

Since beginning graduate programs ordinarily include year courses in analysis, topology, and algebra, our recommendations depart from the norm only in ways intended to enhance the ability of the student to teach lower-division mathematics.

Effective exposition is a skill of major importance to any prospective mathematician, whether he expects his principal professional emphasis to be research or teaching. However, it is unrealistic to assume that a beginning graduate student is qualified to teach well, even in introductory undergraduate courses. Therefore, we propose that he be required to complete an apprenticeship in teaching under the thoughtful direction of experienced members of the faculty. Suggestions for such a program are discussed later in this report.

To complete the program, we suggest that every student be required to take a comprehensive examination designed specifically to test the breadth and depth of the candidate's understanding of mathematics relevant to the undergraduate curriculum. Whenever feasible, the examination should be scheduled so that students have several weeks devoted exclusively to preparation for it. We believe that a truly comprehensive examination is a more appropriate requirement than the traditional master's thesis, principally because preparation for such an examination demands that the student regard his subject as a whole rather than a collection of parts.

In summary, the first graduate component, as described here, consists of the following work:

(1) Completion (if necessary) of a strong undergraduate major program which includes these upper-division courses: three semesters of real and complex analysis, a full year of abstract algebra (the equivalent of Mathematics 11-12, 13, D-E).

(2a) A year of graduate topology, including differential forms.

(2b) A year of graduate analysis: measure and integration and functional analysis.

(2c) A year of work at the advanced undergraduate or beginning graduate level, emphasizing the applications of mathematics: e.g., a year of probability and statistics; or a semester of differential equations followed by a semester of numerical analysis, a seminar in applications, or a "model building" course.

(3) A year or more of work focused on problems of teaching undergraduates.

For a student whose undergraduate preparation does not meet the standards described in (1) and (2c), completion of the first graduate component may require two years of study beyond his bachelor's degree. For example, if his undergraduate preparation in algebra and in analysis is weak, his program for the first graduate component might be as follows:

<u>First Year</u>

Analysis (Mathematics 11)	Analysis (Mathematics 12)
Complex Analysis (Mathematics 13)	Applied Mathematics (Mathematics 10)
Algebra D	Algebra E
Apprenticeship in Teaching	Apprenticeship in Teaching

Second Year

Analysis P	Analysis Q
Topology S	Topology T
Probability (Mathematics 7A) and	Statistics (Mathematics 7B)
OR	
Differential Equations X and	Numerical Analysis Y or Applications Seminar Z
Apprenticeship in Teaching	Apprenticeship in Teaching Comprehensive Examination

Most students will have completed some of the undergraduate courses in this program and thus will be able to substitute electives for some of the subjects listed. A student who has a very strong undergraduate major in mathematics will be able to complete the program in one year, for example, by taking the second year of the preceding schedule.

Graduate departments are urged to give careful attention to the proper placement of entering graduate students and to continue to advise them regarding course selections.

COURSE OUTLINES

Analysis

The following section includes suggested outlines for three one-semester graduate courses in analysis:

P. Measure and Integration (two suggested outlines are offered)

Q. Functional Analysis

R. Complex Analysis

Each student should include courses P and Q in his program of study.

P. Measure and Integration

This course provides an introduction to and essential background for Course Q, can be used in Course R, and is naturally useful in more advanced courses in real analysis. We present two outlines, which represent different approaches and a somewhat different selection of material. If presented in the right spirit, a course in Measure and Integration provides insights into the material of lower-division courses that the student will have to teach.

First Outline

1. The limitations of the Riemann integral. Examples of a series that fails to be integrable term-by-term only because its sum is not integrable; of a differentiable function with a nonintegrable derivative. Limitations of integration in general: there is no countably additive, translation-invariant integral for all characteristic functions of sets (the usual construction of a nonmeasurable set will serve).

2. Lebesgue integration on the line. Outer measure; definition of measurable sets by means of outer measure. Measurability of sets of measure 0, of intersections and unions, of Borel sets. Countable additivity. Application: the Steinhaus theorem on the set of distances of a set of positive measure. Measurable functions, Borel measurability, measurability of continuous functions. Egoroff's theorem. Definition of the integral of a bounded measurable function as the common value of $\inf \int \Psi(x)\,dx$ for simple majorants of Ψ of f and $\sup \int \varphi(x)\,dx$ for simple minorants φ. Riemann integrable functions are Lebesgue integrable. Bounded convergence and applications (necessary and sufficient condition for Riemann integrability; $\log 2 = 1 - 1/2 + 1/3 + \ldots$). Integrability of nonnegative functions, Fatou's lemma, monotone convergence, integrability of general functions. A nonnegative function with zero integral is zero almost everywhere.

3. L^p spaces, with emphasis on L^2; motivation from orthogonal series. Schwarz inequality; with little extra effort one gets (via convex functions) the Hölder, Minkowski, and Jensen inequalities. L^∞ as a formal limit of L^p via $(\int f^p)^{1/p} \to \text{ess sup } f$ as $p \to \infty$. Parseval's theorem, Riesz-Fischer theorem. Rademacher functions;

proof that almost all numbers are normal. Convergence of $\Sigma \pm \frac{1}{n}$ and other series with random signs. Proof (by Bernstein polynomials or otherwise) that continuous functions on an interval are uniformly approximable by polynomials; hence, continuous functions are dense in L^p.

4. Differentiation and integration. Proof that an indefinite integral is differentiable almost everywhere and its derivative is the integrand; the Lebesgue set. Equivalence of the properties of absolute continuity and of being an integral.

5. Lebesgue-Stieltjes integral with respect to a function of bounded variation. A rapid survey pointing out what changes have to be made in the previous development. Applications in probability, at least enough to show how to treat discrete and continuous cases simultaneously. Riesz representation for continuous linear functionals on C[a,b].

6. General measure spaces. Definition of the integral and convergence theorems in the general setting; specialization to n-dimensional Euclidean space. Fubini's theorem. Application to convolutions and to such matters as gamma-function integrals and $\int e^{-x^2}\,dx$. The one-dimensional integral as the integral of the characteristic function of the ordinate set.

7. (If time permits) Complex measures. Decompositions. Radon-Nikodým theorem.

Second Outline

1. Lebesgue integration on the line. F. Riesz's step function approach. Definition of the integral for simple step functions and extension to the class of functions which are limits almost everywhere of monotone sequences of simple step functions. Definition of summable function and fundamental properties of the integral. Extension to complex-valued functions. The basic convergence theorems, including monotone, bounded, and dominated convergence theorems. Fatou's lemma and convergence in measure. Illustrations and

applications: (a) justifications, by bounded convergence, of term-by-term integration of series leading to formulas such as $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots$; (b) use of dominated convergence to perform operations such as $\frac{\partial}{\partial\alpha}\int f(x,\alpha)\,dx = \int \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$; (c) proof of the analogue of Fatou's theorem for series: $\underline{\lim}_i \Sigma_n a_{in} \geq \Sigma_n \underline{\lim}_i a_{in}$ when $a_{in} \geq 0$. Comparison of the Riemann and Lebesgue integral.

2. Measure and absolute continuity. Measurable functions and measurable sets. Properties of measurable sets. Egoroff's theorem. Cantor's function and the relationship between Lebesgue and Borel sets. Nonmeasurable sets. Proof that the integral of a summable function is a countably additive set function. Almost everywhere differentiability of monotone functions. Review of basic properties of functions of bounded variation. Absolutely continuous functions. Fundamental theorem concerning differentiation of the integral of a summable function. Proof that an absolutely continuous function on an interval is of bounded variation and that its total variation is equal to the L^1-norm of its derivative. Helly's theorem on compactness of families of normalized functions of bounded variation.

3. L^p spaces and orthogonal expansions. Convex functions and the inequalities of Hölder and Minkowski. Proof that the L^p spaces are complete. Theorem: If $\{f_i\}$ is a sequence of measurable functions, if $f_i \to f$ pointwise almost everywhere, and if $\underline{\lim}_i \int |f_i| = \int |f|$, then $\underline{\lim}_i \int |f_i - f| = 0$. Lusin's theorem: f measurable and finite almost everywhere and $\delta > 0$ implies there exists a continuous function φ such that $\varphi = f$ except on a set of measure less than δ. Hence, continuous functions are dense in L^p, $1 \leq p < \infty$. Representation of continuous linear functionals on L^p. Orthonormal systems in $L^p(a,b)$. Bessel's inequality, Parseval's inequality, and the Riesz-Fischer theorem. Proof that the Cesáro means of the Fourier series of a function f in $L^p(0,2\pi)$ converge to f in the L^p-norm $(1 \leq p < \infty)$ and uniformly, provided that f is periodic and continuous. From this latter fact deduce the Weierstrass theorem on polynomial approximation of continuous functions on an interval. The trigonometric functions form a complete

orthonormal system in $L^2(0,2\pi)$.

4. Integration on product spaces. Integration and measure in R^n. Theorems of Fubini and Tonelli. Applications to nonlinear change of variable in multiple integrals.

5. Convolution (optional). If f is summable, then $f(t - x)$ is a measurable function on the plane. By Fubini's theorem, f and g in L^1 implies that $f*g(t) = \int f(t - x)\, g(x)\, dx$, the convolution of f with g, is finite for almost all t, $f*g$ is in L^1, and $\|f*g\| \leq \|f\|\,\|g\|$. In fact, for $p > 1$, $q > 1$, and $0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, we have $L^p*L^q \subset L^r$ and $\|f*g\|_r \leq \|f\|_p\,\|g\|_q$. (Actually, $L^1*L^p = L^p$ for $1 \leq p < \infty$.) Also, if $1 < p < \infty$ and $p' = p/(p-1)$ and $f \in L^p$, $g \in L^{p'}$, then $f*g$ is bounded. L^1 under convolution is an algebra without unit. Proof that $\widehat{f*g} = \hat{f}\hat{g}$, where $\wedge$ denotes the Fourier transform. Riemann-Lebesgue lemma.

6. General measure theory (optional). Set functions and introduction of abstract measure spaces. Definition of the integral and rapid review of standard theorems. Total variation of measures, regularity properties of Borel measures. Identification of Borel measures on the line with functions of local bounded variation. Absolutely continuous and mutally singular measures and consequences of the Radon-Nikodým theorem. Riesz representation for $C(X)$, X compact.

Q. Functional Analysis

The purpose of this course is to develop some of the basic ideas of functional analysis in a form suitable to applications and to deepen the student's understanding of linear methods in undergraduate mathematics. Whenever possible, topics should be treated and applied in a setting with which the student has some familiarity; main theorems should be supported with concrete and meaningful examples.

1. Metric spaces. Review of topology and metric spaces if necessary. Completion of metric spaces. Method of successive approximations. Proof that a contraction operator on a complete

metric space has a unique fixed point. Application to existence of the solution of a system of linear equations, polynomial equations, initial value problems for ordinary differential equations and integral equations.

2. Normed linear spaces. Examples (not all complete) from sequence spaces, function spaces, and finite-dimensional spaces. Completion of $C[a,b]$ under the L^1-norm. Proof that the unit ball in a normed linear space is compact if and only if the space is finite-dimensional. Equivalence of norms in finite-dimensional spaces.

3. Linear functionals. The dual space of a normed linear space. Computation of the dual for spaces R^n, c_o, ℓ_p $(1 \leq p < \infty)$, and $C[a,b]$. Contrast with algebraic dual. Convex sets and separation of convex sets by linear functionals. Support functionals. Analytic, geometric, and complex forms of the Hahn-Banach theorem. Applications of the Hahn-Banach theorem, such as (a) computation of the distance from a point to a subspace in terms of the linear functionals which vanish on the subspace; (b) the existence of a function in $L^\infty(0,1)$ of minimal L^∞-norm which satisfies the $N + 1$ relations $\int_0^1 t^n f(t)\, dt = a_n$, $n = 0, 1, \ldots, N$; (c) solution of the Hausdorff moment problem for $C[0,1]$; (d) the existence of Green's function for Laplace's equation in the plane for a domain with sufficiently smooth boundary. Principle of uniform boundedness and applications, such as (a) existence of a continuous periodic function on $[-\pi,\pi]$ whose Fourier series fails to converge; (b) the Silverman-Töplitz conditions for a regular matrix summability method; (c) existence of the Riemann-Stieltjes integral $\int_0^1 f\, d\alpha$ for every continuous f implies that α is of bounded variation on $[0,1]$. Weak (not weak*) convergence of sequences in normed linear spaces. Proof that weakly convergent sequences are bounded but not necessarily norm convergent. Characterization of weakly convergent sequences in spaces such as ℓ_p $(1 \leq p < \infty)$ and $C[a,b]$. Elementary introduction to distribution theory.

4. Linear operators. Examples from matrix theory,

differential and integral equations. The closed graph theorem and the interior mapping principle. Notion of an adjoint operator. Inversion of linear operators near the identity. The spectrum and resolvent of an operator.

5. Hilbert spaces. Inner products, orthogonality, orthogonal systems. Fourier expansions, Bessel's inequality, and completeness. Representation of linear functionals. Self-adjoint operators on a real Hilbert space as a generalization of symmetric linear transformations on R^n. Eigenvalues, eigenvectors, invariant subspaces, and projection operators. The spectral theorem for completely continuous self-adjoint operators. (Here the goal is the formula $Ax = \Sigma \lambda_k (x,x_k)x_k$, where A is a compact and self-adjoint operator, x is any point, $\lambda_1, \lambda_2, \lambda_3, \ldots$ is the sequence of nonzero eigenvalues, and $x_1, x_2, x_3, \ldots$ is the corresponding set of eigenvectors.) Construction of a one-parameter family of projections E_λ which allows representation of the action of A in terms of a vector-valued Riemann-Stieltjes integral

$$Ax = \int \lambda \, dE_\lambda x.$$

Description (without proof) of the corresponding theorem for the unbounded case. Application of the theory of compact, self-adjoint operators to Sturm-Liouville systems or integral equations with symmetric kernels.

R. Complex Analysis

The amount of material that can be covered in this course depends very much on the amount of knowledge that can be assumed from Mathematics 13. The outline assumes that the student knows this material quite well, but some of the more advanced topics may have to be omitted or treated in less depth. Such topics are enclosed in brackets.

1. Holomorphic functions. (Much of this should be review.) Cauchy's integral theorem in a more general setting than was used in Mathematics 13. According to circumstances, this may be for unions of star-shaped regions, for C^1 Jordan curves, for singular

cells, etc., but not for general rectifiable Jordan curves. Cauchy's integral formula. Taylor and Laurent series. Residue theorem. [Evaluation of some definite integrals which are more sophisticated than those of Mathematics 13.] Classification of isolated singularities, Casorati-Weierstrass theorem. Liouville's theorem. Fundamental Theorem of Algebra. $\int f'/f = 2\pi i \cdot$(number of zeros minus number of poles), with applications to some special cases (number of zeros of a polynomial in a quadrant, for example). Maximum modulus theorem. Schwarz's lemma. Rouché's theorem with some concrete applications (Fundamental Theorem of Algebra again; zeros of $e^z + z$ and other special functions). [Montel's theorem, Phragmèn-Lindelöf theorems.]

2. Harmonic functions. Cauchy-Riemann equations. Mean value property for harmonic functions. Poisson formula and Dirichlet problem for the circle and annulus. Connection with Fourier series and Poisson summability of Fourier series at points of continuity. [Other problems on functions holomorphic in a disk: Abel's theorem, elementary Tauberian theorems.] [Fatou's theorem on radial limits.] [Positive harmonic functions, Herglotz's theorem on the integral representation of holomorphic functions with positive real part in a disk. Harnack's theorem.]

3. Holomorphic functions as mappings. Mapping properties of the elementary functions. Nonconstant holomorphic functions are open. Conformality at points where the derivative is not zero. Only holomorphic functions produce conformal maps. Conformal automorphisms of the disk and the half-plane. Normal families. Proof of the Riemann mapping theorem. [Schwarz-Christoffel formula. Conformal representation of a rectangle on a half-plane. Elliptic functions. Proof of the small Picard theorem.]

4. Analytic continuation. Schwarz reflection principle. Analytic continuation. Permanence of functional equations. Monodromy theorem. [Multivalued functions. Elementary Riemann surfaces.]

5. Zeros of holomorphic functions. Infinite products. Entire functions. Meromorphic functions. The Weierstrass factorization theorem. Mittag-Leffler theorem. Gamma function. [Jensen's formula and Blaschke products.]

[6. Approximation. Runge's theorem and approximation by polynomials. Mergelyan's theorem.]

Topology

The following section includes suggested outlines for a sequence of three one-semester graduate courses in topology:

S. Topology

T. Homology and Multivariable Integration (two outlines are offered)

U. Topology and Geometry of Manifolds

Each student should include courses S and T in his program of study. Students who plan to elect course U must study the first (preferred) outline of T.

S. Topology

We assume that the students have made a brief study of metric spaces, Euclidean spaces, and the notion of continuity of functions in metric spaces. (This material is covered in Sections 4, 5, and 6 of Mathematics 11-12.)

1. Basic topology. Topological spaces, subspace topology, quotient topology. Connectedness and compactness. Product spaces and the Tychonoff theorem. Separation axioms, separation by continuous functions. Local connectedness and local compactness. Metric spaces, completion of metric spaces, uniform continuity. Paracompactness, continuous partitions of unity.

2. Applications to calculus. Use the above results to prove again the basic topological results needed for calculus and the Heine-Borel and Bolzano-Weierstrass theorems.

3. Fundamental group. Homotopies of maps, homotopy equivalence. The fundamental group π_1, functional properties, dependence on base point. Show that $\pi_1(S^1) = Z$.

4. Applications of the fundamental group. Brouwer fixed point theorem for the disk D^2. R^2 is not homeomorphic with R^3. Relevance of the fundamental group to Cauchy's residue theorem. Fundamental Theorem of Algebra.

5. Covering spaces. Covering spaces, homotopy lifting and homotopy covering properties. Regular coverings, existence of coverings, universal covering. Factoring of maps through coverings. Relation with Riemann surfaces.

T. Homology and Multivariable Integration

Preferred Outline

1. Manifolds. Topological manifolds. C^k and C^∞ functions on R^n. Differentiable structure on a topological manifold. Diffeomorphisms. C^∞ partitions of unity for paracompact manifolds.

2. Functions on manifolds. The ring $\mathcal{Q}(U)$ of C^∞ real-valued functions on an open set U, the ring $\mathcal{Q}(x)$ of germs of C^∞ functions at a point x. Pullbacks of these rings via a C^∞ function. Tangent bundle and cotangent bundle. Bases for tangent and cotangent spaces in a coordinate system. Vector fields, Poisson bracket, flows. Inverse and implicit function theorems. Frobenius' theorem.

3. Applications to differential equations. Relation of vector fields to ordinary differential equations and of Frobenius' theorem to partial differential equations.

4. Differential forms. Differential forms, elementary forms. Exterior multiplication of forms, the differential operator d on forms; $dd = 0$ and d of a product.

5. Applications to classical vector analysis. The algebra of forms on R^3 contains vector algebra and with d contains vector analysis.

6. deRham cohomology. Pullback of forms via a C^∞ map commutes with d. Closed and exact forms, deRham groups as a cohomology theory.

7. Simplicial homology. Simplicial complexes, simplicial

homology. Barycentric subdivision, simplicial approximation theorem. Calculation of π_1 for a simplicial complex. Singular homology and cohomology of a space.

8. Applications of simplicial homology. The Brouwer fixed point theorem for D^n, invariance of domain, and the Jordan curve theorem. (Recall use of the Jordan curve theorem in complex analysis.)

9. Stokes' theorem. Integral of a p-form over a singular p-chain. Proof of Stokes' theorem. This implies that integration induces a bilinear map from singular homology and deRham groups to R. Green's theorem as a special case of Stokes' theorem.

Second Outline

Note: If a student does not plan to take course U, then the following easier version of T may be desirable. This is carried out by working in R^n instead of in general differentiable manifolds, and the result is still a fairly general version of Stokes' theorem.

1. Simplicial homology. Simplicial complexes, barycentric subdivision, simplicial maps, and the simplicial approximation theorem. Simplicial homology theory, functional properties of homology groups. Calculation of homology groups for simple complexes.

2. Differential forms. Differential forms on open sets of R^n. Properties of differential forms, the operator d on forms. Pullback of forms via a C^∞ function. Application to vector algebra and vector calculus. Closed and exact forms, the deRham groups.

3. Singular homology and applications. Singular homology theory. Applications: the Brouwer fixed point theorem, R^n and R^m are homeomorphic if and only if $n = m$, invariance of domain, Jordan curve theorem.

4. Stokes' theorem. Integration of p-forms over differentiable singular p-chains. Proof of Stokes' theorem.

U. Topology and Geometry of Manifolds

1. Chain complexes. Chain and cochain complexes (examples from T), derived groups. Exact sequences, ladders, the 5-lemma. Exact sequences of chain complexes, Bockstein exact sequence. Chain homotopies. Poincaré lemma and cone construction; derived groups of a contractible open set are zero.

2. Riemannian metrics for manifolds. Riemannian metrics for paracompact manifolds. Geodesics: existence and uniqueness. A paracompact C^{∞} manifold may be covered with a star-finite covering by geodesically convex sets (so that all sets in the covering and all intersections are contractible).

3. Comparison of homology theories. A lattice L of subsets of X containing $\emptyset$ and X gives a category S with elements of L as objects and inclusions as morphisms. A cohomology theory h on S is a sequence of cofunctors h^q from S to abelian groups, along with natural transformations

$$\delta: h^q(A \cap B) \to h^{q+1}(A \cup B)$$

such that the Mayer-Vietoris sequence is exact. Proof that if L is a star-finite covering of X by open sets, then cohomology theories h, $\hat{h}$ which agree on finite intersections agree on X. Use of these results to deduce deRham's theorem and to prove that simplicial and singular theories agree on a simplicial complex.

4. Global differential geometry. The remainder of the course is devoted to surfaces. Gaussian curvature, spaces of constant curvature. Gauss-Bonnet theorem for surfaces, non-Euclidean geometries.

Algebra

The following section includes suggested outlines for two one-semester graduate courses in algebra:

V. Galois and Field Theory

W. Ring Theory and Multilinear Algebra

These courses are independent of one another and should be offered as electives. Each course outline includes a basic minimal list of topics as well as a list of optional topics from which the instructor is invited to choose.

V. Galois and Field Theory

Note: 1 and 2 are reviews of topics that should have been covered in the previous one-year algebra course D-E outlined on page 453.

1. Review of group theory. The third isomorphism theorem. Definition of simple group and composition series for finite groups. The Jordan-Hölder theorem. Solvable groups. Simplicity of the alternating group for $n > 4$. Elements of theory of p-groups. Theorems: A p-group has nontrivial center; a p-group is solvable. Sylow theory. Sylow theorem on the existence of p-Sylow subgroups. Theorems: Every p-subgroup is contained in a p-Sylow subgroup; all p-Sylow subgroups are conjugate and their number is congruent to 1 modulo p.

2. Review of elementary field theory. Prime fields and characteristic. Extension fields. Algebraic extensions. Structure of $F(a)$, F a field, a an algebraic element of some extension field. Direct proof that if a has degree n, then the set of polynomials of degree $n - 1$ in a is a field; demonstration that $F(a) \cong F[x]/(f(x))$, where f is the minimum polynomial of a. Definition of $(K:F)$, where K is an extension field of F. If $F \subset K \subset L$ and $(L:F)$ is finite, then $(L:F) = (L:K)(K:F)$. Ruler-and-compass constructions. Impossibility of trisecting an angle, duplicating the cube, squaring the circle (assuming π transcendental).

3. Galois theory. The group $G(M/K)$ of K-automorphisms of a field M containing K. Fixed field H' of a subgroup H of $G(M/K)$. Subgroup F' of $G(M/K)$ leaving an intermediate field F fixed. Examples like $Q(\sqrt[3]{2})$ to show that $G(M/K)'$ may be bigger than K. An object is closed if it equals its double prime. If $M \supset F \supset F_1 \supset K$ and $G(M/K) \supset H \supset H_1$, then $[F_1':F'] \leq (F:F_1)$ and

$(H_1':H') \leq [H:H_1]$. All finite subgroups of $G(M/K)$ are closed. M/K is galois if $G(M/K)' = K$. Fundamental theorem. Artin's theorem: If M is a field and G is a finite group of automorphisms of M, then M is galois over G'. Extension of isomorphisms theorem. Applications to elementary symmetric functions. Galois subfields and normal subgroups.

4. Construction of galois extension fields. Splitting fields, several characterizations. Uniqueness. Separability. Galois if and only if separable and splitting. Galois closure of intermediate field. Galois group as a group of permutations of the roots. Examples of splitting fields. Explicit calculations of galois groups of equations. Roots of unity. Cyclotomic polynomials. Irreducibility over the rationals. Construction of regular polygons by ruler and compass.

5. Solution of equations by radicals. Definition of radical extension fields. In characteristic 0, if M/K is radical, then $G(M/K)$ is solvable. Tie-up between radical extensions and solving equations by radicals. Insolvability of general equations of degree ≥ 5. If f is irreducible over Q, of prime degree p, and has exactly 2 real roots, then its galois group is S_p. Explicit examples. Hilbert's Theorem 90. Form of cyclic extension if ground field contains roots of unity. If $G(M/K)$ is solvable, then M/K is radical.

6. Finite fields. Recall $GF(p)$. A field has p^n elements if and only if it is the splitting field of $x^{p^n} - x$. $M \supset K$, finite fields, implies M is galois and cyclic. Examples from elementary number theory. The normal basis theorem.

Optional Topics

The following topics are listed with no preferential order. They are to be used at the instructor's discretion.

7. Simple extensions and separability. A finite-dimensional extension field is simple if there are only finitely many intermediate fields. M separable and finite-dimensional over K implies M is simple. Purely inseparable extensions and elements. Maximal separable and purely inseparable subfields. Splitting fields are

generated by these. Transitivity of separability.

8. Algebraic closure and infinite galois theory. Definition of algebraically closed field. Existence and uniqueness of algebraic closure (use Zorn's lemma). Point out that one cannot get the Fundamental Theorem of Algebra this way. Infinite algebraic extensions, Krull topology on galois group. Galois group is compact and totally disconnected. Inverse limit of finite groups. Fundamental Theorem of Galois Theory for this case.

9. Transcendental extensions. Algebraically independent subsets of field extensions. Purely transcendental extensions. Transcendental extensions. Transcendence bases treated so that the proof could be used for bases of vector spaces. Usual properties of transcendence bases and transcendence degree. Transcendence degree of composite. Separable generation. MacLane's criterion.

W. Ring Theory and Multilinear Algebra

1. Categories and functors. Introduce the category of sets. Definition of a category. Examples of categories: the category of groups, the category of rings, the category of fields, the category of vector spaces, the category of modules; epimorphisms, monomorphisms, isomorphisms, surjections, injections. Examples to show that an epimorphism is not necessarily surjective and a monomorphism is not necessarily injective. A group as a one-object category whose morphisms are all isomorphisms; similar ways of looking at groupoids and other algebraic systems. Dual of a category, duality, examples. Additive and abelian categories with examples. Functors and natural transformations with many examples, for instance viewing modules as functors. The Yoneda lemma: Nat(Hom(A,-),T) = T(A). Illustrations and examples of universal objects. Definition and elementary properties of adjoint functors. (The language of categories will be useful throughout the course and elementary categorical notions can simplify many proofs.)

2. Introduction to algebraic number theory. Noetherian rings

and their modules. The Hilbert Basis Theorem. Definition of integral elements. Integral closure. Integers in a number field. Examples of quadratic fields. Units. The integers of $Q(i)$, $Q(\omega)$ form a UFD, but the integers in $Q(\sqrt{-5})$, $Q(\sqrt{10})$ do not form a UFD. Fermat's last theorem for $p = 3$ using $Q(\omega)$.

3. Valuation and Dedekind rings. Definition of a discrete valuation ring as a PID with exactly one nonzero prime ideal. Valuation of quotient field associated with discrete valuation ring and converse. Examples of rank one discrete valuations. Various characterizations of discrete valuation rings including: R is a discrete valuation ring if it is a Noetherian domain which is integrally closed and has exactly one nonzero prime ideal. The ring of fractions of a domain with respect to a multiplicative semigroup. R_p for p a prime ideal. A Dedekind ring is a ring R such that R_p is a discrete valuation ring for all prime ideals p of R. Unique factorization of ideals in Dedekind rings. Other characterizations of Dedekind rings. Approximation lemma. If $M \supset K$ are fields with M finite-dimensional and separable over K, and if K is the field of quotients of a Dedekind ring A, then the integral closure of A in M is Dedekind. Integers of a number field are Dedekind.

4. Tensor products. Definition of one-sided module over a ring R. Examples. Free modules. Submodules, quotient modules, exact sequences. Tensor products defined via universal properties. Uniqueness. Existence. Examples, $Z/2Z \otimes Z/3Z = 0$. If R is commutative, tensor product is again an R-module. Tensor product of maps. Behavior of tensor products with regard to exact sequences and direct sums. Examples. Tensor products of free modules and matrix rings. Associativity of tensor product. Tensor product of n modules over a commutative ring, multilinear maps. Tensors. Tensor product of p copies of a free module and q copies of its dual, components in notation of physics. Tensors as defined in physics: R is the ring of C^{∞} functions on R^n and M is the R-module of derivations of R. M is free, generated by the usual partials. Transformation of coordinates. Express elements of $M^p \otimes_R M^{*q}$ in terms of two coordinate systems to get usual transformation rules. The tensor algebra and its universal property.

5. Exterior algebra. Multilinear alternating maps. Antisymmetric maps. Definition of the exterior algebra as a homomorphic image of the tensor algebra. Universal property of the exterior algebra. p-vectors. Exterior algebra of a free module over a commutative ring, explicit calculation of a basis and dimension of the module of p-vectors. Prove invariance of vector-space dimension once more. Determinants via exterior algebra. Usual formula for determinant, determinant of transpose = determinant.

6. Structure theory of noncommutative rings. Ring means ring with unit. Simple left module is ring modulo a maximal left ideal. Primitive ideals. Division rings and vector spaces over them. The ring of all linear transformations, both finite- and infinite-dimensional case. Schur's lemma. Density theorem. Wedderburn-Artin theorem. Uniqueness of simple modules. Structure of semisimple Artinian rings. Structure of semisimple modules.

7. Finite group representations. The group algebra. Maschke's theorem: The group algebra of a group of order n over a field of characteristic prime to n is semisimple. Representations and characters. Connection between the decomposition of the group algebra over the complex field and the simple representations. The characters determine the representation.

Optional Topics

The following topics are listed with no preferential order. They are to be used at the teacher's discretion.

8. Radicals of noncommutative rings. Radical = intersection of all primitive ideals = intersection of all left maximal ideals. Equivalent definition of radical. Examples. Behavior of radical under homomorphisms and subring formation. Nakayama's lemma. Artinian rings. Radical is nilpotent in Artinian ring. A ring modulo its radical is a subdirect sum of primitive rings. Connection with semisimple rings.

9. Further group theory. Permutation groups. Linear groups. Structure theory of linear groups. Examples of finite simple groups. Groups defined by generators and relations. Further work on representations of finite groups: one-dimensional representations, the

number of simple characters, orthogonality relations, applications, and examples.

10. Further algebraic number theory. Infinite primes. The product formula. The Dirichlet unit theorem and finiteness of the class number.

Applications

It is essential that the prospective teacher of college mathematics know and appreciate some of the honest applications of the calculus, linear algebra, and probability. Merely as a matter of expediency a teacher of these subjects will have need of convincing examples and illustrations; but, more important, a knowledge of some applications will enable him to know best how to present mathematics and will add an extra dimension to his exposition.

There are many different ways in which the prospective teacher can acquire a background in applications of mathematics. We list here several possibilities which seem highly appropriate; each requires at least one year of course work.

1. Probability and Statistics. Some students will wish to pursue the study of probability and statistics at the advanced undergraduate or graduate level. The year-long course Mathematics 7 [page 79] will serve our purpose well, provided that due emphasis is placed upon applications of these subjects.

2. Differential Equations--Applications. In the pages that follow, three one-semester courses at the advanced undergraduate or beginning graduate level are described:

X. Advanced Ordinary Differential Equations with Applications

Y. Problem-oriented Numerical Analysis

Z. Seminar in Applications

As a source of material in applied mathematics, perhaps no subject is richer than differential equations. Hence, our alternative recommendations for a year's study in applications begin with course X.

A second semester can be chosen from several possibilities. Perhaps the best is the course Mathematics 10 [page 92], using one of the three outlines given in the 1972 CUPM publication

Applied Mathematics in the Undergraduate Curriculum [page 705]. Course Y, if taught in the proper manner, will also be suitable for this purpose. Another alternative for this second semester would be for the mathematics department to offer a seminar (course Z) presenting applications of the calculus, linear algebra, and probability to the physical, biological, and social sciences.

In summary, the suggested requirement for a year of study in applications of mathematics is one of the four sequences: 7A-7B; X-10; X-Y; X-Z. Because of the demand on students' time, we have been compelled to limit this requirement to one year. Nevertheless, we hope that many students will have to opportunity and interest to elect a third semester from among 7A, 7B, 10, X, Y, Z.

X. Advanced Ordinary Differential Equations with Applications

This course is designed to provide background for teaching the topics in differential equations that occur in the lower-division GCMC courses; to give further exposure to applications via one of the most intensively used classical routes; and to provide a first course for students who may be interested in specializing in this area. Because of the nature of the subject, many different good course outlines are possible, but, in any case, emphasis should be put on efficient ways of obtaining from differential equations useful information about their solutions, as distinguished, say, from methods for finding baroque solution formulas of little practical value.

1. Fundamentals. The vector differential equation $\dot{x} = f(t,x)$; prototypes in physics, biology, control theory, etc. Local existence (without uniqueness) by the Cauchy construction, when f is continuous. Prolongation of solutions and finite escape times. Properties of integral funnels (e.g., Kneser's theorem); extreme solutions when $n = 2$. Jacobian matrix of f locally bounded $\Rightarrow$ Lipschitz condition $\Rightarrow$ uniqueness $\Rightarrow$ continuous dependence on initial values and parameters. Effects of stationarity.

2. Numerical integration. Euler, Runge-Kutta, and other methods; elements of error analysis for these methods. Practical machine computation.

3. Linear equations. Discussion of physical and other real-world models leading to linear equations. Linearization. Structure

of the solution set of the vector equation (*) $\dot{x} = A(t)x + b(t)$; variation of parameters formula; the fundamental matrix. Matrix exponentials; thorough treatment of (*), using Jordan canonical form, when A is constant. Applications in engineering system theory. Floquet's theorem.

4. Sturm-Liouville theory. The two-point boundary value problem for second-order self-adjoint equations and how it arises. Existence of eigenvalues. Comparison, oscillation, and completeness theorems. Orthogonal expansions. Green's function. Applications to diffusion and wave equations. Some special functions.

5. Stability. Liapunov, asymptotic, and orbital stability; uniform properties. Basic theorems of Liapunov's direct method. Extensive treatment of the linear case. Applications in control theory.

6. Phase-plane analysis. Geometric treatment of second-order stationary systems. Classification of simple equilibrium points. Closed orbits and Poincaré-Bendixson theory.

Optional Topics

7. Power series solutions. Classification of isolated singularities of linear equations; formal solutions; Frobenius' method. Asymptotic series.

8. Carathéodory theory. (Prerequisite: Lebesgue integration).

Y. Problem-oriented Numerical Analysis

Although the course we have in mind overlaps with standard courses in numerical analysis in some of its material, it differs fundamentally in spirit from such courses. The traditional course in numerical computation is intended to train the student to be able to compute certain specific quantities, such as the approximate value of definite integrals, roots of polynomial and transcendental equations, or solutions of ordinary differential equations, by applying known algorithms to well-formulated specific numerical problems. Courses in contemporary theoretical numerical analysis have tended to emphasize the technical aspects of specialized topics, such as the theory of approximation, spline interpolation, numerical linear algebra, or discrete variable techniques for differential equations; here the stress is on widely applicable computational techniques, their underlying theory, and the errors arising in their application.

A problem-oriented course in numerical analysis starts with real-life problems (from physics, economics, genetics, etc.), develops mathematical models (often in the form of differential or other types of functional equations), analyzes the models, and develops and applies numerical methods to the models in order to get some answers. The student's knowledge of analysis, linear algebra, or differential equations is called upon in the analysis of the model; techniques of numerical analysis are studied and sifted through in the search for applicable methods; specific numerical computations are performed by the student, using a computer; and, finally, the numerical answers are examined in two ways: by means of a theoretical analysis of the errors inherent in the algorithm and in the machine computation, and by a comparison with the original problem to see whether the "answer" (often a table of values of some unknown functions) is a reasonably good approximation to reality.

It seems clear that text materials for Mathematics Y should include books or journal articles on applications (as a source of real problems) and numerical analysis texts (as a source of numerical methods). Sample topics and associated texts are:

a. Problems in the theory of flight. Here one can find mathematical models and their analyses in works such as Theory of Flight Paths by Angelo Miele (Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962). One can apply to the ensuing systems of differential equations techniques found in Discrete Variable Methods in Ordinary Differential Equations by Peter Henrici (New York, John Wiley and Sons, Inc., 1962). One sample problem on these lines can be found in Section 10.9 of Numerical Methods and Fortran Programming by Daniel D. McCracken and William S. Dorn (New York, John Wiley and Sons, Inc., 1964). Although these authors pull a refined model of a simplified flight problem out of a hat--which the instructor in course Y must not do--they examine at length the implications of the properties of the numerical solutions for the behavior of the physical system and use the flexibility of their computer program to vary parameters and do some interesting mathematical experimentation.

b. Control theory. Selected models and analyses from an applied text such as Optimum Systems Control by Andrew P. Sage (Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1968) can lead to problems of numerical solution of partial differential equations, two-point boundary value problems, and problems of numerical linear algebra. There are several suitable sources for numerical methods.

Z. Seminar in Applications

As another approach to applications, we suggest a seminar devoted to applications of the calculus, linear algebra, and probability to the physical, biological, and social sciences. Fortunately, there are now several books which contain a wealth of readily accessible examples. Many of these are referenced in the 1972 CUPM report Applied Mathematics in the Undergraduate Curriculum.

The students would participate in the formulation of scientific problems in mathematical terms and in the interpretation and evaluation of the mathematical analysis of the resulting models. Due emphasis should be given to problems whose analysis rests on the use of the computer. It might be appropriate for the instructor to invite guests who could expose the student to the attitudes of users of mathematics. While such an arrangement would, perhaps, not be a traditional course in applied mathematics, it would allow the students to come into contact with a variety of serious applications of the usual mathematics of the first two undergraduate years. The following illustrate the type of examples we have in mind:

a. The formulation and analysis of a system of differential equations which serves as a model for (i) the interdependence of two species, one of which serves as food for the other, or (ii) a time-optimal navigation problem which requires that a boat be transferred from a given initial position to a given terminal position in minimal time.

b. The formulation and analysis of waiting line and traffic problems involving simple calculus and probability.

c. Elementary matrix analysis associated with chemical mixture problems and mechanical equilibrium problems; matrix eigenvalue problems arising from electrical circuit analysis.

d. The "transportation problem" of making optimal use of a given shipping network to obtain a specified redistribution of commodities. This is, of course, a special case of linear programming.

e. Game theory as applied to games of timing ("duels") in which rewards to competing strategists depend on when certain acts are performed.

APPRENTICESHIP IN TEACHING

Every mathematician is a teacher in the sense that he must explain mathematical ideas to other people--to students, to colleagues, or to the mathematical community at large. For this reason the graduate education of every mathematics student should include a program designed to develop skill in oral and written communication of mathematics. This program should begin as soon as the student enters graduate school and continue at increasing levels of responsibility.

Ultimately, the attitude of the graduate faculty will determine the success of any such program. If effective teaching is regarded as an important and nontrivial function of the department, and if senior mathematicians encourage excellent exposition by precept and personal interest, graduate students and younger faculty will respond accordingly. Every instructor of a graduate class should realize that his course can have a profound effect upon his students in the way it serves to strengthen the attributes of a good teacher.

Because the conditions of undergraduate and graduate instruction vary widely from one university to another, the suggestions given below offer a variety of ways in which the mathematics departments might stimulate more interest in good teaching. Each university is encouraged to create its program individually, seeking to establish an intellectual environment in which teaching and learning flourish together.

Some universities have experimented recently with special programs which bring new teaching assistants to the campus before the start of classes in the fall. Sessions are devoted to a general orientation to graduate and undergraduate study at that university and to the role of the graduate assistant. At least one program runs for the entire summer term and includes an initial involvement with graduate mathematics besides activities in preparation for teaching.

During the first stage of his training, the teaching assistant should be given limited duties, but he should be made to feel that he is a junior colleague in a profession rather than a hired hand in a work crew. At a pace which is adjusted individually to his rate of development, he should progress through a sequence of teaching assignments, acquiring more responsibility and independence as he gains in experience and confidence. He can mark homework papers, conduct office hours for undergraduates, prepare questions for tests, and assist in marking tests.

A prospective teacher can learn much by observing a skillful teacher in an undergraduate class in a subject familiar to the apprentice. This is of particular value in a class of selected students, such as freshman or sophomore honors sections, where the interchange between students and the instructor is lively and challenging.

Regular consultation between an apprentice and his supervisor is essential. Each course supervisor should arrange meetings of all assistants for that course; at these meetings there should be free exchange of ideas concerning problems of instruction, alternate suggestions for presenting specific concepts, proposals for future test questions, and planning the development of the course. In addition to formal consultation, however, supervisors should maintain a running dialogue with apprentices, work with them in the marking of tests, and cooperate in performing with them the day-to-day duties which are an integral part of teaching.

After a graduate student has developed competence in these duties and has acquired a basic feeling for classroom instruction, he should be drawn more actively into teaching by conducting discussion sections, by giving occasional class lectures, or by accepting major responsibility for teaching an appropriate course at an appropriate level. His supervisor should maintain good contact through continued consultation, classroom visitation, and informal discussions. As the assistant matures in his teaching role, direct supervision should be relaxed gradually to encourage him to develop his individual classroom style and techniques; the opportunity for consultation should remain open, but the initiative should pass from the supervisor to the assistant.

Special seminars can also be used to assist students to improve their exposition. Many departments require a proseminar in which graduate students present advanced mathematical topics to fellow students and several members of the faculty. It would be equally appropriate to require each first-year graduate student to present a short series of talks on some phase of undergraduate mathematics which is outside his previous course of study. The objective should be to present the topic at a level suitable for undergraduates, emphasizing clarity in organization and expression rather than making the occasion a mathematical "tour de force."

Another possibility is to assign a few graduate students to experimental projects in undergraduate mathematical instruction instead of assigning them regular classroom duties. For example, they could help to prepare a collection of classroom examples for a calculus course, develop problems to be solved on the computer, or plan and evaluate alternative approaches to specific topics in lower-division undergraduate mathematics.

As indicated in the Program Description, the apprenticeship in teaching should constitute approximately one fourth of the total work load of a student during his first graduate component. It is conceivable that some of these activities, such as seminars, can qualify for academic credit. But whether or not academic credit is granted for this phase of graduate work, the student's performance as a teacher should be evaluated, and an informal departmental record should be kept in sufficient detail to show the work done and the level of competence attained.

Finally, any program of increased attention to the teaching role of prospective mathematicians has budgetary implications which cannot be ignored. One additional cost is for increased faculty time devoted to supervising teaching assistants. Another is for stipends for graduate students if the number of apprentice teachers is expanded. But if the quality of mathematics instruction improves in the future as a result of such efforts, the money will have been well spent. Fortunately, there is reason to believe that imaginative proposals to improve the quality of teaching by graduate students can attract the additional financial support needed to make them effective.

Although many graduate students welcome an opportunity to teach and thereby to become self-supporting, the stipend itself is not an adequate incentive for good teaching. This incentive can best be provided by the persistent concern of established mathematicians that teaching be excellent throughout the department.

QUALIFICATIONS FOR TEACHING UNIVERSITY-PARALLEL MATHEMATICS COURSES IN TWO-YEAR COLLEGES

A report of

The *ad hoc* Committee on Qualifications for a Two-Year College Faculty in Mathematics

August 1969

INTRODUCTION

CUPM has published reports on the qualifications needed by teachers of the GCMC curriculum and on the adaptation of that curriculum to the circumstances of university-parallel programs in two-year colleges (Qualifications for a College Faculty in Mathematics (1967) and A Transfer Curriculum in Mathematics for Two-Year Colleges (1969)). The present report is an effort to describe the qualifications desirable for faculty members teaching courses in the university-parallel or transfer programs in two-year colleges.

Our comments and recommendations are addressed to administrators of two-year colleges, to university mathematics departments, to mathematics teachers in two-year colleges, and to those contemplating careers as mathematics teachers in two-year colleges. The concluding section of our report offers specific advice to each of these four groups.

We discuss the qualifications of teachers of the following set of courses, whose subject matter can be thought of as a working definition of university-parallel mathematics.

Mathematics 0. Elementary Functions and Coordinate Geometry. A one-semester course in coordinate geometry and the properties of the elementary functions.

Mathematics A. Elementary Functions and Coordinate Geometry (with Algebra and Trigonometry). A more slowly paced version of Mathematics 0 in which are embedded some topics from high school algebra and trigonometry. This course is to be thought of as extending over more than one semester.

Mathematics B. Introductory Calculus. An intuitive one-semester course covering the basic concepts of single-variable calculus.

Mathematics C. Mathematical Analysis. A two-semester course completing the study of elementary calculus.

Mathematics L. Linear Algebra. A sophomore-level one-semester introduction.

Mathematics PS. Probability and Statistics. An elementary one-semester course (not having calculus as a prerequisite) suitable for students in business and social sciences.

Mathematics NS. The Structure of the Number System. A two-semester course recommended by the CUPM Panel on Teacher Training for beginning the preparation of elementary school teachers. [Since this report was written, the recommendations on teacher training have been revised. See the 1971 publication Recommendations on Course Content for the Training of Teachers of Mathematics, page 158.]

The detailed discussion of these courses will be found in the CUPM report A Transfer Curriculum in Mathematics for Two-Year Colleges, mentioned above. In addition, this report suggests that, under certain circumstances, it may be advisable for a two-year college to offer additional courses and suggests a selection from among the following: further courses for elementary school teachers; finite mathematics; a calculus-based course in probability; numerical analysis and intermediate differential equations (or differential equations with topics from advanced calculus). For suggested work in computing, see the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563.

Our recommendations are intended to apply to all instructors who teach any such university-parallel courses. We are aware of the great importance in two-year colleges of courses in mathematics for students in occupational and technical curricula and of courses designed for students lacking even basic mathematical skills. We are also aware of the existence of difficult and challenging pedagogical and curricular questions related to such courses. We have chosen to wait until there is a better resolution of these questions before seeking to formulate recommendations about the proper qualifications for teaching courses that are not parallel to those commonly offered by four-year colleges and universities. [Some of these questions are discussed in A Course in Basic Mathematics for Colleges.]

The university-parallel role of the two-year college is of increasing importance in the educational system. For example, in California 86% of all freshmen in publicly supported institutions in 1966-67 were in two-year colleges. The percentage of college students enrolled in two-year colleges has been increasing rapidly nationwide. There are already a number of universities in which the junior class is larger than the freshman class. Moreover, a majority of students entering two-year colleges intend to continue their education at least to the bachelor's degree. Thus, university mathematics departments must recognize that the university-parallel courses taught in two-year colleges are becoming an integral part of the university program in mathematics.

Conversely, recent trends in four-year institutions are placing new demands on teaching of mathematics in two-year colleges. As students come to colleges with better preparation in mathematics, many courses are moving downward toward the freshman year. It should be recognized that those now being trained or hired as teachers in two-year colleges must be prepared to deal at some time in the near future with subjects that are now thought of as belonging to the junior or senior years.

The degree most commonly held by mathematics teachers in two-year colleges is the master's degree, but this degree is of such varying quality that it is scarcely useful as a measure of qualification for appointment, promotion, or tenure. We feel it necessary to make recommendations which are independent of degrees held or of total credit hours earned in mathematics courses but which deal, rather, with the substance of the mathematical training of prospective faculty members.

It should be understood that no academic program or degree in itself qualifies an individual to teach effectively at any level unless this preparation is accompanied by a genuine interest in teaching and by professional activities reflecting continuing mathematical growth. These activities may assume many forms:

a. taking additional course work

b. reading and studying to keep aware of new developments and to explore new fields

c. engaging in research for new mathematical results (even if unpublished)

d. developing new courses, new ways of teaching, and new classroom material

e. publishing expository or research articles

f. participating in the activities of professional mathematical organizations

This list reflects our conviction that an effective teacher must maintain an active interest in the communication of ideas and have a dedication to studying, learning, and understanding mathematics at levels significantly beyond those at which he is teaching.

A two-year college mathematics department, whose staff members are engaged in activities such as those described above and have the academic qualifications to be described below, should have confidence in its ability to provide the quality of teaching required of it.

THE FORMAL EDUCATION OF MATHEMATICS TEACHERS IN TWO-YEAR COLLEGES

The university-parallel courses in mathematics that a teacher in a two-year college should be able to teach effectively have been described in the previous section. We shall now set forth our recommendations for the mathematical qualifications for the teachers of these courses.

This mathematical background falls into two distinct components: a basic component which consists of a strong mathematics major program and a graduate component which embodies the requirement that a teacher at a two-year college must have a knowledge of mathematics well beyond that which he will be asked to teach.

Basic Component

The basic component of mathematics courses for the two-year college teacher is most succinctly described as a solid grounding in analysis and algebra, with additional courses in geometry, computer science, and probability providing greater breadth of knowledge.

We assume that the prospective teacher has mastered the following lower-division undergraduate material, as described in the CUPM publication Commentary on A General Curriculum in Mathematics for Colleges.

a. Calculus courses in one and several variables, including an introduction to differential equations (Mathematics 1, 2, 4)

b. The fundamentals of computer science, including experience in programming as well as the use of a computer [See, for instance, the course C1, page 563]

c. A semester course in linear algebra employing both matrices and a basis-free, linear transformation approach (Mathematics 3)

In addition, the prospective teacher should attempt to obtain as many of the following upper-division courses as he can at the undergraduate level.

a. A semester course in advanced multivariable calculus, covering differential and integral vector calculus, including the theorems of Green and Stokes, and an introduction to Fourier series and boundary value problems (Mathematics 5, first version)

b. A year's work in abstract algebra, treating the important algebraic systems (groups, rings, modules, vector spaces, and fields) and thoroughly developing the basic concepts of homomorphism, kernel, and quotient construction, with applications and consequences of these ideas. (This course is described in the CUPM report Preparation for Graduate Study in Mathematics, page 453. See also the courses Mathematics 6M and 6L in Commentary on A General Curriculum in Mathematics for Colleges, page 65 .)

c. A thorough year's course dealing with the important theorems in real analysis, with emphasis on rigor and detailed proofs. The treatment should use metric space notions and should lead to a detailed examination of the Riemann-Stieltjes integral. (Mathematics 11-12)

d. A semester course in complex analysis, covering Cauchy's theorem, Taylor and Laurent expansions, the calculus of residues, and analytic continuation, with application of these ideas to transforms and boundary value problems (Mathematics 13)

e. A semester course in applied mathematics. The student should be introduced to applications of mathematics in order that his teaching might better reflect the relevance of mathematical ideas. [The courses described in the 1972 publication Applied Mathematics in the Undergraduate Curriculum are suitable.]

f. A semester course in which the student studies some geometric subject such as topology, convexity, affine and projective geometries, differential geometry, or a comparative investigation of Euclidean and non-Euclidean geometries (Mathematics 9)

g. A year's course in probability and statistics that reflects the growing importance of this subject to engineering and the biological, social, and management sciences [Mathematics 7 or the course outlined in Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 642]

If the student has not completed all of the upper-division courses of this strong mathematics major as an undergraduate, then he should cover material comparable to that of the omitted courses during his graduate training.

Graduate Component

Graduate (one-semester) courses which are especially appropriate for the graduate component are:

P Measure and Integration

Q Functional Analysis

R Complex Analysis

S General Topology

T Homology and Multivariable Integration

U Topology and Geometry of Manifolds

V Galois and Field Theory

W Ring Theory and Multilinear Algebra

X Advanced Ordinary Differential Equations with Applications

Y Problem-oriented Numerical Analysis

Z Seminar in Applications

Of these, P, S, and X should be in the program of every prospective two-year college teacher.

Detailed descriptions of these courses are given in the CUPM report A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, page 113. The program presented there is designed to prepare teachers to function in the first two years of a four-year college with occasional teaching of upper-division courses. It does not differ greatly from our program: In the four-year college report, instead of P, S, and X, the courses P, Q, S, and T are regarded as essential, and the material on probability and statistics, applied mathematics, and differential equations serves as a pool of courses on the applications of mathematics from which a year sequence is to be elected by the student. These differences are due to the fact that a two-year college teacher has less access to the services of experts in specific fields and, consequently, needs a somewhat broader training.

It should be emphasized that course X in differential equations is not a second undergraduate course in the subject, but is to be a genuine graduate course at least on the same level as the course in measure and integration. The graduate course in applied mathematics which the Committee most strongly favors is one (not yet commonly offered) which stresses the formulation and analysis of mathematical models in diverse fields, using the calculus, probability, and linear algebra of the first two undergraduate years. Course Z is of this type.

Students who plan to continue into advanced graduate work and to specialize in some area of pure mathematics are advised to take as many as possible of the other courses in the list. Other students may substitute electives to obtain a deeper knowledge of some other area of mathematics or computer science.

Undergraduate mathematics, especially in the lower division, is heavily slanted toward real analysis; courses in general topology and measure theory provide essential background for teaching courses in calculus and probability. If further work in analysis is elected, we recommend a study of functional analysis (course Q) in preference to complex analysis (course R) as a sequel to measure theory, as the former will further develop the ideas of linear algebra and the concept of uniform convergence. In spite of its importance for more advanced work in pure mathematics, a second year of abstract algebra to follow the strong undergraduate algebra course described in the basic component is not recommended as essential for teachers of mathematics in a two-year college.

The graduate component of courses should be augmented by two particular features to prepare the prospective teacher for the two-year college mathematics faculty.

First, a year's work focused on the problems of lower-division undergraduate teaching, such as an apprenticeship in teaching as described below, preferably carried out at a nearby two-year college.

Second, a comprehensive examination designed specifically to test the breadth and depth of a candidate's understanding of mathematics relevant to the undergraduate curriculum.

A student who has a strong undergraduate major in mathematics will be able to complete the program in one year, even if he has not completed all of the courses listed in the basic component. For a student whose prior training is not substantially that of the basic component, the completion of the graduate component may require two years of study beyond his bachelor's degree. For example, if his undergraduate major does not include strong preparation in algebra and analysis, his program might be as follows:

First Year (both semesters)
- Abstract Algebra
- Real Analysis (Mathematics 11-12)
- Probability and Statistics
- Apprenticeship in Teaching

Second Year

<u>First Semester</u>
- Measure and Integration (P)
- Complex Analysis (Mathematics 13)
- Topology (S)
- Apprenticeship in Teaching

<u>Second Semester</u>
- Applied Mathematics (Z or one of the courses outlined in <u>Applied Mathematics in the Undergraduate Curriculum</u>)
- Advanced Differential Equations (X)
- Apprenticeship in Teaching
- Comprehensive Examination

The mathematical background in the graduate component, if satisfactorily completed, will permit the student to teach with confidence the university-parallel courses of the two-year college. Moreover, new courses, as they arise, should be well within his competence to prepare.

Apprenticeship in Teaching

An important component in the training of teachers of mathematics for two-year colleges is an understanding of the teaching and learning processes as they apply to these institutions. One of the best ways for the potential instructor to gain this kind of knowledge and experience is through a supervised teaching activity. This activity preferably should take place in a two-year college, but it can, if necessary, be carried out in a four-year institution in appropriate courses.

Most value will be obtained if apprentice teachers receive experience in a variety of courses involving a heterogeneous group of students with differing career aspirations, comparable to the situation that they will encounter in most two-year colleges.

The success of an apprenticeship program will depend significantly upon the attitude of the graduate faculty. If effective teaching is regarded as an important function of the department, and if senior mathematicians encourage excellence in teaching by precept and by example, the apprentice teachers will respond accordingly.

The work assignment of the apprentice should be carefully graduated and should always involve close contact with and supervision by a senior colleague. The apprentice should have frequent opportunities to go over purposes, methods, and content with his supervisor. Arrangements should be made for frequent post-teaching conferences in which the teaching and learning problems encountered are reviewed and solutions suggested. This can be done individually or in a group for all apprentices in the program. Valuable contributions can be made to such seminar sessions by mathematics instructors from two-year colleges and by experts in curricular construction and evaluation.

In total, the apprenticeship in teaching should constitute approximately one quarter of the work load of the student during his graduate experience.

Adequate budgetary provisions should be made for the extra burden of the apprenticeship program on the senior mathematicians, as well as financial support for the apprentices.

An apprenticeship system has a great potential for preparing two-year college mathematics teachers having a real attachment to the discipline and an understanding of the values and the rewards of the teaching profession. Done poorly, it will discourage candidates from the field. Done well, it will attract and retain competent and interested persons.

COMPOSITION OF A TWO-YEAR COLLEGE MATHEMATICS FACULTY

Although mathematics teachers at two-year colleges are called upon to teach specialized courses for a variety of students (remedial, general education, technical-occupational), our attention in the present report continues to be focused upon qualifications of persons who teach courses in the university-parallel curriculum.

It is our recommendation that all teachers of university-parallel courses at a two-year college have the mathematical preparation equivalent to our graduate component. Although the university-parallel courses that a two-year college teacher may be called upon to offer today are principally like those described in the introduction under 0, A, B, and C, it seems reasonable to expect that courses in finite mathematics, linear algebra, probability and statistics, and mathematics for prospective elementary school teachers will be standard offerings in two-year colleges in the near future. Our recommendation reflects a belief that the teacher of university-parallel courses should have mathematical training well beyond the course he is teaching. Moreover, the mathematical background we recommend will permit all faculty members to participate in knowledgeable discussions of curricular changes, both internally and with faculty members of four-year colleges and universities. The mathematical preparation we recommend will permit a faculty member to prepare new courses with confidence. Moreover, it will provide the individual faculty member with a basis for effective participation in mathematical organizations, which in turn will help him to maintain the intellectual curiosity and interest in mathematics that is essential to a successful mathematics teacher.

The Committee believes that a universally well-qualified faculty for the university-parallel courses is most important, with each member able to make a contribution in all of the ways already indicated. We do not, however, envision that all two-year college staff members will have exactly the same mathematical background. The choice available for individual preferences in the graduate component allows a staff which includes people with varying interests, and hence people especially well prepared to teach linear algebra or probability and statistics or computer science.

A two-year college may not be able at this time to recruit all such staff members from candidates with preparation equivalent to our recommended graduate component. In this case, they might seek on a temporary basis new candidates who have the mathematical preparation equivalent to our basic component; these candidates could be assigned to teach courses 0, A, and B. New faculty members whose qualifications are not equivalent to our graduate component should be required to augment their mathematical background so that in time they will be better prepared to have responsibility for any of the university-parallel courses.

RECOMMENDATIONS TO FOUR GROUPS

a. To Administrators of Two-Year Colleges

These recommendations are addressed to those who appoint and promote two-year college faculty members and to those who, through accreditation and certification, influence the setting of qualifications for such teachers.

Although it has been traditional for college policies on the appointment, promotion, and tenure of faculty to include certain degree requirements, it is a fact that the course requirements for a particular degree in mathematics vary considerably from one institution to another; even minimum standards are not well defined. This is especially true of the master's degree. The Committee strongly encourages those concerned to note that this report recommends a set of courses which prospective members of a mathematics faculty should have taken. Successful completion of these courses should insure that the faculty member is adequately prepared, in terms of subject matter, to teach university-parallel courses.

The Committee urges all administrators to recognize proficiency in the content of the courses recommended in this report rather than academic degrees as a basis for faculty appointments and advancement. For example, graduate mathematics training of secondary school teachers, customarily and properly, differs from the training we have described. The Committee suggests that faculty members in mathematics be relied upon to determine the degree of proficiency possessed by those under consideration. Furthermore, it is recommended that orientation programs be developed for new faculty members who have had no previous experience teaching in two-year colleges.

b. To University Mathematics Departments

University mathematics departments should realize from the preceding sections that the major role in the training of mathematics instructors for two-year colleges is theirs. They must accept responsibility for establishing formal programs for the training of new instructors for two-year colleges and for retraining instructors who are now teaching in these institutions.

We believe that this can be done within existing frameworks of mathematics departments whose course offerings approximate in depth the detailed outlines to which we have referred. For such departments, this will not require extensive changes in curricula, except possibly for the introduction of a program for apprenticeship in teaching. However, it is necessary that the mathematics faculty be fully aware of the particular complexion, problems, and status of two-year colleges throughout the country. The mobility of instructors suggests the need for a national point of view. Moreover, in order to fulfill their responsibility, the university faculty must recognize and respect the basic role of two-year colleges and be mindful of the problems that will be faced by mathematics instructors in two-year colleges.

c. To Those Currently Teaching Mathematics in Two-Year Colleges

All college teachers of mathematics, at one time or another, find it necessary to supplement their own mathematical training. Rapid changes are taking place in college mathematics. Hence, increasing numbers of college teachers are continuing their mathematical development by individual study and additional formal course work in mathematics.

Teachers of mathematics in two-year colleges should find that the course outlines referred to in this report provide useful guidelines for individual study, faculty seminars, and additional course work.

To serve the mathematical needs of two-year college students, a faculty member must maintain an awareness of contemporary curricula in both secondary schools and four-year colleges. He will find that the recommended courses provide a basis for effective communication with staff members of mathematics departments of four-year colleges. Personal knowledge of mathematics courses at four-year colleges is needed in order to be aware of the demands that will be made upon students after they transfer. This knowledge and the recommended strong preparation in mathematics make possible the necessary continuous evaluation and development of mathematics courses in two-year colleges.

d. To Prospective Teachers of Mathematics in Two-Year Colleges

The two-year college teacher of university-parallel mathematics courses has the responsibility for training students with a wide variety of goals. Some could be mathematicians, some scientists or engineers; there are others who will use mathematics in economics, psychology, or other social sciences. One who intends to teach mathematics in a two-year college could well use our description of a program of mathematics courses and apprenticeship in teaching as a guide in planning his own graduate work. He should also be aware that the program we have outlined is substantially different both in nature and extent from what we would regard as an optimal graduate program for teachers in secondary schools.

RECOMMENDATIONS ON COURSE CONTENT

FOR THE TRAINING OF TEACHERS OF MATHEMATICS

A Report of

The Panel on Teacher Training

August 1971

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THE PURPOSE OF THIS REPORT

This report presents an outline of the Panel's recommendations for the minimal college preparation for teachers of school mathematics based upon its assessment of those significant changes that have taken place or can be expected to take place in school curricula during the 1970's.

The nature of school mathematics is, of course, far from static, and the forces for change are many. The past 25 years have produced a phonomenal increase in the quantity of known mathematics, as well as in the variety and depth of its applications. This growth has been reflected in our total culture, which has become increasingly mathematical, a trend which is certain to continue. It is inevitable and proper that these changes will be reflected in the content of school mathematics, as well as in the way it is taught.

Thus, in the past ten years we have seen a flurry of activity directed toward improving the mathematics curricula in our schools. The pace of change alone demands that those engaged in such activity periodically review their efforts. A decade seems to be an appropriate period for such a review.

It is reasonable to ask what specific changes in mathematics and mathematics education during the past decade impel us to modify our previous recommendations.

The dependence of western civilization on technology has long been evident, and it has been recognized that mathematics supports the physical and engineering sciences upon which technology thrives. More recently new applications of mathematics to the biological, environmental, and social sciences have developed. Statistics and probability have emerged as important tools in these applications. Indeed, as ordinary citizens we frequently encounter surveys and predictions that make use of probability and statistics, so that an intelligent existence demands our understanding of statistical methods. Consequently, many secondary schools have begun to teach probability and statistics and, as we shall show, there are compelling reasons to teach these subjects in the elementary grades.

But aside from the specific mathematics, e.g., statistics, which is brought to bear upon applications, the applications are interesting in themselves, so that it is pedagogically sound to incorporate them in the mathematics program. Thus, our new recommendations emphasize the applications of mathematics.

Many of these new applications have been aided, perhaps even made possible, by modern computers, and the teaching of computer science and computational mathematics are becoming commonplace in our colleges. Computer programming is now a part of many junior and senior high school mathematics programs, and it has been discovered that the notion of a flowchart for describing an algorithmic process is a useful pedagogical device in the teaching of elementary mathematics, as well as an efficient device for prescribing a computer program. Thus, computers are influencing mathematics education at all levels, and we have attempted in preparing this report to assess this influence and to recommend measures for increasing it.

It is a fact that change induces change. For instance, an important aspect of curricular change over the past decade has been emphasis on the understanding of mathematical concepts. As a result, we have learned that abstract concepts can be assimilated at a much earlier age than was previously thought possible. Thus, we are less reluctant today to suggest that elementary notions of probability may be useful in explaining ideas about sets and rational numbers than we were a decade ago to suggest that elementary ideas about sets might be useful in helping children to understand the process of counting. Furthermore, the curricular revisions of the past decade have led to improved training programs which have produced elementary school teachers who are more confident about presenting mathematical topics. It is our purpose in this report to take advantage of today's teachers' new attitudes and skills in order to meet new challenges.

Finally, our recommendations are intended to reflect improved preparation over the past decade of entering college freshmean.

THE OBJECTIVES OF TEACHER TRAINING

The Panel believes that the following objectives of mathematical training are important:

1. Understanding of the concepts, structure, and style of mathematics

2. Facility with its applications

3. Ability to solve mathematical problems

4. Development of computational skills

These statements deserve amplification.

It is our belief that the disciplined, rational man has the best chance of becoming independent, mature, and creative, and that the development of these qualities is a lifelong process. The intellectual discipline of mathematics contributes in a unique way to this development. We identify two reasons why this is so. First, mathematical concepts are necessarily rooted in man's awareness of the physical world. Understanding mathematics allows him to relate more efficiently to his environment. Second, a person's understanding of a concept depends upon its meaningful relation to and firm grounding in his personal experience, as well as upon his awareness of its role in a system of interrelated ideas. As he learns to relate concepts to one another in an orderly fashion he becomes better organized and he improves his ability to abstract and to generalize, that is, to recognize a concept in a variety of specific examples and to apply this concept in differing contexts. We believe that the study of mathematics can directly benefit this process of personal organization. We therefore regard it as essential that mathematics be taught at all levels in such a way as to emphasize its concepts, structure, and style.

It is possible, of course, to study, to appreciate, and even to practice mathematics by and for itself, but people who can and wish to do this are rare. For most of us an important value of mathematics is its applicability to other scientific disciplines. The recent fruitfulness of mathematics in this regard has already been mentioned, but this is really in the tradition of mathematics, which has repeatedly responded to other disciplines that seek to apply its theories and techniques. It should also be recognized that the sciences in their turn have stimulated the development of new fields of mathematics. Thus, we believe that students of mathematics should acquire an understanding of its wide applicability in various fields, and for this reason, applications should be emphasized in every course.

When we speak of facility with applications we mean the ability to recognize and delineate a mathematical model of a physical, social, biological, or environmental problem. Being able to solve the related mathematical problem is a skill which we also regard as important. Effective mathematics instruction must include development of the ability to attack problems by identifying their mathematical setting and then bringing appropriate mathematical knowledge to bear upon their solution.

Finally, computational skill is essential. Without it the student cannot learn to solve mathematical problems, to apply mathematics, or to appreciate even its simplest concepts and structures. Although we normally think of this skill as including speed and accuracy in applying the common algorithms of arithmetic and algebra, we should keep in mind the fact that it also includes the ability to estimate quickly an approximate result of a computation. Each course should not only provide systematic practice in computation but should also inculcate in the student the skill and habit of estimating.

This report recommends courses that we believe prospective teachers should study in order to help them achieve these objectives, both for themselves and for their future students. First we explain briefly why we believe that teachers need much more mathematical education than most of them are now getting, and why their training needs to be of a special kind in certain cases.

There have been recent improvements in the certification requirements for elementary school teachers, but in our opinion they continue to be inadequate or inappropriate in many cases. Some states require a semester or a year of "college mathematics" without indicating what sort of mathematics this should be. These practices appear to be based on the assumption that little or no special training in mathematics is needed to teach in an elementary school. This assumption has always been unrealistic, and in the present context of rapidly changing and expanding curricula it is wholly untenable.

In some elementary schools the rudiments of algebra, informal geometry, probability, and statistics are already being taught in addition to arithmetic. But even if only arithmetic is taught, the teacher needs sound mathematical training because his understanding affects his views and attitudes; and in the classroom, the views and attitudes of the teacher are crucial. An elementary school teacher needs to have a grasp of mathematics that goes well beyond the content and depth of elementary school curricula.

Similarly, a Level II or III teacher's understanding of mathematics must exceed, both in content and depth, the level at which he teaches. Within the next decade it is to be expected that secondary school teachers will be asked to teach material which many of our present teachers have never studied.

We therefore recommend courses for all teachers which will not only insure that they thoroughly understand the content of the courses they must teach, but will also prepare them to discuss related topics with able and enthusiastic students. These college courses must also prepare teachers to make intelligent judgments about changes in content, pace, and sequence of mathematics programs for their schools, and to have the flexibility of outlook necessary to adjust to the curriculum changes which will surely take place in the course of their professional careers.

THE RECOMMENDATIONS

These recommendations concern only the preparation of teachers of elementary and secondary school mathematics. Whereas in 1961 these teachers were classified into three groups, we find it convenient to use four classifications:

LEVEL I. Teachers of elementary school mathematics (grades K through 6)

LEVEL II-E. Specialist teachers of elementary school mathematics, coordinators of elementary school mathematics, and teachers of middle school or junior high school mathematics (roughly grades 5 through 8)

LEVEL II-J. Teachers of junior high school mathematics (grades 7 through 9)

LEVEL III. Teachers of high school mathematics (grades 7 through 12)

These classifications are to be taken rather loosely, their interpretation depending upon local conditions of school and curricular organization. It will be noted that the various classifications overlap. This is a deliberate attempt to allow for local variations.

The reader should note that the training for Level I teaching is a separate program, while, except for their Level I content, the curricula for the further levels form a cumulative sequence.

The recommendations of this report are not motivated by a desire to meet the demands of any special program of mathematics education or the goals of any particular planning organization. We consider our recommendations to be appropriate for any teachers of school mathematics, including teachers of low achievers.

Level I Recommendations

The applications of mathematics, the influence of computers, and the changes wrought in the 1960's in the teaching of mathematics prompt us to revise our 1961 Recommendations for the Training of Teachers of Mathematics at all four levels. At Level I in particular we are aware that new teaching strategies designed to facilitate and enrich learning are being adopted or are the subject of experimentation. Such strategies impose on elementary school teachers the necessity of a deeper understanding of the school mathematics curriculum than is required by conventional teaching methods and increase the teacher's need for knowledge of mathematics well beyond the level at which topics are treated in the elementary classroom.

We believe that this deeper understanding can be better achieved if mathematics is taught, and understood, from the earliest stages as a unified subject. The function concept, for instance, should serve as a unifying thread in elementary school mathematics, and elementary intuitive geometry should be taught for its connections with arithmetic as well as for its own sake. The applications of mathematics reflect its unity and offer an opportunity to illustrate its power. For instance, the notion of a finite sample space in probability can be used at a very elementary level to illustrate the idea of a set (of outcomes), and the probability of an event can motivate the need for rational numbers. Simple statistical problems yield practice in computing with both integers and rational numbers as well as in applying probability theory to practical situations. Finally, the use of flowcharts helps to explain the elementary algorithms of arithmetic as well as to prepare the student for later study of computer programming.

Thus, while the development of the number system should remain the core of the elementary school curriculum and of the content of Level I courses, there are other crucial topics which ought to be contained in the Level I sequence. We stress this point by listing these topics in the following recommendations.

Recommendations for Prospective Level I Teachers:

> We propose that the traditional subdivision of courses for prospective elementary school teachers into arithmetic, algebra, and geometry be replaced by an integrated sequence of courses in which the essential interrelations of mathematics, as well as its interactions with other fields, are emphasized. We recommend for all such students a 12-semester-hour sequence that includes development of the following: number systems, algebra, geometry, probability, statistics, functions, mathematical systems, and the role of deductive and inductive reasoning. The recommended sequence is based on at least two years of high school mathematics that includes elementary algebra and geometry.
>
> We further recommend that some teachers in each elementary school have Level II-E preparation. Such teachers will add needed strength to the elementary school's program.

Our suggestion of an integrated course sequence represents a very important change which these recommendations are intended to bring about, but there are certain to be questions on how this can be accomplished. We attempt in the sequel to provide our answer to such questions.

There are many ways in which to organize the appropriate material into integrated course sequences, and we encourage experimentation and diversity. Two possible sequences of four 3-semester-hour courses are described in detail in the course guides [page 175]. For reference we list their titles here:

Sequence 1	Sequence 2
1. Number and Geometry with Applications I	1. Number Systems and Their Origins
2. Number and Geometry with Applications II	2. Geometry, Measurement, and Probability
3. Mathematical Systems with Applications I	3. Mathematical Systems
4. Mathematical Systems with Applications II	4. Functions

These sequences differ in the ordering of topics and in the degree of integration, yet both conform to our idea of an integrated course sequence. It may be helpful to discuss briefly, without any attempt at being comprehensive or any desire to be prescriptive, the rationale which led us to these integrated sequences.

The dual role of numbers in counting and measuring is systematically exploited in each of the four-course sequences. In one direction we are led to arithmetic, in the other to geometry. The extension of the number system to include negative numbers may then be explained by reference to both the counting and measuring models. The study of rational numbers may likewise be motivated through measurement and counting, and the arithmetic of the rationals finds justification and natural applications in elementary probability theory. Geometrical considerations lead to vector addition on the line and in the plane--and then in space--and the Pythagorean Theorem leads naturally to irrational numbers.

The arithmetic of decimals can be presented as the mathematization of approximation. Also, the algorithms of elementary arithmetic lead naturally to flowcharts and to a study of the role of computers.

Length, area, and volume have computational, foundational, and group-theoretical aspects. Extensions of ideas from two to three dimensions constitute valuable experience in themselves, are useful in developing spatial intuition, and also help in understanding the nature of generalization in mathematics.

Graphs of functions of various kinds, theoretical and empirical, may be studied, incorporating intuitive notions of connectedness and smoothness.

The function concept plays an important role in an integrated curriculum. Counting, operations on numbers, measurement, geometric transformations, linear equations, and probability provide many examples of functions, and common characteristics of these examples should be noted. Ideas such as composition of functions and inverse functions may then be introduced and illustrated by algebraic and geometric examples. Of course, detailed formal discussion of the function concept should come only after examples of functions have been mentioned in various contexts in which it is useful to do so.

Similarly, the elementary notions of logic such as logical connectives, negation, and the quantifiers should be treated explicitly only after attention has been called informally to their uses in other mathematical contexts. Indirect proofs and the use of counterexamples arise naturally and may be stressed when the structure of the number systems is examined. However, in the final stages of a prospective elementary teacher's training it is useful to return to logic in a more explicit way for the purpose of summarizing the roles of inductive and deductive reasoning in mathematics and providing examples of deductive systems in geometry, algebra, number theory, or vector spaces.

Finally, the references to algorithms in the foregoing paragraphs emphasize the pervasive role of computing and algorithmic techniques in mathematics and its applications. The use of flowcharts for describing algorithms is becoming commonplace in the elementary school. Moreover, flowcharts are proving to be an important educational tool in teaching elementary and secondary school students to organize their work in problem solving. These ideas should therefore be encountered by a prospective teacher in his mathematics training. We recommend that computing facilities be made available, so that he will also have an opportunity to implement some algorithms and flowcharts on a high-speed computer using some standard computing language.

The course sequence should include many references to applications outside mathematics. This is self-evident for probability theory, but it is important to stress this over the whole spectrum of topics studied. In particular, the function concept itself provides many opportunities to underline the significance of mathematical formulations and methods in our study of the world around us.

At the conclusion of the course sequence the prospective teacher should understand the rational number system and the necessity, if not the method, of enlarging it to the real number system. He should be familiar with elementary linear geometry in two and three dimensions. In his study of the integers and the rational numbers, he should understand the essential role played by the properties of the addition and multiplication operations and the order relations in justifying and explaining the usual computational algorithms, the factorization theory of whole numbers, and the methods of solution of equations. He should thereby, and through experience with algebraic structures encountered in geometry, acquire an appreciation of the importance of abstraction and generalization in mathematics.

He should know something of the basic concepts and the algebra of probability theory, and he should be able to apply them to simple problems. He should grasp the idea of an algorithmic process and understand a bit about computers and how one programs them. We expect him to appreciate something of the role of mathematics in human thought, in science, and in society. We hope finally that he can learn all this in such a way that he will enjoy mathematics and the teaching of it, and that he will desire to continue to study mathematics.

Level II Recommendations

In the years since the first set of recommendations was made, dramatic changes have taken place in the mathematics of junior high school. These changes are in depth, as witnessed by greater emphasis on logic and mathematical exposition, and in breadth, as witnessed by the increased amount of geometry and probability. They make it necessary to re-examine the background needed by a teacher at this level. Moreover, the intermediate position of the junior high school requires of teachers at this level an appreciation of the mathematics of the elementary school as well as knowledge of the mathematics of the high school.

Finally, it seems desirable that there be two kinds of teachers in the middle school or the junior high school: those who concentrate on the transition from the elementary school and those who concentrate on the transition to the high school. For this reason we give two sets of recommendations for this level.

Level II-E Recommendations

These recommendations are for students who begin with Level I preparation and pursue further training to qualify them to be either specialist teachers of elementary school mathematics, or coordinators of elementary school mathematics, or teachers of middle school and junior high school mathematics. There should be some teachers with Level II-E preparation in each elementary school. The recommended program is:

A. The Level I program. (A student who is already prepared for calculus may omit the course on functions of the second sequence of courses listed on page 165 .)

B. An elementary calculus course (e.g., Mathematics 1 [page 44]). At this level all teachers need an introduction to analysis and an appreciation of the power that calculus provides.

C. Two courses in algebra. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, page 171 .

D. A course in probability and statistics. This course is identical to the first course under D of the Level III recommendations.

E. Experience with applications of computing. This recommendation is identical to that under F of the Level III recommendations.

F. One additional elective course. For example, a further course in calculus, geometry, or computing.

Level II-J Recommendations

These recommendations provide a special curriculum for the training of junior high school teachers which is slightly less extensive than that for Level III. The recommended program is:

A. <u>Two courses in elementary calculus</u> (e.g., Mathematics 1 and 2, page 44). Greater emphasis on calculus is desirable for this level because teachers at the upper level of the junior high school must see where their courses lead.

B. <u>Two courses in algebra</u>. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, page 171.

C. <u>One course in geometry</u>. Either of the two courses described under E of the Level III recommendations will suffice.

D. <u>A course in probability and statistics</u>. This course is identical to the first course described under D of the Level III recommendations.

E. <u>Experience with applications of computing</u>. This recommendation is identical to that under F of the Level III recommendations.

F. <u>Review of the content of courses 1 and 2 of Level I (either sequence) through study or audit</u>. There is a problem with the interface between the elementary school and the junior high school. In part this is caused by the fact that, traditionally, junior high school teachers are prepared for secondary school teaching and hence are little aware, at first, of their students' capabilities and preparation. We therefore believe that some sort of orientation to the mathematical content and spirit of the elementary school mathematics program is necessary to equip the Level II-J teacher properly. Two means have been considered to meet this need. One method would be to give an additional course in college to the prospective Level II-J teacher, a course which would be a streamlined version of courses 1 and 2 of the Level I program. On the whole, we prefer this solution although it makes the plan of study rather long. The second method would be to encourage schools to supply the new Level II-J teacher with elementary school texts to read, and to require him to visit classes and to talk with elementary school teachers, especially those in grades 4 to 6. A combination of both of these methods might prove most effective.

G. <u>Two elective courses</u>.

Items A through E supply the bare essentials. Greater breadth and greater depth are both to be desired. In order to give the teacher freedom to pursue his interests, electives are suggested, with further courses in computing, analysis, algebra, and geometry having high priority. Teachers with Level III preparation can meet the requirements listed above by fulfilling the intent of F.

Level III Recommendations

Although the mathematics of the senior high school has not changed as dramatically in the past ten years as has that of the elementary school, yet there are significant directions of change which make new recommendations desirable. These are: (1) a gradual increase in the volume and depth of mathematics taught at the secondary level which brings with it an increased occurrence of calculus (with the Advanced Placement program), (2) an increasing use of computers in mathematics courses and as an adjunct in other courses, and (3) an increasing realization that applications should play a more significant role.

Our recommendations, while designed primarily to specify minimum requirements for prospective high school teachers, have also been constructed with a view to maintaining, as far as possible, comparability of standards between prospective teachers and prospective entrants to a graduate school with a major in a mathematical science. We want to maintain a freedom of choice for the student to go in either direction. While the program we recommend for prospective teachers will leave the student with a deficiency in analysis and in algebra in order to meet the CUPM recommendations for entry to graduate school, the prospective graduate student in mathematics would normally need courses in geometry and in probability and statistics to meet our recommendations for teachers. <u>We regard it as a matter of great importance that a program for teachers should be identical to the one offered to other mathematics majors, except for a few courses peculiarly appropriate to prospective high school teachers</u>.

Before detailing the recommendations, some remarks on the role of applications, the computer, and on the problem of teaching geometry are in order.

Every experienced teacher knows that mathematics must begin at the concrete level before it can proceed to a more theoretical or abstract formulation. It is assumed that topics in the courses under discussion will contain a judicious mixture of motivation, theory, and application. A purely abstract course for teachers would be madness, but a course in calculation with no theory would not be mathematics. In addition to including applications where possible in mathematics courses, there is a need for introducing some specific study of the lore of mathematical model building, in order to provide the framework of ideas within which specific applications can be placed in their proper perspective. The idea of a mathematical model of a "real" situation and the associated techniques and rationale of the model building process have developed as a sort of folk knowledge among mathematicians and users of mathematics, and now an effort is being directed toward making these ideas more explicit and including them in the curriculum. The course Mathematics 10 described in <u>A General Curriculum in Mathematics for Colleges</u> (1965) [see page 92] was such an effort, but only now are detailed descriptions of such a course appearing (see <u>Applied Mathematics in the Undergraduate</u>

Curriculum, page 705). As these efforts begin to affect the high school curriculum, where much of the material belongs, it becomes more urgent that the future high school teacher receive appropriate preparation. Conversely, the preparation of teachers to communicate these ideas will accelerate the improved treatment of applications in high schools.

Computers have already had a phenomenal impact on the high school mathematics curriculum in supplementing, and in part replacing, traditional formal methods by algorithmic methods. As access to digital computers becomes more common, one can expect both the flavor and content of high school mathematics courses to change dramatically. In schools where such facilities are already available, it has become clear that opportunities for experimentation and creative outlet, using the computer as a laboratory device, are within the reach of many students whose mathematical ability, motivation, or background would preclude any comparable experience in a formal mathematical setting. Moreover, it has been found that certain abstract mathematical ideas are understood and appreciated more completely when experience is first obtained through the use of a computer. Algorithmic and numerical techniques should therefore be given strong consideration in all courses in which they are appropriate; and, wherever computing facilities are available, use of the computer should be a routine part of these courses.

The nature of high school geometry continues to change. Changes over the past decade have mainly been toward remedying the principal defects in Euclid's Elements that are related to the order, separation, and completeness properties of the line, but more recently there has developed an entirely new approach to geometry that links it strongly to algebra. This approach is now finding its way into the high school geometry course. A teacher should be prepared to teach geometry either in the modern Euclidean spirit or from the new algebraic point of view. Thus we are recommending that he take two geometry courses at the college level.

The minimum preparation of high school teachers of mathematics should include:

A. Three courses in calculus. Mathematics 1, 2, and 4 [page 44] are suitable. This recommendation assumes that the student has the necessary prerequisites. It is also desirable to take advantage of the growing role of computers in introducing mathematical concepts.

B. One course in real analysis. Mathematics 11 [page 93] would be satisfactory provided that the instructor is aware that his students' primary interest is teaching.

C. Two courses in algebra. One of these should treat those topics in linear algebra that are essential for the understanding of geometry and that have become crucial in applications, especially to the social sciences. Mathematics 3 [page 55], with careful attention to examples, would suffice. The second algebra course should be an abstract algebra course approximating Mathematics 6M [page 68]. Again, opportunities should be found to incorporate geometrical ideas that motivate and illustrate various algebraic structures (e.g., groups of symmetries, groups of transformations, rings of functions).

D. Two courses in probability and statistics. The first of these should begin with intuitive notions of probability and statistics derived from the real world. Mathematical model building and the relationship of mathematics to the real world should be considered. Calculus may be required in the latter part of this first course. The second course will treat those additional and more advanced topics normally included in a statistics sequence. In Preparation for Graduate Work in Statistics [page 459] the Statistics Panel of CUPM has described two courses that are close to what we have in mind. Throughout the two courses, care should be taken to include an analysis of some statistical studies which have appeared on the public scene and should make explicit some of the misinterpretations that are possible. Applications (in particular, applications to decision theory) should be drawn from such fields as medicine, education, business, and politics. The range and realism of problems can be enhanced if students are able to use computers.

E. Two courses in geometry. One course emphasizes a traditional approach by concentrating on synthetic methods and a careful study of the foundations of Euclidean geometry with a brief treatment of non-Euclidean geometry. The other course is strongly linked to linear algebra, includes an investigation of the groups of transformations associated with geometry, and is explicitly related to other parts of mathematics. Examples of such courses are given on page 86.

F. Experience with applications of computing. This should involve learning the use of at least one higher level programming language such as BASIC or FORTRAN. For this purpose we recommend a formal course such as C1 [page 563], but the experience may also be obtained independently or in other courses that make use of computers.

G. One course in applications. This should place heavy emphasis on mathematical models in the physical or social sciences. Examples of appropriate outlines can be found in Applied Mathematics in the Undergraduate Curriculum, page 705.

Nine of these 12 items should be included in the undergraduate program of every prospective high school teacher, namely A, B, C, F, and one course each from D and E. The remaining courses may be deferred to his post-baccalaureate study, although consideration should be given to including them among the electives in his undergraduate program. A list of possible elective courses is included at the end of the following paragraph. Were it not for our view that an undergraduate program should permit maximum flexibility in choosing a career and as much latitude as possible for every student to express his own interests in acquiring the proper breadth in his area of concentration, we would have specified that all of the 12 items be included in the undergraduate program of a prospective teacher. Fortunately, it is now becoming commonplace for Level III teachers to continue their mathematical education at the graduate level. Indeed, this is mandatory in many instances through permanent certification requirements or through the salary schedules and policies of individual school systems.

In structuring his undergraduate mathematics program, a student will naturally choose his electives after reflection upon his career goals. If, for example, a prospective high school teacher wishes to pursue graduate study in mathematics, he will necessarily choose additional courses in algebra and analysis beyond those which we have mentioned in our recommendations for Level III teachers. We include below a partial list of electives which would suitably extend our recommendations for the training of high school mathematics teachers.

Real Variables (Mathematics 11-12, page 93)

Complex Variables (Mathematics 13, page 97)

Numerical Analysis (Mathematics 8, page 83)

Abstract Algebra (Mathematics 6M, 6L, page 65)

Geometry and Topology

Number Theory

Foundations of Mathematics

Logic and Linguistics

OTHER ASPECTS OF TEACHER TRAINING

In accordance with our charge, we have made recommendations only on the content of the teacher training curriculum. We are, however, well aware that there are other crucial aspects of a teacher's overall preparation. In discussing these matters we hasten to observe that, just as we do not believe in any sharp distinction between teacher-trainees and other students in respect to the content of their mathematics courses, so we insist that these other aspects are relevant to all mathematics instruction. We believe only that they deserve more emphasis for teacher-trainees than for other mathematics majors.

Communication is of the essence in mathematics, and prospective teachers must pay special attention to all of the ways in which mathematics is most effectively communicated. They should be led to regard mathematics as a creative activity--something which one does rather than merely something which one learns. The active participation of the student in the process of discovering and communicating mathematical ideas is crucial for his real understanding. Courses should be taught in ways that foster active student involvement in the development and presentation of mathematical ideas.

Development of skills in writing and reading and speaking and listening should be an explicit part of teacher training at every stage, and not only in mathematics courses. These, like any other skills, can be developed only through constant and active involvement of the student in practices which exercise these skills. Thus, his regular courses, reading courses, clubs, or seminars should stress opportunities for two-way communication of mathematical ideas.

It is also important for teachers to continue to study and to do mathematics throughout their professional lives. This is closely related to, but goes beyond, the processes of communication mentioned above, for a willingness to grow reflects an enthusiasm that often transcends other skills in communicating mathematics.

Other aspects of communication which are not dealt with in this report are those relating to behavioral objectives and to special teaching methods and aids. While the Panel agrees that these are very important matters, it feels that they demand a much more complex effort and a totally different expertise, and might properly be the subject for another study. An excellent volume, which explores "the educational and psychological problems in the selection, organization and presentation of mathematics materials at all levels from the kindergarten through the high school," is the Sixty-ninth Yearbook of the National Society for the Study of Education, entitled "Mathematics Education."

The relationship of mathematics to other studies is another important matter not touched upon in this report except insofar as we have recommended the study of applications and mathematics. Indeed, we believe that every mathematics teacher should develop skill in other subjects which make use of mathematics.

Finally, we share a widespread concern for the special education of the culturally disadvantaged child and of the child whose achievements, for whatever reason, are below accepted standards. Such children require specially trained teachers. We do not know what form this training should take, but we feel that this is a proper concern of CUPM for the future.

COURSE GUIDES FOR LEVEL I

Introduction

Two sequences, each consisting of four courses, are outlined here in detail. One reason for presenting two different sequences is to illustrate our earlier claim that there are various ways of organizing the material. We have no desire to be prescriptive or definitive with respect either to course content or to the ordering of material. The outlines are to be construed as models only of the content and depth of coverage that we believe will be possible in the best circumstances. We do believe that the material presented approximates that which a really first-rate teacher of elementary school mathematics should know.

We regard either sequence as a way of achieving an integrated curriculum. In the first sequence the courses do not emphasize any particular single area of the traditional curriculum. Thus, arithmetic and geometry are both developed throughout the entire sequence. Each figures prominently in all four courses. In the second sequence, on the other hand, the emphasis is on number systems in the first course, geometry in the second, mathematical systems and induction in the third, and functions in the fourth. In both sequences each course contains topics from most of the areas identified as essential in the recommendation on page . By the end of the second course in either sequence the student will have met most of the topics that we consider essential for the elementary teacher, although not at the depth or in the detail preferred. Indeed, throughout both sequences the reader must be careful to interpret the statements of topics to be covered as referring to a treatment appropriate to the level of the student, and not a definitive treatment such as would be accorded to such a topic if encountered at a higher level. Typical places where there is danger of misinterpretation are Section 5 of Course 2 in the first sequence (Operational Systems and Algebraic Structures) and Section 1 of Course 2 in the second sequence (Intuitive Non-metric Geometry).

The unification of the four courses of each sequence requires the use of a common language for the expression of mathematical ideas. For instance, the concepts of set, function, and operation are introduced early and used throughout. Logical terms are introduced and used where appropriate.

Each section of a course guide has a suggested time allocation stated as a percentage of the course. These time allocations indicate first the balance of the sections within the course, second the depth and detail of treatment of the topics listed under each section heading. Thus they should enable the reader to judge the level of treatment and avoid the danger, already referred to, of giving a more comprehensive (or, perhaps, more superficial) treatment than intended.

It must be kept in mind that a prospective teacher is profoundly influenced by what he observes and experiences as a student.

Later his own methods and philosophy of teaching will reflect that experience. Hence, it is of paramount importance that these courses be conducted in a manner which encourages active participation in mathematical discovery. Frequent and substantial assignments which expose and drive home the attendant manipulative and computational skills should also be the rule.

SEQUENCE 1

Sequence 1 consists of four courses:

1. Number and Geometry with Applications I
2. Number and Geometry with Applications II
3. Mathematical Systems with Applications I
4. Mathematical Systems with Applications II

Course 1 begins with some intuitive geometry so that the concept of a number line is available immediately. Then arithmetic and geometry are developed throughout the entire sequence; the interaction of these two areas of mathematics enriches both subjects. Nevertheless, there are occasions when each area is developed within its own context. In particular, the algebra of the rational numbers is applied extensively to the theory of probability and statistics without reference to the geometric aspects of the number line.

A feature of this sequence is the adoption, to a limited extent, of the spiral approach. Thus, certain notions, such as extensions of the number system and the group of rigid motions, reappear several times, each time at a higher level of sophistication and with enhanced mathematical knowledge at the disposal of the student. An important practical advantage of this approach is that the student who takes only two or three courses of the sequence will have met most of the important mathematical concepts. In such cases, however, the depth of understanding is less than desired.

Some simple logic is introduced where appropriate to enhance the student's understanding. At the end of the sequence the student should understand the nature of mathematical reasoning and proof.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction.

Course 1. Number and Geometry with Applications I

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)

2. The Rational Number System and Subsystems (60%)

3. Probability, Statistics, and Other Applications (20%)

Course 1 is concerned with the study of the rational number system. It commences with the most intuitive geometrical notions, from which attention is focused on the number line and its role as a representation of the set of whole numbers. This approach enables the arithmetical and geometrical aspects of elementary mathematics to be developed from an integrated standpoint emphasizing their complementarity. It is a particular feature of the number line that negative integers are thereby immediately suggested and readily studied; this insures that the development of the number system follows a path which is mathematically natural. The twin approach also enriches the scope for interpretation of the operations of arithmetic. At this stage these operations and their properties are motivated through physical models.

Applications of the arithmetic of the nonnegative rationals to the most intuitive ideas of probability and statistics are given. Further applications of the arithmetic should be a feature of the course.

The language should be informal, but it should be such that a transition to precise mathematical language can be naturally effected. In particular, the student should be prepared for the function concept through the use of appropriate language. Those students who have not already met the notions of sets and functions may require more explicit introductions to these concepts.

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)

Intuitive development of geometric figures in the plane and space, first as idealizations of familiar objects and then as sets of points; an intuitive development of incidence relations and some simple consequences; congruence developed by use of slides and flips of models of figures leading to turns as another means of preserving congruence and with attention also to parallelism, perpendicularity, and symmetry; consideration of measurement of segments with various units and the beginning notion of approximation; informal introduction of the number line.

2. <u>The Rational Number System and Subsystems</u> (60%)

Introduction of the set $W = \{0, 1, 2, 3, \ldots\}$* of whole numbers, from sets of objects; addition in W from disjoint union, multiplication in W from cartesian products (treated informally); counting as the link between sets and numbers; place value systems and decimal numeration of whole numbers (reinforced by examples with nondecimal bases); properties of operations in W from observed properties of operations on sets, including order properties; algorithms for computation in W (include use of flowcharts); simple closed and open mathematical sentences, including inequalities.

Coordinatization of the half-line with W; addition in W as a vector sum, using slides of the number line; subtraction in W as a slide to the left to introduce the set Z of integers; properties of addition and order properties in Z; mathematical sentences in Z. (Multiplication by negatives is delayed until the set of rationals is developed.)

Factorization in W, prime factorization, unique prime factorization, some simple divisibility criteria, the Euclidean algorithm; multiplication and division of integers by whole numbers introduced through experiences with the number line; greatest common divisor and least common multiple.

Introduction of the set Q of rationals through division by n, $n \in W$ and $n \neq 0$, as shown on the number line; change in scale of the number line and its use in measurement; equivalence classes of symbols for rationals; coordinatization of the line with Q; introduction to the question of completeness.

Addition in Q suggested by slides of the number line; multiplication of a rational by a whole number suggested by slides of the number line; multiplication by a positive rational suggested by stretching and shrinking; multiplication by a negative integer suggested by multiplication by a whole number followed by a flip; multiplication in Q; algorithms for computation in Q^+, properties of addition and multiplication in Q, and order properties in Q.

* A glossary of symbols is included on page 202.

Decimal numeration of Q; percentages; integer exponents; scientific notation; orders of magnitude; algorithms and flowcharts for computation in Q; mathematical sentences.

3. Probability, Statistics, and Other Applications (20%)

Examples of statistical experiments in finite event spaces and their outcome sets, leading to counting procedures for determining the number of outcomes of various kinds of compound events (use tree diagrams); sampling problems with and without replacements, leading to combinatorial devices for counting samples; relative frequencies; assignment of probabilities to singleton events and to disjoint unions and intersections of events through addition and multiplication in Q^+.

Other applications, e.g., measurement, constant rate, profit and loss, expectation and risk, percentages, estimation, significant figures, and approximation.

Course 2. Number and Geometry with Applications II

1. Functions (5%)
2. The Rational Number System and Subsystems (20%)
3. Geometry (35%)
4. Real Numbers and Geometry (10%)
5. Operational Systems and Algebraic Structures (10%)
6. Probability, Statistics, and Other Applications (20%)

Course 2 is designed to give a genuine mathematical treatment of ideas introduced and studied at a more intuitive level in Course 1. The language of functions is established; this enables the solution of linear equations over Q and Z to be investigated systematically. Then the interrelationship between geometry and algebra again becomes evident in the study of symmetries, rigid motions, and sets with operations. At the same time, questions connected with measurement are studied, thus insuring that the material can be usefully applied; explicit reference to problems of approximate calculation involving large amounts of data can lead to consideration of computer programs.

The Pythagorean relation prepares the way for the introduction of irrational numbers and a preliminary discussion of real numbers.

The ideas here are difficult, and no attempt should be made to give complete proofs; nevertheless, the topic should be explored extensively.

Algebraic structures are defined, but studied only in familiar examples (including modular arithmetic). Further study of probability and statistics is included, beginning with a study of permutations and combinations which employs the function concept and presents systematic counting procedures.

1. Functions (5%)

The function concept (motivated by examples from Course 1), one-one and onto properties of functions; relations (motivated by examples from Course 1) with emphasis on equivalence and order relations. Binary operations as functions on cartesian products of form $X \times X$. Power sets; unions, intersections and complements as operations; the functions $2^X \to 2^Y$ and $2^Y \to 2^X$ induced by a function $X \to Y$.

2. The Rational Number System and Subsystems (20%)

Review of properties of Q with some arithmetical proofs; properties of Z; Z as an ordered integral domain; realization that Z is not closed under division, with the closure of Z leading to Q.

Coordinatization of the line with Z and then with Q; coordinatization of the plane with Z^2 and then Q^2; and coordinatization of space with Z^3 and Q^3.

3. Geometry (35%)

Review of geometric figures as idealizations of familiar objects and as sets of points in space; review of rigid motions and symmetry; review of congruence, parallelism, and perpendicularity. Groups of symmetries of an equilateral triangle and a square.

Rigid motions as functions that preserve lengths of segments; classification of rigid motions; composition and inverses; intuitive understanding of group properties of the group of rigid motions.

Review of measurement of segments with various units; principles of measurement; principles of measurement applied to length, area, angle measurement, volume; approximation; formulas for measurement related to rectangles, triangles, right prisms, pyramids.

Approximate calculation. Pythagorean relation through the formula for the area of a rectangle.

4. Real Numbers and Geometry (10%)

Adequacy of Q for physical measurement; inadequacy of Q for representing lengths of segments demonstrated by construction of segments with irrational measures; distance between points in Q^2 (use of Pythagorean relation). Introduction of real numbers in terms of nested intervals and nonterminating decimals; location of irrational points on the number line; infinite decimals regarded as sequences of approximating rationals; informal definition of addition and multiplication in R by means of approximating terminating decimals; distance in R^2; use of unique factorization to prove the irrationality of $\sqrt{2}$, $\sqrt{3}$, etc.

Coordinatization of the line, plane, and space with the real numbers; distance in R^2. Solution of linear equations and inequalities and graphs of solution sets in R^2; solution of linear equations in R^3 as intersections of planes.

Intuitive treatment of the perimeter and area of a circle; π as an irrational number.

5. Operational Systems and Algebraic Structures (10%)

Review of properties of the operations of addition and multiplication and the order relation in the number systems W, Z, and Q; defintions of group, ring, integral domain, and field, with examples drawn from subsets of Q, groups of rigid motions, groups of symmetries of a figure, power sets.

Arithmetic mod m as an operational system; contrasted with W, Z, and Q (closed under additive inverses, closed under multiplicative inverses when m is prime, divisors of zero when m is not prime, absence of compatible order relation); applications to the arithmetic of W (e.g., Fermat's theorem, Wilson's theorem, casting out 9's).

6. Probability, Statistics, and Other Applications (20%)

Permutations and combinations; randomness of a sample and tests of randomness; examples of applications of random sampling (using random number tables) to estimation of populations, quality control, etc.

Measure of central tendency in lists of data and possible measures of spread of data.

Random walks and their applications; assigning probabilities to compound events with applications; conditional probability.

Other applications, e.g., area, volume, weight, density; constructing an angle whose measure is p/q times the measure of a given angle.

Course 3. Mathematical Systems with Applications I

1. The Rational Number System (15%)

2. The Real Number System (5%)

3. Geometry (35%)

4. Functions (15%)

5. Mathematical Language and Strategy (15%)

6. Probability, Statistics, and Other Applications (15%)

In Course 3 the process of extending the number system from the whole numbers to the rationals, which has been explained and motivated in previous courses from both geometrical and arithmetical considerations, is carried out as a piece of formal algebra. Polynomials are also studied. Then algebra and geometry each provide examples of small deductive systems to illustrate the nature and power of the axiomatic method. The Pythagorean relation is available in this course, so that Euclidean geometry may be carried out in the coordinate plane. Vector notation and methods are studied, and there is a discussion of the generalization of coordinate geometry to three dimensions.

Algebraic concepts are also exemplified by the study of the group of Euclidean motions and certain subgroups, thereby enriching the notion of subgroup with concrete examples. The trigonometric, exponential, and logarithmic functions are defined and studied in their own right and in view of their applications. There are some explicit discussions in this course on mathematical methods, covering such topics as proof, conjecture, counterexample, and algorithms, together with a review of appropriate logical language. Probability theory is itself developed from an axiomatic standpoint, experience of its practical nature having been gained in previous courses.

1. <u>The Rational Number System</u> (15%)

Formal construction of Z from W and of Q from Z.

Polynomials as functions and forms; Q[x] and Z[x] as rings; the degree of a polynomial; substitution; quadratic equations.

Description of proof by induction in W with examples; application to number-theoretic properties of Z, e.g., divisibility properties, Fundamental Theorem of Arithmetic; Euclidean algorithm; similar applications to Q[x]; remainder theorem.

2. <u>The Real Number System</u> (5%)

Review of the coordinatization of the line with R; approximation of reals by rationals; addition and multiplication in R through rational approximation; the field of real numbers.

3. <u>Geometry</u> (35%)

Review of coordinate geometry of the real plane; distance; graphs of linear equations in two variables.

Plane vectors from translations; vector addition as the composition of translations; multiplication by scalars; vector equations of a line (with resulting parametric and general form of the equations); conditions for parallelism and perpendicularity in coordinate representation; appropriate generalization of these ideas to three dimensions.

Examples of groups and subgroups drawn from geometry, e.g., the group of similarities with the subgroup of rigid motions, the group of rigid motions with the subgroup of rotations about a point, the group of rotations with the subgroup of cyclic permutations of a regular polygon.

Similarities and the representation of similarities as composites of magnifications and rigid motions; similar figures. Constructing the points which separate a segment into n congruent parts.

Equations of circles and the beginning notions of trigonometric functions.

Small deductive system in plane geometry, e.g., incidence properties from postulates of incidence or some constructions from postulates of congruent triangles, or some angle-measure properties from postulates of incidence, parallelism, and segment and angle measures.

4. Functions (15%)

Review of real-valued functions and their graphs; inverses of invertible functions; graph of an invertible function and its inverse.

Beginning notions of exponential functions and some of their properties; applications of these ideas, e.g., growth and decay; logarithmic functions; application of logarithmic functions to approximate calculation and construction of a slide rule.

Definitions and graphs of the trigonometric functions with emphasis on periodicity.

5. Mathematical Language and Strategy (15%)

Review of the language of connectives, the common tautologies, and the relation of some of these notions to set union, intersection, complementation, and inclusion.

Universal and existential quantifiers; denial of a mathematical statement, counterexamples.

Examples from previous sections of direct and indirect proof and of proof by induction; selected new topics to illustrate proof by induction (e.g., binomial theorem for positive integer exponents; number of zeros of a polynomial; sums of finite series); explicit contrast with inductive inference, role of hypothesis, conjecture.

Examples of algorithms and flowcharts.

6. Probability, Statistics, and Other Applications (15%)

Postulates for a discrete probability function and some consequences proved for probabilities of compound events; random walks and their applications.

Computation of measures of central tendency and variance; simple intuitive notions of statistical inference, tests of significance, frequency distributions, passage from discrete to continuous variables, normal distribution; application of statistical inference to real-life situations, e.g., opinion polls, actuarial tables, health hazards.

Course 4. Mathematical Systems with Applications II

1. Geometry (30%)

2. The Real and Complex Number Systems (10%)

3. Operational Systems and Algebraic Structures (40%)

4. Probability, Statistics, and Other Applications (20%)

Course 4 consists of a systematic study of precalculus mathematics. Linear algebra in R^2 and R^3, as vector spaces and as inner product spaces, leads on the one hand to matrix algebra and on the other to the standard trigonometric identities. The algebraic method in geometry is contrasted with the synthetic method. The real numbers R are presented as the completion of the rationals, and the extension of R to the field C of complex numbers is motivated and described.

Abstract algebra occurs in the course--a beginning study is made of abstract group theory--but emphasis is on familiar examples of the various algebraic systems, for example, the integral domain of polynomials over Z, Q, Z_p (p prime). The key notion of homomorphism of algebraic structures is introduced; among the examples treated are the logarithmic and exponential functions which are seen to be mutually inverse isomorphisms.

Frequency distributions form the main topic of the probability and statistics component; although the course remains essentially concerned with discrete probability spaces, the normal distribution is mentioned here. Applications of the preceding theory are made to problems of approximation and error.

1. Geometry (30%)

Coordinatization of space with R^3, distance in space, first-degree linear equations in three variables; vectors in space, vector addition, scalar multiples of vectors in R^2 and R^3, description of the vector spaces R^2 and R^3; norms of vectors, inner product, definition of the inner-product (or Euclidean) space R^3, relation of cosine to the inner product; definition of the vector product in R^3, triple scalar product and volume; the idea of closeness in R^3, with some of the simpler topological properties of this metric space.

Definitions of linear transformations of R^2 and R^3; orthogonal transformations; matrix representation of linear transformations of R^2 and conditions for orthogonality; matrix multiplication suggested by composition of linear transformations; representations of rotations in the plane by orthogonal matrices, leading to the standard trigonometric identities.

Invertible linear transformations of the plane with coordinate representations; rigid motions, magnifications, and other subgroups of the group of invertible linear transformations; representation of similarities in R^2 by matrices.

Analysis of the roles of synthetic and analytic methods in geometry, e.g., properties of circles, coincidence properties of triangles.

2. The Real and Complex Number Systems (10%)

Algebraic extensions of Q; algebraic and order properties of R; discussion of the completeness of R.

Extension of the real number system to the field C of complex numbers; failure of order relations in C; graphical representation of C in R^2.

3. Operational Systems and Algebraic Structures (40%)

System of polynomial forms over Q, its integral domain properties, factorization; Euclidean algorithm and appropriate flow diagram; factor theorem; elementary theory of polynomial equations; comparison with theory for polynomials over Z_p (p prime), Z.

Exponents, extension of exponential functions over Q to functions over R, with graphs; rational functions over Q, over Z_p (p prime); Newton's method of approximating zeros of polynomials (no differential calculus) and appropriate flow diagram.

Subgroups; Lagrange's theorem; applications to elementary number theory (Fermat's theorem and Euler's theorem); commutative groups, quotient groups of commutative groups; application to Z_n.

Homomorphisms of algebraic structures with many examples; identification of those which are one-one, onto; defintion of isomorphism as invertible homomorphism; one-one and onto homomorphisms are isomorphisms; isomorphic systems, e.g., Z_4 and rotational symmetries of a square, the positive reals under multiplication and the real numbers under addition.

4. Probability, Statistics, and Other Applications (20%)

Review of sample spaces, probability functions, random walks; discrete binomial distributions; statistical inference and tests of significance; other frequency distributions with applications, e.g., rectangular, Poisson; normal distribution (treated descriptively).

Application of the number system Q to problems in scaling, ratio, proportion, variation; approximation, errors in approximation, errors in sums, errors in products; Bayesian inference.

SEQUENCE 2

Sequence 2 consists of four courses:

1. Number Systems and Their Origins

2. Geometry, Measurement, and Probability

3. Mathematical Systems

4. Functions

Each course contains topics from most of the areas identified in the general description of the Level I Recommendations (algebra, the function concept, geometry, mathematical systems, number systems, probability, deductive and inductive reasoning). While interrelationships among these topics are explored in the spirit of an integrated curriculum, each of the four courses, nevertheless, has a special emphasis or focus. The emphasis in the first course is on number systems, the second on geometry, the third on mathematical systems, and the fourth on functions.

Although the special character of each course can be suggested in a few words, it is a mistake to assume that any course is narrowly defined by its title. In fact, by the end of the second course the student will have met the full breadth of topics considered essential for the elementary teacher. He will not, however, have reached the depth of understanding desired.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction.

Course 1. Number Systems and Their Origins

1. Sets and Functions (15%)

2. Whole Numbers (45%)

3. First Look at Positive Rational Numbers (10%)

4. First Look at Integers (5%)

5. The Systems of Integers and Rationals (25%)

Course 1 features integration of arithmetic and algebra with supplementary assistance from geometry. The number line is thought of as a convenient device for representing numbers, order, and operations. Rational numbers are introduced in the context of comparing discrete rather than continuous sets (though brief reference is also made to the rational line). Algebraic similarities and differences between the number systems are emphasized. In particular, the systems of integers and rationals are studied in parallel. Algorithms, flowcharts, and manipulative rules for the various number systems are not only justified by referring to physical or schematic models but also are seen as consequences of the algebraic structural properties of the number systems. The whole number system receives much attention, as its algebraic properties (and its algorithms) recur in only slightly altered form in the systems of integers, rationals, and reals.

1. Sets and Functions (15%)

Review, at an intuitive level, of the basic concepts associated with sets and functions in order to establish the language and notation that will be used throughout the course. (For most students this will be a review of things they have seen repeatedly since junior high school.) Set concepts covered are: membership, inclusion, and equality for sets; various ways of describing sets (rosters, set-builder notation, Venn diagrams); special subsets that often lead to misunderstanding (empty set, singletons, the universal set); common operations on sets (intersection, union, complementation, cartesian product); illustration of the above concepts in various ways from real objects and from geometry.

The connection between set operations and logical connectives; for example, "or," "and," and "not" are related to union and intersection and complement while the inclusion relation "$A \subset B$" is related to the implication "$x \in A \Rightarrow x \in B$." (In this first introduction of logic, the treatment should be very brief and informal, but the language is necessary for subsequent use.)

The major function concepts to be covered include an intuitive rule-of-assignment definition; various ways of specifying this rule (arrow diagram, table, graph, set of ordered pairs, formula); notions of domain and range; input-machine-output analogy; one-one and onto properties; one-one correspondences between finite sets and between infinite sets; brief look at composition and inverses with a view toward later ties to rational arithmetic. (Real and geometric examples should be used).

2. Whole Numbers (45%)

Whole numbers are motivated by a desire to specify the "size" of finite sets, numerals and numeration systems by the inadequacy of verbal "symbols" for numbers; the Hindu-Arabic numeration system contrasted with historical and modern artificial numeration systems in order to emphasize the roles of base and place value; order among whole numbers related to the process of counting and to the existence of one-one or onto functions between finite sets; order in W represented schematically by the position of points on a whole number line. In the construction of this "line" the concept of congruent point pairs arises naturally.

The operations of addition and multiplication related, with counting as the link, to the set operations of union and cartesian product (e.g., the use of multiplication in determining the area of a rectangle); multiplication also related to repeated addition and to determining the number of outcomes in a multi-stage experiment (the product formulas for C_r^n and P_r^n might be illustrated); addition and multiplication represented schematically in the usual vector fashion (slides and stretches) on the number line.

The algorithms of whole-number arithmetic justified initially (as in the elementary classroom) by reference to manipulating and grouping finite sets. (A flow diagram for division via repeated subtraction can be given.) The whole numbers with their operations and order now viewed as a mathematical system, the algebraic properties of which are motivated by reference to finite sets and set operations; the algorithms of whole number arithmetic re-examined from an internal point of view and justified on the basis of notational conventions and fundamental algebraic structural principles. The importance of estimating products and quotients should be emphasized as the algorithms are studied.

3. First Look at Positive Rational Numbers (10%)

Fractions motivated by a desire to compare two finite sets. If a probabilistic flavor is desired, compare a set of favorable outcomes with a set of possible outcomes; if a more conventional approach is desired, "ratio" situations can be used. With fractions viewed as operators, addition continues to correspond, in a sense, to disjoint union, while multiplication corresponds to composition; fractions represented schematically as points or vectors on a number line, and the operations viewed vectorially; rules for manipulating fractions motivated initially from physical or schematic representations. Rational numbers appear as abstractions of equivalence classes of fractions, and some algebraic properties of the system of rational numbers can be motivated by physical examples; the algebraic structure of Q^+ is not explored in detail. (The embedding of W in Q^+ considered briefly with a light touch.) A careful structural investigation deferred until the full system Q appears.

4. First Look at Integers (5%)

Integers suggested by some real situation, e.g., profit-loss, up-down, etc.; addition corresponds to an operation in the given situation; multiplication by a positive integer considered as repeated addition. Again the number line is used as a schematic representation; the various uses of the symbol " - " clarified; some algebraic properties identified and motivated; the embedding of the whole numbers in Z introduced, but treated only lightly.

5. The Systems of Integers and Rationals (20%)

Following a brief review of the concepts of open sentence, variable, replacement set, truth set, equation, and solution set (or perhaps only the last two), a systematic, parallel exposition of the algebraic structures of Q and Z in terms of solutions to equations is possible (-a represents the unique solution to $a + x = 0$,

$1/a$ represents the unique solution to $ax = 1$ $(a \neq 0)$, $b - a = b + (-a)$, $b/a = b \times 1/a$); all the familiar rules for manipulating minus signs and fractions follow. The two important unique representations of rationals--as fractions in lowest terms and as (all but one type of) repeating decimals--can be illustrated and computational rules for (finite) decimals justified; several non-repeating infinite decimals described. The work on fractions in lowest terms will involve a certain amount of number theory, which should be done on an ad hoc basis. Review of the concepts of divisibility and prime; the Fundamental Theorem of Arithmetic illustrated and then assumed. (In Course 3 this principle may be proved.)

Course 2. Geometry, Measurement, and Probability

1. Intuitive Nonmetric Geometry (25%)
2. Intuitive Metric Geometry (25%)
3. Probability (20%)
4. Further Geometry (20%)
5. The Real Number System (10%)

In Course 2 the system of positive rational numbers reappears in two new contexts: in the context of geometric measurement, where continuous sets are being compared, and in the context of probability, where discrete sets are being "measured." Thus the system of positive rational numbers and the concept of measurement act as unifying threads. But the major blocks of new content covered are in geometry and probability.

Many opportunities for tying together these areas present themselves. For example, in the initial work in geometry which inevitably is concerned with establishing terminology and notation, combinatorial problems can be inserted to make the content more interesting. Later a probabilistic technique for approximating π could be given. Also, while studying probability, geometric representations can be given for many situations. For example, random processes are simulated by spinners, and experiments involving repeated trials are represented by random walks.

The geometry and the probability in this course are presented in a rather intuitive, nondeductive fashion. The main purpose here is to present the elementary facts in these areas, not to investigate their logical structure. Course 3 re-examines both areas from a more rigorous, deductive point of view.

The field of real numbers is also discussed to some extent in this second course.

1. Intuitive Nonmetric Geometry (25%)

Geometry viewed as the study of subsets of an abstract set, called space, whose elements are called points; the subsets are called geometric figures; drawing conventional pictures of points, lines, and planes suggests incidence relations. Some of the logical connections between various incidence properties explored. (The 4-point geometry might be introduced here, but axiomatics should not receive much emphasis in this course.) The standard terminology associated with incidence (collinear, coplanar, concurrent, parallel, skew, ...) reviewed in a combinatorial context (e.g., into how many pieces is the plane partitioned by n lines no three of which are concurrent and no two of which are parallel?). Further geometric figures (half lines; rays; open, closed, and half-open segments; half planes, half spaces, plane and dihedral angles and their interiors and exteriors) defined in terms of the basic figures--points, lines, planes; the set operations; the intuitively presented notions of (arcwise) connectivity and betweenness. The standard notation for these figures reviewed, using combinatorial problems for motivation (e.g., how many angles are "determined" by n points no three of which are collinear?).

The ideas of polygonal and nonpolygonal curves, simple curves, simple closed curves, and the interior of a simple closed curve presented intuitively along with a few other topological and geometrical concepts such as dimension of a figure, boundary of a figure, convexity; polygons, polyhedra, and Euler's formula.

Congruence introduced intuitively in this nonmetric setting as meaning same size and shape. At this first contact, the notions of measurement and distance avoided. The natural development of ideas seems to be: congruence which leads to a process of measuring which in turn suggests the existence of a distance function. Perhaps here, but probably more appropriately in Course 3, a definition of congruence in terms of distance can be given. Congruence of segments, angles, and other plane and spatial figures, with perpendicularity introduced in terms of congruence of adjacent angles.

Congruence in the plane viewed in terms of intuitive notions of rigid motions of the plane (slides, turns, flips, and their compositions). Symmetries of figures in terms of invariant point sets and rigid motions. The composition of rigid motions is a rigid motion and the inverse of a rigid motion is a rigid motion. The group concept introduced to tie together algebra and geometry. A fuller treatment of transformation geometry is suggested in Course 4.

2. Intuitive Metric Geometry (25%)

The process of measuring described in terms of filling up the set to be measured with congruent copies of a unit and counting the number of units used. Illustrated for segments, angles, and certain plane and spatial figures. Integrally nonmeasurable figures (with respect to a given unit) introduced and the positive rationals used as operators endowed with stretching-shrinking or replicating-partitioning powers. The rational number line reinterpreted in terms of segments and lengths, briefly showing the existence of rationally nonmeasurable segments and the real number line; a non-repeating infinite decimal exhibited and the theorem on decimal representation of irrationals recalled. The assignments of numbers to figures viewed as functions; observation that such measure functions are additive and invariant under congruence. The domain of segment measure functions extended to the domain of polygonal curves by additivity; perimeters computed. The additivity property applied to partitioning techniques for finding area; some familiar area and volume formulas derived (triangles, parallelograms, prisms, pyramids). The formula $A = \ell \times w$ for rectangles with irrational dimensions illustrated by drawing inscribed and circumscribed rectangles with rational dimensions. Plausible limiting arguments presented for circles and spheres; irrationality of π. The angle-sum theorem for triangles verified experimentally and then extended to convex n-gons by triangulation; the subsequent results about the various angle measures in regular polygons applied to making ruler-protractor drawings. Use of these measuring instruments suggests investigation of practical versus ideal measurement. Various units of length, area, angle, volume measurements and conversion factors relating them; the inevitability of approximation in practical measurement and the usage of such terms as "greatest possible error," "precision," "accuracy," "relative error." The various notational conventions in use for reporting how good an approximation is: significant digits, ± notation, interval of measure, scientific notation.

3. Probability (20%)

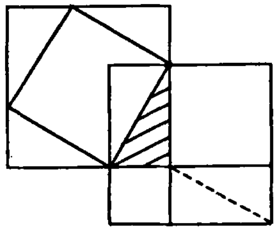
Various single and multi-stage experiments with discrete sample spaces considered and represented geometrically (trees, walks, spinners); large sample spaces and events "counted" using permutation and combination techniques. The terminology--sample space, outcome, event--compared with the terminology of geometry--space, point, figure; the assignment of probabilities to events compared with the assignment of lengths, areas, etc., to geometric figures. Both involve a comparison of two sets; the rational numbers are the indicated algebraic system. *A priori* assignment of probabilities (from shape of die, partitioning of spinner, constituency of urn, ...) compared with *a posteriori* assignment (long-range stability of relative frequency of events). The assignment of probabilities to events in terms of the point probabilities of their constituent outcomes, in the finite case, leading to additivity of probability measures; Venn diagrams used to illustrate the connection between set operations and logical connectives and to suggest the useful formulas

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{and} \quad P(A) = 1 - P(A').$$

Conditional probability and independent events; problems involving multiplication along the branches of a tree. The connection between C_r^n and the number of paths of a certain kind in the plane; the formula $C_r^n p^r (1 - p)^{n-r}$ for r successes in n repeated trials deduced and used; Pascal's Triangle (if not described earlier). The work on probability should include many exercises, as this may be the first contact with the subject for many prospective elementary school teachers.

4. Further Geometry (20%)

The simplest straightedge-compass constructions reviewed and related to the parallel postulate and congruence conditions for triangles (some proving of triangles congruent appropriate here). Each straightedge-compass construction technique compared with a ruler-protractor drawing technique for the same figure. Some plausibility argument for the Pythagorean Theorem, perhaps the one suggested by this sketch.



The simplified congruence condition for right triangles stated. Some work on square roots is appropriate here: a proof, based on the Fundamental Theorem of Arithmetic, that $\sqrt{n}$ is irrational or a whole number; an algorithm or two for computing rational approximations to $\sqrt{n}$; a remark that the existence of square roots in R but not in Q is a tipoff that R must have some extra fundamental property that Q does not have; geometric construction of a segment with irrational length, with respect to a given unit. Projection techniques for drawing similar triangles reviewed. Applications to scale drawing and to straightedge-compass construction of the rational number line. Similarity conditions for triangles analogous to the congruence conditions for triangles; for the case of right triangles the "AAA" similarity condition reduces to just "A." This suggests the calculation of trigonometric ratios and their use in indirect measurement. The Law of Cosines as a generalization of the Pythagorean Theorem.

5. <u>The Real Number System</u> (10%)

A brief review of the properties of the reals in some nonformal way--e.g., R includes Q; enjoys the same basic properties of +, ×, < that Q does but has one more property, namely, the least upper bound property. Use of this property to suggest but not prove the existence of $\sqrt[n]{a}$ for all $a \in R^+$ and all $n \in Z^+$. Some work with rational exponents.

Course 3. Mathematical Systems

1. The System of Whole Numbers (25%)

2. Fields (25%)

3. Geometry (25%)

4. Probability-Statistics (25%)

In this course certain portions of algebra, geometry, and probability are studied more deeply in a systematic, deductive way. Proof receives more emphasis than in Courses 1 and 2. Some of the concepts of logic itself receive explicit treatment, along with new results in algebra, geometry, and probability.

1. The System of Whole Numbers (25%)

The algebraic and order properties of W reviewed; the well-ordering principle introduced. Symbol $a|b$ defined; proofs of some simple divisibility theorems such as $a|b \Rightarrow a|bc$, $a|b$ and $a|c \Rightarrow a|(b + c)$. Various simple divisibility criteria of the base ten numeration system derived. Primes and prime factorization; the sieve of Eratosthenes; checking for prime divisors of n only up to $\sqrt{n}$; Euclid's theorem and Wilson's theorem. Unsolved problems such as Goldbach's conjecture and the twin primes problem. Other interesting odds and ends, e.g., figurate, deficient, abundant, and perfect numbers.

The important concepts of common and greatest common divisor (GCD) introduced and the existence, uniqueness, and linear combination expressibility theorems for GCD derived. Techniques for finding the GCD (by listing all divisors, by the Euclidean Algorithm, from known prime factorizations); the least common multiple (LCM), its existence and uniqueness proved, its relation to the GCD, and several techniques for finding it. (A flow diagram for the Euclidean

Algorithm is appropriate here.) The concept of relative primeness and the lemma stating that $p|ab \Rightarrow p|a$ or $p|b$ (p prime), leading to a proof of the Fundamental Theorem of Arithmetic. Whether a rigorous proof of this theorem should be given using the well-ordering principle or mathematical induction, is debatable. The use of the Fundamental Theorem in reducing fractions and in demonstrating the existence of irrationals. Euler's φ-function might be defined and the theorems of Euler and Fermat illustrated.

2. Fields (25%)

Field defined using Q and R as prototypes. Subtraction and division in a field defined and their usual properties derived; Z_p (p prime) defined and shown to be a field; various properties of subtraction and division illustrated again in this context. Possibly enough group theory interposed (Lagrange's theorem, order of element theorem) to prove the theorems of Euler and Fermat. In the context of solving a linear equation over a field, several concepts of logic can be studied; statement, equality, variable, open sentence, reference set, truth set; the unsolvability of $x^2 = 2$ over Q contrasts with its solvability over R; the completeness property of R recalled. The statement of this property depends on the concept of order; the definition of an ordered field abstracted from familiar properties of Q and R. Simple order properties deduced; Z_p shown to be unorderable (in any decent sense). The logical connectives and their relation to the set operations, within the context of solving inequalities over (say) R. Equivalence of open sentences; equivalence transformations. The Archimedean and density properties for R derived; a short excursion into limits (optional). The intermediate value theorem cited, behavior of polynomials for large $|x|$ illustrated; existence and uniqueness of positive n^{th} roots deduced.

3. Geometry (25%)

Incidence axioms suggested by the Euclidean plane and space stated as abstract axioms and exhibited in a finite model; simple incidence theorems proved and interpreted in both models. Illustrate in the context of segments how the natural genesis of concepts: congruence (superposition) → process of measurement → distance function, can be reversed in a formalization of geometry: postulated distance function → congruence defined in terms of it. The intuitive notion of betweenness used to define rays and segments; betweenness also defined in terms of distance; the ruler postulate; proofs of a few elementary betweenness properties. (The depth to which this Birkhoff-SMSG approach is carried is a matter of taste. For the future elementary teacher it might be more appropriate to do most of the deductive work in the spirit of Euclid, pointing out from time to time an implicit betweenness or existence assumption.) Possible subjects for short deductive chains include: triangle congruence and straightedge-compass constructions; parallels, transversals, and angle sums; area postulates and an area proof of the Pythagorean Theorem.

Some flavor of other modern approaches to geometry, with attention restricted to the plane: coordinatization of the plane, vector addition and scalar multiplication of points, lines as subspaces and their cosets, vector and standard equations for lines; Pythagorean Theorem and its converse, Law of Cosines, perpendicularity, dot product, norm, distance; isometry, orthogonal transformation, matrix representation of linear transformations, classification of orthogonal transformations, decomposition of an isometry into a translation and an orthogonal transformation. Alternatively, a coordinate-free study of transformation groups.

4. Probability-Statistics (25%)

Review of the "natural" development of the terminology and basic concepts of outcome, sample space, event, point probability function, and probability measure; axioms for a probability measure. The possible backward rigorization in probability of the intuitive notion of "equally likely outcomes" as "outcomes having the same probability" compared with the backward rigorization in geometry of "congruent segments" as "segments having the same length." It might be worthwhile to digress in more generality on equivalence relations, partitions, functions, and preimages. After a few deductions from the axioms $[P(\emptyset) = 0,\ P(A') = 1 - P(A),\ P(A \cup B) = P(A) + P(B) - P(A \cap B)]$, less emphasis is placed on deduction and further probabilistic concepts and techniques are presented. The idea of <u>simulation</u> by urn models, balls in cells, spinners, and the use of random number tables illustrated through a wide variety of problems. Permutations and combinations reviewed. More formal attention to combinatorial identities such as

$$2^n = (1 + 1)^n = \sum_{i=0}^{n} C_i^n, \qquad C_{k-1}^{n-1} + C_k^{n-1} = C_k^n,$$

$$\sum_{i=j}^{n-1} C_j^i = C_{j+1}^n.$$

More work on conditional probability, repeated trials, and random walks. Simple problems in hypothesis testing (e.g.: Ten tosses of a coin result in eight heads. With what confidence can you reject the hypothesis that the coin is honest?). Elementary expectation problems where expectation is thought of simply as a weighted average (e.g.: How much should one expect to win on a roll of a die if the payoff when n turns up is n^2 dollars?). The same problem posed using a nonsymmetric spinner instead of a die. Common distribution functions; measures of spread.

Course 4. Functions

1. Real Functions (10%)
2. Algebraic Functions (20%)
3. Exponential and Logarithmic Functions (15%)
4. Transformations and Matrices (20%)
5. Trigonometric Functions (15%)
6. R^2 and the Dot Product (20%)

The purpose of this course is two-fold: to present a satisfying culmination to the four-course sequence and to prepare the student to continue in mathematics with a calculus course such as Mathematics 1 [page 44]. Both of these goals are met in the context of a course centered around the function concept. Preparation for calculus is accomplished by studying special real functions, namely, the elementary functions; study of special functions of the plane (affine transformations) provides an appropriate dénouement of the four-course sequence by bring together arithmetic, algebra, and geometry in a transformation approach to plane geometry.

1. Real Functions (10%)

Brief review of the general concept of function from both the rule-of-assignment and ordered-pair points of view; specialization to the case where the domain and range are real numbers. Simple examples of functions--some artificial, some from science or business. Graphing and reading graphs. Graphs of real functions make it easier to think of them as mathematical objects in their own right, subject to operations as are other mathematical objects. Addition, subtraction, multiplication, division, and composition of functions viewed graphically as well as algebraically. Various additive and multiplicative groups of functions. One could look for rings, vector spaces, or even algebras of functions if that much abstract algebra is available.

2. Algebraic Functions (20%)

Real functions specialized to polynomial functions with emphasis on linear and quadratic functions. Slope and equations of straight lines, zeros of polynomials, and the factor theorem. Graphical interpretation of linear functions in the plane; geometric interpretation of linear functions on the number line in terms of stretches, shrinks, slides, and flips. (This suggests considering analogous functions of the plane and presents a natural opening for the discussion of general mappings of the plane and then the special types of mappings which are important for geometry, namely, translations, rotations, magnifications, and reflections. A discussion of transformation geometry may be included at this point.)

Increasing and decreasing real functions; there is no analogous concept for plane functions since the plane is not ordered. The completeness of the real number system and the Intermediate Value Theorem done at an intuitive level; invertible functions, both in the context of functions of the plane and in the context of real functions.

For real functions this leads to work with roots and rational exponents. Quadratic equations and various explicit algebraic functions.

3. Exponential and Logarithmic Functions (15%)

It is not reasonable to give a rigorous development of exponential functions. After adequate study of a^x for x rational and after some geometric motivation, the existence of a^x for x real should be assumed. The "laws of exponents" need emphasis. Other isomorphisms and homomorphisms recalled. Graphs of exponential functions and combinations thereof; logarithm functions defined as inverses of exponential functions, their properties derived from the properties of exponential functions; a minimal amount of computational work with common logarithms.

4. Transformations and Matrices (20%)

Having just completed computational work with some real functions, one can naturally ask whether various functions of the plane can also be given a concrete numerical representation. This leads to linear algebra and the study (in R^2) of vectors, dependence, independence, basis, linear transformation, matrix representation of linear transformations, matrix multiplication, and transformation composition.

5. Trigonometric Functions (15%)

The sine and cosine functions introduced by recalling the trigonometric ratios (Course 2); their definitions in terms of the winding function. The other trigonometric functions defined, graphs drawn, and questions of periodicity and invertibility entertained. Rotation matrix derivation of the addition formulas for sines and cosines. Proof of other trigonometric identities. The Pythagorean Theorem and its converse recalled and the Law of Cosines proved as a generalization.

6. R^2 and the Dot Product (20%)

The dot product motivated by the Law of Cosines as a measure of perpendicularity. The chain of ideas from dot product through length to metric traced. The central role played by this metric in currently popular axiomatic developments of geometry. The geometric and algebraic significance of the determinant function for 2 × 2 matrices.

Glossary of Symbols

SYMBOL	MEANING
W	$\{0, 1, 2, \ldots\}$, the set of whole numbers
Z	The set of integers
Q	The set of rational numbers
R	The set of real numbers
C	The set of complex numbers
Z^+, Q^+, R^+	The set of positive elements of Z, Q, R, respectively
Z_n	The set of integers modulo n
$x \in A$	The element x belongs to the set A
$A \subset B$	The set A is a subset of set B
$A \times B$	$\{(x,y): x \in A \text{ and } y \in B\}$
A^2	$A \times A$
A^3	$A \times A \times A$
$\Rightarrow$	implies
$a \mid b$	a divides b
GCD	Greatest common divisor
LCM	Least common multiple
C_r^n	$\frac{n!}{r!(n-r)!}$
P_r^n	$\frac{n!}{(n-r)!}$
2^A	The set of all subsets of A
A[x]	The set of all polynomials in one indeterminate with coefficients in A

TWO-YEAR COLLEGES AND BASIC MATHEMATICS

The Panel on Mathematics in Two-Year Colleges was formed in 1966 following some preliminary study of the need and potential in this area for the kind of activities which CUPM had successfully pursued in other areas. The members of the Panel were chosen from two-year colleges, four-year colleges, and universities so that extensive experience in various phases of education would be available. The Panel initially sponsored a series of meetings at which representatives of a wide spectrum of two-year colleges provided much detailed information about local variations, supplementing the Panel's studies of the national scene. The Panel also participated in several other activities related to the problem, such as meetings of the National Science Foundation Intercommission Panel on Two-Year Colleges, meetings of various organizations of two-year college mathematics teachers, individual visits to institutions, and a wealth of personal contacts. During this study phase the Panel was divided into subpanels concentrating on three topics: mathematics for general education, mathematics for technical-occupational programs, and mathematics for four-year college transfer programs (in all disciplines). Many two-year college teachers who consulted with the Panel expressed the opinion that guidance was most needed on the first two topics. However, it became increasingly clear as the study progressed that considerable overlap existed in the problems in these three areas and that an initial concentration on the third topic was most natural, both logically and from the viewpoint of CUPM's customary methods of operation. Thus, the Panel decided to concentrate its initial efforts on the construction of a program for university-parallel mathematics courses in two-year colleges. Its report, A Transfer Curriculum in Mathematics for Two-Year Colleges, was issued in 1969.

Concurrent with the decision of the Panel to restrict itself to the university-parallel curriculum, CUPM appointed an *ad hoc* Committee on Qualifications for a Two-Year College Faculty in Mathematics, whose membership overlapped that of the Panel. The report of this Committee, which appears in the section on TRAINING OF TEACHERS, discusses the qualifications of teachers of university-parallel mathematics courses and makes some general remarks concerning two-year college mathematics faculties.

The Transfer Curriculum report is essentially an adaptation of the first part of the GCMC curriculum [page 33] to the particular circumstances of those students in two-year colleges who intend to transfer to a four-year institution. That report intentionally deferred the consideration of lower-level or nonuniversity-parallel courses as a matter for further study. In 1970 CUPM appointed a Panel on Basic Mathematics to consider the first of these two areas: courses at a level below that of Mathematics A in the Transfer Curriculum. Among the members of this new Panel were persons from the Two-Year College Panel and representatives from developing institutions. The Panel felt that it would be possible to replace many of these courses by a single flexible course which involved a mathematics laboratory and was innovative in its approach. Its recommendations, together with an outline and commentary on the proposed course, appear in the 1971 publication A Basic Course in Mathematics for Colleges.

In 1971 CUPM issued A Basic Library List for Two-Year Colleges. This list was compiled by an ad hoc committee, with the assistance of many teachers from two-year colleges, four-year colleges, and universities.

Having offered suggestions for the improvement of university-parallel and basic mathematics programs, CUPM then turned to the much more complicated area of mathematics for technical-occupational programs in two-year colleges. A reconstituted Panel on Mathematics in Two-Year Colleges laid plans for producing materials designed to improve mathematics instruction for students in these fields. Due to lack of funds, it has not yet been possible to bring these plans to fruition.

A TRANSFER CURRICULUM IN MATHEMATICS

FOR TWO-YEAR COLLEGES

A Report of

The Panel on Mathematics in Two-Year Colleges

January 1969

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I. BACKGROUND

1. Two-Year Colleges.[1]

It is impossible not to be impressed, perhaps a little overwhelmed, by the growth and diversity of the two-year college sector of American higher education. To cite only a few facts indicating the rate of growth: new institutions are being added at the rate of one each week to the approximately 900 that now exist; the largest student body in any Florida educational institution is that of Miami-Dade Junior College; 86 per cent of all freshmen entering California public colleges in 1966 enrolled in two-year colleges; Seattle Community College opened in 1967 with an initial enrollment of 12,688 students. The extraordinary rate of growth in number of institutions and size of individual institutions is matched by their diversity, of size, purpose, make-up of student body, type of institutional control, variety of curricula, etc. A glance at a random selection of two-year college catalogs reveals this diversity quite strikingly. Further aspects of it can be noted in the annual directories of the American Association of Junior Colleges, and others are described and documented in a 1967 report to Congress from the National Science Foundation.[2]

Anyone interested in mathematical education who is accustomed to the comparatively stable national scene in either the elementary and secondary schools or in the four-year colleges and universities may find the two-year college picture bewildering. There is an immense variety of programs, including not only those comparable to the first two years in a university, but also a large assortment of technical, occupational, and semiprofessional programs as well as programs for general education and remedial study. The programs offered vary widely from school to school; the two-year college is often much more responsive to the community it serves and somewhat less responsive to tradition than the other institutions.

The student body attracted by these programs or by the convenience and economy of two-year college education exhibits extremes of age and maturity, background and preparation, and ability and motivation. More than two thirds of these students regard themselves

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1. The name "two-year college" is intended to tie in with the "lower division" at a college or university. Students at any of these schools may spend more or less than two years on their "lower-division" work.

2. The Junior College and Education in the Sciences, U. S. Government Printing Office, Washington, D. C., 1967.

initially as transfer students,[1] but only about one third eventually proceed to a four-year college.[2] This phenomenon is reflected in the figures on two-year college course enrollments in mathematical science compiled by the Conference Board of the Mathematical Sciences:[3]

> "Of these course enrollments 324,000 were classified by the institutions themselves as being in freshman courses and only 24,000 as being in sophomore courses. This is a much sharper drop than in four-year colleges."

This survey and the NSF report contain revealing data on the faculty. They describe a faculty which includes many part-time teachers, has a broad range of academic preparation, is recruited from a variety of sources (graduate schools 24%, colleges and universities 17%, high schools 30%, other sources 29%)[4] and is highly mobile ("About 25 per cent of all junior college professors were new to their particular campuses in 1964-65.").[5]

The situation sketched here has attracted the attention of many organizations concerned with the improvement of American higher education, of which CUPM is but one.

2. CUPM.

The Mathematical Association of America (MAA) is the national professional organization concerned with the teaching of mathematics on the college level. The Committee on the Undergraduate Program in Mathematics (CUPM) is one of the standing committees of the MAA. Simultaneously, CUPM is one of the eight college commissions in the sciences, supported by the National Science Foundation, "to serve as instruments through which leading scientists can provide stimulation, guidance, and direction to the academic community in the improvement of undergraduate instruction."

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1. NSF Report, p. 92.

2. NSF Report, p. 5.

3. Report of the Survey Committee, Volume I. Aspects of Undergraduate Training in the Mathematical Sciences. [For more recent figures, see Volume IV. Undergraduate Education in the Mathematical Sciences, 1970-71.] Available from Conference Board of the Mathematical Sciences, Joseph Henry Building, 2100 Pennsylvania Avenue, N.W., Suite 834, Washington, D.C. 20037.

4. NEA figures for 1963-64 and 1964-65 in all fields (see CBMS survey, p. 76).

5. NSF Report, p. 71.

The early curriculum recommendations published by CUPM dealt with specific aspects of education in mathematics, such as the training of physical science and engineering students, the training of teachers of elementary and high school mathematics, and the undergraduate preparation of graduate students of mathematics. It eventually became clear that CUPM had a responsibility to show how an overall curriculum could be constructed which was within the capabilities of a fairly small college and allowed for the implementation of its various special programs. A study of this problem led to the 1965 report A General Curriculum in Mathematics for Colleges (GCMC).* This report has received wide publicity through CUPM regional conferences and Section meetings of the MAA. Its major features have met with general approval and are having a growing influence on college textbook and curriculum reforms in tangible as well as in many intangible ways. Regarding this influence, it must be stressed that CUPM does not write curricula in order to prescribe what courses a department should teach, but rather to offer generalized models for discussion and to provide the framework for meaningful dialogue (within schools, between schools, and on a broader scale) on serious curricular problems. These models are meant to be both realistic and forward looking.

The present report is a natural extension of these efforts. In addition it provides, together with GCMC, an aid to articulation of two- and four-year programs.

3. The Present Report.

This initial CUPM report on two-year colleges is aimed at the transfer programs only. Some history of the study and reasons for this choice are outlined in the introduction to this section of the COMPENDIUM. Here we mention the following reasons:

Transfer programs are offered at almost all two-year colleges, and they determine the basic mathematics offerings. Local variations are least in this area.

A workable, imaginative solution to the problem of a transfer curriculum would provide the natural first step and could go far towards solution of the problems in the other two areas. It would answer the needs of the great majority of the students (two thirds intend to transfer), especially if it kept open a variety of options for these students, at least through the first year.

* The report presents a curriculum which can be taught by a staff of four or five and which includes courses for the various special programs. (An exception is the special course sequence for elementary teacher training; staff for these courses is not included in the estimate.)

The transfer curriculum lends itself more readily to a curriculum study with follow-up conferences. The other two problems (mathematics for general education and mathematics for technical-occupational programs) involve additional considerations, such as pedagogical techniques and the special needs of students with very specific goals, making this general approach less effective. CUPM's wide experience with the GCMC report and conferences (frequently involving two-year college teachers) had clearly indicated a need and a demand for similar efforts suitably adapted to the realities of two-year colleges.

The present report differs from GCMC in several major respects. Among these is the fact that it includes more detailed course descriptions, discussions of the rationale for choices that were made, and frequent comments on how topics might be taught. It also includes comments on implementation; the use and influence of the computer; articulation; etc. (see Chapter 5). Finally, it includes explicit recommendations on teacher training (see Chapter 3).

4. Staff.

The CBMS Survey mentioned earlier reports the following data on staff. The 167 two-year colleges having enrollments of more than 2,000 employed 44 full-time mathematics staff members with a doctorate in some field (not necessarily a mathematical science); 451 with a master's degree plus one additional year of graduate study in some field; and 439 with a master's degree in mathematics. The corresponding figures for the 543 schools of under 2,000 enrollment are 66, 306, and 513 respectively. There are, of course, many additional staff members with less academic preparation or who teach part time.

From these figures we can compute roughly the average number of staff members with academic preparation in the above range: five or six for the larger schools and one or two for the smaller schools. The corresponding figures when part-time faculty are included are eight and two, roughly.

The present report takes account of these facts by outlining a group of basic university-parallel courses that provide the necessary offerings for normal transfer programs (including elementary teacher training). It was prepared with the small department in mind. A larger department would have the potential for offering some additional courses to supplement these. Several possibilities for such courses are suggested in the report.

The important thing is that we believe our basic courses can be taught by the equivalent of approximately two full-time staff members. In Chapter 5, Section 1, we give an illustration of how this can be done. The actual number of teachers required in a particular case will depend, of course, on class size, teaching load, and other such factors.

It is perhaps the most striking common feature of two-year colleges that their faculties are highly student-oriented, much more so than the more discipline-oriented faculties at the four-year colleges and universities.[1] Nevertheless, it is important not to overlook problems of academic qualifications. In particular, a report such as this or the GCMC report would exist in a partial vacuum if there were no accompanying considered statements concerning the academic qualifications desired of the teachers of the programs. Such a report has been written to accompany the GCMC[2] and another report outlining a graduate program to achieve the proposed training has been prepared.[3] Simultaneously with its decision to begin with the university-parallel study, CUPM organized an ad hoc Committee on Qualifications for a Two-Year College Faculty in Mathematics. The report of this Committee[4] will be a companion to this report.

5. The Proposed Programs.

The courses which CUPM proposes are described in detail in Chapters 2, 3, and 4. We summarize them here under the broad classification of Basic Offerings and Additional Offerings, and discuss some of the programs which such offerings permit. Where appropriate, we draw attention to the comparable course in the report A General Curriculum in Mathematics for Colleges. The latter, of course, serves as an alternative to be considered by those two-year colleges that are structured on purely university-parallel lines.

BASIC OFFERINGS

I. Calculus Preparatory

(a) Mathematics 0. Elementary Functions and Coordinate Geometry.

(b) Mathematics A. Elementary Functions and Coordinate Geometry, with Algebra and Trigonometry.

One or both of these courses should be offered by every two-year college.

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1. See, for example, R. H. Garrison. Junior College Faculty: Issues and Answers. American Association of Junior Colleges, 1967.

2. Qualifications for a College Faculty in Mathematics (1967).

3. A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates (1969).

4. Qualifications for Teaching University-Parallel Mathematics Courses in Two-Year Colleges (1969).

II. Calculus and Linear Algebra

(a) Mathematics B. Introductory Calculus. (An intuitive course covering the basic concepts of single-variable calculus. Similar to Mathematics 1 [page 44].)

(b) Mathematics C. Mathematical Analysis. (A more rigorous course completing the standard calculus topics, as in Mathematics 2, 4 [page 51].)

(c) Mathematics L. Linear Algebra. (An elementary treatment similar to Mathematics 3 [page 55], but parallel to, rather than preceding, the last analysis course.)

Categories I and II constitute the basic pre-science offerings and should be offered by every two-year college with a transfer program.

III. Business and Social Science

Mathematics PS. Probability and Statistics. (An introductory course stressing basic statistical concepts.)

IV. Teacher Training

Mathematics NS. Structure of the Number System. (A year course as recommended by the Panel on Teacher Training for the preparation of elementary school (Level I) teachers. The second year of preparation, algebra and geometry, should also be offered whenever possible (see Chapter 3).)*

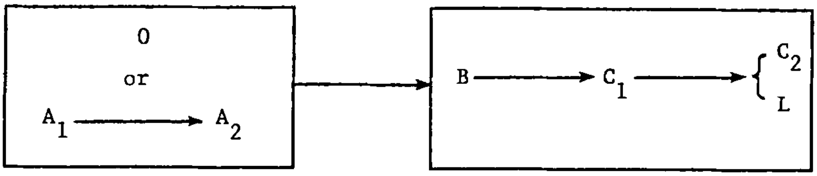
This completes the minimal set of offerings envisioned for a two-year college that embraces the usual range of university-parallel programs.

The courses A, C, and NS are full year courses. If we consider a semester system and use the obvious notation, the basic program is represented by the following diagram.

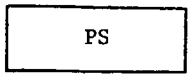
* Since this report was written, the recommendations of the Panel on Teacher Training have been revised. See the 1971 report Recommendations on Course Content for the Training of Teachers of Mathematics, page 158.

BASIC OFFERINGS

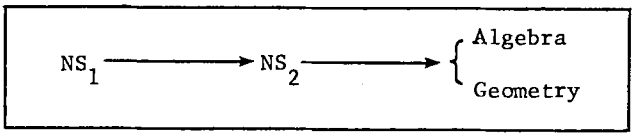
Calculus Preparatory | Calculus-Linear Algebra



Business and Social Science



Elementary Teacher Training



ADDITIONAL COURSE OFFERINGS (Optional)

1. Mathematics FM. Finite Mathematics. (A course of considerable interest and utility, especially for students of non-physical sciences.)

2. Mathematics DE. Intermediate Differential Equations.

3. Mathematics DA. Differential Equations and Advanced Calculus.

4. Mathematics PR. Probability Theory. (A calculus-based course adapted from Mathematics 2P [page 76].)

5. Mathematics NA. Numerical Analysis.

Selections from this list are suggested to round out a department's offerings according to its special needs and interests.

PROGRAMS*

We see this combination of courses as providing a variety of tracks to meet the needs of students with different educational goals and mathematical backgrounds and abilities. Some of the possibilities are described here.

* See page 224, Pace and manner of presentation.

(i) For the student in physical science, engineering, and mathematics (including future secondary school teachers) the program should include B, C, and L. A sequence for well-prepared students is provided by the semester courses

$$B \longrightarrow C_1 \longrightarrow C_2 \longrightarrow L,$$

plus possible additional courses. A slightly less well-prepared student can achieve this by taking the sequence of courses

$$0 \longrightarrow B \longrightarrow C_1 \longrightarrow (C_2 \text{ and } L).$$

The student who is not prepared for Mathematics 0 at the outset requires an extra semester or summer session to complete calculus at the two-year college, as indicated by the program

$$A_1 \longrightarrow A_2 \longrightarrow B \longrightarrow C_1 \longrightarrow (C_2 \text{ and } L).$$

(ii) Students of the biological, management, and social sciences probably will take Mathematics PS. Many of them will want a calculus course. Assuming adequate preparation for Mathematics 0, a reasonable sequence for such students is

$$PS \longrightarrow 0 \longrightarrow B,$$

to which Mathematics FM may be added in the third or fourth semester.

Although Mathematics L is thought of as following half of Mathematics C, it is possible for a bright student to take it after Mathematics B. The resulting sequence

$$0 \longrightarrow B \longrightarrow L$$

would be of great value as a mathematically stronger alternative to the more standard

$$0 \longrightarrow B \longrightarrow FM,$$

or

$$0 \longrightarrow FM \longrightarrow B.$$

(iii) If the primary interest of a student is the inclusion of some mathematics as a part of his general education, then any of the courses PS, A, 0, FM or the sequences already mentioned will, depending on his previous preparation, serve the purpose effectively.

II. THE BASIC UNIVERSITY-PARALLEL COURSES

In this chapter we describe in some detail the courses we propose for the basic offerings and discuss reasons for the choices that have been made.

1. Calculus Preparatory: Mathematics 0 and Mathematics A.

CUPM has taken the position that precalculus mathematics properly belongs in the high school but that many colleges may need to continue teaching courses at this level. In view of this need, a course, Mathematics 0, is described in A General Curriculum in Mathematics for Colleges. The same course is proposed for the two-year college.

In addition, recognizing the presence in two-year colleges of many students whose high school preparation is in need of reinforcement, the Panel suggests another course, Mathematics A. The subject matter of Mathematics A includes that of Mathematics 0, but the intended pace is much slower so that reviews of topics from arithmetic, algebra, and geometry can be introduced and pursued at appropriate stages of the Mathematics 0 outline.

It is hoped that each two-year college will be able to adapt the basic idea of this course to its own needs, perhaps offering both Mathematics 0 for its well-prepared students and a version of Mathematics A to meet the needs of the rest of its students who hope to complete some calculus. The courses, although designed for students who plan to take calculus, should carry credit for general education requirements. Most schools, however, have special general education courses and would advise their use in preference to Mathematics 0 or A for the purpose.

Mathematics 0. Elementary Functions and Coordinate Geometry.

Discussion: The prerequisites for Mathematics B (Introductory Calculus) include the following two components, (a) and (b):

(a) Three years of secondary school mathematics. The usual beginning courses in algebra (perhaps begun in eighth grade) and geometry account for two of these years. The remaining year should include: quadratic equations; systems of linear and quadratic equations and inequalities; algebra of complex numbers; exponents and logarithms; the rudiments of numerical trigonometry; the rudiments of plane analytic geometry, including locus problems, polar coordinates, and geometry of complex numbers; and arithmetic and geometric sequences.

(b) A course such as Mathematics 0 (or Mathematics A below): A course outline for Mathematics 0 is given on page 75.

Note: The proposed first course in calculus (see page 226) treats topics in analytic geometry only incidentally, as the calculus throws new light on the subject. Thus, rather than an integrated analytic geometry and calculus, it is more in the nature of a straight calculus course. It assumes as a prerequisite sufficient command of coordinate geometry for the study of single-variable calculus and some familiarity with the elementary functions and general concepts of function as outlined above.

Mathematics A. Elementary Functions and Coordinate Geometry, with Algebra and Trigonometry.

Discussion: This special course is designed for those students who, because of a weak mathematics background, are not prepared to begin an intense calculus preparatory course such as Mathematics 0. Many two-year colleges are currently meeting the needs of these students with courses in intermediate algebra, trigonometry, college algebra, and analytic geometry. It is felt that the contributions of all such courses to the overall two-year college offerings would be achieved better through the medium of this single course, adapted to local conditions.

A two-year college might choose to offer only Mathematics A or only Mathematics 0 or both, depending on the needs of the student population.

When teaching courses of high school level in a two-year college, it is customary to repeat the material in essentially the same form as it was presented in high school. For students who were not successful in high school, this approach is often no more fruitful the second time than the first. The virtue of Mathematics A is its fresh approach to old topics. Instead of requiring courses that repeat early high school mathematics and are then followed by a course on elementary functions, it is suggested that a repetition of the high school material be interwoven with the topics of the latter course.

The notion of function is given a central and unifying role in Mathematics A. Functions and their graphs serve as a peg on which to hang the review of elementary material. They also provide a new perspective, as well as a method of illustrating the old material. In this way, it is thought that the necessary review can be presented in a manner sufficiently fresh and interesting to overcome much of the resistance that students may carry with them from high school. (Moreover, this new approach should make Mathematics A more pleasant for the faculty to teach.) The development of a sound understanding of the function concept provides a solid cornerstone on which to build additional mathematical concepts in later courses.

This course should serve at least the following three purposes:

(i) To prepare the student for calculus and other advanced mathematics by including the material of Mathematics 0. Rather than being concerned with precisely which topics should be included, however, the emphasis should be on developing the ability to understand and use mathematical methods at least at the level of Mathematics B.

(ii) To review and remedy deficiencies in arithmetic, algebra, geometry, and basic logic:

(a) by means of assignments and class drill associated with Mathematics 0 topics. (Every opportunity should be seized to expose and stamp out abuses of logic and notation, and such common atrocities as $1/2 + 1/3 = 1/5$, $1/2 + 1/3 = 2/5$, $(a + b)^2 = a^2 + b^2$, and $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$, familiar to all teachers.)

(b) by interjecting discussions of review topics as the occasion arises in Mathematics 0 topics (see illustrations, page 221).

(iii) To develop mathematical literacy. By this we mean the ability to read and understand mathematical statements and the ability to translate into mathematical language (making proper use of logical connectives) statements and problems expressed in ordinary English. Continual practice should be given in solving "word problems" and in analyzing mathematical statements, with particular emphasis on developing the ability to understand and to use deductive reasoning.

Because of the limited time available, some material must be slighted. In this instance classical synthetic plane geometry is not covered in the detail or to the extent that is common in high school. However, we compensate for this by including enough analytic geometry to provide sufficient geometric preparation for calculus. Care has been used in choosing the topics in Mathematics A so as to include those topics which the student needs in order to progress successfully to calculus or to calculus-related courses. It nevertheless seems likely that five hours a week will be needed for Mathematics A during the first semester and possibly also during the second semester. It should be a slow course that includes a great deal of problem solving and also lengthy excursions as suggested by student interests and needs. Although Mathematics A serves to prepare students for calculus, it should also make students aware of the power and beauty of mathematics. The teacher should exploit every opportunity to provide examples and applications appropriate to the age and maturity of the students. Extensive use of "word problems" can be very effective in developing the ability to think mathematically and to use mathematics. We have attempted to give some indications along these lines in the course outline. In our view it is far more important to stir and develop the interest of the students than to cover each suggested topic.

It is essential that the students be forced throughout the course to translate English sentences into mathematical ones and vice versa, in order to improve their mathematical literacy.

Unfortunately, there exists at this time no single textbook completely appropriate for a course such as Mathematics A. However, by drawing upon several existing texts, the teacher should be able to find suitable textual material, illustrative examples, and exercises with which to build the course and achieve the remedial objectives.

Finally, we believe that a student who is not capable of handling Mathematics A upon entering a two-year college will in all likelihood be unable to pursue mathematical subjects in a four-year college with profit. Such a student should not be considered a transfer student in mathematical subject fields.

COURSE OUTLINE FOR MATHEMATICS A

Concept of function. Introduction as a rule associating to each element of a set a unique element of another set. (There shouldn't be very much worry about the definition of the word "set." Many examples should be given, with special attention devoted to functions that cannot be easily represented by formulas: for instance, the number of birds in a given locality or on a given tree as a function of the time or of the individual tree; the number of female students at the college as a function of the year; a function whose domain is the set $\{A, B, C\}$ and whose range is the set $\{D, E\}$; etc.) Cartesian coordinate systems and graphs of functions. Review of relevant geometry, such as basic facts concerning perpendiculars and directed line segments. Review of negative numbers, decimals, and the arithmetic of fractions, keyed to the problem of representing functions graphically. Decimals reviewed in this vein. Relative sizes of numbers illustrated on the x-axis. Motivation by means of examples of the definitions of sum, product, difference, and quotient of functions. Review of rational operations with algebraic expressions. Graphing of inequalities. Review of fractions and decimals, with special emphasis on comparison of sizes. Simple probability as a set function, i.e., a brief discussion of possibility sets, truth sets, and the assignment of measures to sets and their subsets. Conditional probability.

Polynomials of one variable. Linear functions of one variable and their graphs. Slope of a straight line. Various forms of the equation of a straight line. The graph of a linear equation as a straight line. Quadratic functions of one variable and their graphs. Completing the square. Use of completion of the square to solve some simple maximum and minimum problems. (This provides an opportunity to introduce some interesting "word problems.") Definition of polynomial functions of one variable. Sum and product of polynomials. Laws of exponents for integer powers. Division of polynomials, with the process first taught purely as an algorithm (stress should be given to the similar process for integers, whereby one integer is divided by another and a quotient and remainder obtained; the formula $m = qn + r$). Examples, pointing out the desirability of finding zeros of polynomial functions and ranges where the functions are positive or negative. Review of factoring: common factor, quadratics and the quadratic formula, difference of squares, sum and difference of cubes. The factor and remainder theorems. Theorems on rational roots. The need for complex numbers. Complex numbers and their algebra (introduced as a natural extension of the real number system). Geometric interpretation of addition of complex numbers, multiplication by real numbers (i.e., the vector operations) and of multiplication (see illustrations, page 221).

Arithmetic and geometric sequences. Definitions and sum formulas. Approximation (noting error term) of $\frac{1}{1 - x}$ by $\sum_{k=0}^{n} x^k$ for $-1 < x < 1$ (both by using the sum formula and by repeating the division algorithm). Intuitive discussion of the meaning of limit. (Application can be made to the fractional representation of an infinite repeating decimal. The limit concept can be illustrated by using both repeating and nonrepeating infinite decimals.)

Combinations and the binomial theorem. Finite sets, their unions and intersections. Combinations and permutations of members of a set. The binomial coefficients $\binom{n}{k}$ as the number of combinations (subsets) with k members chosen from a set with n members. Additional simple probability, using the formulas for numbers of permutations and combinations. More applications and "word problems" using poker, dice, baseball lineups, etc. Review of the manipulation of integer exponents. Direct evaluation of $(x + y)^n$ for $n = 2, 3, 4$. The binomial theorem for natural number exponents. Proof using the formula for number of combinations.

Rational functions and polynomials of more than one variable. Rational functions and their graphs, with particular attention given to factoring the denominators and to the concept of asymptote. Use of asymptotes to strengthen further the intuitive notion of limit. Linear inequalities. Simultaneous linear equations in two or three unknowns. Applications to elementary linear programming. Polynomial functions of two variables. Quadratic polynomials of two variables. Definitions and resulting equations of conics, but only for the axes parallel to the coordinate axes (no discussion of rotation of axes). Simultaneous solution of a linear and a quadratic equation in two variables, of two quadratics, and geometric interpretation of the solutions. (The student should be made aware of the importance of functions of several variables; he should know that the initial restriction to one variable is only for the purpose of handling the simpler problem first.)

Exponential functions. Review of the laws of exponents, with explanations of why, with $a > 0$, a^0 is defined to be 1, a^{-x} to be $1/a^x$, and $a^{p/q}$ to be $\sqrt[q]{a^p}$ or $(\sqrt[q]{a})^p$. Real numbers reviewed and presented as infinite decimals. Discussion of the need to give meaning to arbitrary real power. Discussion of approximations to specific numbers such as $10^{\sqrt{2}}$ or $(1.2)^{\pi}$ and how this concept might be used along with the limit concept to define such exponentials. (This provides a good opportunity to contribute another step toward the understanding of the limit concept. If a computer is available, convergence can be made to seem plausible by actually computing approximations to a number such as $(1.2)^{\pi}$, using the first few places in the decimal expansion of π.) General exponential functions. Sketch of the theory of binomial series for negative and fractional exponents, with more discussion of the limit concept. Use of these in approximations of roots and powers. Applications of exponential functions to selected topics such as growth, interest, or electricity.

Logarithmic functions. Discussion of the general notion of the inverse of a function. Application to the definition of the logarithm to base a as the inverse of the exponential to base a. Review of the use of common logarithms in calculation. The slide rule (only a few lessons; see illustrations, page 221). Some discussion of the idea of approximating exponentials and logarithms by polynomials; no proofs given. (If a computer is available, use it to verify that the approximating polynomials give values of the exponential and logarithm agreeing with those in tables. If time and interests permit, consider $\left(1 + \frac{x}{n}\right)^n$ as n increases, to relate compound interest and exponentials and to motivate polynomial approximations of e^x.)

Trigonometric functions. Review of simplest geometric properties of circles and triangles, especially right triangles (very few formal geometric proofs). Definition of trigonometric functions as ratios (illustrate, but do not belabor, problems of triangle solving). Trigonometric functions defined on the unit circle as functions of a real variable. Graphs of trigonometric functions. Symmetry; even and odd functions. Review of ratios, fractions, and decimals once again. The following trigonometric identities: the elementary identities $\sin x/\cos x = \tan x$, $\sin(x + \frac{\pi}{2}) = \cos x$, etc.; the Pythagorean identities $\cos^2 x + \sin^2 x = 1$, $\sec^2 x = 1 + \tan^2 x$, $\operatorname{cosec}^2 x = 1 + \operatorname{cotan}^2 x$; formulas for $\cos(x \pm y)$, $\sin(x \pm y)$, $\cos x \pm \cos y$, and $\sin x \pm \sin y$. Law of sines and law of cosines. Trigonometric (polar) representation of complex numbers. Roots of complex numbers. Graphs of sums and products of trigonometric and exponential functions. Periodicity. Applications to periodic motion and other periodic phenomena. Inverse trigonometric functions using the general notion of inverse function. Their graphs.

Functions of two variables. Introduction to 3-dimensional coordinate systems, functions of two variables, and graphs of functions of two variables for a few simple cases.

Some illustrations. 1. Elementary probability theory provides an excellent opportunity to develop both the ability to reason deductively and the ability to translate "word problems" into mathematical language. Examples can be chosen from horse racing, tossing of dice or coins, drawing colored balls from bags, etc. The following is one of an endless variety of problems that are useful for developing the ability to analyze a problem and to assign probability measures to sets:

A man tosses two coins and then informs you that the outcome includes at least one head. Determine the probability that the outcome consisted of two heads, given that the rule determining the man's statement is the following (two separate cases are considered):

Case (a). He says, "There is at least one head," precisely when this is true; otherwise he says nothing.

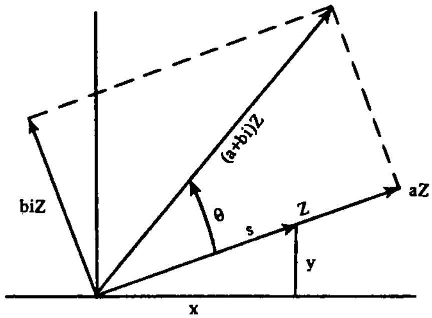
(The fact that he spoke means the outcome TT is eliminated from the four possibilities, leaving three equally likely cases and an answer of 1/3.)

Case (b). He says, "There is at least one head," when the outcome is HH. He says, "There is at least one tail," when the outcome is TT. In the remaining cases, he chooses which of these two statements to make on the basis of a coin toss.

(Under these rules, the outcome TT has probability zero in view of the statement actually made. The remaining elements of the sample space have probabilities 1/2 for HH and 1/4 each for HT and TH. The answer is 1/2.)

2. The student will easily follow the interpretation of complex numbers as vectors and their addition as vector addition. He should also be taught to interpret this addition as a transformation of the plane. Thus $(a + bi) + (x + iy)$, where x and y are variables, defines a <u>translation</u> of the entire plane in the fixed direction of $a + bi$. This remark is preliminary to the more difficult step (for the student) of interpreting $(a + bi)(x + iy)$ as an "operator" on the plane, a point of view which is of prime importance in many technologies.

This is broken down into steps: $a(x + iy)$ extends the vector $(x + iy)$ in the ratio $a:1$ [might be called an "extensor" or "amplifier" (one considers also the case that a is negative)]. Next, $i[x + iy] = ix - y$ is a rotation (of the variable vector and the entire plane) through $+90^\circ$ (i.e., counterclockwise), and $bi[x + iy]$ extends this new vector in the ratio $b:1$. From this the geometric meaning of $(a + bi)(x + iy)$ appears as a vector sum. The length of this sum vector is $s\sqrt{a^2 + b^2}$, where s is the length of $Z = (x + iy)$, its polar angle is as shown in the figure (where θ, the polar angle of $a + bi$, occurs as shown by reason of similar triangles).



Hence, multiplication by $a + bi$ is an operator which rotates the plane through the angle θ and stretches all vectors in the ratio $\sqrt{a^2 + b^2}$ to 1. This gives the geometric interpretation of a product of complex numbers (multiply lengths and add angles); later in the course it gives the addition formulas for sine and cosine (simultaneously). Of course, this geometric interpretation could also follow a discussion of trigonometry, as an application.

3. Word problems involving conjunctions and disjunctions are useful in sharpening logical thinking. For example:

> Thirty students are given grades 3, 2, or 1. If at least 10 got 3's and at most 5 got 1's, what can be said about the average score S for that class? [$3 \geq S \geq 65/30$]. What can be said about the number N of students who scored 2? If at most 10 got 3's and at least 5 got 1's, what can be said about S? [$1 \leq S \leq 65/30$]. What can be said about the number N of students who scored 2?

Other (similar) problems can be introduced in a variety of contexts.

4. A brief discussion of the slide rule is of value in reinforcing the student's understanding of logarithms. Conversely, such an understanding can be the basis for learning the principles upon which the instrument is designed and can give the student an adequate foundation for self-taught skill in its use.

Also, the student's interest in the slide rule can be exploited to great advantage in order to develop through practice his skill at estimating the magnitude of the result of an arithmetic computation (the problem of placing the decimal point).

2. Calculus and Linear Algebra: Mathematics B, C, and L.

This group of courses constitutes the proposed standard offerings for students expecting to major in mathematics, engineering, or the physical sciences, but it is structured in such a way as to serve a variety of other needs. In particular, this is achieved by the course Mathematics B (similar to Mathematics 1 [page 44]). The main feature of this course is the inclusion of the main concepts of calculus (limit, derivative, integral, and the fundamental theorem) in the setting of a single variable, dealing with the elementary functions studied in the earlier course and including computational techniques and applications of these ideas and methods. Thus the course provides a meaningful and usable study of the calculus for students who are unable to pursue the full sequence, as well as a desirable introduction for those who are.

Mathematics B can also be regarded as the calculus of elementary functions and, as such, forms a natural unit with Mathematics 0 or A: a thorough study of the elementary functions.

The remainder of the usual calculus topics, including series, functions of several variables, and elementary differential equations, are described here as a single one-year course, Mathematics C.

Finally, we describe a course in linear algebra, Mathematics L, which is similar to Mathematics 3 [page 55], only slightly less ambitious and employing a strong geometric flavor.

We consider it very important that the two-year college student who starts Mathematics C should complete it at the same school, rather than risk the probable discontinuity of study and loss of time attendant upon transfer before completion of this material. For this reason we have abandoned the feature of GCMC which proposes that linear algebra precede the study of functions of several variables and which allows a somewhat deeper study of the latter. This feature constituted only one of the reasons for the inclusion of linear algebra in the lower-division offerings. The remaining arguments apply as well to the two-year college situation, where the course can be taken simultaneously with the last half of Mathematics C in a two-year sequence starting with Mathematics 0, and so will serve an important group of students.

Pace and manner of presentation.

Without intending to prescribe the exact format in which the proposed courses are to be offered, we nevertheless find it convenient to assume occasionally a system of semesters (roughly 14 weeks of classes each) in order to state relative intensities of presentation in the familiar terms of 3-, 4-, and 5-hour[1] semester courses. In these terms we imagine the basic sequence of courses as embodying an increase of pace, the pace of the first courses taking realistic account of the wide range of student abilities and the last courses matching in pace the comparable courses at the transfer institutions.[2]

To illustrate, a distribution which fits our image of the average, multi-purpose community college is:

Mathematics A_1, A_2, 0, and B--5 hours each

Mathematics C_1--4 hours

Mathematics C_2 and L--3 hours each

Such a progression of pace will, it is hoped, allow for enough additional drill and reinforcement of topics in the earlier courses and for enough attention to mathematical literacy to provide for the student's growth in these areas to a degree of maturity which, at the end of the two-year college experience, puts him on equal standing with his fellow students at the best transfer school, where more severe screening may have taken place.

It is essential that this nurturing and growth process be achieved without weakening the content of the courses to the detriment of the better students. Indeed, for the better of the science-oriented students, it is important that these courses at the two-year college achieve as deep a penetration of the subject matter as comparable university courses so as to prepare the student for more advanced courses in mathematics and related disciplines. A calculus course that fails to teach students to understand and analyze mathematics, that gives too few and too careless demonstrations of mathematical facts, will make later work very difficult for the student who attends a four-year college whose upper-division curriculum presupposes more: students who have learned in high school to think mathematically may become disillusioned; students who have passed such a course may have false impressions of their mathematical knowledge; for all students, such courses mean the postponement of involvement with the true meaning of mathematics.

This, however, most emphatically does not mean that the course should strive for complete "rigor" as exemplified by, say, a detailed

1. These refer to the number of class meetings, not to the amount of credit granted, which we consider a local problem.

2. To be sure, four-year colleges have students of varying abilities also, but there is more of a sink-or-swim philosophy than at many two-year colleges which regard the careful nurturing of students as a greater part of their function.

development of the real number system from axioms, preceded by formal discussions of logic and followed by careful epsilon-delta foundations for the calculus. It is possible to be intuitive and still be correct, simply by following intuitive motivations and plausibility arguments with precise statements and proofs where possible, and with honest omissions or postponements of these when necessary.

It is in this sense that Mathematics B is suggested to be an intuitive course (a certain amount of such omission is certainly indicated for the suggested amount of material to be covered). The gaps left in Mathematics B, the promises for more sound foundations, are intended to be partially filled in Mathematics C where, for example, questions of limit, continuity, and integrability reappear (in the series and several variables settings).

Mathematics C, therefore, is intended to be a more sophisticated course, both in selection and treatment of subject matter, than Mathematics B. For example, we feel that the inclusion of a study of sequences and series in Mathematics C not only is important in itself and for its applications but also is particularly effective for developing mathematical maturity. With the least upper bound property of the real numbers as the principal tool it is possible to give careful proofs of the convergence theorems for monotone bounded sequences and for series of nonnegative numbers with bounded partial sums. Moreover, the study of sequences and series is a mathematical discipline that shows the need for precise definitions and demonstrates the dependence of mathematical understanding and mastery of formal techniques on basic theory. Also, the subject dramatizes the inadequacies of many of the student's prior intuitive ideas. It provides the student with perhaps his first major experience with a mathematical situation for which precise methods are indisputably needed. Indeed, it is necessary that some of the concepts introduced in Mathematics B, for example "limits," be further developed here.

Mathematics B. Introductory Calculus.

Discussion: Mathematics B is designed to follow Mathematics 0 or Mathematics A. The purpose is to introduce the ideas of derivative and integral, motivated by geometrical and physical interpretations, and to study their interrelations through the Fundamental Theorem of Calculus; to develop techniques of differentiation and integration; and to demonstrate the power and utility of the subject matter through frequent and varied applications.

Skills in manipulation need to be stressed in order that the student may learn to solve effectively problems in the applications of calculus, and to facilitate his study of later topics.

Continuing emphasis should be placed upon developing mathematical literacy; the lessons should stress correct interpretations of the written statements that set forth theorems, problem conditions, and proofs.

COURSE OUTLINE FOR MATHEMATICS B

1. Slope of the secant line to a curve. Geometric discussion of the limiting position. Derivative as slope of tangent line. Average rates and instantaneous rates of change as intuitive motivation for derivative. Limit of difference quotient as special case of limit of function. Intuitive discussion of $\lim_{x \to a} f(x)$ including algebra of limits (describe the definition briefly using pictures and mention that the limit theorems hold for this definition). Definition of derivative based on intuitive notion of limit; algebraic properties: sum, product, and quotient formulas with proofs.

2. Derivative of x^n, polynomials, and rational functions. Point out continuity of polynomials (limit obtained by direct substitution). Examples of discontinuities, definition of continuity (illustrate with pictures). Derivatives and continuity properties of trigonometric functions reduced to showing $\lim_{h \to o} \frac{\sin h}{h} = 1$. The existence of $f'(a)$ implies that f is continuous at a.

3. Interpretations and applications: slope, velocity, rates of change, curve sketching--including increasing, decreasing, maxima, and minima (confined to rational functions and simple combinations of trigonometric functions). Need for more differentiation techniques.

4. Chain rule. Its importance to building techniques of differentiation. The obvious "proof"

$$\frac{df(g(x))}{dx} = \lim_{h \to o} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to o} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x)) \cdot g'(x)$$

breaks down because $g(x+h) - g(x)$ might be zero for arbitrarily small h. Difficulty is avoided either by a trick or by showing that in this case both sides are zero, yielding a proof (either do it or give a reference). Comments on the use of limit theorems and differentiability hypotheses in this argument.

General differentiation formulas for rational and trigonometric functions. Drill. Implicit differentiation. Derivatives of algebraic functions (general power formula) and inverse trigonometric functions.

5. More applications: word problems, related rates, curve sketching; need for more information on max, min, etc.

6. Intuitive discussion of Rolle's theorem, Mean Value Theorem. Proofs, assuming existence of maximum for continuous function on closed, bounded interval. Intermediate Value Theorem.

7. Higher derivatives and the differential as best linear approximation.

8. More applied problems. Curve sketching, rate problems and maximum and minimum problems with the more complete methods now available, acceleration, concavity.

9. Integration. (Motivate the integral as a limit of sums by observing how the latter arise naturally in several specific practical problems, e.g., area, work, moments, volume, mass, pressure. The integral is an abstract entity which embraces these and other quantities.) Discussion, with reliance on intuitive understanding of the limit involved. Antiderivative and Fundamental Theorem of Calculus. Plausibility arguments for its validity and stress on its meaning for computation. Practice in finding antiderivatives, need for further techniques.

10. Exponential and logarithm. Definition of $\log x$ as $\int_1^x \frac{1}{t}\,dt$. Derivation of properties of $\log x$ from this formula. Exponential function as inverse of $\log x$. Properties. Drill on complete list of differentiation formulas. Problems on applications involving these functions.

11. Applications of integration to finding areas, volumes of revolution, arc lengths, work done in emptying tanks.

12. If time permits: A more careful development of the definition of integral, discussion of properties of continuous functions, and proof of the Fundamental Theorem.

Mathematics C. <u>Mathematical Analysis</u>.

Discussion: Completing the traditional subject matter of calculus and giving an introduction to differential equations, this course serves also as the vehicle for an increasing sophistication, in both topics treated and the manner of their presentation.

Except for the obvious desirability of including the material on vectors prior to the beginning of Mathematics L (where, in any case, this material is briefly reviewed), no attempt has been made to divide the subject matter to fit quarters or semesters. There are two principal reasons for this. First, it seems important that students who intend to complete the entire calculus sequence (Mathematics B and C) should do it all at the same institution. Otherwise, wide variations in the order of topics can make transfer very difficult. Second, we leave to the particular college the problem of determining how many quarters or semesters and how many meetings per week are needed to present the material.

COURSE OUTLINE FOR MATHEMATICS C

Techniques of integration. Integration by parts. Integration by substitution, including use of the inverse trigonometric functions. Explanation of plausibility of general partial fraction expansion; use of partial fractions in integration. Emphasis on the use of tables of integrals. (Drill on techniques of integration provides useful practice in the manipulation of the elementary functions. Stress should be placed on this rather than the development of ingenuity in integration techniques for its own sake.)

Limits. Limit of f(x) as x approaches a, where each deleted neighborhood of a contains a point of the domain of f. Right and left limits. Limits as x (or -x) increases without bound. Statement of basic theorems about limits with a proof of one or two of them and careful discussions of why all the theorems are reasonable and important. Continuity defined by the use of neighborhoods and then expressed in the language of limits. Continuity of sums, products, quotients, and composites of functions. Discussion of the Intermediate Value Theorem; zeros of a function; Newton's method. Discussion of the Maximum Value Theorem; proofs of Mean Value Theorems; l'Hôpital's rule. Integral as a limit; the Fundamental Theorem via the Mean Value Theorem; numerical integration and integrals.

Sequence and series. Definition of sequence as a function. Limits of sequences. Definition of series and sum of series. Relation to limit of sequence. Convergence of monotone bounded sequences and of series of nonnegative numbers with bounded partial sums. Comparison, ratio, and integral tests for convergence of series. Conditional convergence. Power series and radii of convergence. Taylor's theorem with remainder. Taylor's series.

Vectors. 3-dimensional vectors: addition, multiplication by scalars, length, and inner (dot, or scalar) product. Parametric equations of lines and planes established by vector methods. Inner products used to find the distance between points and planes (or points and lines in a plane). Vector-valued functions and the position vector of a curve. Parametric equations of curves. Differentiation of vector-valued functions. Tangents to curves. Velocity as a tangent vector. Length of a curve. Polar coordinates. Curves in polar coordinates. Area in polar coordinates. Radial and transverse components of velocity for polar coordinates in the plane, with application to determining the length of a curve whose equation is expressed using polar coordinates. Area of surface of revolution. If time permits: vector products, acceleration, curvature, and the angle between tangential and radial lines.

Differentiation. Partial derivatives. Differentials of functions of several variables. The chain rule for functions of several variables. Directional derivatives, gradients, and tangent planes. Theorem on change of order of partial differentiation. Implicit and inverse functions. Taylor's theorem and maxima and minima of functions of several variables.

Integration. Further use of simple integrals for computing such things as work, force due to fluid pressure, and volume by the shell and disk methods. Multiple integrals. Volume. Iterated integrals. Change of variables for polar, cylindrical, and spherical coordinates.

Differential equations.* Discussion of what an ordinary differential equation is; of ways in which ordinary differential

* Many current calculus texts include a chapter which provides a brief introduction to differential equations, and closely related topics are often interwoven through other chapters (e.g., simple growth and decay problems in connection with logarithmic and exponential functions). Such treatments indicate the time allowance, if not necessarily the spirit, of the treatment which we have in mind.

equations arise; and of what it means for a function to be a solution of a differential equation, with verification of solutions in specific cases. Use of tangent fields for equations of type $y' = f(x,y)$ to build more understanding of the meaning of a differential equation and of its solution curves. Integration of first-order equations with variables separable that arise in studying problems of growth and decay, fluid flow and diffusion, heat flow, etc. Integrating factors for first-order linear equations, with more applications. Second-order homogeneous equations with constant coefficients. Use of undetermined coefficients to solve the initial value problems of undamped and damped simple harmonic motion with forced vibrations. If time permits: more study of second-order nonhomogeneous equations; series solutions.

Mathematics L. Linear Algebra.

Discussion: There are perhaps three main reasons why a course in linear algebra should be offered by a two-year college that is able to do so. Such a course will contain the first introduction both to a kind of mathematical abstraction essential to the mathematical maturity of the student and, of course, also to serious mathematical ideas other than those of calculus. The subject is becoming more and more useful in the physical, biological, management, and social sciences. Finally, it provides the proper setting for a deeper understanding of the basic notions of analytic geometry that the student sees in Mathematics 0, A, B, and C.

In this program the linear algebra course is designed to be a course parallel to the last part of Mathematics C. Of course, vector ideas and methods should still be introduced as much as possible in the calculus sequence since they make possible a more meaningful development of that subject. Even though some basic vector ideas will be introduced in Mathematics C, the proposed linear algebra course can still be taken by capable students who have had only Mathematics B. To this end it begins with an introduction to geometric vectors, i.e., the course below has been designed in such a manner as to be accessible to a student who has not worked with vectors before.

In its 1965 publication A General Curriculum in Mathematics for Colleges, CUPM recommended that the student take linear algebra before the last term of calculus. This recommendation is gaining very widespread acceptance in the four-year college. However, the situation in two-year colleges is quite different; it is probably better there to allow the last part of calculus and the linear algebra to be taught in parallel, rather than to require one before the other. There is little question that most four-year colleges and universities would rather see the transfer student come in with the calculus completed than with some linear algebra at the price of having to complete the calculus. To finish calculus before transferring, the student may not have time to take linear algebra first, although he may well be able to take it simultaneously with the last part of calculus. Moreover, those transfer students who will major in the social or management sciences may take only one term of calculus but almost certainly will find linear algebra more useful than any other kind of mathematics. Here again, it is important that good students be able to take both the last part of calculus and the linear algebra.

The course outline below emphasizes a geometrical outlook and adheres closely to the guidelines for linear algebra suggested by the following quotation from A General Curriculum in Mathematics for Colleges (1965):

> When linear algebra is taught as early as the first term of the second year of college one should examine carefully the choice of topics. We need something intermediate between the simple matrix algebra which has been suggested for high schools and the more sophisticated Finite-Dimensional Vector Spaces of Halmos. The pieces of linear algebra which have demonstrated survival at this curriculum level for many years are: systems of linear equations and determinants from college algebra, uses of vectors in analytic geometry, and the calculus of inner products and vector cross products. More modern ideas call for the introduction of matrices as rectangular arrays with elementary row operations, Gaussian elimination, and matrix products; then abstract vector spaces, linear dependence, dimension, and linear transformations with matrices reappearing as their representations. Finally, of all subjects in undergraduate mathematics after elementary differential equations the one which has the widest usefulness in both science and mathematics is the circle of ideas in unitary geometry: orthogonality, orthogonal bases, orthogonal expansions, characteristic numbers, and characteristic vectors.

COURSE OUTLINE FOR MATHEMATICS L

Geometrical vectors. 2- and 3-dimensional vectors, sums, differences, scalar multiples, associative and commutative laws for these. Dot and cross products. Equations of lines in two and three dimensions and of planes, by vector methods. Normals and orthogonality, direction cosines. Components of vectors and vector operations in terms of them. Mention correspondence between plane (space) vectors at the origin and R^2 (R^3).

Matrices and linear equations. Matrices, row and column vectors defined as rectangular arrays and n-tuples of real numbers. Sums of matrices and vectors, products of matrices. Examples and applications to elementary economics, biology, analytic geometry. Linear equations written in matrix form. Gaussian elimination. Row echelon form of matrix. Existence and uniqueness theorems for solutions of homogeneous and nonhomogeneous systems of linear equations. Numerical examples for 2, 3, 4 variables.

Vector spaces. Need for unifying concept to subsume n-tuples and geometric vectors of the previous paragraphs. Definition of abstract vector space over the reals. Examples of vector spaces from the above, including the null space of a matrix (solution space of a system of linear equations), function spaces, spaces of solutions of linear differential equations, space of polynomials of degree $\leq n$, of all polynomials. Linear combinations of vectors. Geometric applications. Linear dependence and independence. Theorem: If $m \geq n$, then m linear combinations of n vectors are dependent. Definition of dimension and proof of uniqueness (finite-dimensional case only). Subspace. Geometric examples and function space examples. Dimension of subspaces: $\dim (S+T) = \dim S + \dim T - \dim (S \cap T)$, with appropriate definitions and examples.

Linear transformation. Definition of linear transformation. Matrices associated with linear transformations with respect to different bases. Sums and products of linear transformations defined. (Show how this leads naturally to definitions of sums and products of matrices given in the second paragraph above.) Range and kernel of linear transformation. dim (kernel) + dim (range) = dim (domain). Elementary matrices. Rank of linear transformation and matrix. Row rank = column rank. Linear equations re-examined from this point of view. Inverses of linear transformations and matrices. Calculation of inverse matrix by elimination. $PAQ = I_r$ if r = rank of A. Matrix of same linear transformation with respect to different bases. Similarity.

For the remainder of the course a choice may have to be made between the following two topics.

Determinants. The determinant as a function from matrices to the reals which is alternating and multilinear on the rows and columns, i.e., the determinant of a matrix is unchanged by adding a multiple of a row (column) to another row (column), the determinant of a diagonal matrix is the product of the diagonal entries, etc. (No attempt at proving existence in the general case.) Derivation of explicit formulas for 2×2 and 3×3 determinants. Calculation of several explicit $n \times n$ determinants for $n > 3$, for example, van der Monde 4×4 and 5×5 determinants. Expansion in terms of cofactors. If A, B are $n \times n$ matrices, then determinant $(AB) =$ determinant $A \cdot$ determinant B. (Last two topics with only indications of proof.) A very brief discussion of the formula $A \text{ adj } (A) = I \det (A)$. Cramer's rule.

Inner product spaces. Definition. Orthogonal and orthonormal bases. Gram-Schmidt process. Distances. Orthogonal complements. Orthogonal expansions. Examples in function and polynomial spaces where inner product is defined via an integral. Applications to 3-dimensional analytic geometry.

3. Probability and Statistics: Mathematics PS.

The discussion and list of topics given here represents tentative recommendations arrived at jointly by some members of the CUPM Panels on Statistics and Two-Year Colleges, in order to elaborate the latter Panel's recommendation that a course of this kind be included in the basic offerings.

Historically, a proliferation of introductory statistics courses, differing not only in content but also in emphasis and method of approach, has developed. Some of the factors responsible for this are the desire by different disciplines to have courses tailored to their needs, the varied backgrounds of persons teaching statistics, and differences in objectives for the basic statistics courses. The Panel on Statistics of CUPM is undertaking a full study of the difficult problem of developing models for such courses. The study will look to the future. It will investigate the role of computers, teaching devices, and laboratories. It will consider special approaches, such as those stressing decision theory, nonparametric methods, Bayesian analysis, data analysis, and others. A major aspect of the study will be consultation with representatives of the various fields whose students are served by courses of this kind.

The study may well indicate that no single statistics course is best suited to fill the variety of existing and anticipated needs; the Panel may eventually recommend a choice from among several different types of statistics courses (hopefully providing at the same time the impetus for the appearance of any new texts they may require).

The projected recommendations will, of course, supersede the suggestions given here in that they will be more explicit and will be based on a more extensive study of the problem.*

Discussion: An important element in the education of transfer students in many fields is a one-term introductory course in statistics, including an adequate treatment of the needed topics from probability. Such a course is often required of students majoring in particular fields: to cite a few examples, business administration, psychology, sociology, forestry, industrial engineering. In addition, it serves as an excellent elective subject for other students.

A course of this kind can be intellectually significant although based on minimal mathematical preparation. In what follows a prerequisite of two years of high school algebra is assumed.

The main function of a first course in statistics which will be terminal for many students and, for others, will provide a basis for studying specialized methods within their majors, is to introduce them to variability and uncertainty and to some common applications of statistical methods, that is, methods of drawing inferences and making decisions from observed data. It is also desirable, as a second objective, for the students to learn many of the most common formulas, terms, and methods.

Faculties of some departments served by this course stress the latter objective and, in addition, wish their students to be knowledgeable in certain specialized statistical techniques. We suggest that it is preferable for the students to learn basic concepts in the first course. (If a specialized technique is needed, it can be taught effectively in the subject matter course in the context in which it is used, at a small cost in time.)

It is quite plausible that an imaginative presentation which develops only a minority of the most common topics could be very effective in providing the students with an intelligent and flexible approach to statistical problems and methods through the development of basic concepts. Therefore, we regard the second objective of the course as secondary in importance.

Since the main objective of the course is understanding of the basic statistical concepts, proofs and extensive manipulations of

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* The report mentioned in these paragraphs appeared in 1972 and is entitled Introductory Statistics Without Calculus. See page 472.

formulas should be employed sparingly. While statistics utilizes these, its major focus is on inference from data. By the same token, the course should not dwell upon computational techniques. The amount of computation should be determined by how much it helps the student to understand the principles involved.

We urge the use of a variety of realistic problems and examples, to help motivate the student and to illustrate the approach to such problems by statistical methods. (The use of a laboratory in conjunction with the courses offers attractive possibilities for these purposes.) Examples drawn from disciplines of common interest to the class members should be sought. Also, the generation of data within the class (e.g., heights, birth dates) can be very useful.

Despite the diversity of basic courses, there has been a common set of topics included in most of them. The list below corresponds in an approximate fashion to this common set. A number of topics have been omitted from the list--some because we feel they would detract from more important topics: combinatorial aspects of probability, skewness, kurtosis, geometric mean, index numbers, and time series, for example. (These need not be avoided if they serve as pedagogical devices in attaining the major objectives of the course.) Other topics have been omitted because their incorporation would make it difficult to cover enough of the topics on the list. These include Bayes' theorem, Bayesian analysis, experimental design, sequential analysis, decision theory, and data analysis (insofar as this refers to an as yet emerging body of knowledge devoted to techniques for detecting regularities from data). These are potentially excellent topics; some of the possibilities for innovation in the basic statistics course are either (1) to stress some of these at the expense of omitting other (perhaps many) topics included in the list, or (2) to make one of these topics the basic focus for the entire course and discuss other topics in the framework thus established.

Finally, we urge that this course be a sound intellectual experience for the students. It should not be just a compendium of terms and techniques.

LIST OF TOPICS FOR MATHEMATICS PS

Note: This is not a course outline. It is not intended that this list of topics be necessarily covered in the order presented or in its entirety; an extremely skillful presentation would be required for a great majority of them to be combined satisfactorily in one course.

1. Introduction. Data: collection, analysis, inference. Examples of some uses of statistics emphasizing data collection, summarization, inference and decisions from data.

2. Probability: sample space, probability of an event, mutually exclusive events, independent and dependent events, conditional probability. (Keep enumeration to combinatorial notation at most. Use tables.) Random variable, expected value, mean and variance of a random variable.

3. Discrete random variable: binomial, use of binomial tables, Poisson (optional).

4. Summarization of data: grouping data. Central tendency: arithmetic mean, median, mode. Dispersion: range, standard deviation (others optional). Graphical analysis: histogram, frequency polygon, cumulative frequency polygon, percentiles.

5. Continuous random variables and distributions. Normal distributions. Use of tables. Normal approximation to binomial.

6. Sampling theory: random sampling, Central Limit Theorem, normal approximation to distribution of $\bar{x}$.

7. Point estimation: estimation of mean of the normal and mean of the binomial.

8. Interval estimation: confidence intervals on the mean of the binomial, confidence intervals on the mean of the normal using the t distribution. Large-sample approximate confidence intervals.

9. Hypothesis testing. Null hypothesis, alternate hypothesis, Type I and Type II errors, power of the test. Test of hypothesis on the mean of the normal using the t test. Test on the mean of the binomial. Large-sample approximate tests.

10. χ^2 tests: goodness of fit, contingency tables.

11. Comparison of two population means. Paired and unpaired cases using the t test (nonparametric or range tests may be substituted).

12. Nonparametric methods. Sign test, Mann-Whitney test, Wilcoxon test. Confidence interval on median.

13. Regression and correlation. Least squares, correlation coefficient. Estimation and tests.

14. One-way analysis of variance. F test, simultaneous confidence intervals.

III. TEACHER TRAINING AT TWO-YEAR COLLEGES

One of the major concerns of CUPM since its inception has been the proper mathematical training of prospective teachers of school mathematics. Although much progress has been made over the years on this serious problem, a great deal remains to be done, and the two-year colleges will play an increasingly important part in these efforts. For this reason we have included in this report a statement from the CUPM Panel on Teacher Training, together with some excerpts from their current recommendations.

[Editor's note: The original report contained a statement from CUPM's Panel on Teacher Training and a course outline for Mathematics NS (The Structure of the Number System). Rather than reprint these, we refer the reader to the latest report of the Panel on Teacher Training, Recommendations on Course Content for the Training of Teachers of Mathematics (1971), and to the course outlines contained therein. See page 177.]

IV. OPTIONAL ADDITIONAL OFFERINGS

Many two-year colleges offer courses for transfer students other than a basic set of courses such as those described in Chapters 2 and 3. With the growing trend for well-qualified and highly motivated students to attend two-year colleges, the demand for such additional courses will increase. The opportunity to teach such courses will certainly be attractive to faculty members and will offer them stimulating challenges. In this chapter we offer some suggestions for the kind of additional courses which we think deserve prime consideration, including sample course outlines and discussions of the circumstances under which the course is deemed desirable.

1. Finite Mathematics: Mathematics FM.

This course is designed primarily for students interested in business, management, social, and biological sciences. Its purpose is to introduce a variety of mathematical topics, showing how these are related to problems in the areas mentioned above. It would be a valuable component of the curriculum only if a substantial amount of time, at least 20% of the course, were devoted to applications of the material developed. Simply taking up mathematical topics of potential relevance is not enough; indeed, unless it includes a substantial unit on applications it would be better not to offer the course.

In finite mathematics the student is exposed to mathematical topics of a different nature from those he has previously studied--topics that apply to new types of problems and topics that are perhaps more closely related to his special interests. It is expected that some students, even if a small proportion, will be attracted to further study of mathematics by this course, either because they find these new topics interesting in themselves or because they recognize the need to learn more for further applications.

The course described, while not requiring all of the topics in Mathematics 0, can profitably build on the mathematical sophistication and maturity of that level of mathematical education in order to develop the ideas needed for the applications. A course certainly can be designed for students who have completed less, although it may then be necessary to have the course meet an extra hour per week in order to have enough time at the end of the course for the desired substantial unit on applications. A course assuming Mathematics PS could proceed somewhat further with applications, either in greater depth or in greater variety. However, we believe finite mathematics will serve a greater need and a larger audience if statistics is not a prerequisite.

The topics suggested are chosen because of their applicability; they include finite sets, probability, and linear algebra. The elements of logic may be, but need not be, treated as a separate topic. All that is needed is an ability to analyze and interpret compound statements formed by the usual connectives "and," "or," "not," "if and only if," "if ... then ...," and familiarity with what is meant by a logical argument. These ideas can be woven into the course. We prefer the latter procedure since it will help to insure time for applications.

Although it is not indicated in the outline, an attractive way of beginning is to present one or two applied problems that will be solved by the end of the course (e.g., one of the several Markov chain examples). As the course progresses and partial solutions are obtainable, the student should be apprised of the progress being made toward the ultimate solution to be given at the end of the course.

COURSE OUTLINE FOR MATHEMATICS FM

Sets. Subsets and set inclusion. The algebra of sets: union, intersection, complementation, difference. The relation with logic of compound sentences constructed from the connectives "and," "or," "not," "if and only if," "if ... then ... ." Tree diagrams as a systematic way of analyzing a set of logical possibilities. The number of subsets of a set. Cardinality of finite sets; card $(A \cup B)$ = card A + card B - card $(A \cap B)$. Partitions, binomial and multinomial theorems and related counting problems.

Probability. The concept of a probability measure: the axioms, and the realization of these axioms for finite sets. (The intuitive basis for this model discussed and illustrated by several examples.) The Law of Large Numbers. Equiprobable measure. Probabilistic independence. Repeated occurrences of the same event and binomial distribution. Approximation of the binomial distribution by the normal distribution as a means of making predictions about the likelihood of outcomes of repeated independent trials. (If time permits, also discuss Poisson distribution as approximation of the binomial.) Introduction to Markov chains (particularly if the course is begun by a discussion of a problem involving Markov chains; otherwise this topic may be delayed).

Linear algebra. Vectors and matrices introduced as arrays of numbers. Addition of vectors and matrices. Products (motivated by such examples as: given a vector of prices of certain commodities and a matrix giving the quantity purchased on each of a set of days, find the total expense on each day). Solution of linear equations, including the cases where the number of equations and number of indeterminates differ, and including the cases of multiple solution and no solution. The definition of the inverse of a matrix. Its computation without using determinants. Applications to Markov chains; classification of chains, with the emphasis on absorbing chains and regular chains. Fixed vectors and eigenvectors. Examples such as: for regular chains, analyze a problem in social mobility, probability of change of job classification, change of residence, change in brand preference among choices for a particular article, probability of change of political party allegiance, genetic heredity; for absorbing chains, many of these same examples with a change in the probabilities, e.g., brand preferences, genetic heredity, as well as matching pennies, learning processes.

Applications. One of the largest classes of applications involves Markov chains, including the long-term development of proportions of dominant and recessive genes, learning models, models on the judgment of the lengths of lines, the spread of a rumor in a housing development, or an extension of any of the items mentioned above.

Another attractive example is linear programming, including: an introduction to convex sets; the fact that a linear function defined on a convex set assumes its maximum and minimum at a vertex; some simple examples that can easily be solved by hand, with indication of the magnitude of the task in case the convex set has many faces and vertices. If time permits: a discussion of means of solving such problems, including the simplex method.

Still another example is the theory of games. Here one can introduce the idea of a zero-sum two-person game, especially a 2 x 2 matrix game, the idea of pure and mixed strategies, pure strategy solutions both for strictly determined games and for those that are not strictly determined. Mixed strategies and the min-max theorem (proved for 2 x 2 matrix games). Examples: one of the simplified poker games; competition between two businesses.

2. Differential Equations and Advanced Calculus: Mathematics DE and DA

Many two-year colleges now offer courses in differential equations. This practice is expected to continue because of the large number of students specializing in fields for which some familiarity with differential equations is important. Some students who take such a course do not transfer to four-year colleges and some of those who do transfer plan to major in fields which do not require many upper-division courses in mathematics at the four-year college. A two-year college might well offer a course in differential equations as an elective for such students, with Mathematics C and L as prerequisites. However, such a course has limited value if it trains students only in formal techniques for special cases without developing understanding of the general nature and meaning of differential equations and their solutions.

We regard linear algebra as being a more basic and important tool than differential equations for most students. For students who will take only one of the courses we strongly recommend linear algebra. Since the concept of vector space is very important when discussing general linear equations, linear algebra should be a prerequisite for differential equations. If students do not have this preparation, the treatment of systems of differential equations will, of necessity, be quite minimal.

The content of a differential equations course should be planned to meet the needs and interests of the students. Therefore, we have outlined a course, Mathematics DE (Intermediate Differential Equations), that has considerable flexibility and can be varied to meet such needs and interests. Certain important topics have been listed first. Other topics are listed from which choices can be made, either as additional topics or as substitutions. For example, the study of existence and uniqueness theorems might be substituted for the study of systems of equations, or numerical and operator methods might be introduced throughout the course.

For some students, further study of calculus is more important than the study of differential equations. This is true for many students planning to take mathematics after transfer. For such students we recommend a course which develops some concepts of advanced calculus and also provides an extension of the study of differential equations begun in Mathematics C. This course, Mathematics DA (Differential Equations and Advanced Calculus), would follow Mathematics C and L. Recognizing varied student and faculty interests and time limitations, we describe the core material and some additional topics from which choices should be made as time permits.

The two courses whose outlines follow are designed to develop good understanding of the mathematics involved and to improve the mathematical maturity and ability of the students. These courses are suitable only for students who have gained from courses such as Mathematics C and L the ability to understand and to readily assimilate mathematics. In particular, it is assumed that the students have been given an introduction to differential equations such as that described in the outline of Mathematics C.

COURSE OUTLINE FOR MATHEMATICS DE

Review. First-order equations with variables separable; integrating factors for first-order equations. Second-order linear homogeneous equations with constant coefficients. The method of undetermined coefficients, with applications to undamped and damped simple harmonic motion with forced vibrations.

Linear differential equations. Superposition of solutions. Homogeneous equations of n^{th} order with constant coefficients; existence and uniqueness theorems for initial value problems. Linear dependence and independence; vector spaces of solutions. Relations between solutions of a nonhomogeneous equation and the corresponding homogeneous equation. Methods of undetermined coefficients and variation of parameters. Applications to initial value problems.

Series solutions. Discussion of term-by-term differentiation and integration of a power series within the interval of convergence; proof that the interval of convergence does not change, with at least heuristic justification of the procedure. Use of series for solving some first-order and other simple cases for which convergence can be verified easily and the procedure justified. Discussion of both the method of undetermined coefficients and the use of Taylor series for determining series of solutions. General theory of series solutions about regular points. Discussion of the indicial equation and of the nature of series solutions about regular singular points, with most results stated and only heuristic justification given. Applications to classical equations such as those of Bessel and Legendre. Some simple nonlinear equations, such as $y' = x^2 + xy^2$.

Systems of equations. Equivalence of general systems to systems of first-order equations. Vector representation of a system of first-order equations. Use of matrices to solve systems with constant coefficients. Description of the nature of the solution when the coefficient matrix can be reduced to diagonal form, e.g., when the eigenvalues are distinct or the matrix is symmetric; description and application of the Jordan canonical form. Discussion of applications to mechanical and electrical problems.

Additional or alternative topics chosen from among the following:

1. Numerical methods. Difference equations and interpolation. Runge-Kutta method. Elementary considerations of stability and error analysis.

2. General linear equations. Wronskians; linear dependence and independence; number of linearly independent solutions of an ordinary linear differential equation.

3. Existence and uniqueness theorems. Convergence of power series solutions. Existence and uniqueness proofs for first-order equations using Picard's method. Generalization to systems of first-order equations.

4. Operator methods. The operator D for linear equations with constant coefficients; factoring and inversion of operators; partial fraction techniques. The Laplace transform applied to linear differential equations with constant coefficients and to simultaneous linear first-order equations with constant coefficients.

5. Nonlinear differential equations. Special nonlinear equations which are reducible to linear equations; local stability; simple phase-plane geometry of trajectories. Self-sustained oscillations of

a nonlinear system. Forced oscillations of a nonlinear system (e.g., $x'' + k^2(x-cx^3) = A \cos ct$) and the corresponding resonance phenomenon.

COURSE OUTLINE FOR MATHEMATICS DA

Topology of the real line. Brief description of the construction of the real numbers from the rational numbers. Open and closed sets. Least upper bound property shown to imply Bolzano-Weierstrass and Heine-Borel properties.

Continuity. Preservation of connectedness and compactness by continuous maps; maximum value, mean value, and intermediate value theorems; continuity of inverses. Uniform continuity.

The above topics should be accompanied by numerous examples and by counterexamples showing the need for the hypotheses.

Linear differential equations. Extension of the methods developed for second-order linear equations in Mathematics C. Linear dependence and independence; number of linearly independent solutions of an ordinary linear differential equation; vector spaces of solutions. Relation between solutions of a nonhomogeneous equation and the corresponding homogeneous equation. Methods of undetermined coefficients and variation of parameters. Applications to initial value problems.

Convergence. Review of tests for convergence of series and sequences of constant terms. Algebraic operations with series and power series. Uniform convergence of sequences and series. Term-by-term integration and differentiation of sequences and series. Existence and uniqueness proofs for first-order differential equations using Picard's method.

Additional or alternative topics chosen from among the following:

1. Series solution of differential equations. Power series. Use of power series to obtain solutions of differential equations. General theory of series solution about regular points. Discussion of the indicial equation and of the nature of series solutions about regular singular points, with most results stated and only heuristic justification given. Applications to classical equations such as those of Bessel and Legendre.

2. Systems of differential equations. Equivalence of general systems to systems of first-order equations. Vector representation of a system of first-order equations. Use of matrices to solve systems with constant coefficients. Description of the nature of the solution when the coefficient matrix can be reduced to diagonal form, e.g., when the eigenvalues are distinct or the matrix is symmetric; description and application of the Jordan canonical form. Discussion of applications to mechanical and electrical problems.

3. Riemann integration. Area and integrals. Properties of definite integrals. Existence of integrals of continuous and monotone functions.

4. Transformations. Review of partial differentiation and of linear transformations and matrices. Jacobians. Inverse transformation and implicit function theorems. Change of variables in multiple integrals. Cylindrical and spherical coordinates.

3. Probability Theory: Mathematics PR.

The importance of probability theory makes it a prime candidate for inclusion early in the program of many transfer students. The usual treatment of this subject requires some background in calculus. Sufficient background for this course is provided by Mathematics B. The course itself will provide additional calculus material as the need arises.

This course should lay stress on problems and, in particular, on problems which provide motivation and develop interest in the conceptual aspects of probability theory. Problems of this kind can be found, for example, in Fifty Challenging Problems in Probability by F. R. Mosteller (Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1965).

The course outline below is an adaptation of Mathematics 2P [page 76]. The outline is brief. It is intended that the topics be treated in depth.

COURSE OUTLINE FOR MATHEMATICS PR

(a) Probability as a mathematical system. Sample spaces, events as subsets, probability axioms, simple theorems. Intuitively interesting examples. Special cases: finite sample spaces, equiprobable measure. Binomial coefficients and counting techniques applied to probability problems. Conditional probability, independent events, Bayes' formula.

(b) Random variables and their distributions. Random variables (discrete and continuous), density and distribution functions, special distributions (binomial, hypergeometric, Poisson, uniform, exponential, normal, etc.), mean and variance, Chebychev inequality, independent random variables.

(c) Limit theorems. Poisson and normal approximation to the binomial, Central Limit Theorem, Law of Large Numbers, some applications.

4. Numerical Analysis: Mathematics NA.

A two-year college should consider offering a course in numerical analysis provided it has

(a) a curriculum of courses including all the "Basic Offerings";

(b) someone on the staff qualified to teach the course;

(c) access to computing facilities with reasonably short turn-around time. A time-sharing system is ideally suited to this purpose, as is a calculator-computer that students can use "hands on."

The first two provisions above are not likely to be challenged. To fulfill the second it may be possible for some institutions to find a part-time teacher from local industry who is interested in the mathematical issues raised. As for computing facilities, while numerical analysis can be studied effectively without a digital computer, the actual computational processes involved can be tedious, and the course would be much less likely to attract students. If the only accessible computing equipment involves long delays between starting a program and getting the answers, the result is frustration and a discontinuity of interest in the problem being solved.

This course has several purposes. It is intended to capitalize on the interest students have in computers in order to spur their interest in mathematics and to kindle in them a desire to study further topics in mathematics, for in it they should find that turning a problem over to a computer eliminates some mathematical problems while raising others. In addition, it provides an introduction to the important techniques of numerical solution to a variety of problems and introduces ways in which digital computers can be helpful in problem solution.

It should be recognized that this is not a course in computer science, but rather it is a mathematics course. It contains material closely related to the curriculum of any computer science program, however, and hence is appropriate for those who wish to pursue a major in computer science at a four-year college.

The course begins with a discussion of some of the problems that will be treated. Most of these will be problems with which the student is already acquainted in some form: definite integrals that cannot be expressed in closed form, differential equations that he has not learned how to solve, as well as large systems of linear equations with coefficients containing more than two significant figures, and nonlinear equations in a single variable which in theory he may know how to solve, but which in practice are quite impossible. The process of obtaining numerical solutions to these problems will stimulate the study of additional topics in analysis, linear algebra, probability, and statistics. The student will become concerned with approximations, rapidity of convergence, and error analysis.

Selection of topics from the outline will depend on how much, if any, attention to numerical techniques appeared in the calculus course, and on the tastes and interests of the instructor. It should in any case include examples of interpolation, approximation, and solutions of systems of equations.

COURSE OUTLINE FOR MATHEMATICS NA

Prerequisite: At least one year of calculus, including infinite series.

Introduction. Some typical numerical problems; the theoretical aspects of numerical analysis such as convergence criteria and error estimates, versus the algorithmic aspects such as efficiency of an algorithm and error control. (Most of these topics can be nicely introduced through examples, e.g., the solution of quadratic equations or the problem of summing an infinite series.)

Interpolation and quadrature. The linear and quadratic case of polynomial interpolation; basic quadrature formulas; numerical differentiation and its attendant error problems. If time permits, discussion of the general Lagrange and Newton interpolation formulas, the Aitken method, and Romberg integration.

Solution of nonlinear equations. Bisection method, successive approximations, including simple convergence proofs, Newton's method, method of false position. Application to polynomial and other equations and to special interesting cases such as square and cube roots and reciprocals.

Linear systems of equations. The basic elimination step, Gauss and Gauss-Jordan elimination. Roundoff error, the ill-conditioning problem in the case $n = 2$. If time permits: brief introduction to matrix algebra and the inversion of matrices.

Solution of ordinary differential equations. Series solutions and their limitation. Euler's method, modified Euler's method, simplified Runge-Kutta.

V. IMPLEMENTATION

This final chapter is devoted to some comments on various aspects of the problem of implementing the proposals of the earlier chapters. It is never the intent of CUPM that its curriculum recommendations be "swallowed whole," but rather that they be used as a focus for the discussion of curricular problems on a national or regional scale and ultimately that they serve on the local level as a starting point for improvements tailored to local needs. It is in this sense that CUPM wants to see this program implemented. We point out below how it can be considered in small pieces; how it can be staffed economically; how problems of articulation with high schools, four-year colleges, and other two-year programs can be approached; and how such factors as the computer and the existing choice of textbooks may affect implementation.

We aim the following remarks at a generalized audience in full recognition of the fact that the actual readers represent diverse schools and that each will need to interpret them for his own needs.

1. Implementation by Stages.

CUPM does not, of course, recommend a total alteration of current practices, but rather an evolution in certain directions.

The curriculum seems to divide naturally into parts, reasonably independent from the viewpoint that changes in one part need not force major changes in the others. Thus the proposed structure can be approached in stages, beginning wherever a department recognizes the greatest need:

(1) The introduction of Mathematics 0 and (or) Mathematics A in place of current calculus-preparatory courses.

(2) The revision of the calculus courses along the lines of Mathematics B and C.

Remarks:

(a) If current practice includes highly integrated analytic geometry-calculus courses, these two stages are not quite independent. Nevertheless, the problem of proceeding with one of these stages and making appropriate adjustments in the other courses does not seem to be a major one.

(b) If stage 2 has been carried out successfully, the course Mathematics B should serve the students of nonphysical sciences who need an introduction to calculus. Thus, some need for special-purpose courses, such as business calculus, might be eliminated.

(3) Introduction of Mathematics L.

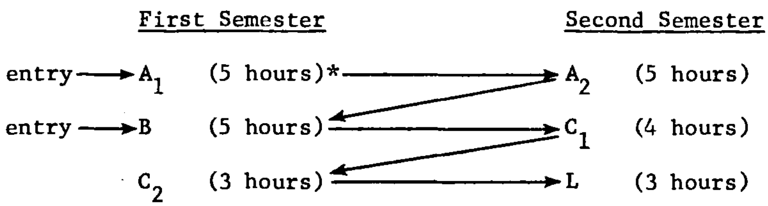
(4) Introduction of Mathematics PS.

Remark: Although courses of this kind are offered in most schools, frequently in departments other than mathematics, a well-designed course given in one department can help avoid duplication of effort within a school.

(5) Introduction or extension of the offerings for elementary school teachers.

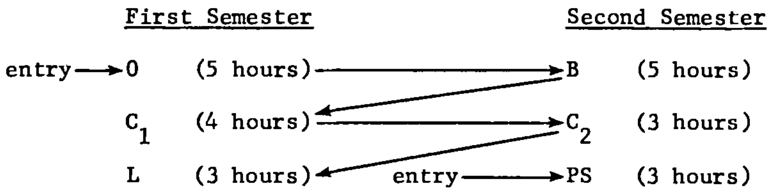
(6) Selection of appropriate additional courses.

We believe that none of these changes would create unusual demands in terms of the teaching staff they require. Indeed, that the entire basic program calls for only slightly more than the equivalent of two full-time staff members is indicated by the following sample patterns.



This provides the full science sequence with two points of entry and uses approximately the equivalent of one full-time instructor.

* See pages 224 and 225.



This offers the additional natural point of entry and Mathematics PS once each year, again using about one instructor. The addition of the course Mathematics NS (3 hours each semester) and perhaps the second year of teacher-training courses completes the basic offerings.

This simplified model provides considerable flexibility, perhaps even more than is needed by many of the smaller schools where the need to offer each course each semester may not be present. On the other hand, it does not take account of multiple-section courses, of the possibilities for more economy through such devices as a summer school entry to the science sequence, or of the additional flexibility that can perhaps be achieved with the quarter system. Nevertheless, it seems clear that departments working out more detailed models according to their own needs will find this program quite economical of manpower as compared to alternative curricula.

2. Articulation.

Articulation efforts have as their primary goal establishing a procedure of transfer that is as smooth and convenient as possible for the student, thereby enabling him to be cognizant of the required courses in his field and to plan carefully whatever program he is pursuing. Although articulation between two-year colleges and four-year colleges encompasses many areas such as staffing, curricula, administration, and personnel, the main obstacle to be surmounted is that of "communication." Detailed information must be exchanged between the two, and that information must then be disseminated to all concerned.

As indicated in "Guidelines for Improving Articulation Between Junior and Senior Colleges,"* there are five main aspects of the transfer from the two-year college to the four-year college. These are: administration; evaluation of transfer courses; curriculum planning; advising, counseling, and other student personnel services; articulation programs. We have kept these in mind in formulating the present recommendations.

Many states, however, have large, geographically dispersed systems of two-year colleges and four-year colleges in which a slow-down or lack of communication causes some of the items listed in the

* Published by the American Council on Education, 1785 Massachusetts Avenue, N.W., Washington, D. C. 20036

"Guidelines" to become problems. Florida and California, for example, are states having geographically dispersed systems which have experienced or foreseen such problems. They have organized statewide committees to discuss, study, and recommend solutions or guidelines for the solution of these problems.*

The Advisory Council on College Chemistry published a report in January, 1967, entitled "Problems in Two-Year College Chemistry, Supplement Number 20A." This report deals with curriculum and articulation problems in the area of chemistry that are in many ways identical to those in mathematics.

The counseling, guidance, and placement procedures employed by the two-year college serve to minimize the articulation difficulties both between the high school and two-year college and between the two- and four-year college. Complete information about the individual two-year college's mathematics program should be in the hands of local high school counseling staffs. In this manner the high school student, regardless of his background, can be informed as to where he should start his mathematics program in college. For example, a graduating student who suddenly realizes that he needs calculus for his major field, and who is not prepared, should be informed that the local two-year college offers courses such as Mathematics A and Mathematics 0 to correct the deficiency.

In general, the mathematics department is urged to exercise its responsibility for coordinating its work with high schools, general counselors, and transfer institutions. Two particular goals are to improve placement procedures and to establish sound criteria for transfer credit.

Articulation between institutions can cease to be a problem only when complete communication has been established.

3. <u>General Education and Technical-Occupational Programs</u>.

A further study of these topics will be a major part of the continuing work of the Panel. However, because of their great importance in two-year colleges a brief commentary on these subjects is included here.

<u>General Education</u>. A desirable way for a nonscience major to acquire some understanding and appreciation of mathematics may be to take a course or courses in general education mathematics. This is true if the course is carefully planned and taught by skilled and inspiring teachers. Above all else, the course should be so designed

* See Report of the California Liaison Committee, November, 1967, Sacramento, and A Report on Articulation in Mathematics (Florida), June, 1966, Tallahassee.

and taught that the students discuss and do mathematics rather than only hear about mathematics.

No definite outline for such a course is suggested at this time. Each teacher should judge the best type of course to suit his class. CUPM hopes to offer him more specific assistance in this task soon. [See A Course in Basic Mathematics for Colleges, page 314.]

Whatever is done should be meaningful relative to the student's experience and studies; it should be taught in a new light relative to his past experiences with the study of mathematics. In many ways college students are becoming more sophisticated, and hence a new approach is needed to reinforce old mathematical experiences and to develop new ones. It is desirable that the course achieve the minimum levels of understanding, skills, and reasoning needed by all students regardless of their goals or preparation, and that it provide some base for continuation in mathematics for those students who can and should continue. The course should emphasize basic mathematical concepts of recognized importance for an educated person and should be oriented, where possible, toward meaningful applications. The material should cut across the traditional segmentation of arithmetic, algebra, and geometry. The solution of "word problems" should be emphasized throughout.

Mathematics 0, A, PS, and FM as described above can be very effective general education courses for students having the necessary preparation.

Technical-Occupational. Even though the majority of students who enter two-year colleges do so with the intention of transferring, there still exists a large group who seek training and instruction that will prepare them for employment at the end of two years. In order to accommodate the latter group of students, along with the portion of the first group who change their objectives, most two-year colleges provide a wide array of technical, semi-professional, and occupational programs.

The need for mathematics in these career programs varies widely with the program. Some require no more than simple arithmetic while others demand a working knowledge of mathematics ranging from elementary algebra through the calculus and differential equations. We feel that the course Mathematics A, with examples drawn from technical areas, has great potential as a service course for many such programs. If a program requires a more specialized course, it should be taught by a mathematics instructor who is familiar with the curriculum's objectives. The content and sequence in which the topics are taught should be determined by the coordinated efforts of the mathematics and career curriculum faculty members. The "integrated" or "related" mathematics of vocational programs may be taught more effectively by the vocational instructor.

Whenever possible, instruction and practice in the use of the computer for problem solving should be given in engineering- and science-oriented two-year occupational programs and use of the computer in data processing should be taught in business- and health-oriented two-year occupational programs.

4. The Computer in Two-Year Colleges.

The two-year college must be prepared for, and articulate with, the changes taking place in both secondary schools and four-year colleges in the area of computing and in the use of computers in mathematics courses. Computers are rapidly being introduced into secondary schools both physically and as a part of the curriculum. Courses of three types are already being taught.

1) The first type of course serves as an introduction to and emphasizes "computer science" as a distinct discipline incorporating the many facets of modern computers. It is exemplified by the School Mathematics Study Group's course Algorithms, Computing, and Mathematics.

2) The second type of course emphasizes instruction in a computer language. It includes an exposition of the essentials of a programming language and many examples of programs for the solution of various categories of mathematical problems.

3) The third type of course emphasizes the integration of computing into the mathematics curriculum. It uses the computer as a part of the instructional process in selected subject matter areas within mathematics.

To indicate the trend further, we note that the School Mathematics Study Group is preparing a new curriculum intended to go from seventh through twelfth grades which emphasizes the use of algorithmic ideas and computation. Flowcharting, for example, is introduced in the seventh grade. Many colleges have introduced computing into their elementary mathematics courses--calculus, in particular.

The reasons for this far-reaching change appear to be pedagogical, substantive, and social.

Pedagogically, the computer is a mathematical instrument and a laboratory. Computing facilitates, extends, and enriches learning in mathematics; mathematics extends and facilitates and is essential to learning in computing. The mathematics class is the prime place in the two-year college for learning computing. Computing in the mathematics class can aid understanding, help to build concepts, and develop mathematical intuition.

Substantively, the student's education is broadened to include an introduction to computer science and numerical analysis.

Socially, the computer is becoming an essential part of many aspects of the student's life and work and is an excellent means of demonstrating the relevance of mathematics to the needs of society and the individual. In addition, training in the use of the computer is likely to provide the two-year college student with a skill which will open the doors to a variety of employment opportunities otherwise unavailable to him.

Every transfer student, with the possible exception of humanities majors, should at some time in his college career take a course in computer science such as the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics [page 563]. We make this recommendation both because of the direct importance of the computer in many fields of knowledge and because we feel the students should be aware of the capabilities and limitations of the computer.

The two-year college should be prepared to introduce computing into mathematics courses wherever relevant and appropriate. All of our recommended courses can make good use of flowcharts, algorithms, and computer-programmed assignments.

Support for these recommendations can be found in the studies that have been and are being made of the role of the computer in higher education.

5. Implementation of Individual Courses.

(a) Applications.

At all times in teaching these courses, every effort should be made to illustrate and motivate the material in them by showing how it is used in a large variety of applications. Furthermore, the basic orientation should be towards the solution of concrete problems; at all costs the deadly dry "definition, theorem, proof" approach should be avoided. Of course, there should be carefully stated definitions, theorems, and some proofs, especially in Mathematics C and L as well as in the postcalculus courses, but, especially in the earlier courses, PS, O, A, B, and FM, the problem-solving approach should predominate and new concepts should only be introduced after adequate motivation and examples.

One of the main aims of the teacher should be to get as active student participation in these courses as possible, by involving students in class discussions and a great deal of homework, which, of course, must be adequately criticized and discussed. One of the best ways to awaken and stimulate students' interest is constantly to demonstrate the many applications to a large variety of areas of knowledge that their course material has. After all, a great deal of the mathematics in this program originated in attempts at solving very definite real-world problems.

Courses 0, A, and FM offer many opportunities to include applications to elementary probability and combinatorics. In the latter course some mention of questions answered by linear programming techniques can also be made. In the calculus courses the usual applications should, of course, be done but, at all times, the teacher must be on the lookout for unusual applications as, for example, those given in the last part of Calculus in the First Three Dimensions by Sherman K. Stein (New York, McGraw-Hill Book Company, 1967). The technical literature is full of applications of the calculus to subjects such as biology and the social sciences and problems such as traffic flow, pollution, etc. Every calculus teacher has an obligation to try to learn something about them. Mathematics L also has many applications in these areas and the same general remarks apply.

(b) Rigor.

Statements (1) and (2), beginning on page 41 of Commentary on A General Curriculum in Mathematics for Colleges, reflect our views.

(c) Textbooks.

Obviously the textbooks for all the courses will have to be chosen with a great deal of care. The teacher should make a continuing effort to keep abreast of new books as they appear. Moreover, while teaching any given course from a fixed book, the teacher should be consulting several other books for additional problems, extra illustrative material, and different slants on presentation. Clearly, local conditions will often play a determining role in the choice of books, but at all times the instructional staff should be aware of other books existing for the same subjects and should make good use of these additional sources.

The CUPM Basic Library List and Basic Library List for Two-Year Colleges should prove particularly helpful in making choices of texts and reference works for the courses described in this report. Moreover, the book review sections of the American Mathematical Monthly, Mathematical Gazette, Mathematics Magazine, Mathematics Teacher, School Science and Mathematics, and SIAM Review often have helpful comments about new books and should be consulted regularly.

A COURSE IN BASIC MATHEMATICS FOR COLLEGES

A Report of

The Panel on Basic Mathematics

January 1971

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I. INTRODUCTION

1. Background

There is a sizable student population in both two-year and four-year colleges taking courses with titles such as college arithmetic, elementary algebra, intermediate algebra, and introduction to college mathematics. In fact, according to a report issued by the Conference Board of the Mathematical Sciences, approximately 200,000 of the 1,068,000 students enrolled in mathematics courses in four-year institutions in the fall of 1965 were taking courses of this level. In 1966 approximately 150,000 out of 348,000 two-year college mathematics students were enrolled in such courses.* A large portion of these students do not take mathematics courses beyond this level or at best go on to courses of the college algebra and trigonometry type, after which their college-level mathematics training ends. These statements were supported by a survey of a representative sample of two-year colleges conducted by members of CUPM's Panel on Mathematics in Two-Year Colleges in the fall of 1969.

In January, 1970, CUPM formed a Panel on Basic Mathematics to consider the curricular problems involved in mathematics courses of this level. This Panel contained members of the Two-Year College Panel, which had already completed a great deal of preliminary work in this area, as well as additional representatives from two- and four-year colleges and universities. This Panel was charged with making curricular recommendations for the student population described above, whether enrolled in two- or four-year colleges. We will refer to the broad general area of courses below the level of college algebra and trigonometry as basic mathematics.

Many of the students in basic mathematics courses have seen this subject matter in elementary and high school without apparent success in learning it there. It is often the case that a second exposure to essentially the same material, similarly organized, is no more successful even though an attempt is sometimes made to present the subject matter in a more "modern" manner.

There are certainly many complex reasons for this state of affairs. Some of these may be psychological and sociological and may require the work of learning theorists and others trained in the

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* A more recent report of the Conference Board of the Mathematical Sciences (Report of the Survey Committee, Volume IV. Undergraduate Education in the Mathematical Sciences, 1970-71) indicates that approximately 193,000 of the 1,386,000 students enrolled in mathematics courses in four-year institutions in the fall of 1970 were taking courses at this level. In 1970 approximately 272,000 out of 584,000 two-year college mathematics students were enrolled in such courses.

social sciences in order to lessen their influence. However, it is our belief that the type of student described can also be greatly helped by reform in the mathematics curriculum. Evidently there are many ways of doing this; perhaps the most direct would be to allow these students to gain sufficient mastery of arithmetic in a mathematics laboratory by means of various learning aids and only then permit them to begin the elementary-intermediate-college algebra route. We believe that the curriculum described in what follows is another path that promises to be of greater interest and use to the present generation of college students. Of course there are many possible ways of attacking these problems, and we would not want our solution to be interpreted as the only possible one. Nonetheless, we believe that it deserves very serious consideration by the mathematical community and hope that many different kinds of institutions will find our suggestions, wholly or in part, of good use when dealing with the type of students described.

2. The Present Recommendations--Mathematics E

We propose the replacement of some of the currently existing basic mathematics courses by a single flexible one-year course, A Course in Basic Mathematics for Colleges (hereafter referred to as Mathematics E), together with an accompanying mathematics laboratory. The main aim of this course will be to provide the students with enough mathematical literacy for adequate participation in the daily life of our present society.

The adjunct laboratory, which we describe in detail in Section V, will serve to remedy the deficiencies in arithmetic that so many of these students possess. Moreover, the laboratory will offer opportunity for added drill in algebraic manipulation and for instruction in vocationally oriented topics of interest to particular groups of students. This learning should be tailored to each student's individual goals and needs but should not ordinarily require constant supervision by an instructor. It must also be carefully integrated with the material in the course proper.

As a subsidiary aim of the course we hope that the student will gain enough competence in algebraic manipulation, the translation of statements into algebraic formulae, and the careful use of language, to allow him to continue, if he wishes, with courses such as Mathematics A [page 216], intermediate or college algebra, or various vocational mathematics courses, including courses in computing. Thus the proposed course will include material on simple algebraic formulae, handling simple algebraic expressions, the distributive property (common factor), setting up and solving linear equations in one or two variables, the beginning of graphing, and the rudiments of plane geometry. However, we emphasize that the main thrust is to provide the basic literacy spoken of earlier; if there are pressures of time or lack of student interest, then topics should simply be omitted.

This course has been developed to meet a set of circumstances different from those which prevail in the secondary schools. Not only is there the increased maturity due to the higher age level of the students (many of whom may have been out of school for a number of years), but there is the significant fact that the students in question have elected to enter college. Thus, a new approach appropriate to these conditions is needed.

One device for meeting this need is the introduction of flowcharting and of algorithmic and computer-related ideas at the beginning of the course. These ideas permeate the course, encouraging the student to be precise in dealing with both arithmetic and nonarithmetic operations. However, the presentation of computer-related ideas in the course will not depend on the availability of actual machines. Topics of everyday concern, such as how bills are prepared by a computer, calculation of interest in installment buying, quick estimation, analyses of statistics appearing in the press, and various job-related algebraic and geometric problems, are mainstays of the syllabus.

We have tried to make the proposed course coherent; that is, after a topic is introduced, it should be used in other parts of the course and not left dangling. The students must be actively involved throughout the course and should be encouraged to formulate problems on their own, based on their experience. Full advantage should be taken of the playful impulses of the human mind; interesting tricks and seemingly magic ways of solving problems are to be exploited. This point is more fully discussed on page 278.

It should be quite obvious from the foregoing that spirit is more important than content in the proposed course. In order to make as clear as possible the exact intent of the Panel's recommendations, we have included below a detailed discussion of the relationship of the proposed course to certain other courses widely offered at this level, a topical outline of a preferred version of the course, and an extensive commentary on how these topics can achieve our objectives. In addition, we give in Section V a description of the laboratory we propose as a means of dealing with remedial problems and meeting individual goals. In Appendix I there is a brief compilation of problems representative of those that might embody the philosophy of our program; Appendix II contains exercises illustrating the use of flowcharts.

Because the needs of students in the various institutions across the country vary, the Panel believes that a number of different adaptations of the proposed course may be developed. Thus, our topical outline should not be construed as a rigid description of a single course but rather as a flexible model.

II. THE BASIC MATHEMATICS CURRICULUM

There is a large number of well-populated courses in that part of the mathematics curriculum which precedes college algebra and trigonometry. One large group of such courses consists of different versions or repackagings of subjects for special groups of students, especially those in various occupational programs. In addition, there are what might be termed the low-level liberal arts courses, perhaps designed to satisfy a mathematics requirement.

The Panel has gathered anecdotal evidence concerning the reasons students are enrolled in such courses:

1. Some students, despite poor preparation, are ambitious to proceed to more advanced courses in mathematics. We observe that only a small minority of such students do in fact proceed as far as a course in calculus. As described below and in greater detail in Section V, our course would make this possible for the minority of students who will actually go on, even though it may not be as efficient as the traditional route.

2. Many students are advised or required to take certain existing courses in mathematics in the hope that such courses will give the students in some peripheral fashion the kind of mathematical literacy that is the central purpose of our proposed course in basic mathematics.

3. Many mathematics requirements are made in the hope that the mathematics courses prescribed will contain quite specific techniques of value in a student's proposed area of specialization or vocation. Through use of the laboratory, students can be taught such specific material almost on an individual basis.

4. Frequently, there is an institution-wide requirement of one or more courses in mathematics intended primarily for cultural or general education purposes. The basic mathematics course we propose is to be broad and relevant to the actual concerns of students and, therefore, could perhaps serve as a genuine liberal arts course for students of this level of mathematical maturity better than do most of the courses currently taught for that purpose.

Therefore, we believe that the course we propose will meet the objectives listed above better than current courses in the basic mathematics curriculum. Our aim is not to add a new course to the profusion of courses already existing. We wish, rather, to replace several of them by a flexible, more relevant course which will come closer to meeting the genuine requirements of this kind of student. In order to clarify this point, we examine in somewhat greater detail the relationship between the proposed course and certain widely offered courses.

1. Arithmetic. We feel that the courses customarily offered in colleges under this title are subsumed by the course we propose.

2. Elementary Algebra. A student finishing our course should be able to acquire (considering both his work in the classroom and in the laboratory) the computational and technical facility expected of a student finishing a course in elementary algebra. In addition, he should be able to make use of this knowledge in various concrete circumstances. Therefore, it would appear that the introduction of the course we propose could quite properly result in the elimination of courses in arithmetic and elementary algebra.

3. General Education Courses. It is necessary to distinguish between two quite different levels of general education or liberal arts courses in mathematics. The higher-level course may frequently contain material of considerable complexity and depth and might be thought of as an alternative to calculus (or at least to college algebra and trigonometry) for students majoring in the humanities or social sciences. Such courses, because of the relatively high level, are not properly part of the basic mathematics curriculum as defined above and, therefore, lie outside the scope of this report.

The second or lower-level courses, although motivated by much the same philosophy, may be thought of as forming a part of a program of general education for a broader category of students not necessarily restricted to those majoring in the liberal arts. Such courses are sometimes used as a substitute for remedial courses. A general education course is usually intended to develop an interest in and appreciation of mathematics, beginning with the concept of mathematics as an art or as a discipline and working gradually outward to broader issues.

Mathematics E (although it has its remedial aspects) is not primarily a remedial course. From the standpoint of a general education program the proposed course is a broad one; it can be termed the mathematics of human affairs, and as such should be a reasonable alternative to the usual general education mathematics course. Moreover, the prospective students for Mathematics E are likely to be of a pragmatic turn of mind. For them an appreciation of mathematics seems likely to stem from seeing how mathematical ideas illuminate areas in which they have an established interest.

4. Courses for Students in Occupational Curricula. It must not be presumed that all courses designed specifically for students in occupational programs are at a low level. For example, students in physical science related curricula such as Engineering Technology will generally begin their college mathematical training at a higher level and often continue through the calculus. In addition, there are many occupational curricula which by their nature must contain mathematics courses taught in extremely close relationship to the major program. For these categories of students the proposed course, Mathematics E, will not serve. However, there remain large numbers of students in occupational curricula whose mathematical needs are less specialized.

These include students in some programs in business and health professions as well as students in other occupational programs not containing a strong element of scientific training. These programs very likely encompass a large majority of students in occupational curricula. We believe that the principal mathematical need for such students is basic mathematical literacy together with some work in the laboratory directed toward their special needs.

5. Mathematics for Prospective Elementary School Teachers. These students have very special and pressing needs to which our course does not address itself. However, Mathematics E might be needed by some as preparation for the teacher-training courses recommended by CUPM. [See Recommendations on Course Content for the Training of Teachers of Mathematics, page 158.]

6. Mathematics A. This course is described in A Transfer Curriculum in Mathematics for Two-Year Colleges, page 205. Mathematics A is, briefly stated, an improved and extended version of college algebra, trigonometry, and analytic geometry interwoven with certain remedial topics. The Panel believes that Mathematics A should prove as viable in some four-year institutions as in the two-year colleges for which it was originally designed. Mathematics A is the most natural continuation of Mathematics E for the minority of students who continue with further courses in mathematics.

7. Intermediate Algebra. We take intermediate algebra to mean a semester course containing a rather systematic and extensive review of topics normally encountered in some form in elementary algebra, followed by new material on such topics as exponents and radicals, functions and graphs, quadratic equations, systems of equations, and inequalities, with selected topics from among complex numbers, logarithms, permutations and combinations, and progressions. The emphasis is on technical algebraic proficiency with occasional digressions into the applications of specific techniques.

Many of the students for whom the Panel would prescribe the year course in Mathematics E now take a year sequence composed of elementary and intermediate algebra. Mathematics E contains much nonalgebraic material and touches certain broader issues which the corresponding algebra sequence does not cover; Mathematics E, however, cannot be expected to provide as great a degree of technical algebraic proficiency as the conventional algebra sequence.

We suggest that what is appropriate here is a serious and realistic appraisal by the mathematics department at a given institution as to which sequence is more directly related to the real reasons why students take or are required to take courses at this level.

Intermediate algebra is not a more advanced course than Mathematics E, but rather one with different goals. The Panel feels that Mathematics A rather than intermediate algebra is the more natural continuation of Mathematics E. Thus, a school offering Mathematics E and Mathematics A might have no further need for intermediate algebra courses unless it has a significant number of students who (a) have an adequate competence in elementary algebra <u>and</u> (b) need a mastery of specialized algebraic techniques as opposed to a more generalized mathematical competence <u>and</u> (c) are not prepared for Mathematics A.

Finally, the introduction of Mathematics E, depending upon local circumstances, will make possible a considerable economy and simplification in the basic mathematics curriculum, an economy which should have special appeal to small colleges or to colleges having only a small number of poorly prepared students.

III. OUTLINE OF MATHEMATICS E

In this section we present an outline of one sequence of topics which the Panel feels is appropriate for use in implementing the purposes of the course. It should be re-emphasized that coverage of these topics is in itself neither necessary nor sufficient for the course to fulfill the spirit of the Panel's recommendations. One purpose in the presentation of the outline in detail is to demonstrate the existence of at least one sequence of topics which have obvious relevance to the interests and needs of present-day students and yet can be mastered by the students for whom the course is designed.

Although we feel we are recommending something more than coverage of a list of topics, the Panel has given extensive consideration to the question of which topics would best serve our purposes. We feel that others who give equally serious consideration to these questions will arrive at a sequence of topics in substantial agreement with ours.

Although the details of how much time should be allotted to each specific topic can only be determined by actually teaching the course, the Panel has in mind that approximately 50 per cent of the year's work would be devoted to Parts 1 through 4, approximately 25 per cent to Parts 5, 6, and 7, and the remaining 25 per cent to Parts 8 and 9.

The page references in the Outline are to the Commentary which follows.

Part 1 Flowcharts and Elementary Operations

There are two reasons for beginning this course with an introduction to computing. First, there is the rather obvious matter of initial motivation of the students. Second, we wish to introduce early the idea of a flowchart, which will permeate the entire course. (See pages 271-277.)

1-a Brief introduction to the nature and structure of digital computers. Specimens of computer programs and computer output, but no real programming until Part 5. Flowcharting as a preliminary device for communicating with the computer.

1-b <u>Flowcharts</u>. Further illustration of flowcharts by non-mathematical examples including loops and branches. Sequencing everyday processes.

1-c <u>Addition and multiplication of whole numbers</u>. Addition and multiplication as binary operations. The commutativity and associativity properties illustrated by everyday examples. Multiplication as repeated addition, illustrated by examples. Drill in these operations. Flowcharts for these operations, notion of variable, equality and order symbols. Introduction of the number line as an aid in illustrating the above and to provide for the introduction of the coordinate plane.

1-d <u>The distributive property and base 10 enumeration</u>. Distributive law done very intuitively and informally by examples on 2- or 3-digit numbers in expanded form. (See page 276.) Illustrate these two topics by means of simple multiplication.

1-e <u>Orders of magnitude and very simple approximations</u>. Relate order of magnitude to powers of 10. Motivate approximations to sums and products by means of simple examples. Lower and upper bound for approximations, no percentage errors. Introduction of the symbol $\approx$. (See pages 281-283.)

1-f <u>Subtraction of whole numbers</u>. Three equivalent statements:

$$a + b = c; \quad a = c - b; \quad b = c - a$$

Commutativity and associativity fail for subtraction, operation not always possible. Multiplication distributes over subtraction. Approximations as in 1-e. Drill.

1-g <u>Exact division of whole numbers</u>. Three equivalent statements for a and b not zero:

$$ab = c; \quad b = c/a; \quad a = c/b$$

Division is not always possible, division is noncommutative and nonassociative. Flowcharting. Computational practice.

1-h <u>Division with remainder</u>. Informal discussion of division with remainder. Flowchart process as handled by a computer. Approximations as in 1-e.

1-i <u>English to mathematics</u>. Translations of English sentences taken from real-life situations into algebraic symbolism. (See pages 277-281.)

Part 2 Rational Numbers

It is intended that this part shall have an informal and pragmatic flavor. Even though some references are made to certain of the field axioms, we do not wish to approach the number system from a structural point of view.

2-a <u>Extending the number line to the negatives</u>. Absolute value and distance.

2-b <u>Rational operations on the integers</u>. To be derived from plausibility arguments as novel as possible but not from the field axioms. Drill in these operations.

2-c <u>Fractions with the four rational operations</u>. Special case of the denominator 100 as percentage. Simple ratio and proportion. Drill in manipulations with fractions.

2-d <u>Decimals</u>. Use base 10 notation with negative exponents. Relation between fractions and decimals via division. Many practical applications and practice.

2-e <u>Roundoff and truncation errors</u>. (Most computers truncate rather than round off.) Significant digits and scientific notation.

2-f <u>More on English to mathematics</u>. Use the new ideas developed in this Part. More flowcharting with examples drawn from interest computations and financial problems, including the use of the computer. (See page 280.)

Part 3 Geometry I (See also Part 7)

The purpose of this material is to refresh the student's acquaintance with basic geometric vocabulary and then to present that minimal geometric background sufficient for the introduction of coordinate systems. (See pages 285-287.)

3-a <u>Introduction to geometric ideas</u>. Informal discussion of points, planes, segments, lines, angles, parallel and perpendicular lines.

3-b <u>Geometric figures</u>. Circles, triangles, special quadrilaterals, notion of congruence.

3-c <u>Use of basic instruments</u>. Ruler, protractor, compasses, T-square. Error in measurements.

3-d <u>Conversion of units</u>.

3-e <u>General introduction to linearity and proportion</u>. Many examples. Notion of similarity.

3-f <u>The coordinate plane</u>. Points and ordered pairs, road maps, etc.

3-g <u>The graph of $y = mx$</u>. Slope.

Part 4 Linear Polynomials and Equations

In this part there is to be a treatment of algebraic ideas at a level sufficient for the applications that follow, but which stops somewhat short of the technical algebraic competence usually sought in conventional courses in algebra. Students who wish a higher degree of technical competence may obtain it through appropriate work in the laboratory. (See Section V.)

4-a <u>English to mathematics</u>. A few word problems leading to one linear equation in one unknown as motivation for algebraic manipulation. Solve some equations by trial and error. Devise flowcharts for trial-and-error solutions.

4-b <u>Transformations of one equation in one variable</u>. Both identities such as $2x + 3x = 5x$ and $3(x + 2) = 3x + 6$ as well as transformations such as: if $4x + 5 = 11$, then $4x + 3 = 9$.

4-c <u>Flowchart for solving $ax + b = c$</u>. Include a variety of other forms.

4-d <u>Applications</u>. Word problems drawn from many different areas.

4-e <u>Situations leading to one equation in two variables</u>. (Motivation for next section.)

4-f <u>Transformations of one equation in two variables</u>. Leading, for example, to the form $y = mx + b$, being careful not to restrict the names of the variables to x and y.

4-g <u>Graphs of linear equations in two variables</u>. Slope of $y = mx + b$. Relation of $y = mx + b$ to $y = mx$.

4-h Solutions of two linear equations in two variables. Graphical and analytical methods, applications.

Part 5 The Computer

The amount of time devoted to this Part will turn out to be quite short or quite long depending on whether actual use is to be made of a computer. (See pages 16-17.)

5-a General discussion of the computer. Ability of a computer to respond to well-defined instructions. Illustrate with simple programs. Brief discussion of error due to truncation. Memory, operations, speed, with reference to the available equipment.

5-b Uses of the computer in modern society. Many different applications, with limitations of the computer stressed.

5-c Elementary instruction in programming. Language appropriate to the institution; writing programs from flowcharts.

5-d Varied applications. Drawing from material already presented, including more sophisticated financial problems. Run programs on computers when available.

Part 6 Nonlinear Relationships

We have included this material primarily to display the power of a mathematical model and to provide for development of the themes of flowcharting, approximation, and graphing which have been introduced earlier.

6-a Some examples of nonlinear relationships. Repeated doubling, and exponential growth of populations. Compound interest.

6-b The graph of $y = x^2$. Concept of square root and graphical evaluation of square root. Use of tables and approximation of square roots by averaging.

6-c Pythagorean theorem and distance formula. Very brief discussion of irrational numbers and the fact that lines and curves have no gaps.

6-d The graph of $y = ax^2$. Applications.

6-e The graph of $y = ax^2 + bx = x(ax + b)$. Roots and intercepts, maximum and minimum, applications.

6-f Graphing of $y = ax^2 + bx + c$. Use vertical translation from $y = ax^2 + bx$. Note that there may be 0, 1, or 2 roots of the corresponding quadratic equation.

6-g Approximation of roots. Use of the computer. (See page 276.)

6-h Inverse, joint, and combined variation. Applications.

6-i Suitable bounds for accuracy and estimates. Products and quotients, relative and percentage error, graphical illustrations. (See pages 281-284.)

Part 7 Geometry II

Our hope here is that the geometrical material presented will be made relevant and that there will be suitable links with the mathematical ideas introduced earlier. (See pages 285-287.)

7-a Areas and perimeters of plane figures. Rectangles, triangles, parallelograms, and circles. No extensive involvement with theorems and proofs. Perhaps compute area of irregular regions by use of rectangles and Monte Carlo methods.

7-b Surface areas and volumes. Use of formulas for areas and volumes of spheres, cylinders, parallelepipeds.

7-c Applications. Consumer problems, pollution problems, conversion of units.

7-d Elementary constructions. Use of straightedge and compasses. Include special triangles like isosceles right triangles, 30-60 right triangles, etc.

7-e Further extension of work on similar figures.

Part 8 Statistics

Besides the obvious interest and relevance of this material, it offers opportunities for use of virtually all of the ideas previously introduced in the course. We have in mind the use of statistical ideas in making practical decisions among realistic alternatives. (See pages 287-288.)

8-a The role of statistics in society. Problems of interpretation of charts, graphs, percentages.

8-b Descriptive statistics. Various kinds of graphs; mean, median, and mode; range and standard deviation; quartiles and percentiles.

8-c The normal distribution. Informal discussion.

8-d Statistics and the consumer. Informal discussion of bias; choosing samples. Flowcharts and computing should be used whenever appropriate.

Part 9 Probability

This Part represents a rather minimal introduction to the subject, avoiding any heavy involvement with combinatorics, but including one or more applications of complexity sufficient so that the methods of earlier chapters can be displayed to good advantage. (See pages 289-293.)

9-a Empirical probability. Mortality tables, long-run relative frequencies.

9-b A priori probability. Tossing coins, rolling dice, selecting discs from box. Experiments in which relative frequencies are compared with theoretical probabilities.

9-c Elementary counting principles. Emphasis on devising a procedure for listing of outcomes of an experiment, the procedure suggesting a principle or formula for obtaining the count. (See pages 291-292.)

9-d Further a priori probability. Independent trials of an experiment. Examples selected from everyday experiences such as athletics.

9-e Informal decision theory with examples. (See Exercise 15 in Appendix I.)

IV. COMMENTARY ON THE OUTLINE

An outline for a course is a good device for specifying the order of topics but a poor device for emphasizing the ideas which are to receive continuous attention. This section is devoted to making clear the intentions of the Panel concerning certain special topics and certain recurrent themes. Other topics of equal importance have been omitted from this discussion because the kind of treatment they should receive is relatively clear from the outline.

1. Computers and Computing

Mathematics E is to begin with a very brief description of how a digital computer receives its instructions and the operations it is capable of performing, preferably made concrete by exposure to actual computing machinery. The course returns to the subject in somewhat greater depth in Part 5 of the outline. At other places in the course opportunities to discuss and illuminate special topics from a computer point of view are to be exploited as these opportunities arise. The concept of a flowchart as a device for analysis is to be a central theme.

The purpose of introducing and utilizing computer-related ideas is basically the psychological one of getting the students involved with and thinking about mathematics. In a course of the kind we describe, this point is likely to be of crucial importance. The Panel has more than adequate anecdotal evidence that the injection of computing will provide very strong motivation for the student. A secondary reason is the ubiquity of the computer in our present society and the need for the students to understand its potentialities and limitations in order to function as citizens and employees.

The Panel feels very strongly that the basic validity of the course we propose would not be compromised by the absence of computing equipment at the institution where the course is being taught, nor would the lack of computer access lead us to prescribe a significant reduction in the emphasis on computer-related ideas such as flowcharting.

However, the Panel feels equally strongly that even a small amount of direct experience with computers will greatly enhance the motivational powers of these ideas and will result in a more effective course.

At the very least, the following should be done:

a. The students should have one or more guided field trips to a computer center to see how a problem is actually handled by a computer, e.g., they should understand what a computer program is and how it is related to a computer.

b. Students should be involved in team efforts to write quite simple programs.

c. The instructor should demonstrate how these programs are read by the computer and how the computer imparts its results.

It is clear that such a minimal degree of exposure would require only very limited access to computing equipment and would not imply actual possession of a computer.

In Part 5 we suggest further experience in programming. In various places that follow we mention some of the opportunities which exist for making some topics both more realistic and more challenging by actual computer usage.

The question of how extensively such opportunities can be utilized is clearly dependent on costs, even at colleges with extensive computer installations. The Panel feels that computer usage should certainly be carried beyond the minimum outlined above if at all possible. However, neither our experience nor the experience of others seems adequate at present to make any accurate estimates of the educational benefits to be expected as a function of costs.

2. Flowcharts

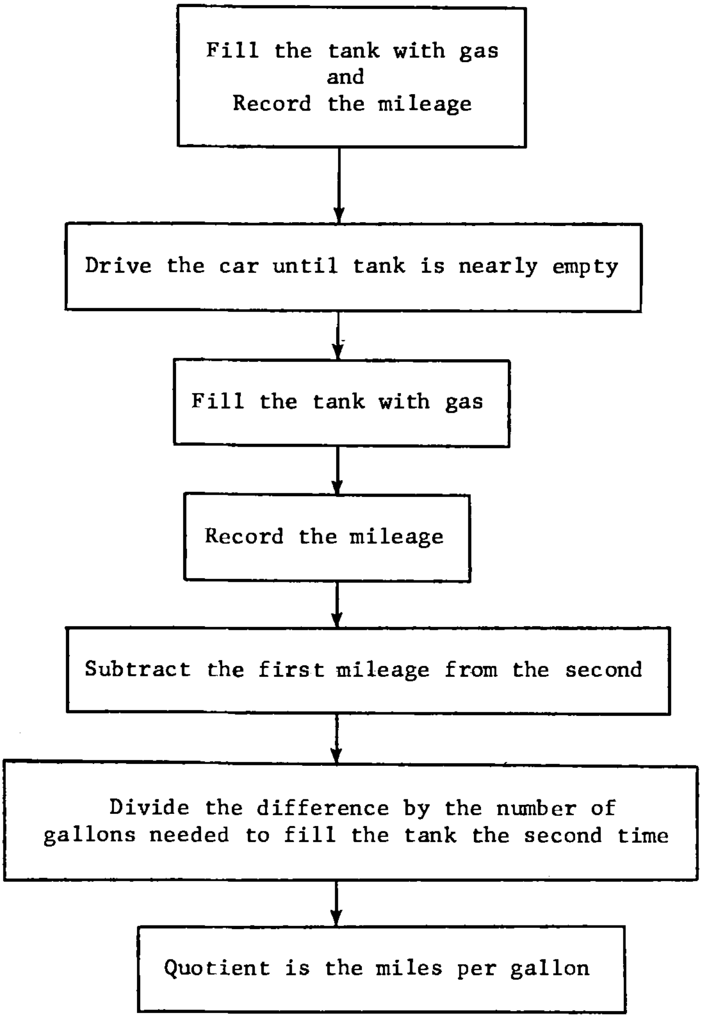
We have chosen to discuss flowcharting in the first part of the course both in order to arouse immediate student interest and because this technique is used extensively throughout the course. It must be made clear to the student that the computer operates only through very specific instructions presented to it in a special language. The idea of flowcharting can be first presented as an essential step in the process leading to the production of such detailed instructions.

The utilization of the construction of flowcharts as a technique in the analysis of problems recurs throughout the course being described. A flowchart is an extremely valuable way of describing a procedure to be followed either by another person or a computer. Its object is to break a problem up into easily managed steps whose interrelationships are clear.

It is not necessary to have a detailed familiarity with the actual capabilities of the computer to begin describing algorithmic processes with flowcharts. Perhaps the best examples to use in introducing flowcharting ideas are those of a nonmathematical nature such as the ones which follow. These first four examples illustrate the four kinds of elementary structure which flowcharts can have.

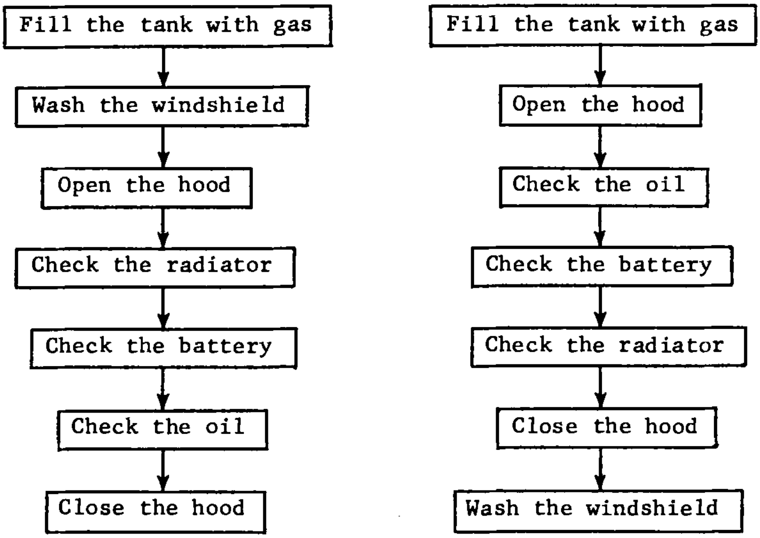
A. Sequential operations with no choice.

To determine the gas mileage for your car.



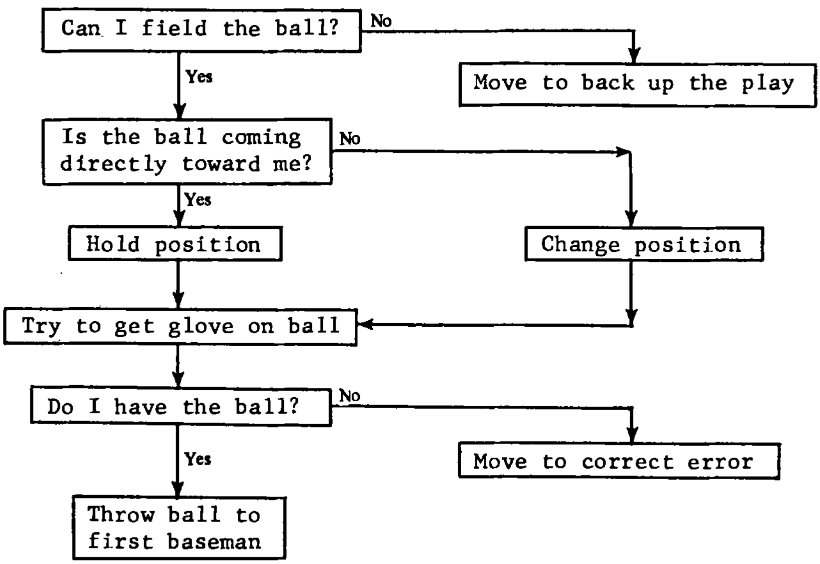
B. Sequential operations with a choice of order.

To service a car at a filling station.



C. Branches.

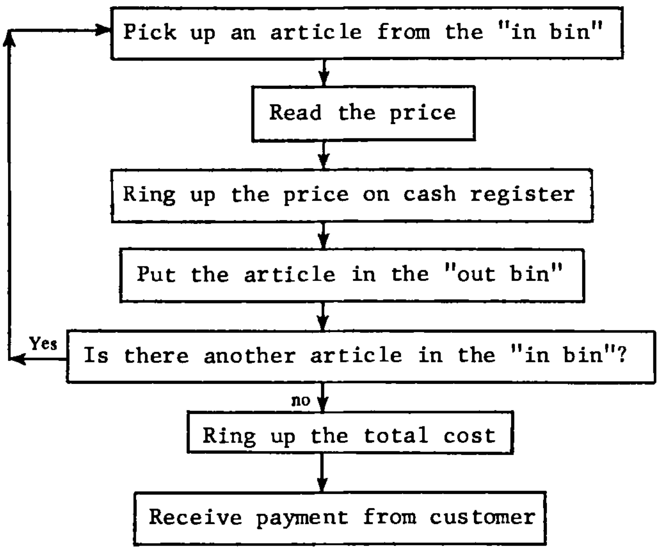
Reaction of a baseball shortstop to a groundball hit toward his position with no runner on base.



D. Loops

An algorithm may call for a process to be repeated any number of times to achieve a desired result. Thus, instead of repeating the same instructions for a set of objects, one may simply call for the same instructions to be applied to the set of objects until the set is exhausted. Representing such an algorithm by a flowchart gives rise to the so-called loop concept.

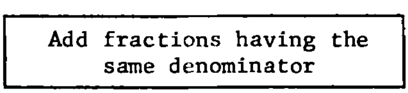
Action of cashier at a grocery store.



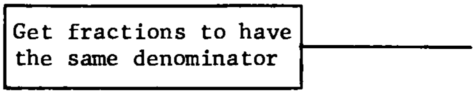
These flowcharts are all incomplete in the sense that the operations described within a given box are susceptible to further analysis into smaller steps. If a flowchart is to result in a computer program, the capabilities of the computer will determine completeness.

The student should be encouraged to construct flowcharts based on his own experience.

We propose the study of flowcharting not only as an end in itself, but also for its usefulness throughout the course to explicate concrete details in situations which may not be purely computational--for example, the standard situation faced by a student unable to solve a problem in which he does not know where to begin. Thus the source of a student's difficulty in adding two fractions may be his inability to break the problem down into simpler parts. Such a student would probably be able to add fractions with the same denominator. Let him then write



It is then apparent that if this could be preceded by

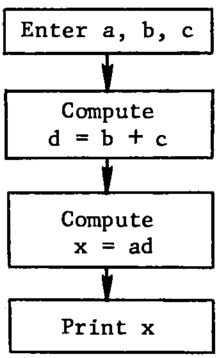


an important first step toward the solution of his problem would have been taken. A further analysis of the contents of this box via flowcharting should result in the solution.

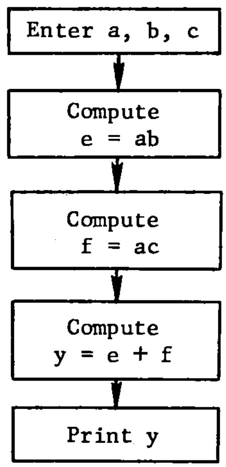
Flowcharts can illuminate mathematical assertions that fail to have the desired impact when presented in the usual manner. For example, the two sequences of operations indicated in the equation for the distributive law can be dramatized by the exhibition of the two flowcharts that describe the sequences:

Compute

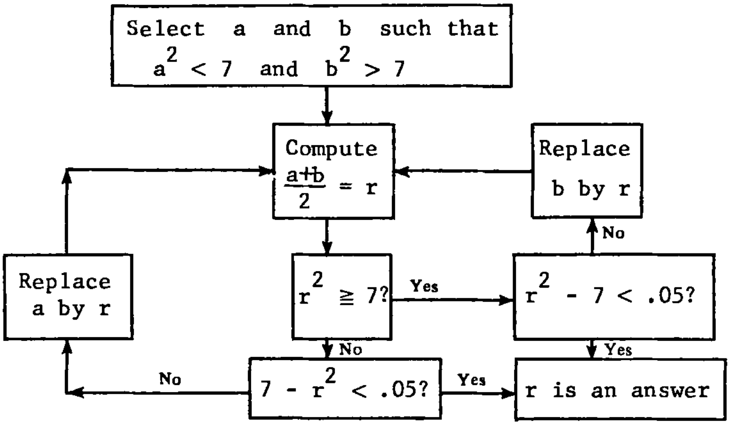
$x = a(b + c)$



$y = ab + ac$



Once the student has been introduced to the graph of the quadratic equation, he should have no difficulty in understanding, for example, that if the graph of $y = 7 - x^2$ is above the x-axis at $x = a$ and below the x-axis at $x = b$, then the square root of 7 lies between a and b. This idea is exploited in the following flowchart of a simple algorithm for approximating the square root of 7. We have chosen this algorithm for its simplicity in spite of the fact that it is not the most efficient one for hand computations. The relative speed with which a computer would perform the work would be impressive to the students.



This flowchart illustrates graphically the idea of an iterative process and suggests the economy of presentation which loops in such charts provide. The example itself affords drill for the student who performs the successive approximations. Of course, access to a computer so that a program based on this chart could actually be run would provide added motivation for the student.

3. The Role of Applications and Model Construction

Since the development of mathematical literacy relevant to participation in human affairs is a major objective of the course, a strong relationship between the topics and their applicability must be established. The purpose of this section is to offer suggestions on the kind of applications that should be emphasized and the points at which they should be included.

Although it would be expected that separate sections devoted to "word problems" might be a part of the course, the mere inclusion of such sections is not sufficient to accomplish the desired goals. To aid in establishing the relevance of the topics, examples of practical problems should be used to motivate the topics. Additional examples need to be interspersed with sufficient frequency to insure that interest is maintained.

In order to help in overcoming one of the major difficulties associated with practical problems, separate sections entitled "English to Mathematics" are listed in the suggested outline. The purpose of those sections is to assure that attention is actually given to the process of translating English statements into mathematical equations and other mathematical models.

Care must be taken in the selection of examples to insure that the majority of the problems are actually practical. Too often in existing courses and texts, problems are "cooked up" to lead to a particular kind of equation. Consequently, some of the motivational value of the examples is lost because of their lack of relation to reality. Problems with sentences such as "John is three times as old as Mary was five years ago," or "Jack has twice as many quarters as dimes and seven more nickels than dimes," may lead to nice equations, but they are uninteresting simply because the situations are never encountered.

It is not being suggested that examples taking advantage of the playfulness of the human mind should be ignored. However, problems in which variables must represent quantities that are likely to be known in the situation being described should occur infrequently and only when a note of whimsy is desired.

Perhaps the most important problem that the instructor will face in teaching the typical Mathematics E student is to arouse his interest and to induce him to participate with some degree of enthusiasm in the course. It is hoped that the computer-related aspects of the course will help to bring this about. However, an appeal to the playful and puzzle-solving impulses of the student should also be used whenever possible. Most of the student audience under consideration is willing to think quite hard, and, indeed, in a mathematical fashion, when it comes to certain pursuits that it regards as pleasurable--for example, when it comes to playing card games. Moreover, a tricky puzzle will capture the attention of many of those students while a straight mathematical question will rarely do so. It is therefore very likely that if these elements are used whenever appropriate in Mathematics E they will jolt the students into awareness. For example, the old puzzle about the bellboy and the five dollars or the three warring species crossing the river two at a time[1] can be used to illustrate logical analysis and step-by-step consideration of problems. The elements of these puzzles should be brought out and used throughout the course in order to point out the intellectual similarity between dealing with them and with solving ordinary mathematical problems. Of course, in Part 9 many references should be given to the games that our students actually play, but even the seemingly more arid parts of the course can surely benefit from this kind of material. The Thirteen Colleges Curriculum Program[2] has made very successful use of this technique and has found it a very useful and efficient way to lead students into a study of mathematics that the students rejected originally. This material contains stimulating problems that have been used successfully by participants in the Thirteen Colleges Curriculum Program to engage

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1. Many such problems can be found in Mathematical Recreations by Maurice Kraitchik (New York, Dover Publications, Inc., 1942).

2. Thirteen Colleges Curriculum Program Annual Report. Analytical and Quantitative Thinking (Mathematics). Florida A and M University, Tallahassee, Florida, 1969.

the attention and reasoning power of students taking courses at approximately the level of Mathematics E.

The following points should be considered in the selection and presentation of examples and exercises: (1) The majority of problems should describe familiar situations in which variables represent quantities that could reasonably be expected to be unknown. (2) To develop the ability to translate English into mathematics, it may be necessary to begin with problems that students are able to solve without the aid of an equation or other model. However, at several points in the course, students will have the background necessary to construct models for problems that they will not be able to solve completely. If full advantage is taken of this situation, then applications can be used to motivate the need for being able to solve equations and otherwise manipulate models to produce solutions. (3) Students should frequently be asked to estimate an answer to an exercise prior to writing the equation or formulating a model, and to explain the process by which the estimation was made. Such explanations are often surprisingly close to being a correct verbal equation which can be more easily described by symbols than could the original problem. (4) Students should be encouraged to describe problems which they have actually encountered and which they would like to be able to solve. The experiences of students with charge accounts, savings accounts, tax problems, other college courses, vocational experiences, and situations that occur in playing cards and other games can be excellent sources of problems at various times within the course.

As an illustration, a sequence of problems that might be used near the beginning of the course to develop the ability to translate English into linear equations is given below. The problems begin with one that could be solved without a formal statement of the equation and progress to the point that the majority of students would need an equation to complete the solution.

1. A student has grades of 65 and 76 on two exams. In order to maintain the average that he desires, he must have accumulated 210 points after the third exam. Write an equation that can be used to find the grade which he must make on the next exam.

2. Suppose that the student in the previous problem wishes his average for the three exams to be 72. Write an equation that can be used to find the grade which he must make on the next exam.

3. Suppose that the third exam in problem 2 is a final exam and will count as two regular exams in computing his average. Write an equation that can be used to find the grade which he must make on the final exam in order that his average will be 72.

4. Suppose that in problem 3 the student may choose to use his textbook on the final exam, but if he so chooses, his final exam score is lowered by 10 points. Write an equation that can be used to find the grade which he must score on the final exam (before deduction) in order that his average will be 72.

Note that the equation for the first problem may be given by a sentence as simple as $P = 210 - 65 - 76$, which the student could certainly solve. In fact he could probably solve this problem without the aid of an equation. An equation for problem 4 is given by

$$\frac{65 + 76 + (X - 10) + (X - 10)}{4} = 72,$$

which the student probably could not solve at this point.

There is a wealth of possibilities for motivating and illustrating various topics throughout the course with consumer problems. Such problems seem to be especially adaptable to the notion of flowcharting. Two examples of such problems are given below. By adding such complicating factors as additional purchases, variable payments, and minimum service charges, one can expand the problems to the point where a flowchart is almost essential for describing the payment process. (See Appendix II.)

1. Suppose that a customer makes purchases on credit totaling \$560.00. Interest and all other charges on the debt make the total balance \$610.00. The customer must pay \$25.00 per month until he owes less than \$25.00. He will pay the remainder as a final payment. Construct a flowchart that contains boxes giving the amount owed and the number of payments remaining after n payments.

2. Suppose that a customer makes purchases on credit totaling \$560.00. On the last day of each month $1\frac{1}{2}\%$ of the unpaid balance is added to his account. The customer plans to pay \$25.00 each month until he owes an amount less than or equal to \$25.00. He will pay the remainder as the final payment. Payments are paid on the first of the month. Describe the payment process with a flowchart that contains boxes giving the amount paid and the amount owed after n payments.

The problems should include some that are more open-ended than is usually the case with textbook problems. Some problems should request a reasoned choice between alternative courses of action, and some should ask what additional information is needed in order for an ill-posed problem to become solvable.

It should be stressed that the Panel intends that "word" problems should be emphasized much more in this course than is usually the case. We do not have in mind teaching artificially neat procedures for solving quite special classes of problems (as, for example, in the rote methods frequently used to drill students in the solution of time-rate-distance problems). We believe that students in a course of this level can learn to create and analyze simple mathematical models. Many students are inhibited by their attitudes toward mathematics and beset by a fear of failure. If the problems are realistic enough to seem significant to the students and initially are simple enough to insure a ready solution by most students, we believe that the students can be helped to develop a regular use of mathematical ways of thinking in the analysis of practical situations.

Appendix I contains a list of problems and examples that illustrate the kind of applications that should be a part of the course. The list is not meant to convey any sort of desirable achievement level for the course. Neither is it meant to be comprehensive with respect to the areas from which applications should be selected. It does represent the spirit of the course relative to meaningful applications.

4. Estimation and Approximation

In a wide variety of practical problems an approximate solution will serve as well as the result of an exact calculation. In other situations (such as that of a shopper in a supermarket) estimation is the only practical course of action available. Even if an answer has been calculated exactly, it is useful as a check to obtain a quick approximate solution.

We hope that a graduate of this course will be an habitual estimator. The topic of estimation, like that of flowcharting, should appear throughout the course and should be presented in as many places as possible, taking advantage of opportunities as they arise. The course should make the student moderately proficient in estimating products, reciprocals and quotients, as well as powers and roots, of 2- and 3-digit numbers. Although the underlying theory is elementary, the approach must be through trial and error leading to the formulation of certain principles which in turn lead to the basic question of tolerance and control.

Because the audience we have in mind is not generally well prepared in arithmetic, one should start slowly, perhaps with relative errors discussed as percentages. These should be taken from everyday life: population figures, sports, betting, etc. Then one needs to present a crude technique of converting relative errors expressed as percentages. Thus, we discuss "48 parts in 99" and point out that it is roughly "50 parts in 100." Also, a relative error of 16/53 is approximately a 30% relative error, since 16/53 is approximately 15/50.

The product 13×27 can be estimated in several useful ways. For example,

(a) We could make each number smaller; replace 13 by 10 and 27 by 25 and obtain

$10 \times 25 = 250$, which we regard as roughly correct. We write

$13 \times 27 \approx 250$,

where the symbol $\approx$ means "is approximately equal to."

(b) We could make only one of the numbers smaller, using 10 and 27 for example, and write

$13 \times 27 \approx 10 \times 27 = 270$.

Clearly (a) and (b) are "underestimates," i.e.,

$13 \times 27 > 250$ and

$13 \times 27 > 270$.

(c) We could make each number larger; replace 13 by 15 and 27 by 30 and obtain

$15 \times 30 = 450$, writing

$13 \times 27 \approx 450$.

(d) We could make only one number larger, using 13 and 30, for example, and write

$13 \times 27 \approx 13 \times 30 = 390$.

Of course (c) and (d) are "overestimates." Thus, we can write

$270 < 13 \times 27 < 390$,

which the student must learn to read as two statements: 270 is less than 13×27, and 13×27 is less than 390. At this stage, we might suggest that an average be used. Thus, the average of 270 and 390 is 330; hence,

$13 \times 27 \approx 330$.

(e) In estimating the product we can make one factor larger and one smaller; this frequently leads to a better estimate. Thus, we could replace 13 by 10 and 27 by 30 and write

$13 \times 27 \approx 10 \times 30 = 300$.

The student will see that this estimate is perhaps better, but that we have lost control in the sense that we no longer know whether the estimate is larger or smaller than the correct answer.

Some students might wish to pursue these ideas further in the laboratory. They might be encouraged to consider questions such as: if a, b, and x are positive, x is small compared to a and b, and $a < b$, then is it the case that

$$\frac{a - x}{b - x} < \frac{a}{b} < \frac{a + x}{b + x} \text{ ?}$$

Such a formulation might be conjectured empirically, and then an algebraic analysis attempted.

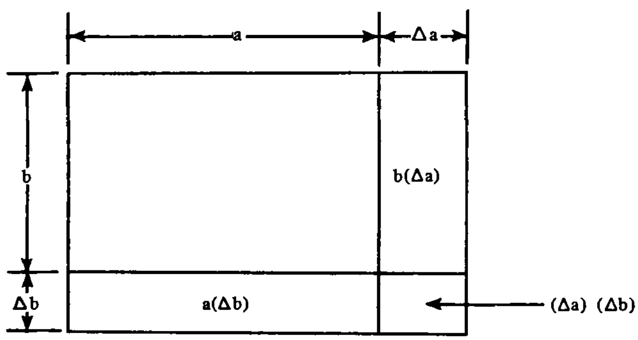
Somewhere in the course it might be well to introduce the student to the use of the slide rule, although the theory should be postponed until the student himself wants it and is ready for it. At the beginning it is just an instrument which serves to estimate products and quotients. Scientific notation would be introduced in discussing the position of the decimal point.

Before any formula about estimation is presented, the students should have been exposed to problems in which a quick rough estimate is useful.

Example: Steve wishes to mail 28 books to his new residence. The prices of the books range from \$6.95 to \$12.50. For how much should he insure the books?

The theory of approximation must, of course, be presented in stages, following the algebraic readiness of the student. Once the student has the necessary algebraic readiness, he might be introduced to the concept of relative error. Thus, if Δx denotes the error in x, then by the relative error $R(x)$ we mean $\Delta x/x$.

After doing a sufficient number of problems involving this concept, a student might be enticed to consider $R(ab)$. This can be motivated by considering the following diagram:



Thus, $\Delta(ab) = b \cdot \Delta a + a \cdot \Delta b + \Delta a \cdot \Delta b$.

Now, dividing by $a \cdot b$, we obtain

$$R(ab) = R(a) + R(b) + R(a) \cdot R(b).$$

From this picture, a student can perceive that if $R(a)$ and $R(b)$ are "small enough" (depending on the situation), then an estimate for $R(ab)$ is

$$R(ab) \approx R(a) + R(b).$$

The student should, of course, have had the opportunity to conjecture this by considering several examples.

Having arrived at this formula, the student might be ready to understand, for example, that if $|R(a)| < 30\%$ and $|R(b)| < 20\%$, then the estimate for $R(ab)$, namely $R(a) + R(b)$, is correct to within 6%.

At the point where the student has learned to graph, we can use the graph of $y = x^2$ to get estimates of squares and of square roots (see also flowcharting). It should be interesting to the student that a smooth graph is a good "estimator." Of course, from the relation

$$R(ab) \approx R(a) + R(b),$$

we obtain

$$R(a^2) \approx 2R(a)$$

or

$$R(\sqrt{a}) \approx R(a)/2,$$

from which we conclude that the relative error in the square root is about half the relative error in the number.

Following this, the interested student could be shown (this should probably be done in the laboratory) that if $R(b)$ is "small enough," then

$$R\left(\frac{1}{b}\right) \approx -R(b),$$

and that if $R(a)$ is also "small enough," then

$$R\left(\frac{a}{b}\right) \approx R(a) - R(b).$$

At this stage, the student will have accepted the usefulness of these relations and will enjoy working examples and seeing how well he can quickly estimate products, quotients, powers, and roots.

5. Comments on Geometry

Geometric ideas should be presented in two stages. At the beginning (Part 3 in the outline) only basic geometric vocabulary and those ideas necessary for introducing the coordinate plane are presented. Once the student has gained some mathematical experience (Part 7 in the outline), he can be exposed to slightly more complicated geometric notions.

As early as in Part 3 it is fairly safe to assume that most of the students of whom we speak will have some notions of the basic geometric concepts of points, planes, segments, lines, angles, parallel lines, and perpendicular lines. It is hoped that the course can reinforce some of the correct notions the student already has, while at the same time pointing out a number of concrete uses of geometry in his daily life. Abstract definitions and an axiomatic approach must be avoided. It is appropriate to give the student experience with specific geometrical figures by performing constructions in the laboratory with the use of the ruler, compass, protractor, and draftsman's triangle. With even rudimentary ideas of similarity, one could discuss problems of indirect measurement.

After the student has been introduced to the idea of a plane, the coordinate plane can be regarded as analogous to the familiar devices employed in designating a specific section of a road map (by letter and number) or some particular locale in an ocean (by latitude and longitude).

The coordinate plane can then be introduced in the usual way, i.e., by reproducing on each axis the number line discussed in Parts 1 and 2, choosing the intersection of the two axes as the origin, and by making use of the concepts of perpendicular and parallel lines discussed earlier.

The graph of $y = mx$ can now be introduced, perhaps with an example such as the following:

Example: The owner of an ornate gift shop in a high-rent district has determined that he can make a reasonable profit and cover all overhead expenses (stock costs, rent, shipping costs, employees' salaries, taxes, utilities, office supplies, advertising, insurance, breakage, etc.) by making the selling price of each item equal to twice its cost. Let x be the cost of an item and y its selling price; then $y = 2x$. A table of corresponding values of x and y should be constructed and these ordered pairs plotted, with appropriate emphasis on the fact that the points are arrayed in a straight line.

The concept of steepness or slope might first be discussed in an already familiar context, such as the "pitch" of a roof or the "grade" of a road, with emphasis on the fact that the pitch, grade, or slope is determined by "rise over horizontal run." The slope of a line can be introduced by graphing functions such as $y = 4x$,

$y = 2x$, and $y = \frac{1}{2}x$ in the same plane and inviting students to compare the three lines. Such comparisons should make it difficult for them to avoid the notion that the constants 4, 2, and $\frac{1}{2}$ have something to do with the relative steepness of the lines.

Taking specific points on each of the lines $y = 4x$, $y = 2x$, and $y = \frac{1}{2}x$, a series of right triangles should be constructed by "walking away" from each point a varying horizontal distance and then "walking up" the required vertical distance to the line. Hopefully, the student will have done enough laboratory work on similarity (in Parts 3-5) to understand that all right triangles with hypotenuse on the same line and one leg horizontal are similar. Thus, successive computations of corresponding ratios will provide a simple illustration of the fact that the ratio of rise to horizontal run remains constant for the same line, and that the slope of each line is equal to the coefficient of x in $y = 4x$, $y = 2x$, and $y = \frac{1}{2}x$. Similar demonstrations with different examples can be used to show that the slope is positive if the line slants up to the right, is negative if the line slants down to the right, is nonexistent if the line is vertical, and is equal to the constant m in the equation $y = mx$.

In Part 7 the student should be introduced to the basic formulas for areas and perimeters of rectangles, triangles, parallelograms, and circles. He should also learn the formulas for surface areas and volumes of parallelepipeds, cylinders, and spheres. This, of course, provides an opportunity to do more work on approximations. The student will find this work more interesting if it is presented through useful examples such as the following:

<u>Example 1</u>: A man wants to cover his front yard with top soil before he plants his lawn. The yard is 100 ft. by 30 ft. and he wants to have the top soil 6 inches deep. How many cubic yards of top soil should he order?

<u>Example 2</u>: An oil discharge of fixed volume may spread over a very large area as the thickness of the slick decreases. Find an algebraic expression for the area covered by a given volume of oil in terms of the thickness of the slick.

<u>Example 3</u>: A certain noxious substance is discharged into a river as a by-product of a factory. Suppose the rate of flow of the stream in winter is at least 1,500,000 gallons per day and the rate of discharge of the pollutant is variable but may be as great as 2,000 gallons per day. If a concentration of 150 parts per million of the pollutant is the maximum that is considered "safe," what conclusion can be drawn as to the safety of the stream near the site of the factory? If the stream is safe at the factory site, how much additional pollutant can be discharged without exceeding a safe concentration? If the stream is unsafe at the factory, consider the situation downstream where additional inflow from tributaries may be thought of as diluting the concentration. How much inflow from tributaries will be needed before the augmented stream will be "safe"?

How does the situation change when the spring rains double the flow of the stream?

Some work on conversion of units, making use of mensuration formulas, should also be given--for example, the following:

Exercise: The base of a rectangular container of water is 2 ft. by 5 ft. Suppose that a metal ball 13 inches in diameter is dropped into the water and is completely submerged. How much will the level of the water rise?

The usefulness of similarity can be exemplified by pointing out that when a house is built, the contractors use plans to help them. These plans may be drawings or blueprints. In these the length of a line segment is much smaller than the actual length it represents. The ratio of the length of a line segment to the actual length it represents is called a scale. Maps are other scale drawings that students are familiar with. Many examples can be given of proportion using the idea of such scale drawings.

6. Remarks on the Material on Probability and Statistics

These two Parts can be considered as an opportunity to synthesize and apply many of the ideas of the preceding Parts. In addition, a case can be made for these topics as being among the most important mathematical concepts a citizen can acquire. He needs some knowledge of statistics in order to understand the news he hears on radio or television. Many of the problems facing the various governmental agencies in regard to taxes, welfare, education, and other public matters are comprehensible only to one who has some knowledge of statistics. Without an understanding of these problems, it is difficult, if not impossible, for a citizen to vote intelligently. The advertising of consumer goods is frequently couched in statistical or pseudo-statistical terms, and it has a direct bearing upon how one allocates one's income. In short, some knowledge of statistics seems important for everyone. Since the real payoff from a knowledge of statistics lies in using statistics to make decisions, and because most decisions of the sort considered daily are made in the presence of uncertainty, a basic grasp of elementary probability seems equally important.

What is proposed in Part 8 is a treatment, at a level accessible to the students to which the course is addressed, of the most fundamental notions of statistics. The main thrust of the Part should be the preparation of students to become intelligent consumers of statistical information rather than to be statisticians. Accordingly, they should be alerted to the pitfalls of interpretations based on some rather commonly encountered misuses of statistical information. Distortion of the scales of a statistical chart, faulty use of percentages, and poor sampling techniques are examples of causes of misinterpretations. An excellent and very readable book (for students, too) on this topic is How to Lie with Statistics by Darrell Huff and Irving Geis (New York, W. W. Norton and Company, Inc., 1954).

For example, one might pose the following question:

> Statistics show that in 1954 among fatal accidents due to automobiles, 25,930 occurred in clear weather, 370 in fog, 3,640 in rain, and 860 in snow. Do these statistics show that it is safest to drive in fog?

Students should become accustomed to analyzing and interpreting sets of numerical data. As a first step in this process, they need to be familiar with the construction and use of histograms, bar charts, line graphs, and pie diagrams. The World Almanac is a good source of material, e.g., census figures on U. S. population over a period of years, current distribution of population by age groups, personal income per capita, etc. Of course, daily newspapers and magazines contain other examples. The computation and use of measures of central tendency (mean, median, mode) and the use of percentile or other similar ranking indices should be presented with examples for actual hand-computation chosen so as to avoid an excessive amount of tedious labor and restricted to a list of numbers rather than grouped data. If a computer or a desk calculator is available, it would be possible in the laboratory to have interested students compute some of these measures using more meaningful data which they have collected or might be expected to encounter in vocational areas of particular interest to them.

Examples relevant to these matters might include discussions raised by questions such as:

> Is it safe for an adult who doesn't swim to step into a pool whose average depth is 4 feet?
>
> A newspaper reports that the average American family consists of 3.6 persons. What, if anything, does this mean? Is your family average?
>
> City A has average daily temperature of 75 degrees, as does city B. In city A, temperatures range from 10 to 99 degrees during the year. In city B, temperatures range from 60 to 80 degrees. For someone preferring moderate climate and hence considering moving to city A or city B, is there any reason to prefer one over the other and why?

An informal discussion of the normal distribution can be based upon the study of approximately bell-shaped histograms such as might be obtained, for example, from recording heights of a large number of college men. It may be observed that for such data the mean, median, and mode are approximately equal, so that phrases like "the average height is 69 inches" are not ambiguous. That this is not generally the case should also be noted. For instance, a distribution of family incomes would probably not have this property. The notion of standard deviation as a measure of dispersion and its relationship to the shape of distributions can be briefly touched upon--e.g., approximately 2/3 of the cases lie within one standard deviation of the mean, 95% within two standard deviations of the mean, and practically all are included within three standard deviations of the mean. In Part 9, after the notion of probability has been studied, it is possible to return to these properties in discussing problems of statistical inference.

An informal discussion of bias in sampling and the idea of a random sample might be enlivened by an experiment involving use of random number tables to select a random sample. A flowchart could be constructed to describe the steps in such a procedure.

The probability concepts proposed for Part 9 are simply those that bear upon the students' ability to make judgments in the presence of uncertainty. Little is proposed beyond introducing the general notions of the meaning of a priori and empirical probability, and developing an ability to understand the application of these notions in common-sense ways. These concepts are detailed in the outline, but their development should focus on the establishment of confidence in general impressions and interpretations rather than on the manipulation of formulas.

Empirical probability might be introduced by an example such as the American Experience Mortality Table. This table is based on birth and death records of 100,000 people alive at age 10. Entries give the number of these people living at various ages; for age 40 the entry 78,106 means that of the original group this many were still alive at age 40. The ratio 78,106/100,000 or .78 is called the empirical probability that a child of 10 will live to be 40. Students should be given some practice on the use of this table and caution on its misuse (e.g., probability .8 of living from age 10 to age 40 does not mean that in every group of ten individuals two will die before age 40).

The notion of a priori probability should be introduced from the point of view of an experiment performed with a set of objects in which certain outcomes are of interest. Some simple examples are: (1) A penny is tossed to see which of the outcomes, heads or tails, results. (2) One of several discs, each painted red or white, is drawn from a box and its color noted. (3) An individual is selected from a group of people and his opinion asked concerning a specific proposal of current interest to students.

It should, of course, be pointed out that experiments (1) and (2) are idealizations of certain processes having an intrinsic interest, such as the interaction of genes or the selection of a tax return for audit by the Internal Revenue Service.

It would be natural for these experiments to be performed in the laboratory with the results used to motivate the fundamental definitions. For example, in experiment (2), if it is known that the box has 50 white and 50 red discs, students will probably agree to assigning probability 1/2 for outcome "white disc." What if, instead, it is known that the box has 10 white and 90 red discs? What if the number of red and the number of white discs is unknown to the student? Experiments in the laboratory using boxes of discs with different proportions of the two colors might be undertaken with each student performing the experiment of drawing out a disc, then replacing it, say, 20 times, and recording the color of disc on each draw. A histogram of the relative frequencies collected by the class for outcome "white disc" illustrates the idea of variability and can lead to a discussion of how much confidence can be placed in information from a single, small sample. Combining the relative frequencies of groups of two students and then of five students, one can study the effect of sample size on variability. From an analysis of several histograms the class might be led to agree on a sample size when the composition of the box is unknown.

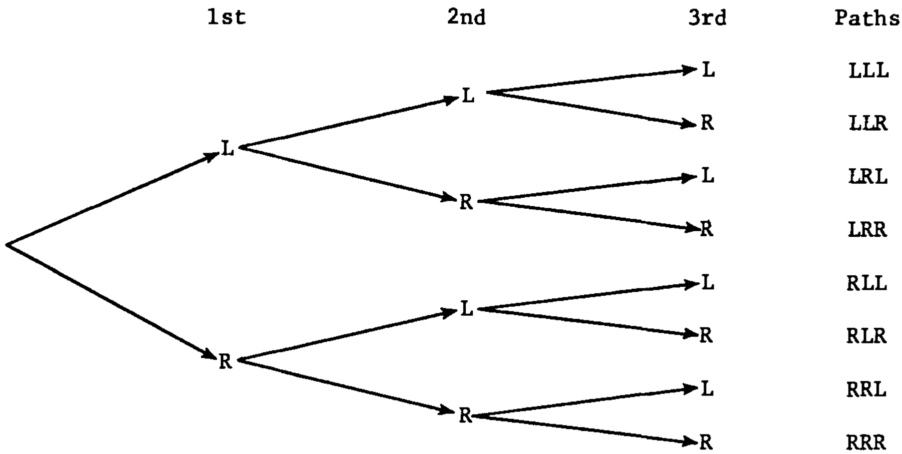
In experiment (3), if the percentage of the population in favor of the proposal is unknown and the problem is to estimate the unknown percentage, one should be careful to obtain a random selection from the population. For example, the percentage of the population in favor of building a new sports arena should not be estimated from those people who are leaving the old arena after a game. Estimation of the percentage of a population who favor a proposal is an example of statistical inference in which probability models are used to evaluate the likelihood with which assertions based on simple data are valid.

After some preliminary excursions into the computation of probabilities, it soon becomes apparent that a few guidelines to counting are needed. These should be based on the notion of first attempting a listing of (equally likely) outcomes. For problems involving a sequence of actions in succession--first do this, second that, etc.--the tree diagram provides a natural vehicle for presenting one of the so-called counting principles, the multiplicative one.

For example, in a psychology experiment a rat is placed at the entrance of a T-maze from which he runs either to the left arm, L, or the right arm, R.

(a) Suppose that the experiment is performed 3 times. List the possible paths of the rat on the three trials.

(b) If the food is always placed at L, how many of these paths could the rat take to receive the food at least two of the three times?



(a) See last column above; (b) 4: LLL, LLR, LRL, RLL.

With this basis for counting, the notion of the set of permutations of r objects chosen from n distinguishable objects, $n \geq r$, can be pictured as an r-stage tree. Starting with many numerical examples, one arrives at the formula for the number P(n,r). Simple discussion of factorials arising from the case of P(n,r) is needed to facilitate arriving at the formula for the number of combinations (subsets) of r objects selected from a set of n objects. Students should compute a few factorials to see how fast they grow. A flow diagram for computing n! might be developed. The formula for the number of combinations or subsets of r objects from a set of n should be derived after considering lists of permutations and of combinations in cases such as $n = 4$, $r = 2$; $n = 4$, $r = 3$. It should be stressed that this formula and the one for permutations are treacherous and should not be used until one has first ascertained (by an attempt at listing) that one of them is appropriate. Special combinatorial problems (e.g., circular permutations, permutations with repetition of symbols, etc.) should be avoided.

The formal treatment of probability theory should culminate in a discussion of the binomial distribution patterned after and illustrated by coin-tossing, die-throwing, etc. It is here that the convenience of the counting principles and formulas is noticeable. Some simple binomial tables should be available for student use in sampling (e.g., for $n = 10$, with various cases for p).

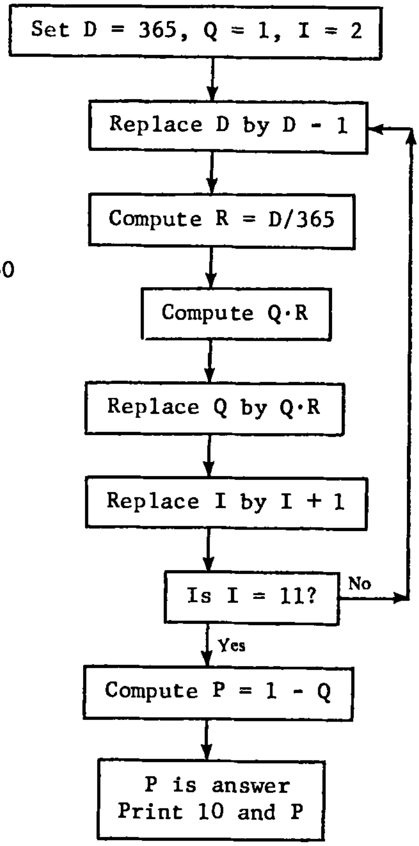
At this point let us consider an example to show the uses of the material of earlier chapters. (See Exercise 15 in Appendix I.)

Example: A famous probability example which interests beginning students is the so-called "birthday problem." The question is, "In a room containing N people, how large a number should N be so that the odds are 50-50 that at least two persons have the same birthday (month and day, not year)?" The answer, $N = 23$, defies the intuition of most people. The problem might be introduced as stated, various guesses given, and obvious extreme answers discarded. For example, for birthdays occurring in an ordinary year, if there are 366 people in the room, then it is certain that at least two have the same birthday. No doubt some will suggest that one half of 366 or, say, 180 is a good guess for the answer to the question, but the probability of coincidence of two or more birthdays for 180 people is still indistinguishable from 1! At this point the class can be encouraged to find the probability of coincidence in the case of small numbers N. For example, if N = 2, the number of possible (ordered) birthday pairs is 365·365, while the number of differing ordered pairs among these is 365·364, so that the required probability is $1 - \frac{365 \cdot 364}{365 \cdot 365}$ or 1 - 364/365 = 1/365, indicating a small probability of two birthdays on the same day. Similarly, for N = 3 we find the probability of at least two birthdays falling on the same day to be $1 - \frac{365 \cdot 364 \cdot 363}{365 \cdot 365 \cdot 365}$, etc. It will be clear that the computation becomes prohibitive in a short time as one increases the size of N. Also, some remarks on approximating the fractions arising in such computations should tie in with previous discussions. Actual multiplication of numerator factors followed by division by the product of denominator factors would certainly lead to overflow on most computers. Writing, for example, the fractional part in the case N = 3 as $1 \cdot \frac{364}{365} \cdot \frac{363}{365}$ suggests a more hopeful procedure. A computer program (in BASIC language), with a corresponding flowchart, to compute and print out the values of this probability in the cases N = 10, 22, 23, 24, 25, 30, 40, 50, 60 is given below. Even with 60 people in the room, it is practically certain (probability .99) that at least two will have matching birthdays!

Birthday Problem BASIC Program

```
10  Read N
15  Let D = 365
20  Let Q = 1
25  For I = 2 to N
30  Let D = D - 1
35  Let R = D/365
40  Let Q = Q*R
45  Next I
50  Let P = 1 - Q
55  Print "N = "N; "P = "P
60  Go to 10
65  Data 10,22,23,24,25,30,40,50,60
70  End
```

Flowchart for case N = 10



The printout is given below:

N = 10	P = .116948
N = 22	P = .475695
N = 23	P = .507297
N = 24	P = .538344
N = 25	P = .5687
N = 30	P = .706316
N = 40	P = .891232
N = 50	P = .970374
N = 60	P = .994123

Now the student could be asked to consider just how significant the above 6-digit numbers are. What is the underlying assumption in the mathematical model? Is it true that each day of the year is equally likely for a birthday?

V. THE MATHEMATICS LABORATORY

The term "laboratory" will be used to indicate arrangements for teaching other than classroom instruction or undirected individual study. The word "laboratory" is also to mean a place (or places) in which would be located programmed materials, various audio-visual devices, books, and perhaps computer terminals. It might also include facilities for individual or group conferences. Laboratory facilities and organization may differ widely from school to school, and no prescriptive suggestions seem warranted.

Although we have not studied the role of mathematics laboratories in general, some form of a laboratory is an integral part of Mathematics E. Without a laboratory it is difficult to see how the flexibility necessary to deal with individual differences in preparation, ability, and goals can be achieved. Whatever form the laboratory may take, or whatever kinds of materials or devices are used, the total laboratory program must have three distinct goals:

(1) to correct deficiencies in preparation
(2) to make provision for individual goals
(3) to reinforce and extend the classroom instruction

We will discuss each of these three goals in turn.

1. The Remedial Goal

It is false to assume that any student knows all about mathematics up to a certain point in the standard curriculum, after which he knows nothing. Both the knowledge and the deficiencies of the students in the course will be scattered throughout their previous mathematical work, probably in small pieces. Small gaps in prerequisite knowledge and techniques can be diagnosed and treated in the laboratory.

The approach we have in mind is to determine as specifically as possible the individual deficiencies of a student and to prescribe as specifically as possible materials which the student can use individually to correct these deficiencies <u>in advance</u> of the time when these ideas and techniques are needed in the course.

From the outline one can see that Part 1 is essentially without prerequisites and that the prerequisites for Part 2 are minimal. Thus, while Part 1 is being taught in class the student can use the laboratory to master, by as many cycles of diagnosis and treatment as are necessary, the skills and concepts which are needed for Part 2. This leads in general to the idea that while one Part is being taught in class the student will be using the laboratory program to identify and correct any deficiencies he may have in the prerequisites for the next Part.

The problem of determining student needs for the remedial work of the laboratory is not trivial. Clearly, one requirement is the use of existing or teacher-constructed diagnostic tests. These tests must be highly discriminating and yet easily administered. They must pinpoint the student's deficiencies and indicate appropriate remedial activity.

Persons concerned with the design of the tests should also plan to minimize the degree of professional skill necessary to evaluate the results. Hopefully, the tests will be such that persons other than professional teachers can administer them and use the outcomes to assign appropriate remedial work.

Diagnostic tests should not constitute the only avenue for referring students to the laboratory for remedial work. If, on the basis of routine homework performance or class test results, it seems necessary for some students to repeat instruction in certain areas, instructors should have a means of referring these students to the laboratory for such reinforcement.

2. Provision for Individual Goals

Students will take (or be assigned to) this course for quite varied reasons. Beyond the mathematical literacy which is our fundamental purpose, some students who intend to continue in further mathematics courses will require more thorough grounding in or a broader coverage of fundamental algebraic skills. Others enrolled in technical or occupational curricula may need a further development of certain particular aspects of mathematics relevant to their curricula as well as applied problems directed specifically to their proposed major fields.

We imagine that corresponding to a given Part in the classroom presentation there might be several special-interest blocks composed of applications or illustrations of the same basic principles in different fields. For example, in connection with Part 8 on statistics, the prospective nurse might be studying statistics relative to public health, the prospective businessman the statistics of income and employment, and the prospective police administrator the statistics of crimes. Similarly, even such a straightforward subject as linear equations has applications to many subjects.

In addition to variant versions of an application, there may be need for optional blocks which endeavor to teach mathematical techniques not embodied in the main sequence of the course. For example, it is likely that a student interested in drafting might profit from some work in numerical trigonometry. We can also imagine that a student intending to take a course in chemistry might well profit from specific instruction in certain special techniques commonly used in chemistry courses. Finally, a student who does not possess any specific vocational objectives might even elect special-interest blocks in such subjects as number theory or more algebra.

We believe it is important that the student be allowed to choose freely from among the special-interest blocks that may appeal to him. Not only would this enhance the appeal of the laboratory material, but it might even be important to the student that he has been allowed to make some choice in order to adapt the course to his own goals as he sees them.

The availability of adaptation of the course to individual needs might well go far toward eliminating the demand for separate courses for special groups of students. In many local situations this flexibility might be good for the diplomatic relations between the mathematics faculty and the rest of the institution. In this connection, faculty members in other disciplines could well be invited to give advice or even aid in the construction of special-interest blocks for students interested in specializing in their fields.

An important class of students will be those who initially desire to use Mathematics E as preparation for Mathematics A or science or vocational courses having algebra as a prerequisite. We have previously remarked that the majority of students will not, in fact, go on. However, it would be desirable for this to be possible for those students with the requisite ability. Therefore, we suggest that at the beginning of the second semester a student who has such a goal and who is not heavily involved in the remedial aspect of this course could elect to devote the major portion of his laboratory time to studying programmed material on the techniques of algebra. In this way a course which will be terminal for the majority of students can become a nonterminal course for those willing and able to continue.

3. Reinforcement of Classroom Instruction

The laboratory may also be used to provide common extensions of the classroom work. As examples of such use of the laboratory we mention (a) the performance of certain geometric constructions by means of more effective devices than straightedge and compass, (b) computer programming and the use of computer terminals, (c) experiments in probability and statistics.

4. Management of a Mathematics Laboratory

The complexity of a mathematics laboratory may vary greatly, depending on the number of students at the level of Mathematics E which a given institution must serve. If only one or two sections of students are involved, procedures of diagnosis and prescription of suitable materials may reduce to conversation between the professor and the student. On the other hand, if several thousands of students and a large number of professors are involved in the basic mathematics program, the management of a laboratory may be a quite complex matter.

It may even be the case that some assistance from a computer is needed in order to keep records, to schedule conferences, and to give some feedback to the professor from the laboratory. If a laboratory reaches this degree of complexity, it would be natural for its services to be offered also to students whose mathematical deficiencies reveal themselves in other courses.

In some institutions the laboratory may be combined with similar endeavors in other disciplines and housed in a central learning resources center, thus relieving the mathematics faculty of the details of such an operation, much as a library relieves the mathematics faculty of the details of caring for collections of books. The difficulty with the latter arrangement is that the personnel of the laboratory would normally be expected to serve a tutorial function as well as performing the functions of clerks and librarians. Some colleges with well-developed laboratories have found students to be effective tutors and librarians provided they are properly directed and supervised. It is very important that professors have control over what the students are doing in the laboratory and that there is a method for professors to receive feedback from the laboratory.

There is naturally a question regarding the relative proportion of laboratory and classwork within the course. The answer would depend on local circumstances, and we have no desire to be prescriptive. The more strongly one believes in the importance of providing for individual differences and in the importance of insuring student activity, the greater the proportion of time one would tend to assign to the laboratory.

VI. QUALIFICATIONS FOR TEACHERS OF MATHEMATICS E

CUPM has considered in other reports the mathematical qualifications which it feels are necessary for the teaching of other mathematics courses in two- and four-year colleges. The present Panel feels that it would be unfortunate if teachers of Mathematics E were to constitute a subfaculty separate from those teaching other freshman and sophomore mathematics courses. Compared with other courses such as Mathematics A or Calculus, the teaching of the course would place somewhat smaller demands on depth in graduate mathematics training and somewhat greater demands on breadth in mathematics and related subjects. Thus, we do not see the need for a training program distinct from the usual one as preparation for the teaching of this course. However, teachers having little or no acquaintance with computing or statistics might require some small amount of additional training such as might be obtained at a summer institute or through some form of departmental or inservice training.

The teacher of Mathematics E must have certain attributes. Above all, he must be convinced of the value of the objectives of the course. In addition, he must have the ability to relate to a heterogeneous group of students. He must be willing to acquire an understanding of the applications of mathematics to the varied occupations to which these students may aspire; he must have sufficient understanding of computers, flowcharting, and estimation to weave these threads through the fabric of the course; he must have the versatility and experience in teaching needed to experiment with the laboratory to insure its success.

APPENDIX I

Sample Exercises

The following list of exercises was selected to demonstrate the existence of realistic problems that are likely to seem significant to the majority of students taking this course. The information given in the exercises is of the kind that would normally be encountered in the situation being described, and the questions asked in the problems are pertinent to the situation.

As indicated earlier in this recommendation, the exercises are not meant to convey any sort of desirable achievement level. In fact, the exercises given are generally more difficult than what might be termed an average exercise for the course. Most of the exercises can be reduced to a sequence of shorter and simpler problems. Some are appropriate for classroom discussion and perhaps would culminate in only a partial solution.

1. A builder quotes a prospective customer a price of $18 per square foot to build a certain style of house. The lot on which the house is to be built will cost an additional $4000. The customer knows that he can plan to spend at least $30,000, but no more than $40,000, on land and construction costs. Write an inequality whose solution will yield the range in which the size of the house must fall.

2. A grocery store advertises a peanut butter sale at a price of 2 jars for 85 cents. You notice that the net weight of peanut butter in each jar is 9 ounces. The same brand can be bought at 55 cents a jar containing 12 ounces of peanut butter. Should one buy the jars which are on sale?

3. Suppose you were required to take a 20 per cent cut in wages. If you are then given a 5 per cent increase in wages, does this mean that 1/4 of your cut has been restored?

4. What is wrong with the following procedure? To find the probability that an American citizen, chosen at random from a list of citizens, was born in a specified state, divide the number of favorable cases, 1, by the total number of states, 50, to obtain 1/50.

5. Two teams are playing a basketball game. A supporter of team A is willing to give you 3 to 1 odds and a supporter of team B will give you 2 to 1 odds, each betting on his favorite team. It is possible for you to bet x dollars with the first man and y dollars with the second and be $10 ahead no matter which team wins. Write two equations involving x and y which express that fact.

6. A car salesman receives $75 commission for each sale of one model of car and $100 for each sale of another model. In a certain period of time he would like to receive a commission of $3300. If he sold only cars of the first model, how many would he have to sell in order to earn the desired commission? Answer the same question for sales of only the second model of car. Set up an expression for the commission when cars of both models are sold. List additional items of information we would need in order to determine exactly how many cars of each model he must sell. Can he reach his goal exactly if he sells an odd number of the cars for which the commission is $75? Explain. When three or four pairs of solutions have been found, what do you notice about the number of sales of the $75 commission model? How many possible answers does the problem have?

7. A machinist is using a boring mill to rough-cut a collar for a steel shaft. In this process the speed of the lathe must be set carefully. Cutting too fast could burn the steel, while cutting too slowly is inefficient and produces a ragged edge. The speed in number of revolutions per minute (R) is given by $C = (\pi RD)/12$, where C is the best cutting speed for the materials being used in feet per minute and D is the diameter of the drill in inches. The value of π is approximately 3.14. If the drill is tool steel and the shaft is machine steel, D should be between 50 and 70 feet per minute. If the drill is 1 inch in diameter, what are the slowest and fastest rates of rotation at which the lathe should be set?

8. The current yearly gross salary of a state employee is $7500. Each year he is given a raise equal to the rise in the cost of living. Each month 10% of his salary is withheld for federal income tax, 5% for his state income tax, 4.8% for social security tax, and $9.50 for insurance. The rise in the cost of living during the current year is 6%. Write an equation whose solution will give his monthly take-home pay for the next year.

9. A housewife has a recipe for making brownies which calls for an 8" x 11" pan. By experience she knows that this will make two dozen servings. She is giving a big party and feels she needs 60 servings. She has two 8" x 11" pans and two 4" x 5" pans in the house. How many pans of which size should she use and how should she adjust her recipe? Note: This kind of problem gives an opportunity for some use of estimation. If the pans available have an area, say, 3.1 times that of the pan for which the recipe was written, the students should be made aware of the fact that tripling the recipe is good enough and that they shouldn't worry about the .1.

10. A risky operation used for patients with no other hope of survival has a survival rate of 80 per cent. What is the probability that at least four of the next five patients operated on will survive.

11. Assume that you are considering the purchase of a piece of land which costs $40,000. If a highway being planned passes through the land, the land will be worth $100,000. If, instead, the highway goes through nearby, the land will be worth $20,000. The probability that the highway will pass through the land is estimated to be .30. Evaluate the investment in the light of the probabilities given.

12. The following (with only trivial changes) is taken from the 1958 Boston and Maine Time Table:

Miles			A.M.
0	Dole Junction	Lv.	8:15
3	Hinsdale	Lv.	8:30
6	Ashuelot	Lv.	8:45
8	Winchester	Lv.	9:10
14	Wesport	Lv.	9:25
16	West Swanzey	Lv.	9:35
18	Swanzey	Lv.	9:45
21	Keene	Ar.	10:00

Observe that this trip of 21 miles requires an hour and three quarters and then check that this means that the average speed is 12 miles per hour. However, prove that an engine whose maximum speed is 20 miles an hour could not have made this trip on schedule.

13. Use data from the latest World Almanac giving U. S. population (official census) at 10-year intervals from 1880 to 1970 to make a line graph showing the population growth in the U. S. over this period. On the vertical scale start at 0 even though the population figures start at 50 million. In plotting points round off each population figure to the nearest million and choose scales of years on the horizontal axis in such a way as to make your picture approximately square.

Now make another line graph of the same data starting your vertical scale at 50 million and making the extent of your vertical scale about 4 inches while your horizontal scale is such that it extends across the width of your paper.

Compare the effect of the two graphs in portraying population growth.

14. A typical credit agreement reads:

Within 30 days after the billing date shown on each such monthly statement, Holder agrees to pay (1) the outstanding indebtedness ("New Balance") for "Purchases"; or (2) an installment of not less than 1/20th of such New Balance or $10, whichever is greater, and in addition a Finance Charge on the previous month's New Balance less "Payments and Credits" at the following rates: 1½% per month on so much of such amount as does not exceed $500; 1% per month on the excess of such amount over $500, or if the Finance Charge so computed is less than 50¢, a minimum Finance Charge of 50¢.

(Construct a flowchart for computing monthly payments. Do this for a $750 and a $100 purchase.)

15. Suppose that a person observing a carnival man in a game involving two tosses of a coin suspects the manner of tossing these coins favors both coins landing the same way, i.e., both heads or both tails. Such an outcome is unfavorable to the player. His decision on whether or not to play the game is based on the following rule: on the next throw of the two coins, if both show the same face he will not play; otherwise he will. Express the probability P that this person will play in terms of p, the probability with which the carnival man tosses a head. Determine P for $p = 0, .1, .25, .5, .75, 1$ and interpret the results.

16. One section of the Tax Reform Act of 1969 reads as follows:

"Low Income Allowance--

(1) The low income allowance is an amount equal to the sum of--

(A) the basic allowance, and
(B) the additional allowance.

(2) Basic Allowance--the basic allowance is an amount equal to the sum of--

(A) $200, plus
(B) $100, multiplied by the number of exemptions.

The basic allowance shall not exceed $1,000.

(3) Additional Allowance--

(A) the additional allowance is an amount equal to the excess (if any) of $900 over the sum of--

(i) $100, multiplied by the number of exemptions, plus
(ii) the income phase-out.

(B) Income Phase-out--The income phase-out is an amount equal to one-half of the amount by which the adjusted gross income for the taxable year exceeds the sum of--

(i) $1,100 plus
(ii) $625, multiplied by the number of exemptions."

The preceding statement offers a wealth of possibilities for problem construction including several possibilities for flowcharting problems.

(a) Write an equation that gives the basic allowance for a taxpayer with four exemptions.

(b) Write an equation that gives the basic allowance for a taxpayer with n exemptions.

(c) The statement limits the basic allowance to $1000. Write an inequality whose solution will yield the maximum number of exemptions allowable for computing the basic allowance.

(d) Write an equation that gives the additional allowance when the income phase-out is $400.

(e) Suppose the adjusted gross income for the taxable year is $4400. Write an equation that gives the income phase-out. (See Appendix II.)

17. The following is the procedure for computing social security benefits:

(1) Determine the "number of years" figure. If you were born before 1930, start with 1956. If born after 1929, start with the year you reached 27. Using your appropriate starting year, count that year and each one thereafter up to but not including the year in which you will be 65 if a man, or 62 if a woman.

(2) List the amount of taxed earnings for all years beginning with 1951. List no more per year than the amount subject to social security tax. These amounts have been: $3600 for 1951 through 1954; $4200 for 1955 through 1958; $4800 for 1959 through 1965; $6600 for 1966 and 1967; and $7800 for 1968 and succeeding years.

(3) Cross off this list your lowest earnings until the number remaining is equal to your "number of years" figure.

(4) Using the reduced list, add up all the earnings that are left, and divide by your number of years. This gives "average earnings," a figure which can be used with the aid of a table to determine the social security monthly benefit.

Construct a flowchart that describes the above process. (See Appendix II.)

18. Suppose you wish to buy a new car and you have determined that you do not want to pay more than \$130 per month for 30 months. If the interest is 5% per year on the original amount borrowed, then what price car should you consider? Approximately how many monthly payments are used to pay the interest? If you now decide that you might pay up to \$150 per month for 30 months, can you quickly estimate the price of a car you can consider?

19. The federal tax on capital gains in the stock market is roughly determined as follows: If you hold the stock for more than six months before selling it (a long-term gain), only half your profit is taxed; if you hold your stock six months or less before you sell it (a short-term gain), then your entire profit is fully taxed. For example, suppose you are in the 40% tax bracket and you have a stock which cost you \$1000. If you sell this stock for \$2000 within six months after you bought it, then your tax is 40% x \$1000, or \$400, so that your profit is \$600. If you sell this stock for \$2000 after holding it for more than six months, then your tax is 20% x \$1000, or \$200, so that your profit is \$800. Thus, long-term capital gain is clearly better than an equal amount of short-term gain.

Now suppose that your stock has shown a gain of \$1000 (paper profit), but six months has not lapsed since you purchased it and you fear that the stock will decline in price. Can you afford to risk losing part of your gain while you wait for it to become a long-term gain? Can you determine how much of your profit you can afford to lose and still be as well off after tax because of the lower tax on long-term gain?

If you think about it, you can show that the answer is

$$\frac{40\% - 20\%}{100\% - 20\%} = \frac{1}{4};$$

i.e., you can afford to lose 1/4 of your paper profit to wait for six months to pass and still have the same profit after tax. So a short-term gain of \$1000 is equivalent to a long-term gain of \$750 after tax. If you are in the 30% bracket, then what fraction of the paper profit can you afford to lose while waiting for six months to pass and still have the same profit after tax.

Examples of Artificial Word Problems

The exercises given as examples in this section are listed to demonstrate the kind of problem that is not in keeping with the spirit of the course. These problems are not necessarily considered to be innately "bad." In fact, some of the problems could justifiably be included in the course. However, the problems are obviously artificial and, hence, may fail to stimulate the pragmatically oriented students.

1. John is three times as old as Mary was five years ago. In twelve years he will be exactly twice as old as Mary. How old is Mary?

2. George and Frank can paint a house in 5 days by working together. Frank could paint the house in 8 days by working alone. How long would it take George to paint the house by working alone?

3. The tens digit of a 2-digit number is twice the units digit. The number is 20 more than the units digit. Find the number.

4. John has twice as many quarters as dimes and seven more nickels than dimes. If he has $2.30 in all, how many nickels does he have?

5. A certain room is 6 feet longer than it is wide. If the perimeter of the room is 82 feet, what is its width?

6. Compute three consecutive integers whose sum is 54.

7. Bob leaves city A at noon and drives 60 mph toward city B. Bill leaves city A at 1:30 p.m. and drives at 65 mph toward city B. At what time will Bill overtake Bob?

8. Frank bought thirteen pounds of meat for $10.55. He paid $1.10 per pound for beef and $.55 per pound for pork. How many pounds of each kind of meat did he buy?

APPENDIX II

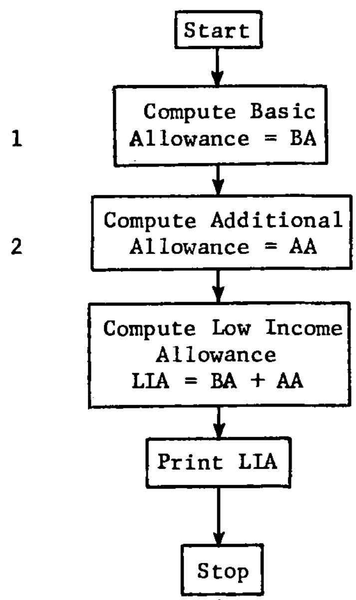
This Appendix contains analyses for exercises 15, 16, and 17.

15. This example, or one like it, might be used to introduce the notion of the uses of probability in decision making. Suppose the man can toss each coin so that for each, independently, p is the probability of heads. (If he tosses each fairly, $p = \frac{1}{2}$.) Then the probability of a head-tail combination is $P = 2p(1-p)$; this is

the probability that the observer will decide to play the game. Of course, p is unknown and can take on any value between 0 and 1. A graph of this quadratic, which students have made before (in the form $y = 2x(1 - x)$, probably), reveals that the maximum P is ½, occurring when p = ½ and the carnival man is tossing the pennies fairly. In the extreme cases of p = 0 or 1 (meaning what on the part of the carnival man?), we have P = 0, i.e., the observer will never play. Some inbetween values of p are of interest; should the carnival man have the finesse to insure always a ¼ chance of heads for each coin, P = 3/8--i.e., about 37½ per cent of the time the observer engages in the game. The value p = .1 would result in P = .18, or less than 1 chance in 5 of the observer's succumbing. The main point is that the computation of P in terms of p, or the graph of P as a function of p, has provided the observer with some (probability) basis of evaluating the consequences of his rule of action. In this case, the decision rule, although based on a minimum amount of data, is a reasonable one.

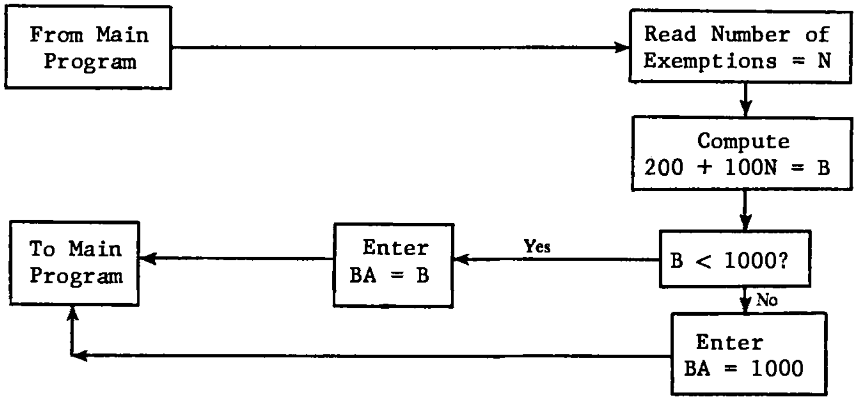
The flowcharts for Exercises 16 and 17 are especially illustrative of how flowcharting may be used to break large problems into sequences of smaller ones. Flowcharts for those two problems are given below:

(A) Flowchart for Exercise 16:

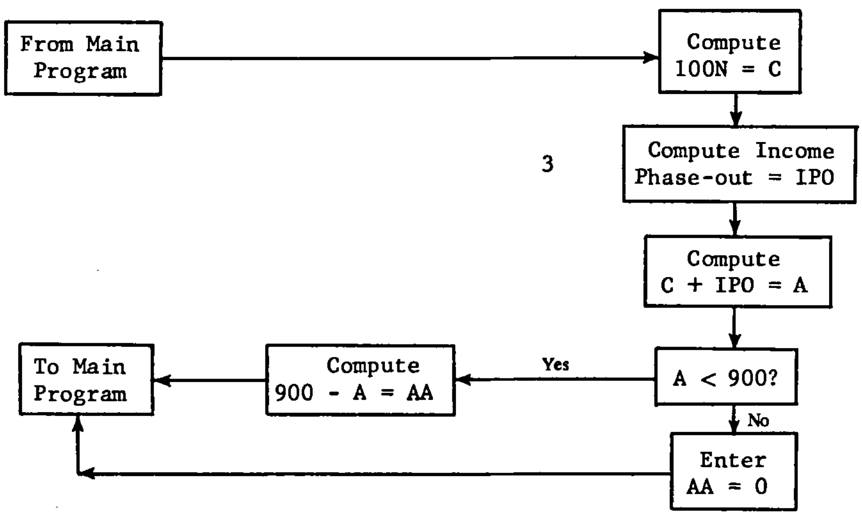


See next pages for expanded flowcharts for boxes 1 and 2.

Expansion of Box 1: Computing Basic Allowance

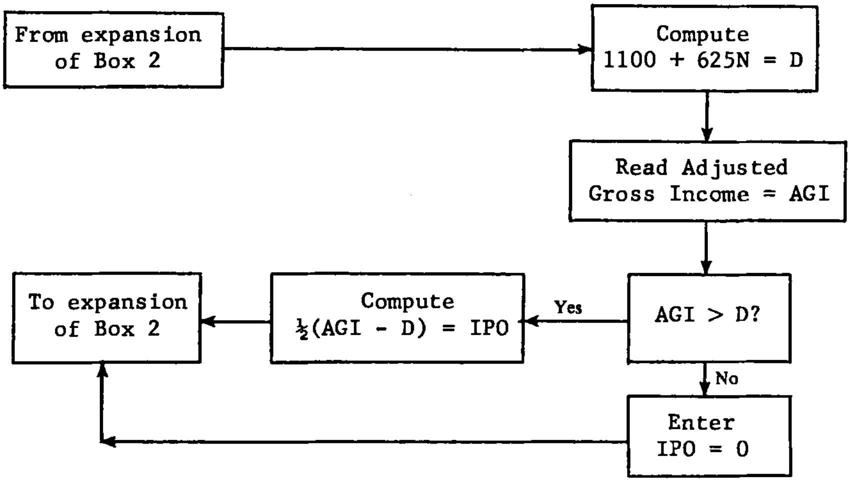


Expansion of Box 2: Computing Additional Allowance



See next page for expanded flowchart for Box 3.

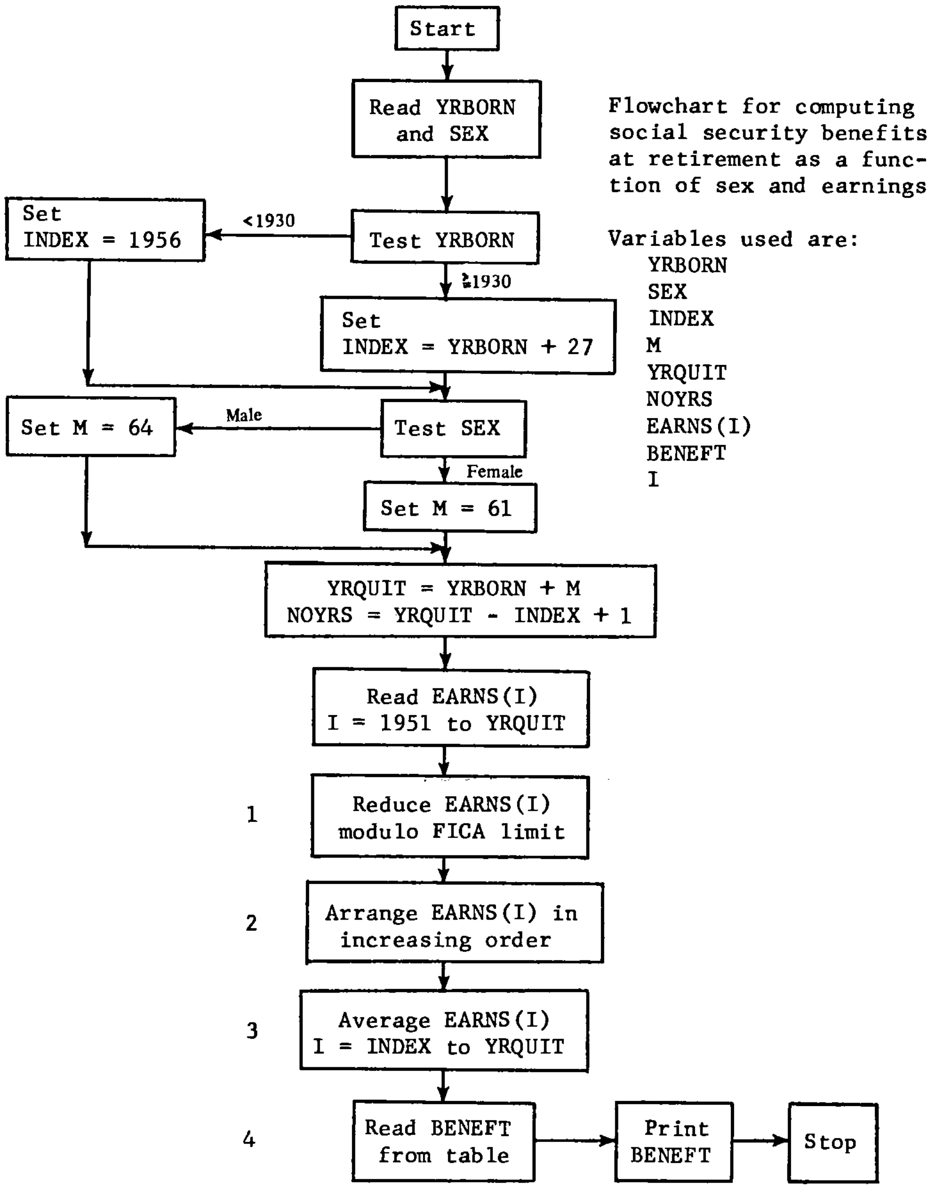
Expansion of Box 3: Computing Income Phase-out



(B) Flowchart for Exercise 17:

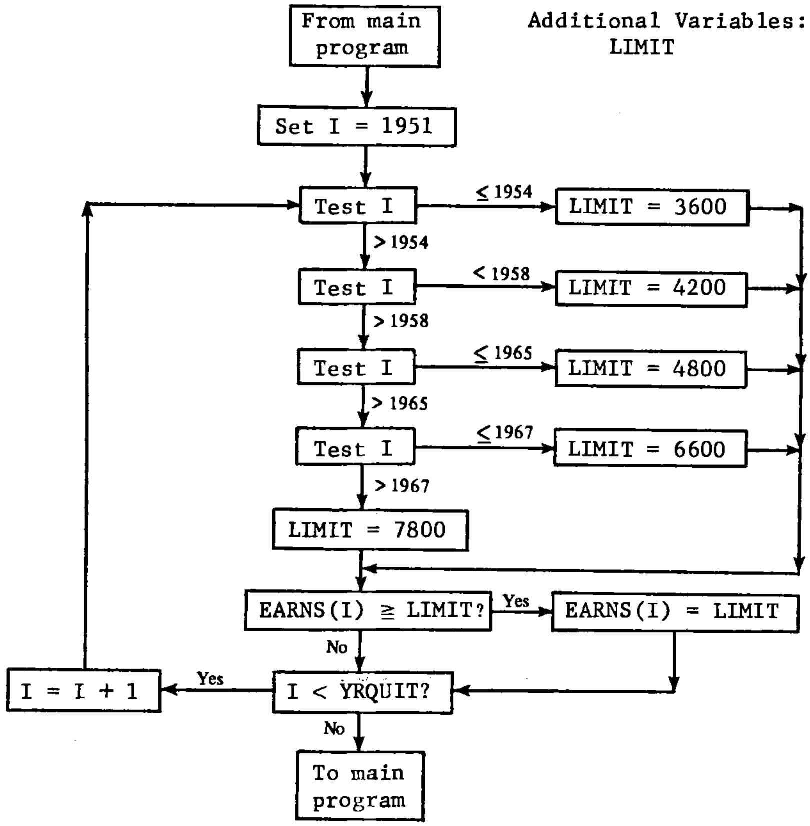
The table referred to in Part 4 of Exercise 17 is quite long. A portion of the table for a retired worker who has reached 65 is given below:

Average Earnings	Benefit (Per Month)
\$899 or less	\$55.00
\$900	\$70.00
\$1800	\$88.40
\$3000	\$115.00
\$4200	\$140.40
\$5400	\$165.00
\$6600	\$189.90
\$7800 or more	\$218.00

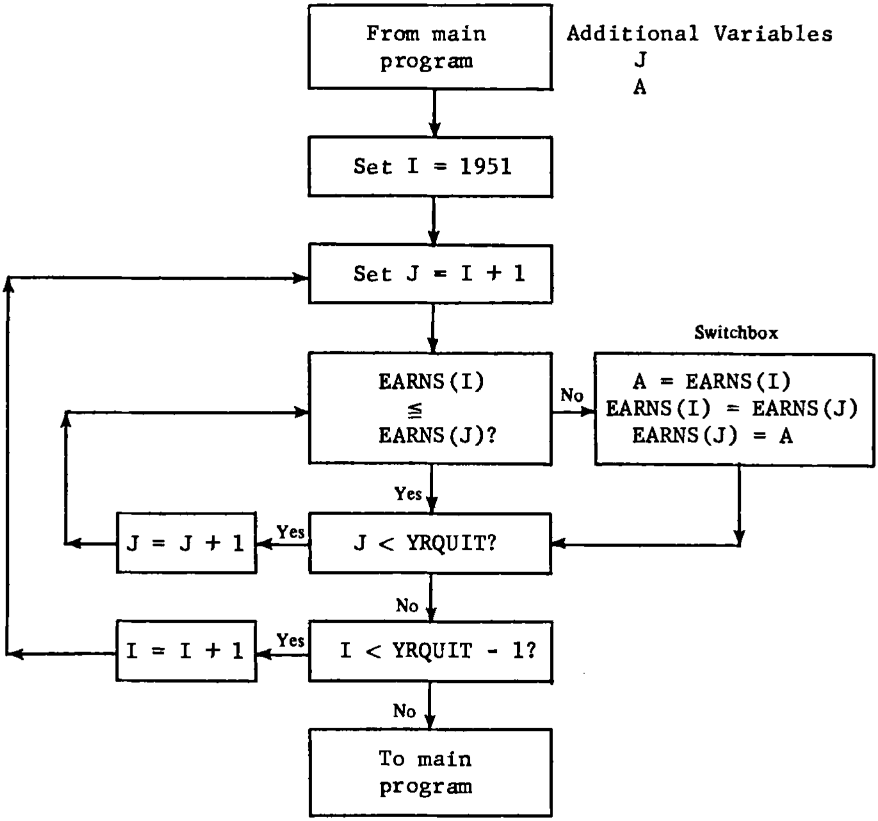


See other pages for expanded flowcharts for Boxes 1, 2, 3 and a discussion of Box 4.

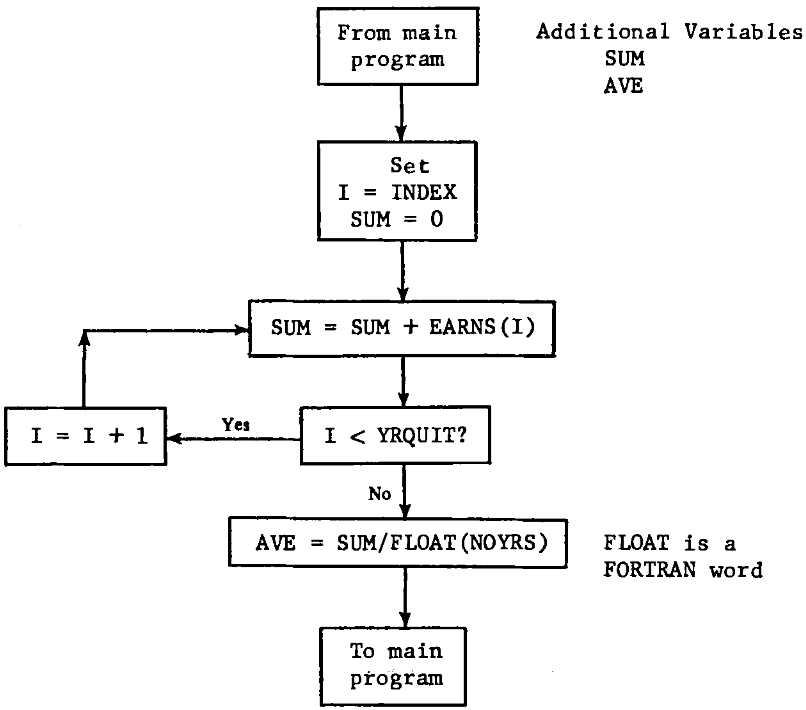
Expanded flowchart for Box 1--reducing actual earnings modulo the FICA limit for each year from 1951 to retirement



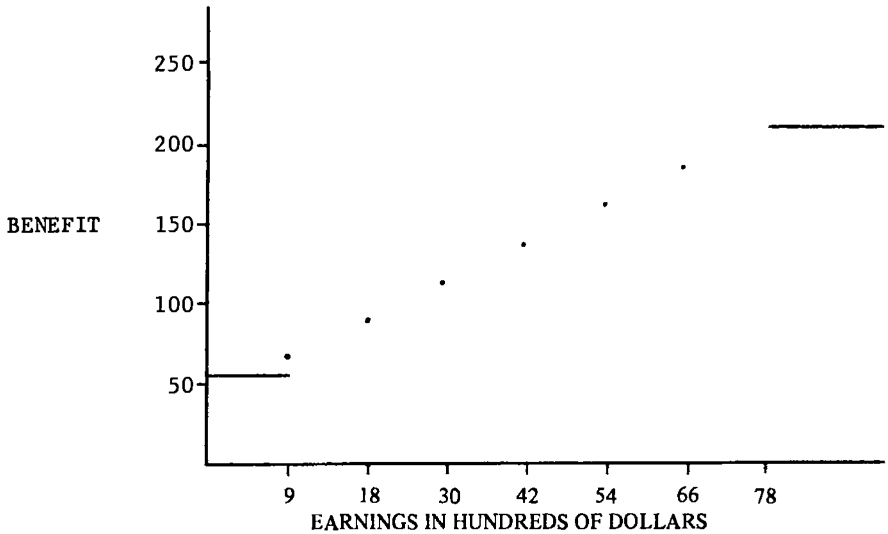
Expanded flowchart for Box 2--arranging FICA earnings in increasing order of magnitude from 1951 to retirement



Expanded flowchart for Box 3--computing average FICA earnings from index year to retirement



Box 4 of the flowchart offers several pedagogical opportunities. First, the student should plot a graph of benefit vs. earnings from the table on page 307. The graph might look like this:



Note that there is a jump of \$15 in benefit between earnings of \$899 and \$900.

Now, several approaches may be used for approximating the benefit associated with earnings between \$900 and \$7800. We list three possibilities that are all related to the fact that the isolated points on the graph seem to lie on a line.

1. The table could be stored in the computer memory and a program provided for interpolating linearly between adjacent points.

2. The student could select a line which actually passes through any two of the isolated points of the graph and determine its equation. There are several ways of doing this. For instance, the line on the points (3000, 115) and (5400, 165) has equation

$$\text{BENEFIT} = 115 + \frac{1}{48}(\text{EARNINGS} - 3000).$$

   This is a "good" line in the sense that it nearly contains the other points of the table, except for the first, the approximation of benefit being within \$2 of the table value in every case.

3. One could use a least-squares fit on some points of the graph. For instance, the equation

$$\text{BENEFIT} = \frac{253}{12000} \times \text{EARNINGS} + \frac{46071}{900}$$

fits the five points with abscissas 1800, 3000, 4200, 5400, and 6600 in this sense. This is a "good" line too. It yields benefits corresponding to tabular values of earnings that are within $1 of tabular benefits in all cases.

The table given in this example was for a man retiring at age 65. A class project could be to find similar information for a woman retiring at age 62 and to complete the program.

A BASIC LIBRARY LIST

FOR

TWO-YEAR COLLEGES

January 1971

TABLE OF CONTENTS

INTRODUCTION

The Basic Library List, [page 1], published by CUPM in 1965, was intended to define a minimal college mathematics library. More recently an ad hoc committee assisted by two-year college, four-year college, and university teachers prepared the present basic library list for two-year colleges. The aims of this list are quite similar to those of the Basic Library List, namely:

1. To provide the student with introductory material in areas of mathematics new to him

2. To provide the interested student with material collateral to the material he is studying in courses

3. To provide the student with material somewhat more advanced than he is likely to encounter in his course work

4. To provide the faculty with reference material, but generally below graduate level

5. To provide the general reader with elementary material in the field of mathematics

6. To provide trainees in various occupations, such as nurses, farmers, technologists, etc., with material designed for their particular needs

A further word concerning item 4 is in order. It is recognized that many faculty members at two-year colleges are still engaged in graduate study; however, it is felt that it is not the responsibility of the two-year college library to provide them reference material for their graduate courses. The reason for this is two-fold: first, such material should be available to them at the institution where they are pursuing graduate work; second, inclusion of such material in a two-year college library might place too heavy a financial strain on the two-year college.

The list is intended as a basic list from which the library can expand according to the needs and interests of the faculty and the students. Needs at different schools will, of course, differ, and the library should reflect the local needs; in this regard see the comment under Sections 6 through 11. There has been a concerted effort to keep the list small; one means of doing this has been to combine under one heading books of a somewhat different character. Alternate choices are listed so that a library can utilize its present holdings to the full. In the interest of keeping the list small, many books of merit have had to be omitted; it is also possible that, despite assiduous searching, the committee has overlooked books which should have been included. Furthermore, books which have been included in the list have been included because of their value as library books; no judgment is made as to their utility as texts for courses.

Some books are mentioned at more than one place in the list. This is not accidental. Since some schools will want to purchase only those portions pertinent to their programs, the committee wanted to be sure that relevant books were covered in each section.

The matter of library books in various remedial areas, i.e., arithmetic, elementary algebra, and the like, has been discussed at length by the committee. It is clear that these subjects are taught at the two-year college level, and that the character of the texts used there varies considerably from those used at lower levels. Despite this, we feel that for reference use by students the two-year college is well advised to include among its books those texts used by the local high schools or texts covering comparable material.

The library committee worked on this list during a two-year period ending in 1970; therefore, books with first publication dates after 1969 are generally not included in the list. Finally, it must be recognized that the list covers a considerable range of sophistication beginning at quite an elementary level. The exposition in some of the more elementary books differs from the sort of presentation one expects at more advanced levels in being more discursive and less axiomatic. The mathematics may occasionally appear not to be in the best tradition of formal practice; however, these books fill a very real need for the audience intended and any solecisms encountered are not so serious as to remove the books from consideration.

After preliminary versions of this list were written, the *ad hoc* committee sought the advice and comments of some 30 reviewers. The reviewers were chosen so that specialists in each of the areas represented in the list would be able to comment. The list thus reflects not only the competencies of the committee but also the informed views of the reviewers.

These recommendations contain about 510 volumes, of which approximately 170 volumes are to be chosen; this does not include journals or series in Sections 22 and 23.

The symbol * indicates that the book has been listed more than once.

1. HISTORICAL, GENERAL, AND RECREATIONAL

History--Both of the following:

1.1 Bell, Eric T. *Men of Mathematics*. New York, Simon and Schuster, Inc., 1961.

1.2 Boyer, Carl B. *A History of Mathematics*. New York, John Wiley and Sons, Inc., 1968.

And at least one of the following:

1.3a Eves, Howard. *Introduction to the History of Mathematics*, 3rd ed. New York, Holt, Rinehart and Winston, Inc., 1969.

1.3b Smith, D. E. *History of Mathematics*, 2 vols. New York, Dover Publications, Inc. Vol. I, General Survey of the History of Elementary Mathematics; Vol.II, Special Topics of Elementary Mathematics.

1.3c Struik, D. J., ed. *A Source Book in Mathematics: Twelve Hundred to Eighteen Hundred*. Cambridge, Massachusetts, Harvard University Press, 1969.

1.3d van der Waerden, B. L. *Science Awakening*. New York, Oxford University Press, 1961; New York, John Wiley and Sons, Inc., paper.

General--All of the following:

1.4 Courant, Richard and Robbins, Herbert. *What is Mathematics?* New York, Oxford University Press, 1941.

1.5 Eves, Howard and Newsom, Carroll V. *Introduction to the Foundations and Fundamental Concepts of Mathematics*, rev. ed. New York, Holt, Rinehart and Winston, Inc., 1965.

1.6 Klein, Felix. *Elementary Mathematics from an Advanced Standpoint*. Vol. 1, Arithmetic, Algebra, Analysis. New York, Dover Publications, Inc., 1968.

1.7 National Council of Teachers of Mathematics. *Enrichment Mathematics for the Grades* (27th Yearbook) and *Enrichment Mathematics for the High Schools* (28th Yearbook). Washington, D. C., National Council of Teachers of Mathematics, 1963.

1.8 Rademacher, Hans and Toeplitz, Otto. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur*. Princeton, New Jersey, Princeton University Press, 1965.

1.9 Sawyer, Walter W. *Mathematician's Delight*. Baltimore, Maryland, Penguin Books, Inc., 1943.

1.10 Steinhaus, Hugo. *Mathematical Snapshots*, 2nd ed. New York, Oxford University Press, 1969.

And at least two of the following:

1.11a Cadwell, James H. Topics in Recreational Mathematics. New York, Cambridge University Press, 1966.

1.11b Court, Nathan A. Mathematics in Fun and in Earnest. New York, Mentor Press, 1961. Out of print.

1.11c Kac, Mark and Ulam, Stanislaw M. Mathematics and Logic: Retrospect and Prospects. New York, Frederick A. Praeger, Inc., 1968.

1.11d Kasner, Edward and Newman, James R. Mathematics and the Imagination. New York, Simon and Schuster, Inc., 1940.

1.11e Lockwood, Edward H. and Prag, A. A Book of Curves. New York, Cambridge University Press, 1961.

1.11f Ogilvy, C. Stanley. Tomorrow's Math: Unsolved Problems for the Amateur. New York, Oxford University Press, 1962.

1.11g Pedoe, Daniel. Gentle Art of Mathematics. New York, The Macmillan Company, 1963; Baltimore, Maryland, Penguin Books, Inc., 1969, paper.

1.11h Sawyer, Walter W. Prelude to Mathematics. Baltimore, Maryland, Penguin Books, Inc., 1955.

1.11i Stein, Sherman K. Mathematics: The Man-Made Universe, An Introduction to the Spirit of Mathematics, 2nd ed. San Francisco, California, W. H. Freeman and Company, 1969.

And one of the following:

1.12a Kline, Morris. Mathematics in Western Culture. New York, Oxford University Press, 1964.

1.12b Scientific American Editors. Mathematics in the Modern World. San Francisco, California, W. H. Freeman and Company, 1968.

Mathematical Recreations--At least one of the following:

1.13a Ball, W. W. R. and Coxeter, H. S. M. Mathematical Recreations and Essays, rev. ed. New York, The Macmillan Company, 1962.

1.13b Kraitchik, Maurice. Mathematical Recreations, 2nd ed. New York, Dover Publications, Inc., 1953.

And at least one of the following (problems and puzzles):

1.14a Bakst, Aaron. Mathematical Puzzles and Pastimes, 2nd ed. New York, Van Nostrand Reinhold Company, 1965.

1.14b Gamow, George and Stern, Marvin. Puzzle-Math. New York, Viking Press, Inc., 1958.

1.14c Gardner, Martin, ed. Scientific American Book of Mathematical Puzzles and Diversions. New York, Simon and Schuster, Inc., 1964.

1.14d Gardner, Martin, ed. Second Scientific American Book of Mathematical Puzzles and Diversions. New York, Simon and Schuster, Inc., 1965.

1.14e Gardner, Martin. Unexpected Hanging and Other Mathematical Diversions. New York, Simon and Schuster, Inc., 1968.

1.14f Graham, Lloyd A. Ingenious Mathematical Problems and Methods. New York, Dover Publications, Inc., 1959.

1.14g Graham, Lloyd A. Surprise Attack in Mathematical Problems. New York, Dover Publications, Inc., 1968.

1.14h Mott-Smith, Geoffrey. Mathematical Puzzles for Beginners and Enthusiasts, 2nd ed. New York, Dover Publications, Inc., 1954

1.14i Phillips, Hubert C. My Best Puzzles in Mathematics. New York, Dover Publications, Inc., 1961.

Various Topics (about mathematics and mathematicians)--All of the following:

1.15 Committee on Support of Research in the Mathematical Sciences. The Mathematical Sciences: A Collection of Essays. Cambridge, Massachusetts, MIT Press, 1969.

1.16 Cundy, Henry M. and Rollett, A. P. Mathematical Models, 2nd ed. New York, Oxford University Press, 1961.

1.17 Hadamard, Jacques. Psychology of Invention in the Mathematical Field. New York, Dover Publications, Inc., 1945.

1.18 Hardy, G. H. Mathematician's Apology, rev. ed. New York, Cambridge University Press, 1967.

1.19 Newman, James R. The World of Mathematics, 4 vols. New York, Simon and Schuster, 1962. Vol. I, Men and Numbers; Vol. II, World of Laws and the World of Chance; Vol. III, Mathematical Way of Thinking; Vol. IV, Machines, Music and Puzzles.

1.20 Pólya, Gyorgy. How to Solve It, 2nd ed. New York, Doubleday and Company, Inc., 1957.

1.21 Pólya, Gyorgy. Mathematical Discovery on Understanding, Learning and Teaching Problem Solving, 2 vols. New York, John Wiley and Sons, Inc., 1962.

Sets and Collections of Books

1.22 New Mathematical Library, 22 vols. New York, Random House/Singer School Division.

Numbers: Rational and Irrational (NML 1). Ivan Niven

*What is Calculus About? (NML 2). W. W. Sawyer

An Introduction to Inequalities (NML 3). E. Beckenbach and R. Bellman

*Geometric Inequalities (NML 4). Nicholas D. Kazarinoff

The Lore of Large Numbers (NML 6). P. J. Davis

Uses of Infinity (NML 7). Leo Zippin

Geometric Transformations (NML 8). I. M. Yaglom, translated by Allen Shields

Continued Fractions (NML 9). Carl D. Olds

*Graphs and Their Uses (NML 10). Oystein Ore

Hungarian Problem Book I and II (NML 11 and 12). Translated by E. Rapaport

Episodes from the Early History of Mathematics (NML 13). A. Aaboe

Groups and Their Graphs (NML 14). I. Grossman, et al.

The Mathematics of Choice (NML 15). Ivan Niven

From Pythagoras to Einstein (NML 16). K. O. Friedrichs

The MAA Problem Book II (NML 17).

*First Concepts of Topology (NML 18). W. G. Chinn and N. E. Steenrod.

Geometry Revisited (NML 19). H. S. M. Coxeter and S. L. Greitzer.

Invitation to Number Theory (NML 20). Oystein Ore

Geometric Transformations II (NML 21). I. M. Yaglom, translated by Allen Shields

Elementary Cryptanalysis--A Mathematical Approach (NML 22). Abraham Sinkov

2. FINITE MATHEMATICS

Although Finite Mathematics is not well defined, it is generally understood to encompass modern problems in elementary set theory, logic, probability, linear programming, and theory of games solved by methods not involving the calculus. In the following list, all the books deal with these topics.

At least two of the following:

2.1a Crouch, Ralph B. Finite Mathematics and Statistics for Business. New York, McGraw-Hill Book Company, 1968.

*2.1b Kaye, Norman J. Elementary Quantitative Techniques for Business Problem Solving. Belmont, California, Dickenson Publishing Company, Inc., 1969.

2.1c Kemeny, John G., et al. Finite Mathematics with Business Applications. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

*2.1d Kemeny, John G., et al. Introduction to Finite Mathematics, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

2.1e Marcus, Marvin. A Survey of Finite Mathematics. Boston, Massachusetts, Houghton Mifflin Company, 1969.

2.1f Richardson, William H. Finite Mathematics. New York, Harper and Row, Publishers, 1968.

2.1g Wheeler, Ruric E. and Peeples, W. D. Modern Mathematics for Business Students. Belmont, California, Brooks/Cole Publishing Company, 1969.

3. PREPARATION FOR CALCULUS

The following list is at the level of Mathematics 0 [page 75] or Mathematics A [page 216]. It is intended to provide reference material for courses leading to the calculus but does not include programmed materials or books for remedial work.

At least two of the following:

3.1a Dolciani, Mary P., *et al*. *Modern Introductory Analysis*. Boston, Massachusetts, Houghton Mifflin Company, 1970.

3.1b Golightly, Jacob F. *Precalculus Mathematics--Algebra and Trigonometry*. Philadelphia, Pennsylvania, W. B. Saunders Company, 1968.

3.1c Horner, Donald R. *Precalculus: Elementary Functions and Relations*. New York, Holt, Rinehart and Winston, Inc., 1969.

3.1d Hu, Sze-Tsen. *Elementary Functions and Coordinate Geometry*. Chicago, Illinois, Markham Publishing Company, 1969.

3.1e Knight, Ronald A. and Hoff, William E. *Introduction to the Elementary Functions*. Belmont, California, Dickenson Publishing Company, Inc., 1969.

3.1f Marcus, Marvin and Minc, Henryk. *Elementary Functions and Coordinate Geometry*. Boston, Massachusetts, Houghton Mifflin Company, 1969.

At least two of the following:

3.2a Allendoerfer, Carl B. and Oakley, Cletus O. *Principles of Mathematics*, 3rd ed. New York, McGraw-Hill Book Company, 1969.

3.2b Good, R. A. *Introduction to Mathematics*. New York, Harcourt Brace Jovanovitch, Inc., 1966.

3.2c Haag, Vincent H. and Western, Donald W. *Introduction to College Mathematics*, 2nd ed. New York, Harcourt Brace Jovanovitch, Inc., 1968.

3.2d Meserve, Bruce E., *et al*. *Principles of Advanced Mathematics*, rev. ed. New York, Random House/Singer School Division, 1970.

3.2e Pownall, Malcolm W. *A Prelude to the Calculus*. New York, McGraw-Hill Book Company, 1967.

3.2f Shanks, Merrill E., *et al*. *Pre-Calculus Mathematics*, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1968.

3.2g Rosenbloom, Paul C. and Schuster, Seymour. Prelude to Analysis. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

3.2h Zwier, Paul J. and Nyoff, Larry R. Essentials of College Mathematics. New York, Holt, Rinehart and Winston, Inc., 1969.

4. CALCULUS

General Calculus. There are many good calculus books available. Several of these should be in the library. The following represent some of the various possible approaches.

4.1a Bers, Lipman. Calculus. New York, Holt, Rinehart and Winston, Inc., 1969.

4.1b Crowell, Richard H. and Slesnick, William E. Calculus with Analytic Geometry. New York, W. W. Norton and Company, Inc., 1968.

4.1c de Leeuw, Karel. Calculus. New York, Harcourt Brace Jovanovitch, Inc., 1966.

4.1d Johnson, Richard E. and Kiokemeister, F. L. Calculus with Analytic Geometry, 4th ed. Boston, Massachusetts, Allyn and Bacon, Inc., 1964.

4.1e Protter, Murray H. and Morrey, Charles B., Jr. Calculus with Analytic Geometry: A First Course, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

4.1f Sherwood, George E. and Taylor, Angus E. Calculus, 3rd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1954.

4.1g Thomas, George B., Jr. Calculus and Analytic Geometry, 4th ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

Honors Calculus--One or more of the following:

4.2a Apostol, Tom M. Calculus, 2 vols. Waltham, Massachusetts, Blaisdell Publishing Company. Vol. I, One-Variable Calculus with an Introduction to Linear Algebra, 2nd ed., 1967; Vol. II, Multi-Variable Calculus and Linear Algebra, with Applications to Differential Equations and Probability, 2nd ed., 1969.

4.2b Courant, Richard. Differential and Integral Calculus, 2 vols. (translated by E. J. McShane) New York, John Wiley and Sons, Inc. Vol. I, 2nd ed., 1937; Vol. II, 1936.

4.2c Hardy, G. H. A Course of Pure Mathematics. New York, Cambridge University Press, 1959.

4.2d Spivak, Michael. Calculus. New York, The Benjamin Company, Inc., 1967.

Background--At least one of the following:

4.3a Boyer, Carl B. History of the Calculus and Its Conceptual Development. New York, Dover Publications, Inc., 1959.

*4.3b Khinchin, Alexander Y. Eight Lectures on Mathematical Analysis. Lexington, Massachusetts, D. C. Heath and Company, 1965.

*4.3c Sawyer, W. W. What is Calculus About? New York, Random House, Inc., 1961.

4.3d Selected Papers on Calculus, Tom Apostol, editor. Belmont, California, Dickenson Publishing Company, Inc., 1969.

4.3e Toeplitz, Otto. Calculus: A Genetic Approach. (edited by G. Köthe, translated by L. Lange) Chicago, Illinois, University of Chicago Press, 1963.

See also 1.22.

Calculus of Several Variables--At least one of the following:

4.4a Fadell, Albert G. Vector Calculus and Differential Equations, vol. III. New York, Van Nostrand Reinhold Company, 1968.

4.4b Osserman, Robert. Two-Dimensional Calculus. New York, Harcourt Brace Jovanovitch, Inc., 1968.

4.4c Williamson, Richard, et al. Calculus of Vector Functions. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1968.

Advanced Calculus--At least one of the following:

4.5a Apostol, Tom M. Mathematical Analysis: A Modern Approach to Advanced Calculus. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1957.

4.5b Buck, R. Creighton. Advanced Calculus, 2nd ed. New York, McGraw-Hill Book Company, 1965.

4.5c Kaplan, Wilfred. Advanced Calculus. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1952.

4.5d Kreider, Donald L., et al. Introduction to Linear Analysis. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1966.

4.5e Taylor, Angus E. Advanced Calculus. Waltham, Massachusetts, Blaisdell Publishing Company, 1955.

5. STATISTICS AND PROBABILITY

General--At least one of the following:

5.1a Huff, Darrell and Geis, Irving. How to Lie with Statistics. New York, W. W. Norton and Company, Inc., 1954.

5.1b Levinson, Horace C. Chance, Luck and Statistics, 2nd ed. New York, Dover Publications, Inc., 1963.

5.1c Moroney, M. J. Facts From Figures. Baltimore, Maryland, Penguin Books, Inc., 1956.

Elementary Statistics--At least one of the following:

5.2a Blackwell, David. Basic Statistics. New York, McGraw-Hill Book Company, 1969.

5.2b Dixon, Wilfrid J. and Massey, F. J., Jr. Introduction to Statistical Analysis, 3rd ed. New York, McGraw-Hill Book Company, 1969.

5.2c Freund, John E. Modern Elementary Statistics, 3rd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

5.2d Hodges, Joseph L. and Lehmann, E. L. Basic Concepts of Probability and Statistics, 2nd ed. San Francisco, California, Holden-Day, Inc., 1970.

5.2e Hoel, Paul G. Elementary Statistics, 2nd ed. New York, John Wiley and Sons, Inc., 1966.

5.2f Mode, Elmer B. Elements of Probability and Statistics, 3rd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

5.2g Mosteller, Frederick, et al. Probability with Statistical Applications, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970.

5.2h Wallis, Wilson A. and Roberts, Harry V. Statistics: A New Approach. New York, The Macmillan Company, 1956.

5.2i Wolf, Frank L. Elements of Probability and Statistics. New York, McGraw-Hill Book Company, 1962.

Mathematical Statistics--At least one of the following:

5.3a Brunk, H. D. Introduction to Mathematical Statistics, 2nd ed. Waltham, Massachusetts, Blaisdell Publishing Company, 1965.

5.3b Hoel, Paul G. Introduction to Mathematical Statistics, 3rd ed. New York, John Wiley and Sons, Inc., 1970.

5.3c Hogg, Robert V. and Craig, A. T. Introduction to Mathematical Statistics, 3rd ed. New York, The Macmillan Company, 1970.

5.3d Meyer, Paul L. Introductory Probability and Statistical Applications, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970.

5.3e Mood, Alexander M. and Graybill, F. A. Introduction to the Theory of Statistics, 2nd ed. New York, McGraw-Hill Book Company, 1963.

Elementary Probability--At least one of the following:

5.4a Berman, Simeon M. Elements of Probability. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

5.4b Gangolli, R. A. and Ylvisaker, Donald. Discrete Probability. New York, Harcourt Brace Jovanovitch, Inc., 1967.

5.4c Gnedenko, Boris V. and Khinchin, Alexander Y. Elementary Introduction to the Theory of Probability, 5th ed. (translated by Leo F. Boron) New York, Dover Publications, Inc., 1961.

5.4d Goldberg, Samuel. Probability: An Introduction. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1960.

5.4e Scheerer, Anne C. Probability on Discrete Sample Spaces with Applications. Scranton, Pennsylvania, Intext Educational Publishers, 1969.

5.4f Thompson, W. A., Jr. Applied Probability. New York, Holt, Rinehart and Winston, Inc., 1969.

Intermediate Probability

5.5 Feller, William. Introduction to Probability Theory and Its Applications, vol. I, 3rd ed. New York, John Wiley and Sons, Inc., 1968.

And at least one of the following:

5.6a Breiman, Leo. Probability and Stochastic Processes, with a View Towards Applications. Boston, Massachusetts, Houghton Mifflin Company, 1969.

5.6b Parzen, Emanuel. Modern Probability Theory and Its Applications. New York, John Wiley and Sons, Inc., 1960.

5.6c Rozanov, Y. A. Introductory Probability Theory. (translated by M. Silverman) Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

Other Approaches--At least one of the following:

5.7a Chernoff, Herman and Moses, L. E. Elementary Decision Theory. New York, John Wiley and Sons, Inc., 1959.

5.7b Kraft, Charles H. and van Eeden, Constance. Nonparametric Introduction to Statistics. New York, The Macmillan Company, 1968.

5.7c Savage, I. Richard. Statistics: Uncertainty and Behavior. Boston, Massachusetts, Houghton Mifflin Company, 1968.

Applied Statistics--At least one of the following:

5.8a Chorafas, Dimitris N. Statistical Processes and Reliability Engineering. New York, Van Nostrand Reinhold Company. Out of print.

5.8b Cochran, W. G. and Cox, G. M. Experimental Designs, 2nd ed. New York, John Wiley and Sons, Inc., 1957.

5.8c Grant, Eugene L. Statistical Quality Control, 3rd ed. New York, McGraw-Hill Book Company, 1964.

5.8d Hays, William L. Statistics for Psychologists. New York, Holt, Rinehart and Winston, Inc., 1963.

5.8e Mainland, Donald. Elementary Medical Statistics, 2nd ed. Philadelphia, Pennsylvania, W. B. Saunders Company, 1963. Out of print.

5.8f Schefler, William C. Statistics for the Biological Sciences. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

5.8g Schlaifer, Robert. Introduction to Statistics for Business Decisions. New York, McGraw-Hill Book Company, 1961.

5.8h Wasserman, W. and Neter, J. Fundamental Statistics for Business and Economics. Boston, Massachusetts, Allyn and Bacon, Inc., 1966.

5.8i Wine, Russell L. Statistics for Scientists and Engineers. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964.

5.8j Yates, Frank. Sampling Methods for Censuses and Surveys, 3rd ed. New York, Hafner Publishing Company, Inc., 1960.

Tables--At least one of the following:

*5.9a Burington, Richard S. and May, Donald C., Jr. Handbook of Probability and Statistics with Tables, 2nd ed. New York, McGraw-Hill Book Company, 1969.

*5.9b Chemical Rubber Company. Handbook of Tables for Probability and Statistics, 2nd ed. Cleveland, Ohio, The Chemical Rubber Company, 1968.

5.9c Owen, Donald B. Handbook of Statistical Tables. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

6. VOCATIONAL MATHEMATICS

The books under this heading are specific to a particular program offered in various two-year colleges. If that program is not offered, there may be no need for the library to purchase these books.

The listings are reference materials for general shop courses. The books in 6.2 are somewhat more general than the ones in 6.3. For further books that may be useful under this heading, see also those listed in Section 8--Technology.

6.1 Grazda, Edward E., _et al_. _Handbook of Applied Mathematics_, 4th ed. New York, Van Nostrand Reinhold Company, 1966.

At least one of the following:

6.2a Levine, Samuel. _Vocational and Technical Mathematics in Action_. New York, Hayden Book Company, 1969.

6.2b Slade, Samuel and Margolis, L. _Mathematics for Technical and Vocational Schools_, 5th ed. New York, John Wiley and Sons, Inc., 1968.

And at least one of the following:

6.3a McMackin, Frank J. and Shaver, John H. _Mathematics of the Shops_, 3rd ed. New York, Van Nostrand Reinhold Company, 1968.

6.3b Wolfe, J. H. and Phelps, E. R. _Practical Shop Mathematics_, 2 vols., 4th ed. New York, McGraw-Hill Book Company. Vol. 1, Elementary, 1959; Vol. 2, Advanced, 1960.

7. BUSINESS

The books under this heading are specific to a particular program offered in various two-year colleges. If that program is not offered, there may be no need for the library to purchase these books.

At least two of the following:

7.1a Bush, Grace A. and Young, John E. _Foundations of Mathematics with Applications to the Social and Management Sciences_. New York, McGraw-Hill Book Company, 1968.

*7.1b Kaye, Norman J. _Elementary Quantitative Techniques for Business Problem Solving_. Belmont, California, Dickenson Publishing Company, Inc., 1969.

7.1c Locke, Flora M. and Dehr, D. _College Mathematics for Business_. New York, John Wiley and Sons, Inc., 1969.

7.1d Roueche, Nelda W. Business Mathematics: A Collegiate Approach. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

7.1e Snyder, Llewellyn R. Essential Business Mathematics, 5th ed. New York, McGraw-Hill Book Company, 1967.

Mathematics of Finance--At least two of the following:

7.2a Cissell, Robert and Cissell, Helen. Mathematics of Finance, 3rd ed. Boston, Massachusetts, Houghton Mifflin Company, 1969.

7.2b Curtis, Arthur B. and Cooper, J. (Revised by W. McCallion) Mathematics of Accounting, 4th ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961.

7.2c Freund, John E. College Mathematics with Business Applications. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

7.2d Hart, William L. Mathematics of Investment, 4th ed. Lexington, Massachusetts, D. C. Heath and Company, 1958.

7.2e Rider, P. R. and Fisher, C. H. Mathematics of Investment. Ann Arbor, Michigan, Ulrich's Books, Inc., 1951.

7.2f Rosenberg, R. Robert. College Mathematics, 4th ed. New York, McGraw-Hill Book Company, 1967.

Mathematics of Management--At least one of the following:

7.3a Corcoran, A. Wayne. Mathematical Applications in Accounting. New York, Harcourt Brace Jovanovitch, Inc., 1968.

7.3b Dean, Burton V., et al. Mathematics for Modern Management. New York, John Wiley and Sons, Inc., 1963.

7.3c Goetz, Billy E. Quantitative Methods: A Survey and Guide for Managers. New York, McGraw-Hill Book Company, 1965.

7.3d Springer, Clifford H., et al. Mathematics for Management Sciences. Homewood, Illinois, Richard D. Irwin, Inc. Vol. I, Basic Mathematics, 1965; Vol. II, Advanced Methods and Models, 1965; Vol. III, Statistical Inference, 1966; Vol. IV, Probabilistic Models, 1968.

7.3e Stern, Mark E. Mathematics for Management. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1963.

Every library should have the following:

7.4 Minrath, William R. Handbook of Business Mathematics, 2nd ed. New York, Van Nostrand Reinhold Company, 1967.

8. TECHNOLOGY

The books under this heading are specific to a particular program offered in various two-year colleges. If that program is not offered, there may be no need for the library to purchase these books.

Engineering Technology--At least one of the following:

8.1a Blakeley, Walter R. Calculus for Engineering Technology. New York, John Wiley and Sons, Inc., 1968.

8.1b Placek, Ronald J. Technical Mathematics with Calculus. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1968.

8.1c Rice, Harold S. and Knight, Raymond M. Technical Mathematics with Calculus, 2nd ed. New York, McGraw-Hill Book Company, 1966.

8.1d Washington, Allyn J. Basic Technical Mathematics with Calculus, 2nd ed. Menlo Park, California, The Cummings Publishing Company, 1970.

Electronics and Electricity--At least two of the following:

8.2a Adams, Lovincy J. and Journigan, R. P. Applied Mathematics for Electronics. New York, Holt, Rinehart and Winston, Inc., 1967.

8.2b Barker, Forrest I. and Wheeler, Gershon J. Mathematics for Electronics. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

8.2c Herrick, Clyde N. Mathematics for Electronics. Columbus, Ohio, Charles E. Merrill Publishing Company, 1967.

8.2d Korneff, Theodore. Introduction to Electronics. New York, Academic Press, Inc., 1966.

8.2e National Radio Institute Staff. Mathematics for Electronics and Electricity. New York, Holt, Rinehart and Winston, Inc. Out of print.

8.2f Nunz, Gregory J. and Shaw, William L. Electronics Mathematics, 2 vols. New York, McGraw-Hill Book Company, 1967. Vol. 1, Arithmetic and Algebra; Vol. 2, Algebra, Trigonometry and Calculus.

8.2g Singer, Bertrand B. Basic Mathematics for Electricity and Electronics, 2nd ed. New York, McGraw-Hill Book Company, 1965.

8.2h Westlake, John H. and Noden, Gordon E. Applied Mathematics for Electronics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1968.

8.2i Zelinger, G. Basic Matrix Analysis and Synthesis. Elmsford, New York, Pergamon Press, Inc., 1966.

Chemical Technology--One of the following:

8.3a Bard, Allen J. Chemical Equilibrium. New York, Harper and Row, Publishers, 1966.

8.3b Freiser, Henry and Fernando, Quintus. Ionic Equilibria in Analytic Chemistry. New York, John Wiley and Sons, Inc., 1963.

8.3c Margolis, Emil J. Chemical Principles in Calculations of Ionic Equilibria. New York, The Macmillan Company, 1966.

8.3d Robbins, Omer, Jr. Ionic Reactions and Equilibria. New York, The Macmillan Company, 1967.

One of the following:

8.4a Andersen, Laird B. and Wenzel, L. A. Introduction to Chemical Engineering. New York, McGraw-Hill Book Company, 1961.

8.4b Anderson, H. V. Chemical Calculations. New York, McGraw-Hill Book Company, 1955.

8.4c Hamilton, L. F., et al. Calculations of Analytic Chemistry, 7th ed. New York, McGraw-Hill Book Company, 1969.

8.4d Nyman, Carl J. and King, George B. Problems for General Chemistry and Qualitative Analysis. New York, John Wiley and Sons, Inc., 1966.

8.4e Peters, M. S. Elementary Chemical Engineering. New York, McGraw-Hill Book Company, 1954.

Health Sciences--At least one of the following:

8.5a Asperheim, Mary K. Pharmacology for Practical Nurses, 2nd ed. Philadelphia, Pennsylvania, W. B. Saunders Company, 1967.

8.5b Lipsey, Sally Irene. Mathematics for Nursing Science. New York, John Wiley and Sons, Inc., 1965.

8.5c Sackheim, George I. Programmed Mathematics for Nurses, 2nd ed. New York, The Macmillan Company, 1961.

8.5d Sisson, Harriet E. Applied Pharmaceutical Calculations. Minneapolis, Minnesota, Burgess Publishing Company, 1966.

Other Technologies

Standard reference books and handbooks in other specialized technologies and vocations, e.g., mechanical engineering, agricultural engineering, etc., should be in the library. Since most of these books deal more with the specific field than with the mathematics involved in that field, it is felt that the choice of such books should be left to those intimately involved in the field, rather than to members of the mathematics staff.

9. DATA PROCESSING

The books under this heading are specific to a particular program offered in various two-year colleges. If that program is not offered, there may be no need for the library to purchase these books.

Overview--At least two of the following:

9.1a Allen, Paul. Exploring the Computer. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1967.

9.1b Boore, William F. and Murphy, G. R. Computer Sampler: Management Perspectives on the Computer. New York, McGraw-Hill Book Company, 1968.

9.1c Davis, Gordon B. Computer Data Processing. New York, McGraw-Hill Book Company, 1969.

9.1d Moursund, David G. How Computers Do It. Belmont, California, Wadsworth Publishing Company, Inc., 1969.

9.1e Sanders, Donald H. Computers in Business: An Introduction. New York, McGraw-Hill Book Company, 1968.

9.1f Swanson, Robert W. Introduction to Business Data Processing and Computer Programming. Belmont, California, Dickenson Publishing Company, Inc., 1967.

9.1g Wheeler, Gershon J. and Jones, Donlan F. Business Data Processing: An Introduction. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1966.

9.1h Withington, Frederick G. Use of Computers in Business Organizations. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1966.

Unit Record Operations--At least one of the following:

9.2a Claffey, William J. Principles of Data Processing. Belmont, California, Dickenson Publishing Company, Inc., 1967.

9.2b Levy, Joseph. Punched Card Data Processing. New York, McGraw-Hill Book Company, 1967.

9.2c Micallef, Benjamin A. Electronic Accounting Machine Fundamentals. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

9.2d Salmon, Lawrence J. IBM Machine Operation and Wiring, 2nd ed. Belmont, California, Wadsworth Publishing Company, Inc., 1966.

Assembly Language--At least one of the following:

9.3a Cashman, Thomas J. and Shelly, Gary B. IBM System/360 Assembler Language. Fullerton, California, Anaheim Publishing Company, 1969.

9.3b Chapin, Ned. 360 Programming in Assembly Language. New York, McGraw-Hill Book Company, 1968.

9.3c Computer Usage Company. Programming the IBM System-360. New York, John Wiley and Sons, Inc., 1966.

9.3d Golden, James T. and Leichus, Richard M. IBM 360 Programming and Computing. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

9.3e Struble, George L. Assembler Language Programming: The IBM System-360. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

COBOL--At least one of the following:

9.4a Raun, Donald L. Introduction to COBOL Computer Programming for Accounting and Business Analysis. Belmont, California, Dickenson Publishing Company, Inc., 1966.

9.4b Wendel, Thomas M. and Williams, William H. Introduction to Data Processing and COBOL. New York, McGraw-Hill Book Company, 1969.

10. COMPUTING--PROGRAMMING LANGUAGES

The books under this heading are specific to a particular program offered in various two-year colleges. If that program is not offered, there may be no need for the library to purchase these books.

It is assumed that one or more books will be obtained concerning the particular computing systems which are available to the institution. Consequently, no books have been listed which apply to a particular system. This is not meant to indicate that no machines are referred to in the listed books, but rather that the book would not be listed if its applications were only to a particular system. While books are listed for the more widely used programming languages, it is presumed that primary attention would be given to books concerning those languages used within the institution. It should be noted that additional books on computing are listed in Section 9; in particular, books on assembler languages and COBOL are listed therein.

Introductory--At least one of the following:

10.1a Forsythe, A. I., et al. Computer Science: A First Course. New York, John Wiley and Sons, Inc., 1969.

10.1b Hull, Thomas E. and Day, D. D. F. Introduction to Computers and Problem Solving. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

And at least one of the following:

10.2a Arden, Bruce W. Introduction to Digital Computing. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

10.2b Galler, Bernard A. Language of Computers. New York, McGraw-Hill Book Company, 1962.

10.2c Rice, John K. and Rice, John R. Introduction to Computer Science: Problems, Algorithms, Languages, Information and Computers. New York, Holt, Rinehart and Winston, Inc., 1969.

Digital Computing--General References--At least one of the following:

10.3a Bartee, Thomas C. Digital Computer Fundamentals, 2nd ed. New York, McGraw-Hill Book Company, 1966.

10.3b Conway, Richard W., et al. Theory of Scheduling. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1967.

10.3c Desmonde, William H. Computers and Their Uses. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964.

10.3d Klerer, Melvin and Korn, Granino A. Digital Computer User's Handbook. New York, McGraw-Hill Book Company, 1967.

10.3e Maisel, Herbert. Introduction to Electronic Digital Computers. New York, McGraw-Hill Book Company, 1969.

10.3f Schriber, Thomas J. Fundamentals of Flowcharting. New York, John Wiley and Sons, Inc., 1969.

Programming Languages--At least one for each language available in the institution.

FORTRAN IV

10.4a Dimitry, Donald L. and Mott, Thomas H., Jr. Introduction to FORTRAN IV Programming. New York, Holt, Rinehart and Winston, Inc., 1966.

10.4b McCammon, Mary. Understanding FORTRAN. New York, Thomas Y. Crowell Company, 1968.

10.4c McCracken, Daniel D. Guide to FORTRAN Programming. New York, John Wiley and Sons, Inc., 1961.

10.4d Organick, Elliot I. FORTRAN IV Primer. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1966.

10.4e Rule, Wilfred P. FORTRAN IV Programming. Boston, Massachusetts, Prindle, Weber and Schmidt, Inc., 1968.

PL/I

10.5a Bates, Frank and Douglas, Mary L. Programming Language: One. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

10.5b Lecht, Charles Philip. The Programmer's PL/I: A Complete Reference. New York, McGraw-Hill Book Company, 1968.

10.5c Pollack, S. V. and Sterling, T. D. Guide to PL-I. New York, Holt, Rinehart and Winston, Inc., 1969.

10.5d Sprowls, R. Clay. Introduction to PL/I Programming. New York, Harper and Row, Publishers, 1969.

10.5e Weinberg, Gerald M. PL/I Programming Primer. New York, McGraw-Hill Book Company, 1966.

BASIC

10.6 Kemeny, John G. and Kurtz, T. E. Basic Programming. New York, John Wiley and Sons, Inc., 1967.

COBOL--See 9.4

Analog and Hybrid Computing--At least one of the following:

10.7a Johnson, Clarence L. Analog Computer Techniques, 2nd ed. New York, McGraw-Hill Book Company, 1963.

10.7b Korn, Granino A. and Korn, Theresa M. Electronic Analog and Hybrid Computers. New York, McGraw-Hill Book Company, 1964.

10.7c Stice, James E. and Swanson, Bernet S. Electronic Analog Computer Primer. Waltham, Massachusetts, Blaisdell Publishing Company, 1965.

11. TEACHING

The books under this heading are specific to a particular program offered in various two-year colleges. If that program is not offered, there may be no need for the library to purchase these books.

The chief function of the two-year college with reference to teacher training seems to be the providing of the subject matter foundations for future teachers. Library books for this purpose are found elsewhere in this publication. It is only at the elementary school teacher level that there appears to be a call for special courses and related library books in the junior college. The following list suggests a few books dealing with pedagogy at both the elementary and secondary levels for the use of teachers who might use the college's facilities or students who are interested in teaching careers.

Elementary School Teacher Preparation in Mathematics

At least one of the following:

11.1a Brumfiel, Charles F. and Krause, Eugene F. Elementary Mathematics for Teachers. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

11.1b Fehr, Howard F. and Hill, Thomas J. Contemporary Mathematics for Elementary Teachers. Lexington, Massachusetts, D. C. Heath and Company, 1966.

11.1c Garstens, Helen L. and Jackson, Stanley B. Mathematics for Elementary School Teachers. New York, The Macmillan Company, 1967.

*11.1d School Mathematics Study Group. Studies in Mathematics. Vol. IX, A Brief Course in Mathematics for Elementary School Teachers. Pasadena, California, A. C. Vroman, Inc., 1963.

And at least one of the following:

11.2a Moise, Edwin E. The Number Systems of Elementary Mathematics: Counting, Measurements and Coordinates. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1965.

11.2b National Council of Teachers of Mathematics. Topics in Mathematics for Elementary School Teachers (29th Yearbook) and More Topics in Mathematics for Elementary School Teachers (30th Yearbook). Washington, D. C., National Council of Teachers of Mathematics. 29th Yearbook, 1964; 30th Yearbook, 1969.

11.2c Peterson, John A. and Hashisaki, Joseph. Theory of Arithmetic, 2nd ed. New York, John Wiley and Sons, Inc., 1967.

And at least one of the following:

11.3a Keedy, Mervin L. and Nelson, Charles W. Geometry, A Modern Introduction. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1965.

11.3b Ringenberg, Lawrence A. Informal Geometry. New York, John Wiley and Sons, Inc., 1967.

11.3c Smart, James R. Introductory Geometry: An Informal Approach. Belmont, California, Brooks/Cole Publishing Company, 1967.

Teaching of Mathematics--At least one of the following:

11.4a Fehr, Howard F. and Phillips, Jo M. Teaching Modern Mathematics in the Elementary School. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1967.

11.4b Riedesel, C. Alan. Guiding Discovery in Elementary School Mathematics. New York, Appleton-Century-Crofts, 1967.

And at least one of the following:

11.5a Johnson, Donovan A. and Rising, Gerald R. Guidelines for Teaching Mathematics. Belmont, California, Wadsworth Publishing Company, Inc., 1967.

11.5b Willoughby, Stephen S. Contemporary Teaching of Secondary School Mathematics. New York, John Wiley and Sons, Inc., 1967.

And at least one of the following:

11.6a Butler, D. H. and Wren, F. L. The Teaching of Secondary Mathematics, 5th ed. New York, McGraw-Hill Book Company, 1969.

11.6b Dubisch, Roy. Teaching of Mathematics, 2nd ed. New York, John Wiley and Sons, Inc., 1963.

12. NUMERICAL ANALYSIS

Introductory Texts--At least one of the following:

12.1a Dodes, Irving A. and Greitzer, S. L. Numerical Analysis with Scientific Applications. New York, Hayden Book Company, Inc., 1964.

12.1b Dorn, William S. and Greenberg, Herbert J. Mathematics and Computing: With FORTRAN Programming. New York, John Wiley and Sons, Inc., 1967.

Texts Combined with Introductory Programming--At least one of the following:

12.2a James, Merlin L., et al. Applied Numerical Methods for Digital Computation with FORTRAN. Scranton, Pennsylvania, Intext Educational Publishers, 1967.

12.2b McCormick, John M. and Salvadori, M. G. Numerical Methods in FORTRAN. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964.

12.2c McCracken, Daniel D. and Dorn, William S. Numerical Methods and FORTRAN Programming. New York, John Wiley and Sons, Inc., 1964.

Intermediate Texts--At least one of the following:

12.3a Conte, Samuel D. Elementary Numerical Analysis: An Algorithmic Approach. New York, McGraw-Hill Book Company, 1965.

12.3b Fox, Leslie and Mayers, D. F. Computing Methods for Scientists and Engineers. New York, Oxford University Press, 1968.

12.3c Macon, Nathaniel. Numerical Analysis. New York, John Wiley and Sons, Inc., 1963.

12.3d Moursund, David G. and Duris, C. S. Elementary Theory and Application of Numerical Analysis. New York, McGraw-Hill Book Company, 1967.

12.3e Stiefel, E. L. An Introduction to Numerical Mathematics. (translated from the German by W. C. Rheinboldt) New York, Academic Press, Inc., 1963.

Intermediate Numerical Linear Algebra Texts--At least one of the following:

12.4a Forsythe, George E. and Moler, C. B. Computer Solution of Linear Algebraic Systems. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

12.4b Fox, Leslie. Introduction to Numerical Linear Algebra. New York, Oxford University Press, 1965.

Advanced Texts--At least one of the following:

12.5a Fröberg, Carl E. Introduction to Numerical Analysis, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

12.5b Henrici, Peter K. Elements of Numerical Analysis. New York, John Wiley and Sons, Inc., 1964.

12.5c Householder, Alston S. Principles of Numerical Analysis. New York, McGraw-Hill Book Company, 1953.

12.5d Isaacson, Eugene and Keller, H. B. Analysis of Numerical Methods. New York, John Wiley and Sons, Inc., 1966.

12.5e Ralston, Anthony. First Course in Numerical Analysis. New York, McGraw-Hill Book Company, 1965.

Some books with a reference character--At least one of the following:

12.6a Carnahan, Brice, et al. Applied Numerical Methods. New York, John Wiley and Sons, Inc., 1969.

12.6b Handscomb, David C., ed. Methods of Numerical Approximation. Elmsford, New York, Pergamon Press, Inc., 1966.

12.6c Kelly, Louis G. Handbook of Numerical Methods and Applications. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1967.

12.6d Mathematical Methods for Digital Computers. Edited by A. Ralston and H. S. Wilf. New York, John Wiley and Sons, Inc. Vol. I, 1960; Vol. II, 1967.

13. MATHEMATICS FOR THE PHYSICAL SCIENCES

General Works--At least one of the following:

13.1a Boas, M. L. Mathematical Methods in Physical Sciences. New York, John Wiley and Sons, Inc., 1966.

13.1b Collins, R. E. Mathematical Methods for Physicists and Engineers. New York, Van Nostrand Reinhold Company, 1968.

13.1c Page, Chester H. Physical Mathematics. New York, Van Nostrand Reinhold Company, 1955. Out of print.

Engineering Case Studies

13.2 Noble, Ben. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967.

Applied Algebra--At least one of the following:

13.3a Hall, George G. Applied Group Theory. New York, American Elsevier Publishing Company, Inc., 1967.

13.3b Hohn, Franz E. Applied Boolean Algebra, 2nd ed. New York, The Macmillan Company, 1966.

Applied Analysis--At least one of the following:

13.4a Brouwer, Dirk and Clemence, Gerald M. Methods of Celestial Mechanics. New York, Academic Press, Inc., 1961.

13.4b Churchill, Ruel V. Operational Mathematics. New York, McGraw-Hill Book Company, 1958.

13.4c Lanczos, Cornelius. Applied Analysis. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1956.

13.4d Lawden, Derek F. Mathematics of Engineering Systems, 2nd ed. New York, Barnes and Noble, Inc., 1959.

13.4e Sokolnikoff, Ivan S. and Redheffer, R. M. Mathematics of Physics and Modern Engineering, 2nd ed. New York, McGraw-Hill Book Company, 1966.

13.4f Urwin, Kathleen M. Advanced Calculus and Vector Field Theory. Elmsford, New York, Pergamon Press, Inc., 1966.

13.4g Von Kármán, T. and Biot, M. A. Mathematical Methods in Engineering: An Introduction to the Mathematical Treatment of Engineering Problems. New York, McGraw-Hill Book Company, 1960.

13.4h Wylie, Clarence R. Advanced Engineering Mathematics, 3rd ed. New York, McGraw-Hill Book Company, 1966.

Operations Research

13.5 Kaufmann, Arnold. The Science of Decision Making. (translated by R. Audley) New York, McGraw-Hill Book Company, 1968.

And at least one of the following:

13.6a Hillier, Frederick S. and Lieberman, Gerald J. Introduction to Operations Research. San Francisco, California, Holden-Day, Inc., 1967.

13.6b Kaufmann, Arnold and Faure, R. Introduction to Operations Research. (translated by H. C. Sneyd) New York, Academic Press, Inc., 1968.

13.6c Sasieni, Maurice W., et al. Operations Research: Methods and Problems. New York, John Wiley and Sons, Inc., 1959.

See also 14.10

14. MATHEMATICS FOR THE SOCIAL AND LIFE SCIENCES

General Books--At least one of the following:

14.1a Kemeny, John G. and Snell, J. Laurie. Mathematical Models in the Social Sciences. Waltham, Massachusetts, Blaisdell Publishing Company, 1962.

14.1b Lazarfeld, Paul T. and Henry, Neil W., eds. Readings in Mathematical Social Science. Cambridge, Massachusetts, MIT Press, 1968.

14.1c Massarik, F. and Ratoosh, P., eds. Mathematical Explorations in Behavioral Sciences. Homewood, Illinois, Richard D. Irwin, Inc., 1965. Out of print.

Elementary Mathematics for Social and Biological Sciences--At least one of the following:

14.2a Gelbaum, Bernard R. and March, James G. Mathematics for the Social and Behavioral Sciences. Vol. 1, Probability, Calculus and Statistics. Philadelphia, Pennsylvania, W. B. Saunders Company, 1969.

*14.2b Kemeny, John G., et al. Introduction to Finite Mathematics, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1957.

Economics--At least one of the following:

14.3a Archibald, George C. and Lipsey, Richard G. An Introduction to a Mathematical Treatment of Economics. London, Weidenfeld and Nicholson, 1967.

14.3b Beach, E. F. Economic Models. New York, John Wiley and Sons, Inc., 1957.

14.3c Boot, Johannes. Mathematical Reasoning in Economics and Management Sciences: Twelve Topics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

14.3d Bushaw, Donald W. and Clower, Robert W. Introduction to Mathematical Economics. Homewood, Illinois, Richard D. Irwin, Inc., 1957. Out of print.

Sociology--At least one of the following:

14.4a Bartos, Otomar J. Simple Models of Group Behavior. New York, Columbia University Press, 1967.

14.4b Berger, Joseph, et al. Types of Formalization in Small Group Research. Boston, Massachusetts, Houghton Mifflin Company, 1963.

14.4c Coleman, James S. Introduction to Mathematical Sociology. New York, Free Press, 1964.

Psychology--At least one of the following:

14.5a Atkinson, Richard C., et al. Introduction to Mathematical Learning Theory. New York, John Wiley and Sons, Inc., 1965.

14.5b Luce, Robert D., et al. Handbook of Mathematical Psychology, 3 vols. New York, John Wiley and Sons, Inc., 1963-1965.

14.5c Luce, Robert D., et al. Readings in Mathematical Psychology, 2 vols. New York, John Wiley and Sons, Inc., 1963.

14.5d Miller, George. Mathematics and Psychology. New York, John Wiley and Sons, Inc., 1964.

Political Science--At least one of the following:

14.6a Alker, Hayward R., Jr. Mathematics and Politics. New York, The Macmillan Company, 1965.

14.6b Riker, William H. Theory of Political Coalitions. New Haven, Connecticut, Yale University Press, 1962.

14.6c Saaty, Thomas L. Mathematical Models of Arms Control and Disarmament. New York, John Wiley and Sons, Inc., 1968.

14.6d Tullock, Gordon. Toward a Mathematics of Politics. Ann Arbor, Michigan, University of Michigan Press, 1967.

Biological Sciences--At least one of the following:

14.7a Keyfitz, Nathan. Introduction to the Mathematics of Population. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

14.7b Lotka, Alfred J. Elements of Mathematical Biology. New York, Dover Publications, Inc., 1957.

14.7c Nahikian, Howard M. Modern Algebra for Biologists. Chicago, Illinois, University of Chicago Press, 1964.

Game Theory--At least one of the following:

14.8a Luce, Robert D. and Raiffa, H. Games and Decisions. New York, John Wiley and Sons, Inc., 1957.

14.8b Owen, Guillermo. Game Theory. Philadelphia, Pennsylvania, W. B. Saunders Company, 1968.

14.8c Rapoport, Anatol. Fights, Games, and Debates. Ann Arbor, Michigan, University of Michigan Press, 1960.

14.8d Rapoport, Anatol. Two-Person Game Theory: The Essential Ideas. Ann Arbor, Michigan, University of Michigan Press, 1966.

14.8e Williams, John D. Compleat Strategyst. New York, McGraw-Hill Book Company, 1965.

Programming--At least one of the following:

14.9a Dantzig, George B. Linear Programming and Extensions. Princeton, New Jersey, Princeton University Press, 1963.

14.9b Gass, Saul I. Linear Programming, 2nd ed. New York, McGraw-Hill Book Company, 1969.

14.9c Glicksman, Abraham M. Linear Programming and the Theory of Games. New York, John Wiley and Sons, Inc., 1963.

14.9d Haley, K. B. Mathematical Programming for Business and Industry. New York, St. Martin's Press, Inc., 1967.

14.9e Vajda, S. Mathematical Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

Mathematical Topics of Special Interest to the Social and Life Sciences--At least two of the following:

14.10a Allen, Roy G. Mathematical Economics, 2nd ed. New York, St. Martin's Press, Inc., 1959.

14.10b Ash, R. B. Information Theory. New York, John Wiley and Sons, Inc., 1965.

14.10c Beckenbach, Edwin F., ed. Applied Combinatorial Mathematics. New York, John Wiley and Sons, Inc., 1964.

14.10d Goldberg, Samuel. Introduction to Difference Equations. New York, John Wiley and Sons, Inc., 1958.

14.10e Harary, Frank, et al. Structural Models: An Introduction to the Theory of Directed Graphs. New York, John Wiley and Sons, Inc., 1965.

14.10f Pierce, J. R. Symbols, Signals and Noise: The Nature and Process of Communication. New York, Harper and Row, Publishers, 1962.

14.10g Saaty, Thomas L. Optimization in Integers and Related Extremal Problems. New York, McGraw-Hill Book Company, 1970.

See also 13.5 and 13.6

15. ANALYSIS AND DIFFERENTIAL EQUATIONS

Differential Equations--At least one of the following:

15.1a Agnew, Ralph P. Differential Equations, 2nd ed. New York, McGraw-Hill Book Company, 1960.

15.1b Boyce, William and DiPrima, R. C. Elementary Differential Equations and Boundary Value Problems, 2nd ed. New York, John Wiley and Sons, Inc., 1969.

15.1c Coddington, Earl A. Introduction to Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961.

15.1d Ford, Lester R. Differential Equations, 2nd ed. New York, McGraw-Hill Book Company, 1955.

15.1e Rainville, Earl D. and Bedient, Phillip E. Short Course in Differential Equations, 4th ed. New York, The Macmillan Company, 1969.

15.1f Ross, S. L. Differential Equations. Waltham, Massachusetts, Blaisdell Publishing Company, 1964.

15.1g Spiegel, Murray R. Applied Differential Equations, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

15.1h Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations. New York, Harper and Row, Publishers, 1963.

Partial Differential Equations--At least one of the following:

15.2a Berg, Paul W. and McGregor, James L. Elementary Partial Differential Equations. San Francisco, California, Holden-Day, Inc., 1966.

15.2b Broman, Arne. Introduction to Partial Differential Equations: From Fourier Series to Boundary Value Problems. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970.

15.2c Churchill, Ruel V. Fourier Series and Boundary Value Problems, 2nd ed. New York, McGraw-Hill Book Company, 1963.

Infinite Series--At least one of the following:

15.3a Green, James A. Sequences and Series. (edited by W. Ledermann) New York, Dover Publications, Inc., 1958.

15.3b Knopp, Konrad. Infinite Sequences and Series. (translated by F. Bagemihl) New York, Dover Publications, Inc., 1956.

Fourier Series--At least one of the following:

*15.4a Jackson, Dunham. Fourier Series and Orthogonal Polynomials. LaSalle, Illinois, Open Court Publishing Company, 1941.

15.4b Rogosinski, W. Fourier Series, 2nd ed. New York, Chelsea Publishing Company, Inc., 1959.

15.4c Seeley, Robert T. Introduction to Fourier Series and Integrals. New York, The Benjamin Company, Inc., 1966.

15.4d Tolstov, Georgy P. Fourier Series, 2nd ed. (translated by Richard A. Silverman) Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

Real Variables

*15.5 Boas, Ralph P., Jr. A Primer of Real Functions. New York, John Wiley and Sons, Inc., 1960.

And at least one of the following:

15.6a Randol, Burton. Introduction to Real Analysis. New York, Harcourt Brace Jovanovitch, Inc., 1969.

15.6b Royden, H. L. Real Analysis, 2nd ed. New York, The Macmillan Company, 1968.

15.6c Rudin, Walter. Principles of Mathematical Analysis, 2nd ed. New York, McGraw-Hill Book Company, 1964.

Complex Variables--At least one of the following:

15.7a Ahlfors, Lars V. Complex Analysis, 2nd ed. New York, McGraw-Hill Book Company, 1966.

15.7b Churchill, Ruel V. Complex Variables and Applications, 2nd ed. New York, McGraw-Hill Book Company, 1960.

15.7c Knopp, Konrad. Theory of Functions, Parts I and II. Problem Books I and II. (translated by F. Bagemihl) New York, Dover Publications, Inc., 1947. Part I, Elements of the General Theory of Analytic Functions; Part II, Applications and Continuations of the General Theory.

General--At least one of the following:

15.8a Gelbaum, Bernard and Olmsted, John. Counterexamples in Analysis. San Francisco, California, Holden-Day, Inc., 1964.

15.8b Smirnov, Vladimir I. Course of Higher Mathematics, 5 vols. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1964. Vol. I, Elementary Calculus; Vol. II, Advanced Calculus; Vol. III, Part 1, Linear Algebra; Vol. III, Part 2, Complex Variables, Special Functions; Vol. IV, Boundary Value Problems, Integral Equations and Partial Differential Equations; Vol. V, Integration and Functional Analysis.

16. ALGEBRA

Theory of Equations--At least one of the following:

16.1a Conkwright, N. B. Introduction to the Theory of Equations. Waltham, Massachusetts, Blaisdell Publishing Company, 1957.

16.1b Dickson, Leonard E. A New First Course in the Theory of Equations. New York, John Wiley and Sons, Inc., 1939.

16.1c MacDuffee, Cyrus C. Theory of Equations. New York, John Wiley and Sons, Inc., 1954.

16.1d Uspensky, James V. Theory of Equations. New York, McGraw-Hill Book Company, 1948.

Elementary Linear Algebra--At least one of the following:

16.2a Beaumont, Ross A. Linear Algebra. New York, Harcourt Brace Jovanovitch, Inc., 1965.

16.2b Curtis, Charles W. Linear Algebra: An Introductory Approach, 2nd ed. Boston, Massachusetts, Allyn and Bacon, Inc., 1968.

16.2c Shields, Paul C. Elementary Linear Algebra. New York, Worth Publishers, Inc., 1968.

16.2d Zelinsky, Daniel. First Course in Linear Algebra. New York, Academic Press, Inc., 1968.

Intermediate Linear Algebra--At least one of the following:

16.3a Finkbeiner, Daniel T. Introduction to Matrices and Linear Transformations, 2nd ed. San Francisco, California, W. H. Freeman and Company, 1966.

16.3b Hohn, Franz E. Elementary Matrix Algebra, 2nd ed. New York, The Macmillan Company, 1964.

16.3c Schneider, Hans and Barker, George P. Matrices and Linear Algebra. New York, Holt, Rinehart and Winston, Inc., 1968.

16.3d Staib, John H. Introduction to Matrices and Linear Transformations. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969.

Advanced Linear Algebra--At least one of the following:

16.4a Halmos, Paul R. Finite-Dimensional Vector Spaces, 2nd ed. New York, Van Nostrand Reinhold Company, 1958.

16.4b Hoffman, Kenneth and Kunze, Ray. Linear Algebra, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971.

16.4c Mostow, George D. and Sampson, Joseph H. Linear Algebra. New York, McGraw-Hill Book Company, 1969.

16.4d Nering, Evar D. Linear Algebra and Matrix Theory, 2nd ed. New York, John Wiley and Sons, Inc., 1970.

Applied Linear Algebra

16.5 Noble, Ben. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

Introductory Abstract Algebra--At least one of the following:

16.6a Andree, Richard V. Selections from Modern Abstract Algebra. New York, Holt, Rinehart and Winston, Inc., 1958.

16.6b Weiss, Marie J. and Dubisch, Roy. Higher Algebra for the Undergraduate, 2nd ed. New York, John Wiley and Sons, Inc., 1962.

Elementary Abstract Algebra

16.7 McCoy, Neal H. Introduction to Modern Algebra, rev. ed. Boston, Massachusetts, Allyn and Bacon, Inc., 1968.

And at least one of the following:

16.8a Dean, Richard A. Elements of Abstract Algebra. New York, John Wiley and Sons, Inc., 1966.

16.8b Fraleigh, John B. First Course in Abstract Algebra. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1967.

Intermediate Abstract Algebra--At least one of the following:

16.9a Birkhoff, Garrett and MacLane, Saunders. Survey of Modern Algebra, 3rd ed. New York, The Macmillan Company, 1965.

16.9b Herstein, I. N. Topics in Algebra. Waltham, Massachusetts, Blaisdell Publishing Company, 1964.

16.9c Lewis, D. J. Introduction to Algebra. New York, Harper and Row, Publishers, 1965.

16.9d Paley, Hiram and Weichsel, Paul M. First Course in Abstract Algebra. New York, Holt, Rinehart and Winston, Inc., 1966.

Advanced Abstract Algebra

16.10 van der Waerden, B. L. Modern Algebra. (translated by T. J. Benac) New York, Frederick Ungar Publishing Company, Inc. Vol. I, 1949; Vol. II, 1950.

17. NUMBER THEORY

General and Historical--At least one of the following:

17.1a Fraenkel, Abraham A. Integers and Theory of Numbers. New York, Academic Scripta Mathematica Studies (Yeshiva University: Scripta Mathematica Studies, No. 5), 1955.

17.1b Ogilvy, C. Stanley and Anderson, John T. Excursions in Number Theory. New York, Oxford University Press, 1966.

17.1c Ore, Oystein. Number Theory and Its History. New York, McGraw-Hill Book Company, 1948.

Elementary--At least one, preferably two, of the following:

17.2a Barnett, I. A. Elements of Number Theory. Boston, Massachusetts, Prindle, Weber and Schmidt, Inc., 1969.

17.2b Davenport, H. Higher Arithmetic: Introduction to the Theory of Numbers, 3rd ed. New York, Hillary House Publishers, 1968.

17.2c Dudley, Underwood. Elementary Number Theory. San Francisco, California, W. H. Freeman and Company, 1969.

17.2d Jones, Burton W. Theory of Numbers. New York, Holt, Rinehart and Winston, Inc., 1955.

17.2e McCoy, Neal H. Theory of Numbers. New York, The Macmillan Company, 1965.

17.2f Stewart, Bonnie M. Theory of Numbers, 2nd ed. New York, The Macmillan Company, 1964.

17.2g Uspensky, James V. and Heaslet, M. A. Elementary Number Theory. New York, McGraw-Hill Book Company, 1939.

Advanced--At least one of the following:

17.3a LeVeque, William J. Topics in Number Theory, vol. 1. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956.

17.3b Nagell, Trygve. Introduction to Number Theory, 2nd ed. New York, Chelsea Publishing Company, Inc., 1964.

17.3c Niven, Ivan and Zuckerman, H. S. Introduction to the Theory of Numbers, 2nd ed. New York, John Wiley and Sons, Inc., 1966.

Larger Reference Works--At least one of the following:

17.4a Dickson, Leonard E. History of the Theory of Numbers, 3 vols. New York, Chelsea Publishing Company, Inc.

17.4b Hardy, Godfrey H. and Wright, E. M. Introduction to the Theory of Numbers, 4th ed. New York, Oxford University Press, 1960.

17.4c Shanks, Daniel. Solved and Unsolved Problems in Number Theory, vol. 1. New York, Spartan Books, Inc., 1962.

17.4d Sierpiński, Wacław. Elementary Theory of Numbers. (translated from the Polish by A. Hulanicki) Polska Academia Nauk Monografie Matematyczne, Tom. 42. New York, Hafner Publishing Company, 1964.

18. LOGIC, FOUNDATIONS, AND SET THEORY

Philosophy

18.1 Barker, Stephen F. Philosophy of Mathematics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964.

General

18.2 Wilder, Raymond L. Introduction to the Foundations of Mathematics, 2nd ed. New York, John Wiley and Sons, Inc., 1965.

And at least one of the following:

18.3a Fraenkel, Abraham A. Set Theory and Logic. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1966.

18.3b Meschkowski, Herbert. Evolution of Mathematical Thought. (translated by J. H. Gayl) San Francisco, California, Holden-Day, Inc., 1965.

18.3c Nagel, Ernest and Newman, James R. Gödel's Proof. New York, New York University Press, 1958.

18.3d Stoll, Robert R. Sets, Logic and Axiomatic Theories. San Francisco, California, W. H. Freeman and Company, 1961.

18.3e Vilenkin, N. Ya. Stories About Sets. New York, Academic Press, Inc., 1968.

Elementary Logic--At least one of the following:

18.4a Dinkines, Flora. Elementary Concepts of Modern Mathematics. Part 2, Introduction to Mathematical Logic. New York, Appleton-Century-Crofts, 1964.

18.4b Exner, Robert M. and Rosskopf, Myron S. Logic in Elementary Mathematics. New York, McGraw-Hill Book Company, 1959.

18.4c Kenelly, John W. Informal Logic. Boston, Massachusetts, Allyn and Bacon, Inc., 1967.

18.4d Suppes, P. and Hill, S. First Course in Mathematical Logic. Waltham, Massachusetts, Blaisdell Publishing Company, 1964.

Mathematical Logic--At least one of the following:

18.5a Copi, Irving M. Symbolic Logic, 3rd ed. New York, The Macmillan Company, 1967.

18.5b Kalish, Donald and Montague, Richard. Logic: Techniques of Formal Reasoning. New York, Harcourt Brace Jovanovitch, Inc., 1964.

18.5c Quine, Willard Van Orman. Mathematical Logic, rev. ed. New York, Harper and Row, Publishers, 1951.

18.5d Suppes, P. C. Introduction to Logic. New York, Van Nostrand Reinhold Company, 1957.

18.5e Tarski, Alfred. Introduction to Logic and to the Methodology of Deductive Sciences, 3rd ed. New York, Oxford University Press, 1965.

Elementary Set Theory--At least one of the following:

18.6a Breuer, Joseph. Introduction to Theory of Sets. (translated by H. Fehr) Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1958.

18.6b Dinkines, Flora. Elementary Concepts of Modern Mathematics. Part 1, Elementary Theory of Sets. New York, Appleton-Century-Crofts, 1964.

18.6c Kamke, E. Theory of Sets. (translated by F. Bagemihl) New York, Dover Publications, Inc., 1950.

*18.6d Lipschutz, Seymour. Set Theory and Related Topics. (Schaum's Outline Series) New York, McGraw-Hill Book Company, 1964.

Advanced Set Theory

18.7 Halmos, Paul R. Naive Set Theory. New York, Van Nostrand Reinhold Company, 1960.

Number Systems--At least one of the following:

18.8a Cohen, Leon W. and Ehrlich, Gertrude. Structure of the Real Number System. New York, Van Nostrand Reinhold Company, 1963.

18.8b Feferman, S. Number Systems: Foundations of Algebra and Analysis. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1964.

18.8c Hamilton, Norman T. and Landin, J. Set Theory: The Structure of Arithmetic. Boston, Massachusetts, Allyn and Bacon, Inc., 1961.

Foundations of Computer Science--At least one of the following:

18.9a Arbib, Michael A. Brains, Machines and Mathematics. New York, McGraw-Hill Book Company, 1964.

18.9b Knuth, Donald E. Art of Computer Programming. Vol. 1, Fundamental Algorithms. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

18.9c Minsky, Marvin. Computation: Finite and Infinite Machines. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.

19. GEOMETRY

General--All of the following:

19.1 Coxeter, H. S. M. Introduction to Geometry. New York, John Wiley and Sons, Inc., 1961.

19.2 Eves, Howard. Survey of Geometry. Boston, Massachusetts, Allyn and Bacon, Inc. Vol. I, 1963; Vol. II, 1965.

19.3 Hilbert, David and Cohn-Vossen, Stephan. Geometry and the Imagination. (translated by P. Nemenyi) New York, Chelsea Publishing Company, Inc., 1952.

Elementary Geometry--At least one of the following:

19.4a Moise, Edwin E. Elementary Geometry from an Advanced Standpoint. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

19.4b Prenowitz, Walter and Jordan, Meyer. Basic Concepts of Geometry. Waltham, Massachusetts, Blaisdell Publishing Company, 1965.

19.4c Wylie, Clarence R. Foundations of Geometry. New York, McGraw-Hill Book Company, 1964.

Vector Geometry--At least one of the following:

19.5a Hausner, Melvin. Vector Space Approach to Geometry. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1965.

19.5b Schuster, Seymour. Elementary Vector Geometry. New York, John Wiley and Sons, Inc., 1962.

Non-Euclidean Geometry--At least one of the following:

19.6a Kulczycki, Stefan. Non-Euclidean Geometry. Elmsford, New York, Pergamon Press, Inc., 1961.

19.6b Wolfe, Harold E. Introduction to Non-Euclidean Geometry. New York, Holt, Rinehart and Winston, Inc., 1945.

Projective and Affine Geometry--At least one of the following:

19.7a Blumenthal, Leonard M. Modern View of Geometry. San Francisco, California, W. H. Freeman and Company, 1961.

19.7b Coxeter, H. S. M. Projective Geometry. Waltham, Massachusetts, Blaisdell Publishing Company, 1964.

19.7c Fishback, William T. Projective and Euclidean Geometry. New York, John Wiley and Sons, Inc., 1969.

Differential Geometry--At least one of the following:

19.8a O'Neill, Barrett. Elementary Differential Geometry. New York, Academic Press, Inc., 1966.

19.8b Willmore, Thomas James. Introduction to Differential Geometry. New York, Oxford University Press, 1959.

Special Topics--Any of the following:

19.9a Albert, A. Adrian and Sandler, Reuben. Introduction to Finite Projective Planes. New York, Holt, Rinehart and Winston, Inc., 1968.

19.9b Dorwart, Harold L. Geometry of Incidence. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1965.

19.9c Jeger, Max. Transformation Geometry. (Mathematical Studies Series, Vol. I) New York, American Elsevier Publishing Company, 1966.

19.9d Kaplansky, Irving. Linear Algebra and Geometry: A Second Course. Boston, Massachusetts, Allyn and Bacon, Inc., 1969.

*19.9e Kazarinoff, Nicholas D. Geometric Inequalities. New York, Random House, Inc., 1961.

19.9f Weyl, Hermann. Symmetry. Princeton, New Jersey, Princeton University Press, 1952.

19.9g Yaglom, I. M. and Boltyanskii, V. G. Convex Figures. New York, Holt, Rinehart and Winston, Inc. Out of print.

20. TOPOLOGY

Intuitive Approaches to Topology--At least one of the following:

20.1a Arnold, Bradford Henry. Intuitive Concepts in Elementary Topology. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

20.1b Bing, R. H. Elementary Point Set Topology. Slaught Memorial Paper No. 8. Washington, D. C., Mathematical Association of America, 1960.

20.1c Fréchet, Maurice and Fan, Ky. Initiation to Combinatorial Topology. (translated from the French by Howard Eves) Boston, Massachusetts, Prindle, Weber and Schmidt, Inc., 1967.

20.1d Lietzmann, Walter. Visual Topology. New York, American Elsevier Publishing Company, 1965.

A somewhat more rigorous approach with many of the classical theorems:

*20.2 Chinn, William G. and Steenrod, Norman E. First Concepts of Topology. New York, Random House/Singer School Division, 1966.

Algebraic Topology--At least one of the following:

20.3a Aleksandrov, P. S. Combinatorial Topology, 3 vols. Baltimore, Maryland, Graylock Press. Vol. I, Introduction, Complexes, Coverings, Dimensions, 1956; Vol. II, Betti Groups, 1957; Vol. III, Homological Manifolds, Duality, Classification, and Fixed Point Theorems, 1960.

20.3b Blackett, Donald W. Elementary Topology: Combinatorial and Algebraic Approach. New York, Academic Press, Inc., 1967.

20.3c Massey, William S. Algebraic Topology: An Introduction. New York, Harcourt Brace Jovanovitch, Inc., 1967.

20.3d Wallace, Andrew Hugh. Introduction to Algebraic Topology. Elmsford, New York, Pergamon Press, Inc., 1957.

General Topology--At least one of the following:

20.4a Baum, John D. Elements of Point Set Topology. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964.

20.4b Bushaw, Donald. Elements of General Topology. New York, John Wiley and Sons, Inc., 1963. Out of print.

20.4c Kuratowski, Kazimierz. Introduction to Set Theory and Topology. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962. Out of print.

20.4d Mendelson, Bert. Introduction to Topology, 2nd ed. Boston, Massachusetts, Allyn and Bacon, Inc., 1968.

20.4e Pervin, William J. Foundations of General Topology. New York, Academic Press, Inc., 1964.

Graph Theory

*20.5 Ore, Oystein. Graphs and Their Uses. New York, Random House, Inc., 1963.

21. TABLES AND DICTIONARIES

The library should contain at least one mathematical dictionary and one or more sets of tables, both numerical and functional.

Following is a list of several such dictionaries and tables; there are others equally good available.

Abramowitz, Milton and Stegun, Irene A., eds. Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. New York, Dover Publications, Inc., 1964.

Burington, Richard S. Handbook of Mathematical Tables and Formulas, 4th ed. New York, McGraw-Hill Book Company, 1965.

*Burington, Richard S. and May, Donald C., Jr. Handbook of Probability and Statistics with Tables, 2nd ed. New York, McGraw-Hill Book Company, 1969.

*Chemical Rubber Company. Handbook of Tables for Probability and Statistics, 2nd ed. Cleveland, Ohio, Chemical Rubber Company, 1968.

Chemical Rubber Company. Standard Mathematical Tables, 19th ed. Cleveland, Ohio, Chemical Rubber Company, 1971.

Davis, Harold T. Tables of Mathematical Functions, 2 vols. San Antonio, Texas, Trinity University Press, 1963.

Davis, Harold T. and Fisher, Vera. Tables of Mathematical Functions, vol. 3. San Antonio, Texas, Trinity University Press, 1962.

Dwight, Herbert B. Mathematical Tables of Elementary and Some Higher Mathematical Functions, 3rd ed. New York, Dover Publications, Inc., 1961.

Dwight, Herbert B. Tables of Integrals and Other Mathematical Data, 4th ed. New York, The Macmillan Company, 1961.

James, Glenn and James, Robert C. Mathematics Dictionary, 3rd ed. New York, Van Nostrand Reinhold Company, 1968.

Karush, William. Crescent Dictionary of Mathematics. New York, The Macmillan Company, 1962.

Larsen, Harold. Rinehart Mathematical Tables, Formulas and Curves, enl. ed. New York, Holt, Rinehart and Winston, Inc., 1953.

Marks, Robert W. New Mathematics Dictionary and Handbook. New York, Grosset and Dunlap, Inc., 1964.

Newman, J. R. The Universal Encyclopedia of Mathematics. New York, New American Library, Inc., 1965.

Nielsen, Kaj L. Logarithmic and Trigonometric Tables to Five Places, rev. ed. New York, Barnes and Noble, Inc., 1961.

The Universal Encyclopedia of Mathematics. New York, Simon and Schuster, Inc., 1964.

Weintraub, S. Tables of Cumulative Binomial Probability Distribution for Small Values of p. New York, Free Press, 1963.

22. JOURNALS

The American Mathematical Monthly. Mathematical Association of America, Inc., 1225 Connecticut Avenue, N.W., Washington, D. C. 20036 Ten issues per year.

The Arithmetic Teacher. National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D. C. 20036 Eight issues per year.

The Mathematical Gazette. G. Bell and Sons, Ltd., Portugal Street, London, W.C. 2, England. Five issues per year.

Mathematics Magazine. Mathematical Association of America, Inc., 1225 Connecticut Avenue, N.W., Washington, D. C. 20036 Five issues per year.

The Mathematics Teacher. National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D. C. 20036 Eight issues per year.

The Two-Year College Mathematics Journal. Prindle, Weber and Schmidt, Inc., 53 State Street, Boston, Massachusetts 02109 Two issues per year.

23. SERIES AND COLLECTIONS

A number of excellent series of monographs on various topics in mathematics exist. Quality varies somewhat within each series. Listing of a series here by no means implies that every book in every series should be purchased, for some volumes cover topics not appropriate to the two-year college.

Blaisdell Scientific Paperbacks. Waltham, Massachusetts, Blaisdell Publishing Company, Inc. A series of six pamphlets that are translations of the Russian series "Popular Lectures in Mathematics." Out of print.

Korovkin, P. P. Inequalities. 1961

Kostovskii, A. N. Geometrical Constructions Using Compasses. 1961

Smogorzhevskii, A. S. The Ruler in Geometrical Constructions. 1961

*Sominskii, I. S. The Method of Mathematical Induction. 1961

Uspenskii, V. A. Some Applications of Mechanics to Mathematics. 1961

*Vorobev, N. N. Fibonacci Numbers. 1961

Carus Mathematical Monographs. Washington, D. C., Mathematical Association of America, Inc.

No. 1. Calculus of Variations. G. A. Bliss

No. 2. Analytic Functions of a Complex Variable. D. R. Curtiss

No. 3. Mathematical Statistics. H. L. Rietz

No. 4. Projective Geometry. J. W. Young

*No. 6. Fourier Series and Orthogonal Polynomials. Dunham Jackson

No. 7. Vectors and Matrices. C. C. MacDuffee

No. 8. Rings and Ideals. N. H. McCoy

No. 9. The Theory of Algebraic Numbers. Harry Pollard

No. 10. The Arithmetic Theory of Quadratic Forms. B. W. Jones

No. 11. Irrational Numbers. Ivan Niven

No. 12. Statistical Independence in Probability, Analysis and Number Theory. Mark Kac

*No. 13. A Primer of Real Functions. R. P. Boas, Jr.

No. 14. Combinatorial Mathematics. H. J. Ryser

No. 15. Non-Commutative Rings. I. N. Herstein

No. 16. Dedekind Sums. Hans Rademacher and Emil Grosswald

Mathematics: Its Content, Methods, and Meaning, 3 vols. Edited by A. D. Aleksandrov, et al. Translated by S. H. Gould. Cambridge, Massachusetts, MIT Press.

MAA Studies in Mathematics. Washington, D. C., Mathematical Association of America, Inc.

Vol. 1. Studies in Modern Analysis. R. C. Buck, editor

Vol. 2. Studies in Modern Algebra. A. A. Albert, editor

Vol. 3. Studies in Real and Complex Analysis. I. I. Hirschman, Jr., editor

Vol. 4. Studies in Global Geometry and Analysis. S. S. Chern, editor

Vol. 5. Studies in Modern Topology. P. J. Hilton, editor

Vol. 6. Studies in Number Theory. W. J. LeVeque, editor

Vol. 7. Studies in Applied Mathematics. A. H. Taub, editor

Schaum's Outline Series. New York, McGraw-Hill Book Company.

Advanced Calculus. Murray R. Spiegel

Analytic Geometry. Joseph H. Kindle

Calculus, 2nd ed. Frank Ayres, Jr.

College Algebra. Murray R. Spiegel

Complex Variables. Murray R. Spiegel

Descriptive Geometry. Minor C. Hawk

Differential Equations. Frank Ayres, Jr.

Elementary Algebra. Barnett Rich

Finite Mathematics. Seymour Lipschutz

First Year College Mathematics. Frank Ayres, Jr.

General Topology. Seymour Lipschutz

Group Theory. B. Baumslag and B. Chandler

Laplace Transforms. Murray R. Spiegel

Linear Algebra. Seymour Lipschutz

Mathematical Handbook of Formulas and Tables. Murray R. Spiegel

Mathematics of Finance. Frank Ayres, Jr.

Matrices. Frank Ayres, Jr.

Modern Algebra. Frank Ayres, Jr.

Numerical Analysis. Francis Scheid

Plane Geometry with Coordinate Geometry. Barnett Rich

Projective Geometry. Frank Ayres, Jr.

Real Variables. Murray R. Spiegel

*Set Theory and Related Topics. Seymour Lipschutz

Statistics. Murray R. Spiegel

Theory and Problems of Probability. Seymour Lipschutz

Trigonometry. Frank Ayres, Jr.

Vector Analysis. Murray R. Spiegel

School Mathematics Study Group. Studies in Mathematics. Pasadena, California, A. C. Vroman, Inc.

Euclidean Geometry Based on Ruler and Protractor Axioms (SM-1)

Protractor Axioms (SM-2)

Structure of Elementary Algebra (SM-3)

Geometry (SM-4)

Concepts of Informal Geometry (SM-5)

Number Systems (SM-6)

Intuitive Geometry (SM-7)

Concepts of Algebra (SM-8)

*Brief Course in Mathematics for Elementary School Teachers (SM-9)

Applied Mathematics in the High School (SM-10)

Mathematical Methods in Science (SM-11)

A Brief Course in Mathematics for Junior High School Teachers (SM-12)

Inservice Course for Primary School Teachers (SM-13)

Introduction to Number Systems (SM-14)

Calculus and Science (SM-15)

Some Uses of Mathematics (SM-16)

Mathematical Concepts of Elementary Measurement (SM-17)

Puzzle Problems and Games Project (SM-18)

Reviews of Recent Research in Mathematics Education (SM-19)

Slaught Memorial Papers. (not all available) Washington, D. C., Mathematical Association of America, Inc.,

No. 3 Proceedings of the Symposium on Special Topics in Applied Mathematics.

No. 5 The Conjugate Coordinate System for Plane Euclidean Geometry. W. B. Carver

No. 6 To Lester R. Ford on His Seventieth Birthday. A collection of 14 articles.

No. 7 Introduction to Arithmetic Factorization and Congruences from the Standpoint of Abstract Algebra. H. S. Vandiver and M. W. Weaver

No. 11 Papers in Analysis. A collection of 23 articles.

No. 12 Differentiation of Integrals. A. M. Bruckner

Topics in Mathematics. Lexington, Massachusetts, D. C. Heath and Company.

Configuration Theorems. B. I. Argunov and L. A. Skornyakov, 1963

What is Linear Programming? A. S. Barsov, 1964

Equivalent and Equidecomposable Figures. V. G. Boltyanskii, 1963

Mistakes in Geometric Proofs. Ya. S. Dubnov, 1963

Proof in Geometry. A. I. Fetisov, 1963

Induction in Geometry. L. I. Golovina and I. M. Yaglom, 1963

Computation of Areas of Oriented Figures. A. M. Lopshits, 1963

Areas and Logarithms. A. I. Markushevich, 1963

Summation of Infinitely Small Quantities. I. P. Natanson, 1963

Hyperbolic Functions. V. G. Shervatov, 1963

How to Construct Graphs. G. E. Shilov, 1963

Simple Maxima and Minima Problems. I. P. Natanson, 1963
(The above two are bound as one volume.)

*The Method of Mathematical Induction. I. S. Sominskii, 1963

Algorithms and Automatic Computing Machines. B. A. Trakhtenbrot, 1963

*Fibonacci Numbers. N. N. Vorobev, 1963

An Introduction to the Theory of Games. E. S. Venttsel, 1963

Translations from the Russian: A Survey of Recent East European Mathematical Literature. Lexington, Massachusetts, D. C. Heath and Company.

Multicolor Problems. E. B. Dynkin and V. A. Uspenskii, 1963

Problems in the Theory of Numbers. E. B. Dynkin and V. A. Uspenskii, 1963

Random Walks. E. B. Dynkin and V. A. Uspenskii, 1963

*Eight Lectures on Mathematical Analysis. Alexander Y. Khinchin, 1965

Convex Figures and Polyhedra. L. A. Lyusternik, 1966

Infinite Series. A. I. Markushevich, 1967

PREGRADUATE TRAINING

The Panel on Pregraduate Training was appointed in 1959 to study the needs of, and to recommend programs for, undergraduate students who intend to study mathematics at the graduate level. The initial efforts of the Panel were concentrated upon the construction of an ideal curriculum for students of outstanding ability. Course outlines designed to lead the undergraduate rapidly toward the frontiers of mathematical research and the Ph.D., purposely overlooking local problems which might be caused by inadequate preparation at the secondary level or by lack of staff at the college level, appeared in the 1963 publication Pregraduate Preparation of Research Mathematicians.

Despite many misunderstandings regarding its assumptions and intent, this report served effectively as a basis for discussion and planning at many institutions. It was reprinted in 1965, together with some additional comments on constructive use of the booklet and admonitions to the effect that misinterpretation of the spirit of the outlines might result from a lack of knowledge of the Panel's basic assumptions and objectives. Perhaps the report's chief value lies in showing what is regarded as ideal preparation for graduate study in pure mathematics by a very distinguished group of mathematicians.

Having completed its work on an ideal undergraduate program for the future research mathematician, the Panel turned to the urgent practical task of recommending specific undergraduate curricula for colleges which would be unable, for any of a variety of reasons, to achieve quickly the goals of its original report. These recommendations were drawn up after consultation with representatives of about 25 of the leading graduate mathematics departments. They appear in the 1965 document Preparation for Graduate Study in Mathematics, together with outlines for upper-division courses in Abstract Algebra and Real Analysis.

PREGRADUATE PREPARATION

OF

RESEARCH MATHEMATICIANS

A Report of

The Panel on Pregraduate Training

Goals for pregraduate programs in mathematics. Schools for which these goals are not immediately attainable are referred to the Panel's companion pamphlet, Preparation for Graduate Study in Mathematics.

May 1963

TABLE OF CONTENTS

ABOUT THIS BOOKLET

In the first year of publication over 7,000 copies of these recommendations have been distributed to departments of mathematics and individual college teachers. Reaction from readers indicates a need for several comments on the constructive use of this booklet.

The introductory pages should be read carefully before the course outlines which follow. These passages describe the underlying idealized objectives of the recommendations. Failure to understand these goals can result in misinterpretation of the spirit of the outlines.

The course outlines beginning on page 378 are illustrative samples which may be modified to fit local situations. Taken as a whole, they represent achievement of long-range goals; however, the reader is urged to use them as sources of mathematical ideas out of which to construct his own first steps toward these goals.

The booklet can be used as a guide to several levels of training in mathematics. The material in the outlines for Basic Undergraduate Mathematics together with roughly half of the algebra course on page 414 is generally considered adequate undergraduate preparation for current graduate programs. Departments and students are urged to use the booklet in planning curricula and individual programs of study.

BASIC ASSUMPTIONS

The recommendations presented in this booklet are idealized, as are most educational programs described in print. Hence, it is necessary to reveal some assumptions which were made by the Panel in constructing the suggested program and the course outlines. It is fully realized that the assumptions are somewhat unrealistic in the sense that few pregraduate programs can now include such courses in all detail. However, "honors" programs may well be able now to include some of these proposals. Such developments may lead to an absorption of the ideas into regular curricula, and the next task of the Panel is to provide indications as to how this may be done.

The program set forth here is designed for the first four years of a sequence of formal course study leading to the Ph.D. It is hoped that full-time students seeking a career in research mathematics will obtain the Ph.D. in a total of seven years, with the last year spent mainly in seminars and completion of the dissertation. Clearly, the number of beginning college students now possessing the knowledge and motivation necessary for entry into such a program is small indeed. If they realize their ambitions, these people can be expected to contribute to society as producers of mathematics--hence the title of the booklet.

We have thus assumed that the students will begin the program with some prior appreciation of mathematical proof, with a secondary school background conforming to the highest recommendations of such groups as the School Mathematics Study Group. More significantly, the program is directed toward all students who have profited fully from the mathematical opportunities afforded in their formal study, who hunger for deeper insight and more powerful techniques, who are intellectually curious and are capable of appreciating the elegance, scope, and excitement of mathematics.

Many persons may feel that no action need be taken to help this restricted group of students and, indeed, that it is impossible to prevent them from becoming creative, gifted mathematicians. Admittedly, this assertion is most often made by those who themselves have survived despite all adversity. The counterexamples, of course, do not survive to testify.

But the present document does much more than merely recommend a program for gifted students. It will also serve as a guide and source of ideas to persons interested in evaluating and modifying mathematical curricula. The Panel has spent more than two years in a critical examination of the basic structure of college mathematics, in relation to present mathematical research. It has attempted to discern important underlying patterns and to effect some unity, both of viewpoint and of technique, within a four-year curriculum. It has attempted to make use of the simplification of concept and technique resulting from recent discoveries, without sacrificing intelligibility.

The Panel has also agreed upon certain broad objectives for the college mathematics program. The student should be introduced to the language of mathematics, both in its rigorous and idiomatic forms. He should be able to give clear explanations of the meaning of certain fundamental concepts, statements, and notations. He should acquire a degree of facility with selected mathematical techniques, know proofs of a collection of basic theorems, and have experience with the construction of proofs. He should be ready to read appropriate mathematical literature with understanding and enjoyment. He should learn from illustration and experience to cultivate curiosity and the habit of experimentation, to look beyond immediate objectives, and to make and test conjectures.

The student should by these means be led to seek an understanding of the place of mathematics in our culture--in particular, to appreciate the interplay between mathematics and the sciences. The proposed program, in the portions dealing with analysis, exhibits the traditional role of the physical sciences as a source of mathematical ideas and techniques. The outlines offered for courses in probability and statistics also indicate the emergence of parts of mathematics from problems in the biological and social sciences.

A list of objectives such as these is largely independent of the content of courses and cannot be implemented completely by even the best collection of texts. This only emphasizes the obvious point that the quality of the mathematical education of the nation rests finally upon the caliber and initiative of teachers. We hope that the present document will stimulate widespread interest in a continuing examination and reformulation of mathematics programs in the college.

THE PROGRAM

The proposed program of pregraduate mathematical studies falls naturally into two parts: Introductory Undergraduate Mathematics occupies the first two years, Higher Undergraduate Mathematics the last two. As to subject matter, the former naturally is focused on the differential and integral calculus, and the latter is devoted mainly to basic material in the fields of analysis, algebra, and geometry. Each of these terms, however, is stretched beyond its traditional meaning to take account of contemporary mathematical development. Each of these principal areas is to be supplemented by related studies. And the two parts of the program are to be distinguished by their method of presentation as well as by their subject matter. In particular, the first two years must be shaped so as to lead gradually to an appreciation of the nature and role of definitions and proofs and an ability to employ mathematical language with precision. The last two years must be designed to merge smoothly with beginning graduate study, forming a period in which the most basic mathematical concepts, results, and methods are secured so as to provide a firm base for subsequent specialization and concentrated research.

We have recognized the pervasive character of linearity by recommending the early introduction of linear algebra and the recurrent use of linearity in the analysis courses. At the same time the analysis courses reflect the traditional role of the empirical sciences as a source of mathematical concepts and methods.

Geometry is construed so broadly as to contain differential geometry, differential topology, and algebraic topology, among others, but the spirit of classical geometry has been retained by emphasizing theorems having a geometric formulation.

Breadth, including a knowledge of fields of application, is of great value for the most significant mathematical research. Although difficult to achieve during the undergraduate years, at least a beginning is essential. For those students able to fit in a considerable variety of courses it is important to attempt to achieve as integrated a picture of mathematics as can be assimilated. In the present program the unity of mathematics is illustrated by emphasizing algebraic and topological ideas throughout.

It is important to emphasize that the ability to follow and formulate rigorous proofs must be balanced with the development of a free-ranging intuition in each mathematical field. Generally speaking, a rigorous treatment of some elements of the material should appear even in the earliest courses, and the rigorous segments should increase in length as the student advances to more recondite material. But failure to nurture intuition at any level can be stifling.

The idealized assumptions adopted by the Panel and described in the previous section, while simplifying the problem of formulating curriculum by abstracting from the host of practical problems which beset individual mathematics departments, are still far from sufficient to characterize a unique solution. Members of the Panel found themselves with continuing differences of opinion concerning such questions as the extent to which courses should deal with applications outside of mathematics, the order and relative emphasis to be given to certain groups of loosely associated topics, and the nature of subject development which is required by pedagogical considerations. These differences within the Panel are reflections, of course, of the varied attitudes, often strongly held, which prevail within the broader mathematical community.

In order to convey a fairly concrete idea of the nature of the program conceived by the Panel, course outlines are furnished in the Appendices which follow. To accommodate the variety of viewpoints which prevail, more than one outline is presented for certain basic courses. Each outline is prefaced by an indication of its scope and intended context.

INTRODUCTORY UNDERGRADUATE MATHEMATICS

Under the assumption of a student audience with strong mathematical training in high school and with excellent motivation, a unified two-year sequence of what might be called "vector space calculus" is recommended as a proper basis for the pregraduate program. Consistent with the historical development of calculus and with the flavor of modern mathematics, the program suggests that calculus be presented so as to introduce and utilize significant notions of linear algebra and geometry in the construction of analytic tools for the study of transformations of one Euclidean space into another. This demands that the material be arranged and presented in such a manner that students are ever mindful of mathematics as an interrelated whole rather than a collection of isolated disciplines. The presentation also needs a healthy balance of well-formulated mathematical arguments, of opportunity for discovery through independent work in solving problems and proving theorems, and of mathematical and physical motivation. The student must learn early that a highly significant aspect of mathematics is that of posing the right question.

This program of Introductory Undergraduate Mathematics comprises approximately 15 semester hours. Since the standard subjects are integrated, only a rough estimate of their proportions can be indicated: about nine semester hours of analytic geometry and calculus, with the remainder divided between linear algebra and differential equations.

Presented in Appendix A are three course outlines which display the embodiment of these ideas. The subdivisions in these outlines are in terms of topics and not in terms of days or weeks. The major differences between the outlines are explained at the outset in Appendix A.

HIGHER UNDERGRADUATE MATHEMATICS

This part of the program builds upon the foundation laid by the Introductory Undergraduate Mathematics curriculum. Assuming the ability to appreciate and handle rigor and abstraction, it is intended to broaden the areas of the student's mathematical knowledge with sufficient depth to provide a firm basis for later research and to allow for the formation of individual mathematical taste.

Courses appropriate for this part of the program should be considered as at the "undergraduate-graduate level," for the same type of course will be needed at the beginning of graduate study. Indeed, it is an historical "accident," certainly not related to intrinsic mathematical considerations, that the undergraduate degree is granted after four years of post-high school study. For this reason institutions with limited facilities should strive to provide courses with a full degree of depth and challenge even if this entails offering a narrower range of subjects. The student who comes to graduate school with one solid course behind him is ready to take a second in another field; but the one who comes with the equivalent of two half-courses is often forced either to repeat material or to proceed to more advanced work with a deficient background.

The advanced part of the program will reflect the interest of the faculty, as well as the needs of the student. Small institutions will concentrate on courses in the mathematical areas of primary interest to their professors. And the selection of materials and modes of presentation within each course will reflect the way in which the individual instructor looks at the subject.

Every college department undertaking this program should provide courses relevant to the central areas of mathematics:

Real analysis

Complex analysis

Abstract algebra

Geometry-topology

Probability or mathematical physics

Not every student will take courses in all of these areas; choices will depend on the student's intent. (See Appendix B for sample course outlines.)

In addition, to achieve a richer and more comprehensive program, a department should offer, as far as its resources will permit, a balanced selection of courses in:

Algebra

Analysis

Applied mathematics (in both the natural and social sciences)

Foundations and logic

Geometry (algebraic, differential, projective)

Mathematical statistics

Number theory

Topology

(See Appendix C for sample outlines of some of these courses.)

For the student, we recommend the following principles:

(a) For the upperclass years, at least three of the following four categories should be represented in the course program: (1) algebra, (2) analysis, (3) applied mathematics, (4) geometry-topology.

(b) Included in the program there should be, in order to achieve depth, at least two full-year courses--that is, courses in which the first semester is an essential prerequisite to the second.

(c) A major in mathematics should have at least seven semester-courses beyond our suggested Introductory Undergraduate Mathematics.

INITIATIVE AND INDEPENDENCE

So far in presenting the program, the greatest attention has been paid to describing the mathematical content of an idealized undergraduate curriculum in contemporary mathematics. In view of our assumptions, students participating in the program will not be satisfied to participate in a passive fashion, and so methods to engage the student as an active partner in scholarship should be devised. Indeed, independent intellectual activity of the student must be nurtured in preparation for the time when he will be independent of his professors and join them as a colleague. Thus, the student must increasingly take the initiative, not only to construct proofs by himself, but to develop his imaginative powers so that he can make conjectures for proof or disproof, perhaps even going on to contribute by creating new concepts or theories.

This process has its beginning in a small way when the student solves textbook problems. Another component is added when the student learns to read the textual material by himself, later making the passage to the reading of papers in the journals, which are more compactly written and therefore more difficult to read. There are other ways in which the undergraduate student can develop his initiative. Without attempting an exhaustive list, we mention a few common patterns.

There are seminars and colloquia wherein the student makes reports. There is the undergraduate thesis in which a student makes a contribution, original for him but not usually original in the larger sense. There is the developmental course, a version of the Socratic method, in which the student is led to develop a body of mathematical material under the guidance of the professor. The number of teachers who employ the developmental method completely is not large, but they are an enthusiastic band of people in their devotion to the procedure; a modified use of the developmental method is employed widely. Other devices have been developed to take advantage of special local conditions, or in line with experimental ideas reflecting special interests.

It would be desirable for all schools to give attention to the problem of enrolling the student actively in the study of mathematics, and so we urge that every pregraduate curriculum be designed to include some appropriate scheme.

Appendix A

INTRODUCTORY UNDERGRADUATE MATHEMATICS

Contained in this Appendix are three outlines for the proposed Introductory Undergraduate Mathematics program, each displaying an initial two-year sequence of idealized college mathematics. The three course outlines which follow have different points of view, and so they differ in emphasis and arrangement of material. The major mathematical differences are these:

(1) Outline I includes a self-contained section on linear algebra in the calculus but a separate course in differential equations. Outline II has a separate course in linear algebra but includes topics in differential equations in the main body. Outline III has no separate courses. It has a section on differential equations and develops linear algebra topics as they are needed to solve various problems in analysis.

(2) Outlines I and II introduce integration before differentiation while Outline III does the opposite.

(3) Outlines I and II treat integration via step functions while Outline III approaches the integral as a linear functional on the space of continuous functions.

(4) Outline II is more ambitious than Outline I. It includes, for example, a good deal of elementary point set topology in the second year.

Aside from such explicit differences, there are some subtler distinctions in attitude between Outline III and the other outlines. A prime motivation of Outlines I and II is concern for the internal structure of calculus and of linear algebra; applications are made when appropriate. In these outlines the generalized Stokes theorem is a fitting climax because of the merging of concepts in algebra, topology, and analysis needed in reaching it and because of its important applications in mathematics and physics. The approach in Outline III is to develop mathematical concepts directly as needed for the solution of important problems that arise in mathematics and physics. In particular, linear algebra is so treated. In addition, Outline III is oriented more in the direction of classical analysis: more emphasis on inequalities; Stokes' theorem is thought of as but one of a number of important theorems beyond the traditional calculus. Outlines I and II are more rigid than Outline III, the attitude being that these are the things juniors should know, and here is a reasonable order of doing it. Outline III is more flexible, the attitude being: it is more important to learn how mathematics is developed to solve problems than to insist that the students know a given amount of mathematics.

INTRODUCTORY UNDERGRADUATE MATHEMATICS

Outline I, First Year

Functions of One Variable and Linear Algebra

1. Review of function concept, the algebra of real numbers, order. Algebra of functions.

2. The historical background of the calculus: the problem of areas, the problem of tangents, the problem of instantaneous velocity. Heuristic discussion of area as an additive set function whose value is determined on rectangles. Transition to the integral of a function via negative areas. Definition of the integral of a step function. Uniqueness. The integral as a positive linear functional on the family of step functions on an interval.

3. Extension of the integral to other functions via upper and lower approximation by step functions. The family of integrable functions on a bounded closed interval. Show this family closed under addition and multiplication by scalars. The integral as a finitely additive interval function. Integrability of polynomials and the sine function (using summation formulae and trigonometric identities--the trigonometric functions are used here as learned in secondary school; precise definitions will come during the second semester). What other functions are integrable?

4. Definition of continuity in terms of neighborhoods (open intervals). Statement that a continuous function on a bounded closed interval is integrable (proof postponed). The continuous functions are closed under addition and multiplication by real numbers. Statement of uniform continuity of a continuous function on a bounded closed interval. Derivation of this from axiom that such intervals are compact (defined in terms of coverings by open intervals).

5. Proof that continuous functions are integrable (using uniform continuity). Some applications of the integral: moments, energy, work, etc.

6. Approximate integration (piecewise constant, linear, and quadratic approximation). Examples. The problem of a better method

of calculation awaits solution. Ways in which functions depart from continuity; kinds of discontinuity. This leads to definition of limit in terms of deleted neighborhoods.

7. Continuity phrased in terms of limit. Continuity of composites of continuous functions. Algebra of limits. Continuity of products and quotients of continuous functions.

8. Problem of tangents. Heuristic geometric definition of a tangent line to a curve. Calculation of the slope of a nonvertical tangent leads to the derivative of a function. Problem of instantaneous velocity does the same. Examples.

9. Rules for differentiation of sums, products, quotients, composites. Derivatives of identity function and of constant functions give derivatives of rational functions. Derivative of sine function gives derivatives of trigonometric functions.

10. Equations for derivatives from equations for functions give derivatives of algebraic functions (exact definition of fractional exponents next semester). Calculation of tangents to various second-degree curves. Implicit definition of functions.

11. Examples of maxima and minima problems. Attainment of maxima and minima by continuous functions on compact sets. Vanishing derivative test. Rolle's theorem and the Mean Value Theorem. Geometric interpretation.

12. Application of Mean Value Theorem to determine where a differentiable function is increasing, decreasing, constant. Higher derivatives and the second derivative test for maxima (minima). Intermediate Value Theorem on intervals. Applications to graphing. The Mean Value Theorem for integrals.

13. The indefinite integral. Continuity of the indefinite integral of integrable functions. Derivability of same at points of continuity of the integrand, and evaluation of the derivative. Geometric interpretation of this theorem.

14. Reduction of the problem of integration of piecewise continuous functions to that of finding primitives. Applications of this theorem: the practical solution of the problem of integration.

15. New functions. Log defined by indefinite integral. Properties of log. Its derivative. Inverse functions in general. Case

in point: exp function. Its derivative and integral.

16. The number e = exp(1). Arbitrary real powers of e in terms of exp. Arbitrary real powers of positive real numbers. Derivative and integral of x^{α}, α real. Definition of inverse trigonometric functions: the difficulty that trigonometric functions are not on a solid foundation. This leads to problem of arc length.

17. Analysis of arc length in terms of the integral. Definition of arcsine in terms of an indefinite integral. Other inverse trigonometric functions. Reprise of trigonometric functions, their derivatives and integrals, now solidly grounded.

18. Difficulty of integrating log and arcsine leads to integration by parts. Substitution. Certain trigonometric substitutions. Completion of square.

19. Integration of rational functions. Functions not integrable by elementary means leads to idea of uniform approximation. $|\int f - \int g| \leqq \int |f - g|$. Sequences of numbers and their limits. Maximum norm of a continuous function. Uniform convergence of sequences of functions. Pointwise convergence.

20. Taylor's theorem with integral remainder. Same with derivative remainder. Notion of a power series.

21. Series in general. Convergence of geometric series. Archimedean axiom introduced to prove L. U. B. theorem. Corollary: the Monotone Convergence Theorem. Comparison test.

22. Existence of radius of convergence of a power series. Uniform convergence on bounded closed intervals within the interval of convergence. Invariance of the radius of convergence under formal differentiation and integration. Justification of term-by-term differentiation and integration.

23. Parametrized curves in R^2 and R^3. Reprise of function idea. Linearly parametrized lines in R^2, lines and planes in R^3 are functions. Addition of points in R^2 and R^3 and multiplication by real numbers introduced as a notational convenience. Definition of R^n, curves in R^n, vector operations in R^n. Linear and affine functions from R^n to R^m. Linear equations in terms of a single linear function equation.

24. Properties satisfied by the vector operations in R^n. Abstract notion of a vector space over R; linear and affine functions between vector spaces. Examples: function spaces, differentiation, definite and indefinite integral. Vector and affine subspaces: reprise of examples from Section 23, polynomial subspaces, solution space of a system of linear equations, direct and inverse images of affine subspaces under affine functions.

25. Dimension and linear independence, linear span. Basis. The standard basis of R^n. Representation of an n-dimensional vector space as R^n.

26. Representation of linear transformations with respect to bases. Matrix notation. The standard matrix of a linear transformation from R^n to R^m. The algebra of linear transformations and matrices.

27. Change of basis and similarity of matrices. Reprise of systems of linear equations as a single linear transformation equation. Rank and nullity theorem. Matrix notation for linear equations. (Column) rank and nullity of a matrix. General theorems on the dimension of the solution space of a linear system.

28. Elementary column operations and column equivalence of matrices. Echelon form of matrices. The use of this form to solve linear systems explicitly; comments on the numerical problem involved.

Outline I, Second Year

Functions of Several Variables and Linear Algebra

1. Review of cartesian products of sets. Cartesian products of n vector spaces. Multilinear functions with values in a vector space. Sums and real multiples of these.

2. Explicit solution of 2 X 2 and 3 X 3 systems of equations motivates notion of determinant. The determinant as a function of the columns of a matrix: multilinear, alternating, and unimodular on the standard ordered basis of R^n. The permutations of $\{1, ..., n\}$. Theorem: Let $X_1, ..., X_n$ be an ordered basis of V and let $Y \in W$; then there is a unique alternating multilinear function F

such that $F(X_1, \ldots, X_n) = Y$. Corollary: existence and uniqueness of the determinant, and an explicit formula for it. Corollary: the basis-free definition of the determinant of a linear transformation.

3. Multiplication of determinants; the group of nonsingular linear transformations. Theorem: An alternating m-linear function on an n-dimensional vector space, $m > n$, is necessarily zero. The solution of the equation $F(X) = A$ where F is a linear transformation from V into itself. Cramer's rule.

4. Invariant subspace of linear transformations; internal direct sums of subspaces. Consequences for matrix representation. One-dimensional invariant subspaces. The characteristic equation and eigenvalues. Cayley-Hamilton theorem.

5. Discussion of length and angle leads to notion of inner product. Length and norm. Schwarz inequality and definition of angle. Example of the integral inner product on the continuous functions on $[a, b]$. Trigonometric polynomials of order $\leqq m$. Orthonormal basis and the Gram-Schmidt process. The standard form of the inner product.

6. Symmetric and orthogonal linear transformations and matrices. Polar decomposition. Diagonalization theorem.

7. Application to conics and quadrics. Volume of a parallelepiped in terms of determinant. Orientation: defined by alternating function. Cross product in dimension 3, given an inner product and an orientation (use representation theorem for real-valued linear functions).

8. Neighborhoods of points as open spheres. Continuous functions between vector spaces with inner product, or more generally with a norm. Examples, including the continuity of the integral in the uniform norm. A linear transformation between two finite-dimensional inner product spaces is continuous.

9. Open and closed sets. Continuity in terms of open sets. Unions and intersections of open and closed sets. Interior, closure, and boundary of a set. Sequential limits. Characterization of previous notions using sequential limits.

10. Limit points in general. Theorem: An infinite subset of a compact (Heine-Borel) set S always has a limit point in S.

Corollary: compact sets are closed. Closed subsets of compact sets are compact. Continuous images of compact sets are compact; real continuous functions on a compact set attain their maxima.

11. Closed bounded sets in a finite-dimensional normed vector space are compact. Corollary: any two norms on a finite-dimensional normed vector space are equivalent. Cauchy criterion and completeness for a finite-dimensional normed vector space.

12. Uniform continuity of continuous functions on compact sets. Sequences of functions and uniform convergence of same. Completeness of space of continuous functions. Ascoli's theorem.

13. Connectedness; the Intermediate Value Theorem for real-valued functions.

14. Problem of volume in R^n. Heuristic discussion of volume as an additive set function whose value is determined on boxes. Problem of bad boundaries.

15. Volume of a domain with nice boundary. The integral defined in terms of step functions and its connection with signed volume under a hypersurface. The integral as a uniformly continuous positive linear function.

16. The integrability of continuous functions. The integral as a finitely additive set function. Differentiability of set functions.

17. Characterization of uniformly differentiable set functions. Reduction of multiple integration to iterated integration, calculation. Setting up iterated integrals.

18. Functions on open domains. The derivative

$$F(X,Y) = \lim_{t \to 0} \frac{F(X + tY) - F(X)}{t}.$$

Class C^1 and the linearity of $F'(X,\cdot)$. Geometric interpretation. Interpret $F(X_0) + F'(X_0, X - X_0)$ as "best" affine approximation to F at X_0. Matrix representation of $F'(X,\cdot)$ and partial derivatives.

19. Mean Value Theorem. Higher derivatives defined recursively:

$F^{(n)}(X, Y_1, \ldots, Y_n)$

$$= \lim_{t \to 0} \frac{F^{(n-1)}(X + tY_n, Y_1, \ldots, Y_{n-1}) - F^{(n-1)}(X, Y_1, \ldots, Y_{n-1})}{t}$$

The differentiability classes C^n. Symmetry and linearity of $F^{(n)}(X, \cdot, \ldots, \cdot)$ with $F \in C^n$. Taylor's formula.

20. Derivatives of composites and the ordinary chain rule (a matrix equation). Jacobians. Local one-one theorem for $J \neq 0$. Differentiability of invertible F with $J \neq 0$. $J(F^{-1}) = J(F)^{-1}$.

21. Implicit Function Theorem. Critical points. Lagrange multipliers.

22. Formula for volume change under C^1 function. $\int f = \int (f \circ F)|J(F)|$. This leads to notion of a differentiable n-form on R^n: $w(X, Y_1, \ldots, Y_n)$, alternating multilinear in the Y's.

23. Heuristic arguments concerning work and total flux integrals lead to notions of a differential p-form on an n-dimensional vector space and the definition of the integral of same over a parametrized p-cell. Singular simplices. Differential p-forms as functions from differential simplices to the reals.

24. Representation of 0-forms by real functions and (given a scalar product) of 1-forms by vector fields. Representation of $(n - 1)$-forms by vector fields and n-forms by real functions, given a scalar product and an orientation. The integral of a differential p-form over a totally nondegenerate singular p-simplex is independent of the parametrization.

25. d: 0-form $\rightarrow$ 1-form, by taking the derivative. The gradient. The extension of the notion to p-forms by differentiation and skew-symmetrization. $d^2 = 0$. Divergence and ($\dim V = 3$) curl of a vector field. Curl $\circ$ grad = 0; div $\circ$ curl = 0.

26. Multiplication of forms as ordinary multiplication of functions skew-symmetrized. Representation of forms with respect to coordinate functions; d in terms of a coordinate system.

27. Stokes' theorem over standard n-cell in R^n. General case of Stokes' theorem, classical Stokes' theorem, Gauss' theorem,

Green's theorem, and the Fundamental Theorem of the Calculus. The meaning of curl and div in fluid dynamics.

Outline I, Differential Equations

An adequate preparation for the course outlined here is the successful completion of the first one-and-a-half years' work of Outline I.

The course is designed for a semester

(i) to give the student the basic existence and uniqueness results for ordinary differential equations and systems of equations;

(ii) to develop in detail the properties of solutions of some important types of linear systems--constant coefficients, analytic coefficients, and systems with regular singular points--by exploiting the student's earlier preparation in linear algebra; and

(iii) to introduce the student to some topics of current research interest: stability of nonlinear systems, eigenvalue problems, elementary partial differential equations.

1. Complex numbers. Complex-valued functions. Polynomials. Complex series and the exponential function. Complex n-dimensional space and functions defined on it.

2. Examples of problems involving differential equations: Newton's laws of motion, heat flow, vibration problems. Initial value problems and boundary value problems.

3. Local existence of solutions to initial value problems for $y' = f(x,y)$, where x, y real and f real-valued. The method of successive approximations (fixed point theorem), using a Lipschitz condition. The polygon method, using the Ascoli lemma. Nonlocal existence, using a Lipschitz condition on f in a strip $|x - x_0| \leqq a$, $|y| < \infty$. Approximations to, and uniqueness of, solutions. Extension of results to case where x real, y complex, f complex-valued.

4. Existence and uniqueness for systems using vector, and vector-valued, functions. Extension of material in Section 2 to this case. Example: central forces and planetary motion.

Applications to equations of n^{th} order.

5. General results on homogeneous linear systems $y' = A(x,y)$ where A is linear in $y \in C^n$. The space of solutions as a vector space of dimension n. Nonlocal existence in this case. The solution of the nonhomogeneous system $y' = A(x,y) + b(x)$. Application to linear equations of n^{th} order.

6. Linear systems with constant coefficients $y' = A(y)$. Explicit structure of space of solutions using $\exp A$, and assuming Jordan canonical form for A. Explicit form that variation of constants takes in this case. Application to n^{th}-order equations. The case $n = 2$ in detail.

7. Linear systems with analytic coefficients (convergent power series as coefficients). Solutions as convergent power series. Application to n^{th}-order equations. Example: the Legendre equation.

8. Linear systems with regular singular points: $y' = x^{-1}A(x)y$, with A having convergent power series expansion. Structure of solution space using $x^A = \exp(A \log x)$. Application to second-order equations with regular singular points. Examples: the Euler equation and the Bessel equation.

9. Introduction to nonlinear theory. Perturbations of two-dimensional real autonomous systems. Classification of simple critical points. Phase portraits. Stability. Asymptotic stability. Relation of nonlinear case to linear approximation.

10. Poincaré-Bendixson theory (optional).

11. Self-adjoint eigenvalue problems for second-order linear equations--the regular case. The space of continuous functions $\mathcal{C}$ as a linear manifold in L^2. The existence of eigenvalues using complete continuity of the Green's operator in $\mathcal{C}$. Bessel's inequality and the Parseval equality. Expansion theorem.

12. Second-order linear partial differential equations. Classification: hyperbolic, elliptic, parabolic. Equations with constant coefficients. Typical initial and boundary value problems in each case. Application of results in Section 11.

Functions of One Variable

1. Functions and the real numbers. Review of the general concept of function. Characterization of the real numbers, the Archimedean axiom, and suprema. The algebra of real functions defined on a set, polynomial functions from the reals to the reals.

2. The problems of area. Historical background, including the work of Archimedes. Heuristic discussion of the problem of defining area as an additive set function whose value is determined on rectangles. Transition to the integral of a fuction via negative areas. Definition of the integral of a step function, uniqueness. The integral as a positive linear functional on the family of step functions on an interval.

3. Extension of the integral to more general functions. Upper and lower approximations. The family of integrable functions on a bounded closed interval. The observation that this family is closed under addition and scalar multiplication. Discussion of some functions which are integrable, monotone functions, sums of monotone functions. Integration of some explicit functions such as polynomial functions and some of the trigonometric functions, assuming that at least an intuitive definition of these functions together with their principal algebraic and geometric properties has been learned in an earlier study of mathematics. Approximations to the integral and estimates of error.

4. Continuous functions. Definition of continuity in terms of open intervals. Observation that the continuous functions are closed under both addition and multiplication. Continuity of the polynomial functions. Derivation of uniform continuity of a continuous function on a closed bounded interval, assuming such intervals are compact (defined in terms of coverings by open intervals).

5. Integrability of continuous functions. Applications of integrals to problems such as calculating areas, moments, work, and energy.

6. Approximations to the integral. Piecewise constant, linear, and quadratic approximations. Estimates of error. Ways in which

functions depart from continuity; discontinuities. Definition of approximations.

7. The algebra of continuous functions. Continuity in terms of limits. Continuity of composites, products, and (under appropriate circumstances) quotients. The algebra of limits.

8. Historical backgound of the problem of tangents. Heuristic geometric definitions of the tangent to a curve at a point. The problem of velocity. Definition of the derivative of a function. Derived function and its geometric interpretation as the function which to every point assigns the slope of the tangent of the original function.

9. Formal differentiation. Derivation of rules for calculating derivatives of sums, products, quotients, and composites. Numerous calculations to develop techniques. Algebraic functions. Calculation of tangents to various second-degree curves. Implicitly defined functions.

10. Maxima and minima. Proof that a continuous function on a closed bounded interval attains its maximum. Criteria for determination of maxima. Vanishing derivative test. Introduction of the second derivative. Sufficient conditions for local maxima. Graphs, geometric ideas of convexity, and maxima. Interpretation of the second derivative as acceleration. Rolle's theorem and the Mean Value Theorem. Intermediate Value Theorem on intervals. Application of the preceding to problems involving graphing, velocity, and acceleration. The Mean Value Theorem for integrals.

11. Relation between integration and differentiation. The indefinite integral, and continuity of functions defined by integration of integrable functions. Differentiability of such functions at points of continuity of the integral. Geometric interpretation of the preceding. Piecewise continuous functions and reduction of the problem of integrating such functions to the problem of finding primitives. Various calculations via this last result.

12. Functions defined by integrals. The logarithm function and its properties. Inverse functions in general. The exponential function and its properties. The number $e = \exp(1)$. Arbitrary real power of e and hence of any positive real number.

13. Methods of integration. The difficulty of integrating the logarithm function. Integration by parts. Substitutions, including certain trigonometric substitutions. Completion of the square. Integration of rational functions.

14. Uniform approximation. Functions not integrable by elementary methods. Approximations of the integral of such functions by approximating the function itself uniformly by functions with elementary integrals. Sequences of numbers and their limits. The sup norm of a continuous function. Uniform limits of continuous functions. Space of continuous functions closed under uniform limits. Pointwise limits.

15. Taylor's theorem with remainder. Various forms of the remainder. Use of Taylor's theorem to approximate functions by polynomials; estimates of the error of approximation in concrete examples. The idea of a power series.

16. Series in general. Infinite series of real numbers. Various tests for convergence, including the comparison test, n^{th} root test, and the ratio test. Power series. Radii of convergence of power series and their determinations. Uniform convergence of the partial sums on bounded closed intervals within the interval of convergence. Proof of the invariance of the radius of convergence under formal differentiation and integration. Justification of term-by-term integration and differentiation.

17. Further properties of power series. The algebra of power series converging in a fixed radius. Analytic functions, Taylor's theorem, and power series. The possibility of defining functions by means of power series. The power series for certain classical functions, particularly the exponential. Possibility of defining sine and cosine functions by power series and observation that this would eliminate the difficulty that they have not been well-defined until this time.

18. Definition of R^n as n-tuples of real numbers. Distances and limits in R^n, the norm, and perpendicularity. R^2 and R^3 discussed explicitly, together with the physical intuition concerning them. Addition and scalar multiplication in R^n. Linear functions mapping intervals in R^1 into R^n, with particular attention to lines

in R^2 and R^3.

19. Integration of functions from intervals to R^n. Observation that in the definition of the integral of real functions, it was important that the domain be a subset of the line, but that only certain properties of the range entered. Introduction of some examples. Differentiation of functions from intervals to R^n. The derivatives of such functions are defined directly, and then it is observed that they could have been obtained from the coordinate functions.

20. Curves in R^n and their tangents. Newton's laws of motion. Two curves f, g meet at a point t if $f(t) = g(t)$. Their order of contact at such a point is the largest integer n such that

$$\lim_{s \to t} \frac{|f(s) - g(s)|}{|s - t|^n} = 0,$$

or ∞ if no such integer n exists. Tangent lines and the order of contact of a tangent line with a curve.

21. Taylor's theorem for functions from an interval to R^n. Geometric relation of Taylor's theorem with the order of contact of a curve and a "polynomial" curve. Approximation of a curve by "polynomial" curves. Arc length in R^n studied carefully. Inverse trigonometric functions and arc length in R^2. Principal normal to a curve. Curves in R^2 and R^3 and their curvature. The osculating circle as an approximation to such a curve.

22. Velocity, acceleration, Newton's laws of motion. Other physical problems involving differential equations (e.g., vibrating string). Families of curves and differential equations. Solutions of certain simple differential equations. Initial and boundary value problems.

23. First-order differential equations. Approximation to solutions. Lipschitz condition and the existence and uniqueness of solution, both local and nonlocal, by the Picard method. Cauchy's proof of existence. Examples of differential equations with distinct solutions passing through a point. Application to central forces and planetary motion.

24. Power series and linear differential equations with

analytic coefficients. Existence of solutions. The possibility of defining functions as the solution of certain differential equations, illustrated by examples such as sine and cosine. Derivation of properties of the sine and cosine from their defining differential equations.

Outline II, Second Year

Topology and Functions of Several Variables

1. Definition of topological space; continuous mappings in terms of open sets. Definition of metric space, the associated topological space of a metric space, continuity at a point for functions mapping one metric space into another, and the equivalence of continuity with continuity at all points. The examples of Euclidean spaces, spheres, and real projective spaces (defined as quotients of spheres).

2. Subspaces, quotient spaces, and product spaces of topological spaces; restriction of continuous mappings. Subspaces and product spaces for metric spaces, and relation with the same operations on topological spaces. The examples furnished by Euclidean spaces and tori.

3. The notion of Hausdorff space, and proof of its stability under the operations of taking subspaces or products. Observation that metric spaces are Hausdorff. Examples to show that quotient spaces of Hausdorff spaces are not necessarily Hausdorff.

4. The notion of compactness defined for a Hausdorff space by the finite covering property. Closed bounded subsets of Euclidean spaces are compact. Proof that a metric space is compact if and only if it is complete and totally bounded.

5. Tychonoff's theorem for finite products. The notion of local compactness, and proof that finite products of locally compact spaces are locally compact. Euclidean spaces and tori.

6. Introduction of complex numbers. Real and complex topological vector spaces. Uniqueness of the topology on finite-dimensional vector spaces.

7. Real and complex projective spaces as the lines in real or

complex n-space. Equivalence of this definition of real projective space with the earlier definition. Complex projective spaces as quotient spaces of spheres. Properties of quotient spaces when the set of points equivalent to any point is compact. Conditions insuring that the quotient of a metric space is metric.

8. Inner products and norms on topological vector spaces. Equivalence of norms of finite-dimensional vector spaces. Isometry of inner product spaces having the same dimension. Cauchy criterion and completeness.

9. Uniform continuity of continuous functions on compact sets. Sequences of functions; the proof that the set of continuous functions is complete. Ascoli's theorem.

10. Connected spaces and components. Continuous images of connected spaces are connected. The Intermediate Value Theorem for real-valued functions on connected topological spaces.

11. Contraction maps in metric spaces and the fixed point theorem for contractions in complete metric spaces. Relation of this theorem to Picard's method for the existence of solutions of ordinary differential equations in open domains in Euclidean space.

12. General results on homogeneous linear systems of differential equations. The solution space. Nonhomogeneous systems. Applications to linear equations of the n^{th} order.

13. Linear system with constant coefficients. Explicit structure of the solution space using Jordan canonical form and exponential. Application to n^{th}-order equations.

14. Integration and volume in Euclidean spaces. Volume of boxes. Domain with smooth boundaries; difficulties involved with bad boundaries. Integral defined using step function and shown to be a uniformly continuous positive linear function. Integrability of continuous functions. The integral as a finitely additive set function. Reduction of multiple integration to iterated integration. Examples and calculations using iterated integrals.

15. The idea of two functions, defined on a domain in Euclidean space with values in a Euclidean space, touching at a point. Explicitly, two continuous functions f and g have order of contact n at x if

$$\lim_{y \to x} \frac{|f(y) - g(y)|}{|x - y|^n} = 0,$$

and touching means order of contact 1. Proof that if an affine function touches f at x, then it is unique; f is defined to be differentiable at x if such an affine function exists. Continuously differentiable functions.

16. Geometric implications of order of contact with particular emphasis on touching. Connection of differentiability with differentiability along lines and with affine approximations. Examples, formulae, and matrix representation using the standard coordinates in Euclidean space.

17. Mean Value Theorem. Recursive definition of higher derivatives. The class of n-times continuously differentiable functions. Taylor's formula, and proof that an n-times differentiable function f has order of contact n at x with the standard approximation to f obtained using the first n derivatives of f at x.

18. Derivatives of composite functions, the chain rule using linear transformations. The Jacobian matrix; examples involving the Jacobian matrix.

19. The Inverse and Implicit Function Theorems in geometric form. Their formulation using coordinates. Invariance of domain under diffeomorphism.

20. Changes of volume induced by a continuously differentiable function. Calculations for a range of examples. The notion of an n-form on a domain in n-space; connection with volume and volume change.

21. Intuitive discussion of differential forms on Euclidean n-space, their use in Newtonian mechanics. Intuitive description of the integral of a q-form over a differentiable singular q-simplex.

22. Exterior algebras for finite-dimensional vector spaces. Morphisms of same induced by linear transformations. Orientations of real vector spaces via the exterior algebra. Duality between q-forms and (n-q)-vectors in an oriented vector space.

23. Introduction of differential forms and vector fields on domains in Euclidean space. Duality between q-forms and fields of

(n-q)-vectors. Integration of differentials over differentiable singular chains.

24. The star isomorphism in the exterior algebra of a Euclidean vector space and its extension to forms and vector fields on a domain in Euclidean space. Morphisms of forms and vector fields induced by differentiable functions.

25. The connection of integration of n-forms over singular n-chains in domains in n-space with the integration of functions defined earlier. Subdivision of domains with smooth boundary and singular chains. Volume, exterior algebras, determinants, and the idea of a Riemannian metric.

26. The exterior derivative and its properties. Connection with gradients; further geometric ideas. Poincaré lemma for convex regions. Interpretation of Poincaré lemma in terms of existence of solutions of differential equations. Exact equations, integrating factors, and calculations in low dimensions. Special properties of 3-space. Curl, divergence, and the exterior derivative.

27. The general Stokes theorem integrating q-forms over singular q-chains. Classical form of Stokes' theorem, including Gauss' theorem, Green's theorem, and the Fundamental Theorem of Calculus. Physical interpretations: study of flows, charges, etc.

28. Further applications of the calculus of differential forms to physical problems, including Maxwell's equations in both Newtonian and relativistic form. Hamilton's equations in dynamics.

Outline II, Linear Algebra

This one-semester course, as part of Outline II, presents separately the fundamental notions of linear algebra which are recommended as suitable for Introductory Undergraduate Mathematics. The course can be presented during any part of the first two years that is deemed appropriate for the students involved.

It is noted here that Chapters 1 through 13 of this course are designated as prerequisite to the algebra courses outlined in Appendix B. This is consistent with the amount of linear algebra in Outlines I and II of Introductory Undergraduate Mathematics.

1. The complex numbers and subfields of the complex numbers. The integers modulo p.

2. Vector spaces and linear transformations. Examples of vector spaces, particularly spaces of n-tuples and spaces of functions. Subspaces, quotient vector spaces. Linear independence, generating sets, and the notion of basis.

3. Dimension for finite-dimensional vector spaces, invariance of dimension, finite-dimensional subspaces of general vector spaces. Behavior of dimension with respect to subspaces and quotient vector spaces.

4. Inner products for real vector spaces; length and volume. Euclidean vector spaces defined as finite-dimensional vector spaces with inner product. Orthogonal bases. Gram-Schmidt process and its relation to volumes. Subspaces, complementary subspaces, and their relation with quotient vector spaces. Lines, planes, hyperplanes, and distances.

5. Hermitian vector spaces defined as complex vector spaces with a complex (Hermitian) inner product. Length of vector, volume of boxes, the associated Euclidean vector space of a Hermitian vector space. Orthogonal bases. Gram-Schmidt process and its relation to volumes. Subspaces, complementary subspaces; lines, planes, hyperplanes, distances.

6. Recollection of definition of linear transformations. Definition of matrix; representation of linear transformations by matrices. Composition of linear transformations. Change of basis.

7. Orthogonal, symmetric, and skew-symmetric transformations of Euclidean vector spaces; their relation with matrices and bases.

8. Unitary, Hermitian symmetric, and skew-Hermitian transformations of Hermitian vector spaces; their relation with matrices and bases.

9. Inductive definition of the determinant of a matrix. Relation of determinants to volumes of boxes.

10. Permutations. New definition of determinant and equivalence with the old. Multiplicative properties of the determinant. The determinant of orthogonal and unitary matrices, and of the transposed matrix.

11. Inverses, adjoints, elementary matrices, and reduction to diagonal form. Applications to systems of linear equations.

12. Proof that in a Euclidean space any nonsingular linear transformation is the product of a positive-definite symmetric transformation and an orthogonal transformation. The idea of Euclidean geometry. Invariance under rotations and translations.

13. Proof that in a Hermitian space any nonsingular linear transformation is the product of a positive-definite symmetric transformation and a unitary transformation. Hermitian geometry.

14. Decompositions of a vector space into irreducible cyclic subspaces relative to a linear transformation. Jordan canonical form.

15. Definition of minimal polynomial. The characteristic polynomial as a product of certain minimal polynomials of irreducible subspaces of a cyclic decomposition.

16. Characteristic vectors, characteristic values, relations with the characteristic polynomial. Special cases involving orthogonal, symmetric, unitary, and Hermitian symmetric transformations.

Outline III, First Year

Functions of One Variable

The questions of what to teach in calculus and how are notoriously difficult to answer, and the answer has to be reargued by each generation. The main difficulty is that calculus has to be both problem-oriented and theory-oriented. The former means that the student must be made aware of how theories arise to deal with concrete problems, that these concrete problems often originate in the external world, and that the external world is an important source of our intuition (and of our aesthetic criteria). The latter means that the basic concepts should be introduced in the same spirit in which they are used by working mathematicians, and that proofs ought to have the same clarity and elegance which distinguishes all first-rate mathematics.

Fortunately, the two views do not conflict but complement each other: to demonstrate how an abstract theory is developed to deal with a concrete problem and unify what is common in various problems is one of the most valuable lessons for budding young mathematicians, far more valuable than merely presenting the postulates for real numbers or the axioms of linear algebra.

The choice of subjects and their arrangement is not entirely rigid, e.g., the construction of the real numbers, which is put at the beginning, can just as reasonably be done later. The same goes for the uniform continuity of continuous functions over compact intervals.

1. Real numbers. The intuitive notion of the continuum of real numbers. The gaps in the rational numbers (Pythagorean theorem); construction of the real number system, either by nested intervals, Dedekind cuts, or infinite decimals. The topology of real numbers; the algebra of limits.

Three basic theorems: the real numbers are complete; closed bounded intervals are compact; a bounded set of real numbers has a supremum.

Nondenumerability of real numbers, denumerability of rational and algebraic numbers.

Mathematical induction.

2. Analytic geometry. Points of 2-, 3-, and n-dimensional space as ordered n-tuples of real numbers. Addition, multiplication by scalars. Straight lines, convex sets, hyperplanes, linear subspaces. Dimension of linear subspaces.

Euclidean distance, scalar product, Schwarz inequality. Orthogonality. Gram-Schmidt process.

Complex numbers.

3. Differentiation. The concept of a function; illustration, graphical representation. Intuitive notion and rigorous definition of a continuous function. The algebra of continuous functions.

Intuitive notion and rigorous definition of the derivative as slope and instantaneous velocity. Derivatives of polynomials. Algebraic rules for differentiating sums, differences, constant multiples, products, and quotients of functions.

Differentiation of trigonometric functions, based on geometric definition.

Linear approximation to functions; derivation of the chain rule.

Local existence and differentiability of the inverse of a function with nonzero derivative. Newton's method.

4. Integration. The intuitive notion of integral as signed area, work; examples of integrals which can be calculated by a direct passage to the limit.

Existence of the integral of a uniformly continuous function over a finite interval.

Basic properties of the integral: linearity, positivity. The Mean Value Theorem.

The integral as function of its upper limit. Differentiation as the inverse of integration. The log function and its inverse. Statement of the theorem that a function with zero derivative is constant. Integration as antidifferentiation. The inverse trigonometric functions. Techniques of integration, partial fractions, integration by parts, change of variables.

Estimation of integrals, Stirling's formula.

Arc length, surface area, and volume of bodies of revolution.

5. More about continuous and differentiable functions. Three theorems about continuous functions: existence of maximum and minimum over a finite closed interval, existence of intermediate values, and uniform continuity of continuous function in compact intervals.

Proof of the Mean Value Theorem. Proof that if $f' = 0$, then f is constant.

Calculation of maxima and minima.

Higher derivatives; their geometric and physical significance.

Taylor's theorem with remainder (both derivative and integral form). Taylor series for the exponential and trigonometric functions, the logarithm, the binomial series. Examples of functions [e.g., $\exp(-1/x^2)$] which are not represented by Taylor series.

The notion of the maximum norm; uniform convergence. The completeness of the continuous functions under the maximum norm. Continuity of the integral with respect to the maximum norm.

Termwise differentiation of series.

The interval of convergence of a power series; calculus of convergent power series.

Improper integrals.

Outline III, Second Year

Linear Algebra and Functions of Several Variables

Again, the choice of subjects and their arrangement is not entirely rigid. Sections 6 and 7 contain more advanced topics; some fraction of these may be covered if there is time.

1. Vector- and matrix-valued functions and their applications in geometry and mechanics. Linear transformations of the plane and 3-space into themselves; their description with the aid of matrices. Definition of a matrix as a linear transformation of R^n into R^m. The multiplication of matrices via the composition of transformations.

Definition of symmetric, antisymmetric, and orthogonal matrices. Orthogonal matrices form a noncommutative group. Description of orthogonal matrices in two and three dimensions in terms of rotation and reflection.

Curves in n-dimensional space as vector-valued functions. The notion of continuity and differentiability of vector-valued and matrix-valued functions. Algebraic rules for differentiating scalar and matrix products of functions.

Arc length and curvature in 2- and 3-dimensional space.

Orthogonal transformations depending on a parameter; their derivative expressed in terms of antisymmetric transformations. Geometric interpretation as infinitesimal rotation. Introduction of vector product in 3-dimensional space.

Mechanics: Newton's laws for particles. Systems of particles acting on each other by central forces. Center of mass, the moment of forces. Rate of change of momentum, angular momentum, and energy.

Motion of rigid systems of particles. Moment of inertia.

2. Ordinary differential equations; application of some notions from linear algebra. Examples of differential equations from physics, chemistry, and geometry, and their explicit solution in terms of elementary functions.

Radioactive decay, vibrating spring, law of mass action, two-body problem, oscillation of electric circuits, trajectories of simple vector fields, etc. Examples where physical intuition suggests the qualitative behavior of solution: under- and over-damp, etc.

Examples of differential equations which cannot be solved explicitly: the three-body problem, etc. The need for a theory, i.e., existence and uniqueness theorems, qualitative estimates, and methods for finding approximate solutions.

Statement and motivation of the existence and uniqueness theorem for the initial value problem. Proof of uniqueness by conservation of energy in special situations (e.g., vibrating spring). Solution of analytic initial value problems by power series; recovery of the exponential and trigonometric functions.

Difference methods for solving the initial value problem. Comparison of exact and approximate solutions in simple cases which can be handled explicitly.

The fixed point theorem for contracting transformations of a complete metric space. Proof of the existence and uniqueness theorem (do it in the special but typical case of a single first-order equation).

The abstract notion of a linear space over the complex numbers. Dimension, coordinates. Linear transformations.

The notion of an operator mapping a certain class of functions into another. Linear operators, linear differential operators.

The set of solutions of a homogeneous linear differential equation forms a linear space. Calculation of the dimension of this space by the existence and uniqueness theorem.

First-order matrix equations $y' = A(t)y$. The solution operator $U(t)$ defined by $y(t) = U(t)y$. Solution of the inhomogeneous equation $y' = A(t)y + f$ given by $y(t) = U(t) \int_0^t U^{-1}(s)f(s)\, ds$.

The algebra of scalar differential operators with constant coefficients: factorization, commutation. Main theorem: if $L_1, \ldots, L_k$ are pairwise relatively prime, then the nullspace of their product is the direct sum of their nullspaces. Proof based on main lemma about relatively prime polynomials. Solution of $(D - \lambda)^n u = 0$ in terms of exponentials and polynomials.

Existence of a complete set of generalized eigenvectors of a linear transformation of a linear space into itself, based on main lemma about polynomials. Triangular form of a matrix.

Differentiation transforms any solution of a linear differential equation with constant coefficients into another solution. The eigenvectors of this transformation are exponentials times polynomials.

The signature of a quadratic form.

The spectral theory of symmetric matrices. Extremal property of eigenvalues. Positive-definiteness. Spectral theory of unitary matrices.

Small vibrations of mechanical systems. Monotonic dependence of characteristic frequencies on the potential energy.

Sturm separation theorem. Simple two-point boundary value problems. Characteristic frequencies, resonance.

3. Differentiation of functions of several variables. Determinants as alternating multilinear functionals.

Open and closed subsets of n-dimensional space. Compactness of bounded, closed subsets.

Functions defined on subsets of n-space. Continuity. Existence of maxima and minima on compact sets. Uniform continuity on compact sets.

Differentiability at interior points in terms of approximation by linear functions. Chain rule. Partial derivatives of first order. Maxima and minima, stationary points. Geometric interpretation of grad f in Euclidean space as normal to surface f = const. Examples.

Functions k times differentiable; approximation by polynomials of k^{th} order. Higher partial derivatives; commutation of partial differentiation. Classification of stationary points. Examples.

Extreme values under side conditions; Lagrange multiplier.

Vector fields; fields of force, gradient fields, conservation of energy, Newtonian potential.

Mapping of n-space into m-space; Jacobian. Composition of mappings. Implicit and Inverse Function Theorem.

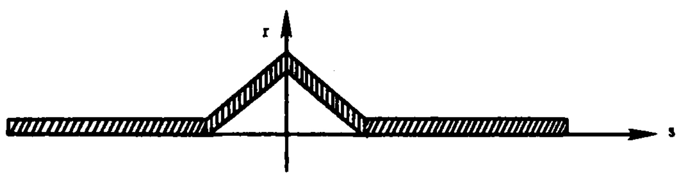
Conformal mapping.

The degree of a mapping, following the method of E. Heinz. Journal of Mathematics and Mechanics, 8 (1959), pp. 231-247.

4. Integration of functions of several variables. The intuitive notion of the integral in Euclidean space: volume, mass, momentum, moment of inertia, potential of mass, etc.

Rigorous definition, starting with the integral of continuous functions with compact support defined by using rectangles in a fixed orthogonal frame. Properties of the integral: linearity, positivity, translation invariance.

Theorem. These properties characterize the integral up to a positive multiple. Proof: Let $I(f)$ be a linear, translation-invariant, positive functional defined for all continuous functions with compact support. Denote by $r(s)$ the "roof" function graphed below:



Denote $r(ms)$ by $r_m(s)$. Every piecewise linear function whose derivatives are discontinuous only at the points i/m, i an integer, can be expressed as a linear combination of $r_m(s)$ and its translates. In particular,

$$r(s) = \sum_{-m}^{m} \left|1 - \frac{k}{m}\right| r_m\left(s + \frac{k}{m}\right).$$

Define now

$$h(x) = \prod_{j=1}^{n} r(x_j)$$

$$h_m(x) = h(mx).$$

Putting $s = x_j$ and multiplying we get

$$h(x) = \sum a_{k,m} h_m\left(x + \frac{k}{m}\right),$$

k a multi-index,

$$\sum_k a_{k,m} = m^n.$$

So, using the first two properties of I, we get

$$I(h) = m^n I(h_m).$$

Every piecewise linear function $\ell(x)$ in R^n whose derivatives are discontinuous only on the hyperplanes $x = i/m$, i an integer, can be expressed as a linear combination of h_m and its translates. This shows that $I(\ell)$ can be expressed in terms of $I(h)$. Since every continuous function f with compact support can be approximated by such piecewise linear functions, it follows that $I(f)$ can be expressed in terms of $I(h)$.

Corollary 1. Volume integral = repeated integral.

Corollary 2. Integral is independent of orthogonal frame chosen (consider functions which depend only on $|x|$).

Corollary 3. Under a linear change of variables, the integral is multiplied by a factor, which is a multiplicative functional of the matrix of the transformation.

Corollary 4. This factor is the absolute value of the determinant of the matrix of the transformation. (Proof by writing the matrix as a product of orthogonal and diagonal transformations and using the first three corollaries.)

The formula for integration by parts:

$$\int f_{x_j} g \, dx = -\int f g_{x_j} \, dx$$

follows from Corollary 1.

Integration over open sets. Intuitive notion of volume of an open set in terms of filling it up with cubes of unequal size. Rigorous definition:

$$V(D) = \text{Sup} \int f \, dx \begin{cases} f \leq 1 & \text{in } D \\ f \leq 0 & \text{outside } D \end{cases}$$

Corollary. Volume is unchanged under rotation and translation. Interpretation of determinant as volume.

Intuitive notion of the integral of a function over an open set in terms of approximating sums over cubes of unequal size.

Rigorous definition:

$$\int_D f\,dx = \text{Sup} \int g\,dx \begin{cases} g \leqq f & \text{in } D \\ g \leqq 0 & \text{outside } D \end{cases}$$

Corollary.

$$\int_D f\,dx \leqq V(D)|f|_{max}.$$

Evaluation of integral over D by repeated integration when f is continuous up to the boundary of D and the boundary of D is nice, i.e., each line cuts it only in a finite number of places (convex domains and unions of convex domains). Examples.

Change of variables in one-to-one multiple integrals: Let $x \to y$ be a mapping of an open set D in x-space onto a set G in y-space with continuous first derivatives and nonzero Jacobian. Let $f(y)$ be a continuous function with support in G. Then

$$(*) \qquad \int_D f(y(x)) \left|\frac{\partial y}{\partial x}\right| dx = \int_G f(y)\,dy.$$

Proof: Let $\sum p_j(y) \equiv 1$ be a smooth partition of unity in y-space, $g_j(y) = p_j(ny)$ a refinement of it. Write the left side of (*) as

$$\sum_j \int g_j(y(x)) f(y(x)) \left|\frac{\partial y}{\partial x}\right| dx.$$

For j fixed, replace the Jacobian by its value at x_j, $y(x)$ by a linear approximation to it. By Corollary 4 the resulting integral equals $\int g_j(y) f(y)\,dy$; the total error committed is easily estimated and tends to zero with increasing n.

Examples: Change to polar coordinates. Evaluation of various integrals, such as the error integral.

Area-preserving maps. Canonical transformations.

Domain with smooth boundary defined by possibility of smooth local parametrization. Proof that $f(x) < 0$ has a smooth boundary if grad $f \neq 0$. Intuitive notion of surface area. Definition by integral; independence of parametrization. Surface integrals.

Integration by parts over domains with smooth boundaries.

Continuous functions form a Euclidean space under scalar product $(f,g) = \int fg\,dx$. Notion of a linear operator. Symmetry and positivity of the Laplace operator under boundary condition $u = 0$

or $du/dn = 0$. Analogy to symmetric positive matrices. Uniqueness of boundary value problem for the Laplace equation and the mixed initial and boundary value problem for the wave equation.

5. Exterior forms. The Gauss and Stokes theorems in special cases. Their interpretation for flows and in the theory of electricity and magnetism.

Definition of exterior form, Grassmann algebra, differential of a function.

Integration of forms over singular chains.

The exterior derivative; gradient, curl, and divergence as special cases. The Poincaré lemma.

The general Stokes theorem. Applications to Cauchy's integral theorem.

6. Introduction to the calculus of variations. Examples of problems in the calculus of variations for functions of one variable. The general problem of finding extrema for

$$\int f(x,y,y')\,dx.$$

The Euler equation; examples where Euler equation can be solved explicitly.

Quadratic variational problems. Proof that the integral is definite if the underlying interval is short enough.

The second variation; examples where the second variation is not positive (catenoid); conjugate points. Geodesics; example of the Poincaré half-plane.

Variational problems for functions of several variables. The Dirichlet integral. Plateau's problem.

7. Harmonic analysis. Fourier transform, Parseval's formula. Convolution.

Appendix B

HIGHER UNDERGRADUATE MATHEMATICS

In this section we have placed one or more outlines for each of the courses which the Panel suggests should be provided by every college department undertaking the program and which are relevant to the central areas of mathematics:

Real analysis

Complex analysis

Abstract algebra

Geometry-topology

Probability or mathematical physics

It is not intended that these outlines shall be construed as completely determining the content of these courses; alternatives will be welcomed by the Panel.

Real Analysis (One Year)

This outline deals with the following three major topics in real analysis:

(1) Various classes of generalized functions such as L_1, L_2, L_∞, distributions, etc.

(2) Measure theory

(3) Nondiscrete decomposition

These topics are basic in a wide variety of fields in analysis, such as the theory of differential equations, the calculus of variations, harmonic analysis, complex variables, probability theory, topological dynamics, spectral theory, and many others.

We advocate presenting this material, notably that listed under (1), within the framework of general topology and functional analysis. The necessary background is developed in Sections 3, 4, and 5; to a certain extent this constitutes a review of material already covered in the Introductory Undergraduate Mathematics. As the outline shows, we believe strongly, as did the founding fathers, in dealing first with special cases and in presenting applications along with the general theory.

Perhaps our most radical departure from tradition is advocating the presentation of the notions of strong derivatives in the sense of Friedrichs and Sobolev, and distributions in the sense of

Schwartz, rather than the Lebesgue theory of differentiation. We feel that this is justified by the simplicity and general usefulness of the newer theories.

There is more material presented here than would fit into a year's course. Sections 8-12 are offered as a variety to choose from. The subjects in the first seven Sections are basic, but the material outlined is a little more than what is strictly necessary for a self-contained treatment.

Fortunately, most of the subjects discussed in the outline are available in textbooks, although not all within the covers of one text.

1. Set theory. Review of the terminology of set theory. One-to-one correspondence, countable and uncountable sets; the uncountability of the real numbers and of other interesting sets. Equivalence relations, order. The Schroeder-Bernstein theorem. The axiom of choice and Zorn's lemma.

2. Real numbers. The construction of real numbers by completion (equivalence classes of Cauchy sequences). The compactness of closed, finite intervals. Hamel basis.

3. Metric spaces. Definition, examples: the continuous functions, L_1, L_2, L_p; the Schwarz and Hölder inequalities. Open and closed sets, dense and nowhere dense sets, separability. Bernstein polynomials and the Weierstrass approximation theorem; Chebyshev's theorem on best approximation.

Completeness and the process of completion. Fixed point theorem and its application. Baire category theorem and its applications. Continuous functions; Tietze's extension theorem.

Compactness and local compactness, Arzelà-Ascoli and Rellich compactness theorems and applications.

4. Topological spaces. Definition, examples. Open and closed sets. Hausdorff spaces. Separability. Compactness; Stone-Weierstrass theorem, Tychonoff's theorem. Topological groups.

5. Normed linear spaces. Hilbert space: definition, orthonormal base, projection theorem, representation of linear functionals. Bounded operators, adjoints, symmetric and unitary operators.

Banach spaces: definition, linear functionals, dual space, bounded linear operators.

Banach-Steinhaus theorem, applications. Hahn-Banach theorem, applications (moment problems).

6. Integrals and measures. There are two competing approaches, neither of which should be slighted. (A) is functional analysis-oriented with applications in classical analysis generally (e.g., orthogonal series, differential and integral equations, classical probability theory). (B) is measure-theoretical, with applications in stochastic processes, ergodic theory, and statistics.

(A) The space C_0 of continuous functions on a complete, locally compact metric space. Signed and complex measures. Relation of measures: absolute continuity, Radon-Nikodým theorem. Convergence theorems. Fubini's theorem. Riesz representation theorem.

(B) Classical general measure-theoretic methods: outer measure, extension of a measure through outer measure, Kolmogorov consistency criterion, conditional measures. Product measure. Mention of finitely additive measures.

7. Differentiation. Functions in n-dimensional Cartesian space. Strong derivatives in the sense of Friedrichs and Sobolev. Sobolev's theorem. Applications to differential equations. Schwartz theory of distributions; applications.

Vitali covering theorem and differentiation almost everywhere.

8. Applications to classical analysis. Orthonormal series, Fourier and other transforms, Riesz-Fischer theorem, Fourier transforms of L_2 and of tempered distributions. Convolution. Classical inequalities based on convexity. Theory of approximation. Applications.

9. Integration on groups. Construction of the Haar measure. Examples.

10. Measure spaces. Definition of an abstract measure space. Measure on the Cartesian product of a countable number of circles. Application: the convergence of random series.

11. Banach algebras. Definition, the Gelfand theorem on the existence of multiplicative linear functionals. Applications to Fourier series and function theory.

12. Spectral resolution of self-adjoint operators. The spectral resolution of bounded, symmetric operators. The discrete,

singular, and absolutely continuous parts of the spectrum.

Complex Analysis, Outline I (One Year)

The first semester of this course, covering Sections 1-8, includes the standard elementary (but basic) topics from the theory of functions of one complex variable. The content of the second semester centers about the conformal mapping theorems for regions of finite connectivity, including the necessary tools for their proof. Other suitable topics for the second semester may be found in Selected Topics in the Classical Theory of Functions of a Complex Variable by Maurice Heins (New York, Holt, Rinehart and Winston, Inc., 1962) and in Banach Spaces of Analytic Functions by Kenneth Hoffman (Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962).

1. Complex numbers as ordered pairs of reals, field properties. Conjugate and absolute value, geometric properties. Polar representation. Stereographic projection and the extended plane.

2. Elementary functions. The derivative. Analytic functions on open connected sets. Detailed treatment of examples: polynomials, rational functions, the group of linear fractional functions, exponential and trigonometric functions.

3. Conformal mapping by elementary functions. Proof that an analytic function is conformal at points where its derivative does not vanish. Specific conformal mappings.

4. Integration along piecewise continuously differentiable curves. Cauchy's theorem for rectangle and circular disk. Integral representation of the derivative. Morera's theorem, Liouville's theorem, and the Fundamental Theorem of Algebra.

5. Taylor series development. Proof that the uniform limit of analytic functions is analytic. Classification of isolated singularities--removable, poles, essential singularities. Zeros of nontrivial analytic functions are isolated. Laurent series.

6. Nonconstant analytic functions are open. The Maximum Modulus Theorem. Schwarz' lemma. The one-to-one analytic maps of the unit disk onto itself.

7. Cauchy's theorem and homology. Simply and multiply connected regions. The Residue Theorem. Argument principle. Rouché's theorem. Evaluation of definite integrals using the Residue Theorem.

Implicit Function Theorem.

8. Analytic continuation.

9. Meromorphic functions. Infinite products; the Weierstrass factorization theorem. Mittag-Leffler theorem.

10. Compact families of analytic functions. Montel's theorem. Conformal equivalence of simply-connected regions. The Riemann mapping theorem.

11. Harmonic functions. Elementary properties: Mean Value Theorem, maximum principle, isolated singularities.

12. The Dirichlet problem for the disk, with continuous boundary values. The Poisson integral.

13. Applications of the Poisson integral: a continuous function having the mean-value property is harmonic, uniform limit of harmonic functions is harmonic. Harnack inequalities and convergence theorem.

14. Subharmonic functions. Elementary properties. Perron's theorem.

15. The Dirichlet problem for a region. Sufficient conditions for existence of a solution. Barriers.

16. Green's function for a region. Relation with conformal mapping of the region. Regions of finite convexity. Harmonic measures.

17. Conformal mappings of regions of finite connectivity onto standard regions.

18. The Hardy H^p-spaces of analytic functions on the unit disk. (This Section assumes some knowledge of Lebesgue integration.) Fatou's theorem; Herglotz's theorem.

Complex Analysis, Outline II (One Year)

Complex analysis offers a unique opportunity to convince the young student who has only a minimal knowledge of algebra and topology that these subjects can interact with analysis in a useful way. The aim in this outline is to present the few key concepts which are remembered by mathematicians of all fields. Simplicity and a hope to excite the student with continuing ideas are emphasized at the expense of an occasional time-honored result or point of view. It is assumed that the student will be familiar with topological concepts appropriate to the plane, including properties of continuous mappings and of arcwise connectivity.

1. The complex field. Characterization of the real field, complex numbers as pairs of real numbers, the complex field C as a field with valuation and conjugation as its automorphism, geometric interpretations, intuitive and rigorous adjunction of a point at infinity as a compactification of the plane.

2. Power series. The ring of formal power series $K[[x]]$ over a field K, operations with series (formal derivative, reciprocal, inverse), convergent series (K = R or C), uniform convergence of a series of functions, radius of convergence of a formal series, operations with convergent power series (differentiation, reciprocal, inverse), exponential function and logarithm functions.

3. Analytic functions. Real and complex analytic functions defined as functions, on open sets, which are locally power series, and the algebra of functions analytic on a region D; principle of analytic continuation (uniqueness of continuation), zeros of an analytic function (discreteness), rational function, poles, the field of meromorphic functions on D.

4. Integration. Differential forms $P\,dx + Q\,dy = \omega$, differential chains γ, integration $\int_\gamma \omega$, exact chains, closed forms, complex forms, homotopy, winding number of closed chain, generalized principle of argument, and Rouché theorem for mappings.

5. Holomorphic functions. f is holomorphic at z_0 if $f'(z_0)$ exists. Cauchy-Riemann equations, Cauchy theorem (if f is holomorphic in D, then $f(z)\,dz$ is a closed form); existence of local primitives that are holomorphic, Cauchy integral representation, holomorphic functions are analytic, Morera's theorem.

6. Applications of integration. Liouville's theorem, algebraic closure of C, Maximum Modulus Theorem, Open Mapping Theorem, Schwarz' lemma, Laurent representation, isolated singular points, residues, calculation of contour integrals, counting of zeros and poles of a meromorphic function, Schwarz reflection, doubly-periodic functions.

7. Functions of several variables. Formal power series in several variables, domain of convergence, operation with series,

analytic functions in several variables, principle of analytic continuation; harmonic functions, holomorphic functions are (complex variable) harmonic functions, every real harmonic function is (locally) the real part of a holomorphic function, harmonic functions are analytic.

8. Global problems. The Riemann sphere, functions holomorphic in regions on the sphere, the fundamental group $\pi(D)$ of a region D, integration as a homomorphism of $\pi(D)$ into the additive group of C, generators of $\pi(D)$, the covering space of D and general solution of the problem $\operatorname{Re}(f) = u$, where u is harmonic in D, subcovering spaces and normal subgroups of $\pi(D)$.

9. Holomorphic function of several variables. Cauchy's theorem, Taylor's theorem, composition of functions and the Implicit Function Theorem, statement of Hartogs' theorem.

10. Spaces of holomorphic functions. The spaces $C(D)$ and $H(D)$ of functions continuous in D and holomorphic in D, fundamental theorems on convergence in compact sets in D, continuity of differentiation in $H(D)$, the univalent functions as a subset of $H(D)$, series of meromorphic functions, the Weierstrass periodic function; infinite products of holomorphic functions, representation of $\sin(\pi z)$ and $1/\zeta(z)$; closed bounded sets in $H(D)$ are compact.

11. Holomorphic mappings. Local properties of a mapping $\omega = f(z)$, special mappings, the conformal automorphisms of a disc, of the plane, and of the Riemann sphere; the Riemann mapping theorem.

12. Analytic spaces. The general notion of an analytic space, holomorphic mappings of analytic spaces, meromorphic functions on an analytic space, fundamental theorem on conformal equivalence of simply connected analytic spaces, differential forms on an analytic space; Riemann surfaces, analytic continuation.

13. Application to differential equations. Existence theorem and uniqueness theorem, dependence upon initial conditions, higher-order equations.

Abstract Algebra (One Year)

The purpose of this course is to introduce the student to basic structures of abstract algebra and to provide an introduction to applications to various branches of mathematics. The prerequisite for this course consists of the material in Chapters 1-13 of the semester course in linear algebra appearing in Outline II of Appendix A.

The main body of the course is divided into nine Sections outlined below. This initial segment should cover approximately 5/3 semesters; for the remaining 1/3 semester, five options are presented, each starting with Section 10 and each representing an introduction to further specialized study.

1. Groups. Definition of groups and morphisms of groups. Notion of subgroup; quotient group. The permutation groups, representation of any group as a group of permutations; groups of some regular solids. Groups of linear transformations. Orthogonal groups, unitary groups, etc. Abelian groups; abelianization of an arbitrary group.

2. Commutative rings. The definition of commutative ring and discussion of examples including the integers, Gaussian integers, and the integers modulo n. Definition of ideal and quotient ring; further examples. The definition of field, integral domains, Euclidean domains, and principal ideal domains. The Euclidean algorithm. Maximal ideals. The problem of unique factorization; proof that principal ideal domains are unique factorization domains. Examples showing that not all integral domains are unique factorization domains.

3. Commutative rings. Definition of a commutative algebra. Polynomial algebras in a finite number of indeterminates, including both existence and universal properties. The polynomial algebra in one indeterminate over a field as a Euclidean ring. Proof that polynomial algebras in a finite number of indeterminates over a unique factorization domain are again unique factorization domains.

4. Modules over commutative rings. Definition of module, morphism, epimorphism, monomorphism, and isomorphism; examples including vector spaces and abelian groups. Sums and products of modules including explicit constructions and universal properties. Observations that finite sums and products coincide. Exact sequences

of modules, submodule, quotient module, modules of morphisms. The fundamental decomposition theorem for modules over principal ideal domains; application such as a review of cyclic decompositions of vector spaces and Jordan canonical form.

5. Graded and exterior algebra. Representation of morphisms of finitely generated free modules by matrices; dual modules, dual bases, duals of morphisms; relation with matrices, transposes, etc. Graded algebras, commutative graded algebras; examples including free associative algebras, polynomial algebras. Exterior algebras, rank, invariance of dimension of free modules, traces, determinants via the exterior algebra.

6. Polynomial algebras and finite-dimensional vector spaces. Given a vector space over a field and a linear transformation, the vector space becomes a module over the polynomial ring in one indeterminate over a field. Cyclic decompositions. Minimal polynomial and characteristic polynomials of linear transformations; the Cayley-Hamilton theorem. The Jordan and other canonical forms of matrices. Trace and determinant via the characteristic polynomial. Eigenvectors and eigenvalues.

7. Field theory. Splitting field of a polynomial, prime factors, finite fields, and fields of fractions. Algebraic extensions, separability, inseparability, norms and traces. Roots of unity, algebraic number field, the theorem of the primitive element. Algebraically closed fields; existence and uniqueness of the algebraic closure.

8. Group theory. Isomorphism theorems for group theory. Composition series; Jordan-Hölder-Schreier theorem. Product of groups. The Remak decomposition for finite groups. Solvable groups, the Sylow theorems; examples. Further study of the permutation groups. Simplicity of the alternating group for $n > 4$.

9. Galois theory. Automorphism of fields, fixed fields of groups of automorphisms of the splitting field of a polynomial as a permutation group. Galois extensions defined using finite automorphism groups; criteria for an extension to be Galois. Fundamental Theorem of Galois Theory from the Artin point of view; discussion of other proofs. Fields of fractions of polynomial rings. Galois

extensions with a symmetric group or Galois group, solvable extensions and relations with solvability of equations by radicals.

Option I: Algebraic Number Theory

10. Rings of integers. Finite extensions of the rational number field; calculation of sample Galois groups. Definitions of rings of integers, calculations of the primes and prime ideal in various examples; examples of rings of integers in number fields which are not principal ideal domains, and examples which are principal ideal domains but not Euclidean rings. Definition of Dedekind ring; proof that the rings of integers in finite separable extensions of the field of fractions of a Dedekind ring are again Dedekind. Various characterizations of Dedekind rings. Study of the rings of integers in quadratic extensions of the rationals.

11. Dedekind rings and modules. Fractional ideals, classical ideal theory for Dedekind rings; examples. Modules over Dedekind rings; fundamental theorem for finitely generated modules. The ideal class group. Finiteness via Minkowski's lemma. Class numbers. Finitely generated torsion-free modules are characterized up to isomorphism by their rank and ideal class, using the exterior algebra to determine the ideal class of the module. Calculations of ideal class groups for a few simple examples.

12. Introductory algebraic number theory. Integral bases, examples, and proof of existence in general. Units, the cyclotomic fields, units in quadratic extensions. The Dirichlet-Minkowski theorem on units. Calculation of various examples.

13. Further introductory number theory. Ramified and unramified primes; examples. Decomposition groups, ramification groups, etc.; examples. Abelian extensions. Cyclotomic and quadratic fields; quadratic reciprocity law.

Option II: Noetherian Rings and Modules

10. Rings with minimum condition. Definition and fundamental properties, equivalence with descending chain condition. Correspondence to rings with no nonzero nilpotent (left) ideals. Reduction to semisimple rings, matrix characterization of simple rings. Modules and their structure as sums of a minimal ideal, fields; quadratic specialization to vector spaces.

11. Noetherian rings. Definition and fundamental properties, equivalence with ascending chain condition. Hilbert basis theorem. Normal decompositions. Correspondence with local rings, decompositions again.

12. Dedekind domains. Definition and some equivalent notions. Characterization of ideals. Finitely generated modules, torsion-free characterization. Torsion modules, connections with matrices. Hilbert zero theorem. Special case: ideal theory for quadratic number fields.

13. Representation theory of groups. Representation of degree n (over an algebraically closed field F whose character does not divide order of the group G), equivalent representations, connections with finitely generated left F(G)-module. Characters, equivalence, direct sum decomposition, irreducible characters. Computation for the symmetric group.

Option III: Geometry of Classical Groups

10. Affine and projective geometry. Affine geometry; synthetic approach and construction of a field. Desargues' theorem, Pappus' theorem and commutativity. Projective geometry, introduction and fundamental theorems. Examples of projective geometries; the projective plane.

11. Quadratic forms. Definition of quadratic forms and their elementary geometry. Orthogonal quadratic forms, orthogonal sum of subspaces. Orthogonal geometry (especially over finite fields). Symplectic forms, symplectic geometry (especially over finite fields).

12. Orthogonal and symplectic groups. Euclidean orthogonal groups. General orthogonal groups, Clifford algebras, spinor norms. Structure of the orthogonal group; structure of the symplectic group.

Option IV: Equations of Fifth Degree

10. Representations of finite groups. Definitions and general properties of representations. Characters; complete reducibility (under the appropriate assumption concerning the characteristic of the field). Schur's lemma; relations on characters (over the complex numbers). Some computations for the symmetric group. Invariants of finite groups.

11. Equations of fifth degree. Lüroth's theorem; group of automorphisms of a rational function field of one variable. Determination of all finite subgroups (and their invariants for character zero); peculiarities of the modular case. The icosahedral equation. Bring's equation. The icosahedral equation as resolvent of the general equation of the fifth degree. Kronecker's theorem on the nonexistence of rational resolvents for general equations of degree greater than or equal to five.

Option V: Elliptic Function Fields

10. Algebraic function fields of one variable. Places and valuations; completion of a field with respect to a valuation. Existence of places; order functions. Divisors and divisor classes. Differentials. Special cases: partial fractions for rational function fields; Riemann-Roch theorem for elliptic function fields. Riemann-Roch theorem for hyperelliptic function fields. Analogies with quadratic number fields.

11. Algebra of elliptic functions. Algebraic group structure of a nonsingular plane cubic curve; the absolute invariant. Addition theorem for elliptic functions, multiplication and division of elliptic functions. Divisor classes of finite order in an elliptic function field. Modular equations (e.g., their Galois groups). Euler's theory of elliptic functions. Gauss' theory of the lemniscate.

Function-theoretic viewpoint.

Geometry-Topology

Aspects of point set topology are prerequisite to most beginning graduate programs--e.g., metric spaces, compactness, connectedness, effects of continuous mappings on such properties, uniformly continuous mappings, Tychonoff's theorem. A substantial part of this material is discussed in the Introductory Undergraduate Mathematics, and it is not regarded as necessary that a separate course in point set topology be in the undergraduate curriculum.

It is, however, highly desirable that every undergraduate take part in some sustained, deep geometric development. Such a variety of significant geometric developments is possible, differing in method and aim from the very start, that the Panel is reluctant to suggest any one or two of them as belonging to every undergraduate's program. Instead, we propose a larger number of courses and recommend that each student take one or two of them. The specific ones of these courses that may be offered in a given college depend on the interests and training of its staff. Hopefully, this will lead to a wide divergence in the types of geometers eventually produced.

We have also felt that the hybrid title "Geometry-Topology" is more descriptive of this area than either title alone.

Outline I. Set-Theoretic Topology (One Year)

The set-theoretic topology included in the calculus course is limited to that needed for the multidimensional calculus. In this course it is developed in a more abstract setting. These techniques and results are then applied to study topological groups, covering spaces, the fundamental group, and 2-dimensional manifolds.

The course outlined is a one-year course. However, Sections 10 through 17 and 18 through 26 are independent. This permits various choices as determined by the interests of the group concerned.

1. Hausdorff spaces. Compactness, local compactness, one-point compactification, sequential compactness. Continuous, open, closed mappings. Uniform continuity.

2. Connectedness, local connectedness, components. Preservation under mappings. Nonlocal connectedness.

3. Product spaces, quotient spaces. The Hilbert cube. Hausdorff maximality principle or the axiom of choice. Product of compact spaces is compact.

4. Separability, 2nd countability. Countability arguments (Brouwer reduction theorem; i.e., irreducibility). Baire category theorem and general method of argument.

5. Metric spaces, equivalent metrics, completeness, topological completeness. G_δ-sets. Baire category theorem in complete metric spaces.

6. Urysohn's lemma, Tietze's extension theorem, metrizability for locally compact 2nd-countable Hausdorff spaces. Paracompactness and Smirnov metrization theorem.

7. Upper semi-continuous and continuous decompositions of compact metric spaces. Hausdorff metric. Relationship of decompositions to mappings.

8. Hahn-Mazurkiewicz theorem, arcwise connectivity.

9. Characterizations of arcs and 1-manifolds.

10. Topological groups, nuclei, quotient spaces.

11. Projection mapping $G \to G/H$ is closed mapping if H is compact. G is compact (locally compact) if H is compact and G/H is compact (locally compact). Examples: orthogonal and unitary groups, Stiefel manifolds.

12. Local isomorphism of topological groups; $G \to G/N$ is a local isomorphism if N is a discrete normal subgroup. If G and G' are locally isomorphic, there exists H with discrete normal subgroups N and N' so that G is isomorphic to H/N and G' is isomorphic to H/N'.

13. Paths, homotopies of paths, fundamental group. Pathspace PX of a topological space X with base point e; continuity of projection map $\pi: PX \to X$.

14. Pathspace of topological group is topological group and π is a homomorphism; π is open, onto if X is a pathwise connected, locally pathwise connected topological group.

15. $\Omega X = \pi^{-1}(e)$, e the unit of X, and $\Omega_0 X$, the identity component of X, are closed normal subgroups of PX; $\tilde{X} = PX/\Omega_0 X$ is the universal covering group of X and $\varphi: \tilde{X} \to X$ induced by π is the covering map. The kernel of φ is $\Omega X/\Omega_0 X$ and φ is open, onto, continuous homomorphism if X is pathwise connected and locally pathwise connected.

16. If X is a pathwise connected, locally pathwise connected, semi-locally simply connected topological group, then $\varphi: \tilde{X} \to X$ is a local isomorphism with kernel $N = \Omega X/\Omega_0 X$ = fundamental group of X; $\tilde{X}$ is simply connected.

17. In the class of all pathwise connected, locally pathwise connected, semi-locally simply connected topological groups which are locally isomorphic to one such group, there is uniquely, up to isomorphism, a simply connected group C^* in the class, and for any C in the class, $C^*/N \cong C$, where N is the fundamental group of C and is in the center of C^*.

18. Fundamental group of the unit circle (as topological group). Winding number of closed path in the plane relative to a point (as element of fundamental group of punctured plane). Fundamental Theorem of Algebra. Simple arc does not disconnect the plane.

19. Jordan curve theorem.

20. Arcwise accessibility of points of arcs and simple closed curves in the plane from their complements. Schoenflies theorem.

21. Simplicial complex, abstract complex, geometric realizations and polyhedra. Imbedding theorem for n-dimensional complexes. Simplicial approximation theorem. Fixed point theorem for n-cells (Hirsch's proof).

22. Manifolds. Triangulability of compact 2-manifolds. Haupt-vermutung for compact 2-manifolds.

23. Cuts and handles. Orientability.

24. Invariance of Euler characteristic. Connectivity of 2-manifolds.

25. Classification of 2-manifolds.

26. Bicollaring, Brown-Mazur theorem.

Outline II. Algebraic Topology (One Year)

This course introduces the student to the tools and techniques of homology theory through a continuation of the study of differential forms as in Outline II of the Introductory Undergraduate Mathematics. For those with a background from Outlines I or III, several topics in Outline II must be studied first.

1. Differentiable manifolds of various classes, charts, atlases. Differentiable mappings. Orientation.

2. Differential forms in coordinate neighborhoods, morphisms, coordinate transformations. Differential forms on manifolds.

3. Exterior derivative, effect on products and transformations and of iteration. The differential forms $F^*(M)$ on a differential manifold M and exterior derivative viewed as a cochain complex, with product. Contravariant homomorphism induced by differentiable mappings of manifolds; compositions.

4. The standard and affine simplices in Euclidean space, face operation. Singular and differential chain group. Boundary of affine and singular chains. Induced mappings and commutation with boundary, $\partial\partial = 0$. Stokes' theorem on compact manifolds.

5. Closed and exact differential forms; cycles and boundaries; cohomology of forms and homology of singular (differentiable) chains. Stokes' theorem establishing dualities between the various classes of forms and chains. Closed forms as linear functionals on homology classes.

6. Definition of singular homology groups and the de Rham groups (i.e., graded quotients of closed by exact forms). Statement of de Rham's theorem in form that the de Rham groups are dual vector spaces to the singular homology groups.

7. Local triviality of the singular and de Rham groups for contractible (or differentially contractible) spaces; in the case of de Rham groups, by integration by parts; introduction of chain and cochain homotopies. Cone construction for singular groups.

8. Singular cohomology groups. Singular cochains with coefficients in an abelian group G as the group of homomorphisms of the group of singular chains into G. Coboundary operator as Hom (∂). Cocycles, coboundaries, cohomology. Properties under mappings. Isomorphism $H^p(X; G) \cong \text{Hom}(H_p(X); G)$ for divisible groups. Restatement of de Rham's theorem as saying the de Rham groups $R^p(M) \cong H^p(M;R)$, where R is the real numbers.

9. Sub-cochain complexes, quotients; homomorphisms of cochain complexes; Bockstein exact sequence for cohomology for a short exact sequence of coefficient groups.

10. System of local coefficients for the singular chain complex; cochains with local coefficients; cohomology. Homomorphisms of local coefficient systems; short exact sequences and the Bockstein sequence.

11. Simplicial complexes, abstract simplicial complexes, polyhedra, geometric realization; simplicial mappings. Oriented simplicial chain complex; alternating simplicial cochain groups. Natural mapping of simplicial homology of an abstract simplicial complex into the singular homology of its geometric realization; same for cohomology; proof later of its isomorphism. Local coefficients for simplicial complexes.

12. Nerve N of a covering; presheaves; cohomology of the nerve with coefficients in the local system of the presheaf. Examples; significance of $H^0(N;G)$.

13. Proof that for a contractible (differentially) covering $\{U\}$ of the connected manifold M, if $\mathcal{C}^p(\mathfrak{F}^p)$ is the presheaf of singular p-cochains (differential p-forms), as a system of local coefficients on the nerve of $\{U\}$, we have

$$0 \to \overset{0}{\mathcal{C}}{}^{p-1} \to \mathcal{C}^{p-1} \overset{\delta}{\to} \overset{0}{\mathcal{C}}{}^{p} \to 0 \text{ is exact}, \quad p > 0, \quad \overset{0}{\mathcal{C}}{}^{0} = G,$$

constant

$$0 \to \overset{0}{\mathfrak{F}}{}^{p-1} \to \mathfrak{F}^{p-1} \overset{d}{\to} \overset{0}{\mathfrak{F}}{}^{p} \to 0 \text{ is exact}, \quad p > 0, \quad \overset{0}{\mathfrak{F}}{}^{0} = R,$$

where $\overset{0}{\mathcal{C}}{}^{p-1}$ is the local system of (p-1)-cocycles, etc. (local triviality). Proof that for a finite open covering of M with nerve N, $H^p(N;\mathcal{C}^q) \cong H^p(N;\mathfrak{F}^q) = 0$ for $p > 0$, $q \geqq 0$. (Use partition [differentiable] of unity subordinate to the covering.)

14. Existence of a differentiably contractible finite open covering on a compact differentiable contractible finite open covering on a compact differentiable manifold (assume some elementary Riemannian geometry). For such a covering, observe by use of Bockstein sequence that

$$R^p(M) \cong H^1(N;\overset{0}{\mathfrak{F}}{}^{p-1}) \cong H^2(N;\overset{0}{\mathfrak{F}}{}^{p-2}) \cong \ldots \cong H^p(N;\overset{0}{\mathfrak{F}}{}^{0}) \cong H^p(N;R)$$

$$H^p(M;G) \cong H^1(N;\overset{0}{\mathcal{C}}{}^{p-1}) \cong H^2(N;\overset{0}{\mathcal{C}}{}^{p-2}) \cong \cdots \cong H^p(N;\overset{0}{\mathcal{C}}{}^{0}) \cong H^p(N;G)$$

and conclude de Rham's theorem. Similarly, obtain isomorphism of simplicial and singular cohomology.

15. Eilenberg-Steenrod axioms for singular homology and cohomology.

16. Cell-complexes; cellular-homology; isomorphism with singular theory; isomorphism with simplicial theory, when defined.

17. Computations; suspensions; complex projective space, real projective space, homology homomorphism induced by the double covering of real projective n-space by the n-sphere.

18. Tensor products of modules; right exactness; homology with coefficients. Bockstein sequence for homology.

19. The functor Tor; universal coefficient theorem for homology.

20. The Eilenberg-Zilber theorem; Künneth sequence for the singular homology of a direct product.

21. Exterior cross-product in cohomology; cup product, properties. Chain approximation to diagonal map for regular cell-complexes, uniqueness.

22. Computation of chain approximation to diagonal for n-sphere; for mod 2 chains on real projective space; for integral chains on complex projective space. Ring structure of $H^*(P_n(R);Z_2)$ and $H^*(P_n(C);Z)$.

23. Borsuk-Ulam theorem; Flores nonembedding examples, invariance of domain.

24. Euler-Poincaré formula; Lefschetz fixed point theorem; application to existence of vector fields on manifolds.

Outline III. Surface Theory (One Year)

This course consists of a year's study of surfaces, their topological, differential geometric, conformal, and algebraic structure. Much modern mathematics consists of partial generalizations of what happens on surfaces. The student should find this material a good source of concrete examples in depth of subjects he will meet as a graduate student. It should develop his geometric insight and show him how analysis and algebra implement geometric intuition. It should solidify his previous mathematical training because it draws heavily on his knowledge of advanced calculus, complex variables, and algebra. Finally, it will display the interplay and overlap of various fields: the genus occurring topologically, geometrically via Gauss-Bonnet, and analytically via holomorphic differentials; or the surfaces of constant curvature, the simply connected complex 1-manifolds, and the non-Euclidean and Euclidean geometries.

1. Combinatorial topology. Homotopy of curves, the fundamental group, covering spaces, deck transformations.

Simplicial complexes, barycentric subdivisions, simplicial approximation theorem.

Simplicial homology. Betti numbers, Euler characteristic, genus.

Classification of triangulable compact 2-manifolds; the only simply connected triangulated 2-manifolds are S^2 and R^2.

The de Rham theorem for triangulated 2-manifolds.

2. Differential geometry. Definition of Riemannian 2-manifold. Bundle of frames. Riemannian connection. Parallel translation--motivation via surface in R^3.

Geodesics, minimizing property of geodesics. Structural equations. Curvature. Exponential map. Gauss' lemma; Gauss-Bonnet theorem for simply connected region bounded by broken curve (as an application of Stokes' theorem); global Gauss-Bonnet theorem for triangulated compact 2-manifold.

Surfaces of constant curvature; Poincaré model for negative curvature; uniqueness theorem for simply connected complete 2-manifolds of constant curvature, constant curvature manifolds as models of hyperbolic and elliptic non-Euclidean geometries.

Surfaces in R^3. 2nd fundamental form and the spherical map. Curvature again. Gauss-Codazzi equation. Uniqueness of the imbedding, given the 2nd fundamental form.

The spherical map for compact surfaces with positive curvature. Rigidity theorem.

Flat surfaces in R^3. The tangent developable. Geometric interpretation of parallel translation via the tangent developable. The only complete flat surface in R^3 is a cylinder.

Minimal surfaces; spherical map for minimal surfaces.

3. Complex manifolds. Definition of a complex 1-manifold. Complex tangent space. Conformal mapping. Reinterpretation of Cauchy-Riemann equations. Review of analytic continuation and examples of complex 1-manifolds as Riemann surfaces of an analytic function element.

Existence of isothermal coordinates in a Riemann 2-manifold;

every Riemannian 2-manifold carries a complex structure.

Riemann mapping theorem; the three different simply connected complex 1-manifolds. Relation with 2-manifolds of constant curvature.

The spherical map of a minimal surface is conjugate conformal. Complete minimal surfaces in R^3.

Potential theory. Hodge's theorem. The dimension of the space of holomorphic differentials is the topological genus.

4. Algebraic geometry. Algebraic function fields of one variable over the complex numbers. Places. The Riemann surface of a function field, the meromorphic functions of this Riemann surface, the meromorphic functions on the Riemann surface of an analytic function element as an algebraic function field.

Algebraic curves in the complex projective plane. Pictures of singularities. The Riemann surface of a nonsingular curve. Birational equivalence of nonsingular curves is the same as conformal equivalence of their Riemann surfaces and the same as algebraic isomorphism of their function fields.

Application of the Hodge theorem to show that any compact complex 1-manifold is the Riemann surface of an algebraic function field.

Divisors as 0-chains. The divisor of meromorphic functions. Bilinear relations. The Riemann-Roch theorem via potential theory. Abel's theorem, and other applications of Riemann-Roch.

Genus zero and the rational functions in the plane. Genus one and the study of the complex structures on the torus. Elliptic functions.

Probability (One Semester)

The development of classical mathematics was principally inspired by problems of physics and engineering. In the usual classical engineering problem the variables of the system are assumed to satisfy a set of well-defined and deterministic relations. These are analyzed, by and large, by the methods of ordinary and partial differential equations and related mathematical techniques. Such deterministic concepts and methods are no longer entirely suitable for treating mathematical problems in the biological and social sciences; furthermore, even in the physical sciences there arise problems which involve uncertainties and variability. Probability theory and stochastic processes provide language and tools by which to analyze such problems.

The course outlined here is designed to develop facility in the language, concepts, motivation, and techniques of probability theory. Stress is put on those stochastic models which are of mathematical importance as well as of interest in other disciplines. The course should aim at rigor in its treatment of both theory and applications.

The subject of probability and stochastic processes combines intuitive and analytical aspects. It draws upon and interacts with much of real analysis, functional analysis, linear algebra, complex variables, etc. It is also a basic subject for many applications. Many of these areas of application signal new directions for pure mathematical research.

1. This Section introduces the basic concepts and terminology, suggesting both an axiomatic and intuitive formulation of the mathematical model underlying probability structure.

Sample space and probability distributions, empirical background, frequency concept, relations amongst events, axiomatic foundations (Kolmogorov formulation).

2. Occupancy problems, random walks, realization of m among N events, coin tossing, run theory.

3. Random variables, conditional probabilities. Stochastic independence, Bayes' theorem, repeated trials, joint and marginal probabilities.

4. This Section seeks to develop certain analytical methods and classical distribution examples.

Expectations, variance, moments of distributions, characteristic functions, generating functions, convolutions, compounding, Chebyshev's inequality, Kolmogorov inequality, three-series theorem, correlation coefficients, classical examples: binomial, Poisson, normal, gamma, t, and F distributions, multivariate distributions, etc.

5. The classical limit theorems of probability theory are the content of the material.

Borel-Cantelli theorem, Law of Large Numbers, Central Limit Theorem, Law of Iterated Logarithm.

6. Introduction to stochastic processes. The structure of stochastic processes is delimited and its classification is outlined.

Time parameter, state space, dependence relations, introduction to Markov processes, independent increments processes, stationary processes, martingales, diffusion.

7. This Section introduces the principal concepts of stochastic processes.

Recurrence and absorption, renewal theorems, first passage probabilities, transient states, arcsine laws, occupation time of a given state.

8. Important categories of stochastic processes.

Random walk, Poisson process, birth and death, Brownian motion, branching processes.

Formulation and analysis of some simple stochastic processes occurring in physics, engineering, biology, and the social sciences (e.g., Ornstein-Uhlenbeck process, gene frequency and population growth models of Wright and Feller, learning models, etc.).

Mathematical Physics (One Semester)*

A large part of analysis originates in problems of the physical sciences; our intuition and our sense of what is important is partly based on experience in dealing with problems of the physical world. This has been so in the past and is likely to remain so in the future, although mathematicians will increasingly look for inspiration to the biological and social sciences and to computing.

It is of greatest importance for the continued vigor of mathematics to keep open the channels of communication with other sciences; colleges should offer courses on a high intellectual plane to accomplish this. The courses must then deal with fundamental ideas as well as techniques, modern analytical concepts and methods should be employed, and subjects of current research interest need to be introduced. Unfortunately, in most American colleges there is no tradition for teaching such courses, there is not a wide enough variety of suitable texts, nor are there enough people inclined or able to teach them. The Panel presents the present outline as a step toward filling the gap.

* Though we are recommending a one-semester course, we include enough material for two or more semesters, to allow for individual variations.

1. Equilibrium problems. Derivation of the Laplace and Poisson equations for: the equilibrium position of a stretch membrane, electrostatic and gravitational potential, steady state incompressible, irrotational flow.

Statement and physical motivation of boundary value problems.

Uniqueness theorems (a) via the maximum principle (proved by Mean Value Theorem) and (b) via the Dirichlet integral.

Invariance of harmonic functions under various groups of transformations: translation, rotation, contraction. Special solutions which are eigenfunctions under these transformations (generalization of the principle that the matrices in a finite set of commuting matrices have common eigenfunctions). Application to the representation of the orthogonal group.

The fundamental solution of Laplace's equation and its physical significance. Green's function. Explicit construction of Green's function for the half-plane, circle, and sphere by the principle of reflection.

Construction of the Poisson kernel for a half-plane by similarity.

Invariance of harmonic functions under conformal map. Conjugate harmonic functions; relation of harmonic and analytic functions. Role of conjugate harmonic functions for flows. The relation of Green's function of a domain to the conformal mapping function.

Dirichlet's principle and the basic principles of the calculus of variations. The Euler equation. The equations of elasticity and of minimal surfaces.

The Laplace difference equation; its relation to random walk.

Free boundary value problems of hydrodynamics.

2. Conservative time-dependent problems (wave propagation). Newton's laws of motion. Derivation from physical principles of the equations governing the motion of a vibrating string and membrane, and the equations of acoustics. The wave equation.

Statement and physical motivation of initial and of mixed initial boundary value problems.

Uniqueness theorems based on the Haar Maximum Principle and on the energy method. The notion of domain of dependence and speed of propagation signals.

Invariance of the wave equation under translation; exponential solutions. Plane waves and the D'Alembert solution. Hamilton's principle. The equations of time-dependent compressible flow; shock waves. The equations governing the flow of traffic. Finite difference approximations to the wave equation.

3. Dissipative time-dependent equations. Derivation of the equations governing heat conduction, diffusion, and viscous flow.

Statement and physical motivation of initial and mixed problems for the heat equation.

Uniqueness theorems based on the maximum principle and on the energy method.

Translation and rotation invariance, exponential solutions, radial separation. The fundamental solution derived by similarity. Uniqueness and existence of solutions to the initial value problem in the entire space.

Relation of the heat equation to probability theory. Finite difference approximation to the heat equation.

4. Introduction to Hilbert space and operator theory. A brief review of linear algebra; Hilbert space. Orthonormal sets, completeness. Bessel inequality, Parseval relation. Projection theorem.

Examples of orthogonal systems: Fourier series and classical orthogonal polynomials. Weierstrass approximation theorem. Gram-Schmidt procedure.

Notion of symmetric operator. Orthogonality of eigenvectors. Completeness of eigenvectors of compact operators. Compactness of integral operators; discussion of the inverse of a differential operator.

Positive-definite operators.

Operational calculus for symmetric operators. Definition of exp A through 1) operational calculus, 2) contour integral, 3) eigenvector expansion, 4) Yosida formula (semigroups).

Theory of Fourier transform; the classes L_2 and $\mathcal{L}$. Application of Fourier series and integral to solve anew the boundary value problem for the Laplace equation in the circle and half-plane, and the initial value problem for the wave and heat equations on the real axis, with and without periodicity.

Determination of eigenfunctions and eigenvalues of the Laplace operator in simple geometries.

5. Existence theorems. Solution of various boundary value problems for the Laplace equation by the method of orthogonal projection, the Hahn-Banach theorem, or one of the many other methods.

Using the existence theory for the Laplace operator and the operational calculus developed in Section 4 to treat the initial value problem for the wave equation $u_{tt} = \Delta u$ and the heat equation $u_t = \Delta u$.

6. Quantum theory and statistical mechanics.

1) The harmonic oscillator in classical and quantum mechanics:

Let ψ_n be the normalized eigenfunction of the system with energy E_n. Then the probability that its position is between a and b is

$$\int_a^b |\psi_n(x)|^2 \, dx.$$

This tends in the classical limit $(h \to 0, E_n \to E)$ to the proportion of time which the classical harmonic oscillator spends between a and b (proof based on asymptotic properties of Hermite polynomials).

2) The motion of electrons in crystals:

Consider the Schrödinger equation

$$\psi'' - V(x)\psi = -E\psi$$

with a periodic potential V. Show that there exist solutions bounded for all x only when E lies in certain intervals, and identify these with conduction bands.

3) The classical and quantum-mechanical partition functions; limiting behavior as h tends to zero. For an ideal gas, we are led to the problem of asymptotic distribution of the eigenvalues of the Laplacian under the boundary condition $u = 0$ on the boundary of the container. Since the thermodynamical properties do not depend on the shape of the container, this suggests that the asymptotic distribution of the eigenvalues depends only on the volume

(Weyl's theorem). Explicit determination for a cube.

Further suggested subjects: the time-dependent Schrödinger equation, the application of group representations in quantum theory.

Appendix C

HIGHER UNDERGRADUATE MATHEMATICS

This Appendix contains some sample outlines of courses in Higher Undergraduate Mathematics which might be considered for inclusion in a program already containing basic courses which cover the fields of Appendix B. Future reports of the Panel will contain additional outlines for this section.

Mathematical Methods in the Social Sciences: Game Theory, Programming, and Mathematical Economics (One Semester)

The desire to formulate quantitative methods for analyzing phenomena in the social, management, and behavioral sciences has led to new types of mathematical problems. The tools needed in dealing with such problems combine principally probabilistic, statistical, and decision-theoretic concepts and techniques. Three specific developments of this kind, *inter alia*, are exemplified by the areas of mathematical research known as game theory, programming, and mathematical economics. The structure of game theory seems suitable for describing some monopolistic practices in addition to providing a norm for certain patterns of rational behavior. The methods of mathematical programming are particularly appropriate for determining optimal policies in a variety of management problems. The formulation of mathematical economics is useful in explaining the workings of some economic systems.

These disciplines appeal to devices from topology (e.g., fixed point theorems), the stability theory of nonlinear differential equations, methods of the calculus of variations, inequalities, linear algebra, convexity, and similar subjects. The intuitive content of the underlying economic interpretation frequently suggests new mathematical theorems. The influence of these disciplines on developments in statistics and probability has also been substantial.

1. Game theory. Classification of games (number of players, zero-sum versus nonzero-sum, personal and chance moves, information structure, utility concepts).

Zero-sum matrix games, minimax theorem, dominance concepts, Snow-Shapley characterization of extremal solution, completely mixed games, dimension relations of solutions, examples.

Infinite zero-sum games (optional material). Separable games (polynomial kernels), convex games, games of timing, bell-shaped games, games over function space, recursive games, games of survival.

n-person games, cooperative and noncooperative games, coalitions, von Neumann solution, simple games, Shapley value, Nash equilibrium point, examples.

2. Linear programming. Formulation of linear and dual linear programming problems. Examples, optimal assignment problem, transportation model, network flow models, etc.

Two principal theorems of linear programming: (i) Existence theorem of solution, (ii) Duality theorem.

Interpretation of dual problem in terms of shadow prices.

Computing algorithms for solutions. Simplex method, primal and dual algorithm, special methods for the transportation problem, application to minimal-cut and maximal-flow theorem.

Equivalence of linear programming and game theory.

3. Nonlinear programming. Equivalence to saddle point theorem. Kuhn-Tucker theorem, Arrow-Hurwitz gradient method. Fenchel formulation of nonlinear programming problem.

4. Methods of mathematical economics and management science. Production, consumption, and competitive equilibrium models.

Frobenius theory of positive matrices. Application to linear production model (Leontief model), Samuelson substitution theorem, formulation of theories of consumer preference. Axiomatic approach. Principle of revealed preference. Derivation of consumer preference relation as a utility maximization. Relation of production theory and nonlinear programming problem. Existence of competitive equilibrium. Formulations of Arrow-Debreu, Wald, McKenzie, and others.

5. Welfare economics, stability theory, and balanced growth. Relation of welfare economics and the vector nonlinear programming problems. Characterization of Pareto optimum solutions. Local and global stability properties of competitive equilibrium. Gross substitutibility, models of balanced growth, von Neumann model of expanding economy, turnpike theorem.

6. Control problems in management sciences and economics. Hohn-Modigliani model of smoothed production, models of optimal inventory analysis, application of Pontryagin maximal principle to two sector growth models, introduction to replacement programs, repairmen problems, queueing theory, reliability models.

Mathematical Logic (One Year)

Many mathematicians think of logic as having for its principal purpose the laying of a firm "foundation" on which the rest of mathematics can be built securely. This can be understood historically, because it was the discovery of paradoxes in set theory which first led a broad segment of mathematicians to take up the study of logic in a quest for consistency proofs.

To make headway toward the twin aims of developing a foundational logic and providing guarantees of consistency, it was necessary to restrict sharply the mathematical methods employed. In particular, early workers laid great stress on the "constructive" character of their work.

As with other branches of mathematics, so with logic: the original aims were partly realized, partly found unrealizable, and partly altered to conform to the broadened perspective arising out of new discoveries. Some logicians began to notice the mathematical structures arising in the earlier work and became interested in these for their own sake. Through the study of these structures, contact has been made with other parts of mathematics at points far removed from the "foundational level" which was the starting point.

As a result of this development, it seems fair to say that the idea ascribed above to "many mathematicians," that the principal purpose of logic is to lay "foundations," does not accurately reflect the spectrum of current activities in the field. Roughly, logicians are now concerned with two large areas of work. One, based on the notion of recursive function, deals with such things as abstract computing machines, nonexistence of decision methods, hierarchical classification of sets of numbers and functions on numbers, and recursive analogues of portions of set theory and analysis. The other, combining Boole's original impulse to algebraize with Tarski's mathematical analysis of semantical notions, includes portions of the field which have come to be known as "algebraic logic" and "theory of models." In both of these principal areas the bulk of the work is carried on without restriction to "elementary" or "constructive" methods. The basic attitude is that any method may be used if it answers a question--and any question may be raised which is interesting! Particularly in model theory, there is generally a heavy use of set theory.

Despite this turn of events, almost all textbook treatments of logic lay great emphasis on the restriction to constructive methods and seem to concern themselves principally with demonstrating how logic can be developed so as to provide a foundation--i.e., to be a beginning--of mathematics. It seems time to attempt a presentation of the subject more closely related to current events. The foundational role of logic is explained as one aspect of the subject, but this is not allowed to restrict and distort the methodology.

In formulating a first course, one might either attempt to give introductions to the concepts in both of the principal areas mentioned above or to go more deeply in one of these directions. Very likely both schemes have merit, but we have preferred to follow the latter. Our judgment has been that for students proceeding toward a Ph.D. in mathematics, serious acquaintance with the ideas of algebraic logic and theory of models is of greatest value.

What do we presuppose of the student entering our course? Competent books on logic are now available for use in the 6th grade; indeed, grade school seems the proper place to compute with truth-functions and thereby learn the mathematical meaning of sentential connectives. High school seems to be the proper place to learn how to formalize sentences employing quantifiers and to learn (in an informal way) some of the elementary rules for handling quantifiers. The early college years will begin to develop the student's ability to apply effectively the basic apparatus of set theory. The proposed course carries on from there.

It is customary to approach mathematical logic by considering first sentential logic and then (first-order) quantifier logic. The course outline given below follows this pattern, except that we interpolate between these parts of the course a substantial section on quantifier-free predicate logic. If we were concerned solely with formal deductive systems for logically valid formulas, this would be ridiculous, since the axiom schemes and rules of inference of (q.f) predicate logic are indistinguishable from those of sentential logic. However, when we deal with model-theoretic aspects of the subject, the situation is quite otherwise. And the section on predicate logic forms a valuable bridge, both from the mathematical and the pedagogical viewpoints, between sentential logic and quantifier logic.

To understand properly the role of logic in mathematics, it is necessary to deal with (i) systems of symbols, (ii) the use of these systems in languages interpreted as referring to mathematical structures, and (iii) the manipulation of symbolic expressions according to formal deductive rules and the relation of such rules to the semantical concepts of (ii). Each section of logical material--sentential, predicate, and quantifier--is subdivided according to the classifications (i), (ii), (iii).

It is possible to treat the high points of sentential and quantifier logic in a single semester. However, to explore the subject in the depth desirable for achieving both a full understanding of the relation of logic to other parts of mathematics and a firm basis for future graduate work, two semesters is not excessive. During 1962-63 an experimental course patterned after the following outline is actually being given, and, despite the encouragement of excellent students, there is difficulty in fitting all of the material into two semesters. But it is felt that after accumulating experience in teaching material so organized, and if a suitable text becomes available, it should be possible to incorporate substantially all of the material in the indicated time.

1. Historical background. Intuitive account of principle concepts such as consequence, deduction; role of sentential connectives in natural languages.

2. Systems of formulas (absolutely free algebraic systems). Axiomatic treatment; various examples and their interrelations; fundamental existence theorem (justifying definition by recursion over formulas); definition of substitution, part, occurrence, and derivation of their fundamental properties from axioms.

3. Truth functions. Relation to connectives; projections, composition of functions; closed sets (examples); generating bases (mention of Post's theorem); proofs of definability and nondefinability (of a given function in terms of a given set of functions); lattice of closed sets; Boolean algebra B_n of n-placed truth-functions, $n = 1, 2, \ldots, \omega$; isomporphisms $B_n \to B_{n+1}$, and the direct limit of $B_1, B_2, \ldots$, as subalgebra of B_ω; infinite sums and products in B_ω; topological aspects of B_ω; compactness.

4. Semantical concepts of sentential logic. (Classical) models (truth-value assignments), associated homomorphism of system of formulas into algebra of truth values, validity, consequence, satisfiability, equivalence, independence; their interrelations; fundamental laws for consequence-relation; connection with substitution; equivalence as congruence relation (replacement law); positive and negative parts of formulas (partial replacement); natural mapping of formulas into B_n; definability by formulas; significance of the consequence relation in B_n; finitary character of consequence from compactness; Boolean algebras as models; the consequence relation determined by a Boolean ideal; normal forms; interpolation theorem; nonclassical interpretations (n-valued, intuitionistic, modal).

5. Deductive aspects of sentential logic. Axioms for derivations;* consequence satisfies these; basic laws obtained from axioms; effective proof of weak completeness (every valid formula derivable from empty set, for any derivation); proof by Zorn's lemma of strong completeness (semantical consequence is the minimal derivation); connection with compactness; characterization of finitary derivations; derivations defined by formal axioms and rules of inference; discussion of deductive logic as foundation for mathematics. Fragments of sentential logic; their deductive interconnections.

6. Systems of open predicate formulas (individual symbols, relation symbols, operation symbols). Terms and formulas; fundamental existence theorem; substitution, part.

7. Semantical concepts of predicate logic. Relational systems; models and variable-assignments; values of terms and formulas; validity, satisfiability, implication, definability, equivalence--with respect to a model and to a class of models; examples; properties of the class of definable relations, characterization of such classes; concept of a Boolean substitution algebra; predicate implication = propositional implication; compactness; decision procedure; Skolem-Löwenheim; implication relative to class of equality-models and its relation to predicate implication; compactness, decision-procedure, and S-L for predicate-equality logic; simple applications of compactness (e.g., condition for abelian semigroup to be imbeddable in group); subsystems, homomorphisms, direct products, direct limits, etc., for relational systems; invariance of validity for sets of equations, and of more general formulas, under these operations; characterization of equational classes and universal classes.

8. Deductive aspects of predicate logic. Formal axioms and rules of inference for predicate logic reduce to those for sentential logic (strong completeness); treatment of predicate-equality logic; complications of formalization for these systems if variables in some of the hypotheses of an implication are treated as universalized;

* The word "derivation" is not in common use. It is employed to indicate any relation (between sets of formulas and formulas) which satisfies certain laws for the consequence relation.

detailed consideration of a mathematical example, such as natural numbers under addition, obtaining complete axiomatization, detailed description of definable relations, decision procedure, strong incompleteness, analysis of nonstandard models.

9. Systems of quantifier-formulas (first order). Free and bound occurrences of variables, complications with substitution; sentences and formulas. Semantical concepts of quantifier-logic: same notion of model as in predicate logic; modified notions of variable-assignment and value-of-formula; same definitions of validity, satisfiability, implication, definability, and equivalence; class of definable relations, characterization of such classes; concept of polyadic algebra; prenex normal forms; reduction of validity and implication for quantifier logic to that of predicate logic via added individual constants or operation symbols; semantical versions of Herbrand's theorem and Skolem normal forms; Skolem-Löwenheim theorem, compactness, applications to algebra; treatment of quantifier-equality logic; equivalence of any formula with one having variables in standard order; simplified description of definable relations; concept of cylindric algebra; reduced products and ultraproducts of relational systems; invariance of validity for quantifier-formulas under latter; characterization of elementary classes; use of ultraproducts to replace compactness arguments in algebraic constructions.

10. Deductive aspects of quantifier logic. Formal axioms and rules of inference (with and without equality); notions of formal proof and formal theorem, formal deduction and formal implication, consistency for sets of sentences; derivation of basic laws of logic (i.e., properties of formal implication); strong completeness (alternative proof of compactness); Craig's interpolation theorem, Beth's theorem on definability, version of A. Robinson; Lyndon's characterization of sentences invariant under homomorphisms; detailed consideration of quantifier theory of natural numbers under addition, axiomatization, method of elimination of quantifiers applied to obtain complete description of definable relations, decision procedure, strong incompleteness, analysis of nonstandard models.

Differential Geometry (One Semester)

This outline of a one-semester course in differential geometry differs from the classical course in these respects: elementary theory of manifolds is presented; some of the classical matrix groups are studied; the Frenet formulas are given in n dimensions; intrinsic Riemannian geometry is studied before imbedded hypersurfaces; and some global theorems are included.

1. Basic facts about smooth manifolds and mappings between them.

2. The rotation group R(n), Euclidean group, and affine group as examples of manifolds. Invariant 1-forms on these groups. The Lie algebras as matrix Lie algebras. Fundamental uniqueness theorem: two maps of a manifold into G differ by a left translation if and only if the left invariant 1-forms of G pulled back by the two maps are equal.

3. Parameterized curves in R^n. Canonical parameterization via arc length. Adapted frames and mapping of curve into Euclidean group. Curvature and higher torsions. Frenet formulas. Application of uniqueness theorem to give determination of curve up to Euclidean motion.

4. Introduction of Riemannian metric. Bundle of frames. Riemannian connection. Parallel translation. Structure equations. Curvature. Geodesics and minimizing property. Exponential mapping. Gauss' lemma. Specialization to 2-manifolds. Gauss-Bonnet theorem for 2-manifolds.

5. Manifolds of constant curvature. Uniqueness for simply connected ones.

6. Hypersurfaces in R^{n+1}. Induced Riemannian metric. 2nd fundamental form and spherical map. Mapping of bundle of frames into Euclidean group. Curvature in terms of 2nd fundamental form. Gauss-Codazzi equation. Application of fundamental uniqueness theorem to give determination of surface up to a Euclidean motion. Principal curvatures, mean curvature, umbilical and parabolic points, curvature, and asymptotic lines, for 2-manifolds in R^3.

7. Rigidity theorem for compact hypersurfaces of positive curvature.

8. Flat hypersurfaces. Tangent developable. Geometric

interpretation of parallel translation using tangent developable. A complete flat 2-surface in 3-space is a cylinder.

9. Isothermal coordinates. A Riemannian 2-manifold as a complex 1-manifold. Minimal surfaces and their spherical maps.

Statistics (One Semester)

Probability theory and stochastic processes provide mathematical tools for a descriptive analysis of certain mathematical models. Statistics gives a means of testing the adequacy of the model. The statistics course outlined below is designed to cover the methodology and basic theory of statistical analysis. The approach is a combination of the classical and the modern, emphasizing at the start the standard procedures of statistics, while the latter part contains the general theory of statistical decisions.

Statistical techniques lean heavily on probability theory, real analysis, and linear algebra. Its content motivates and inspires problems in convexity, inequalities, real analysis, and probability theory. The outline presumes a course in probability theory as prerequisite.

1. Review of probability. Emphasis on theoretical distributions including the important examples of chi-square, t, F distributions of order statistics and functions of order statistics. Multivariate distributions and similarity.

2. Sampling. Description of sample data-means, standard deviation, frequency histogram, etc. Distribution theory of various statistics arising in sampling from normal populations, asymptotic distribution theory of various statistics.

3. Estimation. Formulation of the problem. Discussion of criteria for estimators (unbiasedness, consistency, efficiency, minimizing mean square error, absolute error, etc.).

The concept of sufficiency. Fisher-Neyman characterization, applications to the exponential family of distributions, extremal range distributions, principle of completeness, Rao-Blackwell inequality for improving estimates using sufficient statistics, Cramer-Rao inequality.

Confidence interval estimation. Maximum likelihood estimator and its properties.

4. Testing hypotheses. Formulation of the general problem.

Analysis of the case of a simple hypothesis versus simple alternate hypothesis including the celebrated Neyman-Pearson theorem.

Discussion of the composite hypothesis problem, concept of uniformly most powerful test. Tests derived from likelihood ratio criteria.

5. Statistical decision theory. The preceding approach was classical. Formulation of the Wald statistical decision theory. The concepts of utility, loss, and risk should be discussed. The derivation of the simplest complete class and "admissibility" theorems should be given.

Principles to be explored: Bayes' criteria, minimax, invariance, etc. Comparisons to classical statistical procedures.

Introduction to sequential analysis.

6. Regression theory and design of experiments. The formulation of the general linear hypothesis, linear regression. The Markov principle and the method of least squares. Analysis of variance.

7. Nonparametric statistics. Order statistics and derivation of confidence intervals for percentiles. Tolerance limits, goodness of fit, two sample problem. Kolmogorov-Smirnov statistics, rank procedures.

Number Theory (One Semester)

The theory of numbers has some attractive features which make it a very appropriate topic in the undergraduate curriculum. Having flourished over a very extensive period of time, number theory can be classified among those mathematical topics having a steady appeal. Perhaps this is due in part to its close relationship to algebra and analysis, an aspect of number theory that has been given greater emphasis in recent textbooks on the subject.

Number theory is not related to analysis in quite the same way as it is related to algebra, however. Analysis is used in number theory primarily as a matter of powerful analytic techniques of proof; for example, in the prime number theorem and in various proofs of transcendency of numbers. On the other hand, algebra has found in number theory a rich source of examples for the study of algebraic structures. It is not surprising to find in many books on algebra, therefore, an introductory chapter on the theory of numbers. Indeed, there are mathematical topics that are not easily classified as algebra proper or as number theory; the theory of algebraic numbers is an example of this.

The theory of numbers is an excellent vehicle for clarifying in the mind of the student the nature of proof. It may be that by the junior year in college a student should have no doubt about what is and what is not a proof. Nevertheless, since even some graduate students have occasional difficulty with this, the wide variety of proof techniques used in number theory can serve as excellent models for the student's attention. Furthermore, there is in the theory of numbers much for the student to do over and above examining the basic results. There is an almost unparalleled wealth of problems, including not only applications and examples of the theory, but also extensions and alternative foundations of the theory. Thus, the student has an opportunity both to develop his ingenuity and to discover results for himself through a program of exploration, conjecture, and attempts at proof. This aspect of number theory, often a source of frustration for the average student of mathematics, provides the superior student with much satisfaction and pleasure.

The following topics are presumed to be known by the student at the start of the course: unique factorization of integers, greatest common divisor, least common multiple, simple observations on the distribution of prime and composite numbers.

1. Congruences and residue classes. Congruences as equivalence relations, basic properties of congruences, changes of moduli; residue classes as groups, rings, and fields; theorems of Fermat and Euler on powers of residue; the language of algebra (order of an element of a group, generator, etc.) and the language of number theory (belonging to an exponent, primitive root, etc.); general theorems on solutions of congruences of degree n.

2. Quadratic residues. The Legendre and Jacobi symbols and their properties; the Gaussian reciprocity law.

3. Diophantine equations. The linear case and its relationship to linear congruences and greatest common divisor; $x^2 + y^2 = z^2$, $x^2 + y^2 = n$, $\sum_{i=1}^{4} x_i^2 = n$; impossibility of $x^4 + y^4 = z^4$; $ax^2 + by^2 + cz^2 = 0$.

4. Number-theoretic functions. Euler ϕ-function, divisor function, sum-of-divisors function; multiplicative and totally multiplicative functions; the Möbius function and the inversion formula; estimates of the order of magnitude of various number-

theoretic functions, including lattice points in various configurations; recurrence functions, Fibonacci sequences; the partition function.

5. The approximation of irrationals by rationals. Farey sequences, continued fractions; the best possible theorem (Hurwitz) on approximations; the uniform distribution of the fractional parts of the multiples of an irrational number.

6. Quadratic forms. Definite and indefinite forms; equivalence classes and the class number; questions of representation.

7. Prime numbers. Bertrand's theorem (a prime between n and $2n$); the Prime Number Theorem (either by an analytic proof or the "elementary" proof, or, if there is not the time for either of these, the weaker form of the theorem due to Chebychev).

8. Algebraic and transcendental numbers. Algebraic numbers form a field, algebraic integers and integral domains, quadratic fields, the Euclidean algorithm, unique factorization; the irrationality of π and e, the transcendence of e.

9. Optional topics. Infinitude of primes in an arithmetic progression; arithmetic properties of roots of unity, cyclotomic polynomials.

Geometry: Convex Sets (One Semester)

The study of convex figures is one of the oldest branches of mathematics; indeed, most figures studied by the classical geometers were either convex or stars. On the other hand, many branches of modern mathematics, e.g., functional analysis, game theory, numerical analysis, etc., have found that many problems in their scope are related to problems of convexity. Neither of these points of view of convexity, either as a branch of classical geometry or as a tool for other subjects, catches the essence of the theory. The geometry of convexity is very much alive today. A course in convexity should try to preserve the geometry as much as possible, even though in higher dimensions rigor often demands analytical technique. With this in mind, the course outlined below attempts to develop the subject starting from the simple intuitive notions and ending on the borders of the unknown in such a fashion that the geometric relationships are always in sight.

The course outlined below is intended for one semester. The material listed is more than enough for that length of time. Sections 8, 9, and 10 are all independent of one another and may be considerably shortened and used in any order without serious harm to the course.

1. Elementary properties of E_n. Coordinate systems and vectors. Scalar products, norm, and distance. Limits. Topological notions. Equivalence of topological definition of limit. n-dimensional analogs of the Bolzano-Weierstrass theorem, Heine-Borel theorem, and the theorem that a filter of closed bounded sets with finite intersection property has nonempty intersection. k-flats in parametric form and as solution space of inhomogeneous linear equations. Incidence properties and affine invariance.

2. Properties of individual convex sets. Dimension of a convex set. Interior and boundary, relative interior and boundary. Intersection properties of k-flats with boundary and interior. Preservation of convexity under affine transformations, interior and closure operations, projections, intersection, and lim inf operations. The existence of a support plane at every boundary point and in every direction. Regular points. Separation properties. Hull operator. Equivalence of intersection definition with constructive definition. The existence of exposed points for a closed bounded convex set (CBCS). Every exposed point is extreme. Every point of a CBCS is in the hull of $2n$ extreme points; every interior point of a CBCS is in the interior of the hull of $n + 1$ extreme points. A CBCS is characterized by its extreme points. The hull of a closed bounded set S is a CBCS, namely the intersection of all closed half-spaces containing S.

3. Convex cones and polyhedra; polarity. Support planes, extreme points and rays, and hull formation. Projecting cones and asymptotic cones. Polarity or duality theory for cones and polyhedra, and equivalence of various definitions of polyhedra. Applications to dual systems of linear inequalities, and to game theory and linear programming.

4. The algebra of convex sets. The sum of convex sets is a convex set. Addition is associative, commutative, and satisfies the usual rules for positive scalar multiplication. The sum of closed (open) sets is closed (open). Essential invariance under choice of origin. The same for cartesian products. Relations between sums,

products, and scalar multiplication. Faces, support planes, and diameters of sum in terms of those of summands.

5. Symmetrization operation. Steiner symmetrization in E_2; its relation to area and perimeter, diameter, and width. The symmetral of the sum of two sets. Similar results for Steiner symmetrization about a hyperplane in E_n. Steiner symmetrization about a k-flat in E_n. Central symmetrization

$$K \rightarrow \frac{K - K}{2}.$$

Its relation with volume, area, diameter, etc.

6. Helly's theorem for a finite family of convex sets. Extension to infinite families of CBCS. Further generalizations and relatives of Helly's theorem. Applications: Chebychev's approximation theorem, Jung's theorem, Krasnoselskii's theorem, solutions of convex inequalities.

7. The space of CBCSs, K_n. The inner and outer parallel sets of a CBCS. The distance $\Delta(K_1,K_2)$ between two CBCSs. K_n is a complete metric space. The polyhedra are a countable dense subset. Bolzano-Weierstrass, Heine-Borel, and filter theorems for K_n. Continuity of volume, area, diameter, sum, symmetrization, etc. Applications: surface area problems, isoperimetric problem, etc.

8. Brünn-Minkowski theorem. Linear arrays.

9. Convex functions. Distance and support functions of CBCS, and polar reciprocals. Continuity of convex functions. $f(Z)$ is convex with convex domain in E_n if and only if

$$\{(Z_1, Z_2, \ldots, Z_n, r) | r \geq f(Z_1, Z_2, \ldots, Z_n)\}$$

is convex in E_{n+1}. Preservation of convexity under transformation of domain, sup, composition with monotone increasing convex functions, etc. Differential conditions which imply convexity. The a.e. differentiability of a convex function, i.e., almost all points of the boundary of a convex set are regular (possibly only the case $n = 2$). Extrema of convex functions with convex domain. Helly's theorem for convex functions. The convexity of certain special functions; e.g., Hadamard's 3-circle theorem.

10. Constant width sets. A set has constant width if and only

if it is equivalent in breadth to a sphere. Projection properties. Properties involving area, perimeter, etc. (n = 2).

11. Further refinements and generalizations. Some comments on separation theorems (infinite-dimensional spaces, for open and closed sets). Introduction to convex topologies. Hahn-Banach theorem--applications. Applications to constructive function theory (summability of series, etc.). Derivation of classical inequalities (e.g., Hölder, Minkowski, etc.). Non-Euclidean spaces.

PREPARATION FOR GRADUATE STUDY

IN MATHEMATICS

A Report of

The Panel on Pregraduate Training

November 1965

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FOREWORD: THE GOAL

The task of the Panel on Pregraduate Training is to recommend programs for undergraduates who intend to go on to graduate work in mathematics. The first stage of the Panel's work, to consider an ideal undergraduate program for the future research mathematician, was completed in 1963, and the conclusions have been published in the report Pregraduate Preparation of Research Mathematicians [page 369].

The recommendations in the 1963 booklet are for the first four years of a seven-year program leading to a solid Ph.D. and a career in mathematical research. The booklet contains guidelines and several detailed sample course outlines. Note, however, that in forming these recommendations the Panel made no allowance for the student's possible deficiencies in preparation or tardiness in selecting a goal, for inadequacies of staff, for lack of suitable textbooks, etc. Furthermore, the program was designed for the very gifted undergraduate working under ideal circumstances. Some schools already offer programs compatible with the 1963 recommendations. At many others, however, the recommendations cannot be put into effect very soon. For them, the 1963 booklet provides a goal. The present booklet provides interim guidance.

THE PRESENT RECOMMENDATIONS

The Panel on Pregraduate Training, in considering the CUPM reports Pregraduate Preparation of Research Mathematicians [page 369] and A General Curriculum in Mathematics for Colleges*, makes the following observations:

1. The first two years of the GCMC program include a thoughtful study of calculus through partial derivatives and multiple integrals, and of linear algebra through the elementary theory of vector spaces and linear transformations.

2. If all mathematics students follow the first two years of the GCMC program, those who decide relatively late on graduate work in mathematics will not lose very much, even though early work in the spirit of the 1963 recommendations (Pregraduate Preparation of Research Mathematicians) is very desirable for the future professional mathematician.

* The original GCMC report is not included in this COMPENDIUM. However, the 1972 Commentary on a General Curriculum in Mathematics for Colleges appears on page 33.

3. The first two years of the GCMC program are within the reach of all schools, even if they have to offer a precalculus course as preparation.

The lower division of the GCMC program and its relation to pregraduate training are discussed in more detail later on.

The upper division (last two years). Because of the heterogeneity of the mathematical world, the Panel recognizes that no single curriculum will work for all schools. We therefore recommend the following priorities.

1. A minimal upper-division program for mathematics majors who intend to continue the study of mathematics in graduate school appears in the 1965 GCMC report and is reprinted below:

> A mathematics major program for students bound toward graduate mathematics: Mathematics 1, 2, 2P, 3, 4, 5, 6, 10, 11, 12, 13. A stronger major would be desirable, but this is adequate to enter good graduate schools at the present time [1965]. With two semesters' advanced placement the student still has Mathematics 7, 8, 9 from which to complete a better major.

Any college not already offering a comparable program should take immediate steps to do so.

2. If, however, the college can also supply courses designed especially for the pregraduate student, then it should provide one, or if possible both, of the one-year sequences (analysis and algebra) described below. The choice, if only one can be given, should be determined by the staff's capabilities. The one-semester analysis course (Mathematics 11) of the GCMC recommendations is itself aimed primarily at the pregraduate student; it could be replaced by the one-year course (Mathematics 11-12). On the other hand, the one-semester algebra course (Mathematics 6M) of GCMC serves many purposes; it should be retained for these purposes and the one-year algebra course described below should be added for the pregraduate student. [See page 93 for outlines of Mathematics 11-12 and of Mathematics 11. See page 68 for an outline of Mathematics 6M.]

The Panel has consulted many outstanding graduate mathematics departments; a solid grounding in algebra and analysis is what they most want from incoming students. If time and resources permit, it is, of course, desirable to introduce the pregraduate student to a broader range of material, but not at the expense of depth in algebra and analysis.

Introducing the program. Many departments of mathematics can adopt the present recommendations now. Many departments, indeed, are already offering even more substantial programs; they are referred to the 1963 recommendations and urged to proceed as far and as fast as they can in the directions suggested there. We have no universal advice for departments that feel unable to go as far as the present recommendations now.

Objectives of the program. Our concern is with all prospective graduate mathematics students regardless of their destination in today's diversified mathematical profession. The mathematical involvement of the professional mathematician, heavily influenced by the rapid developments in computer science, continues to broaden and deepen in education, in industry, and in government.

The subject matter recommended for the pregraduate program is discussed briefly above and in more detail below. This subject matter is, of course, very important; but equally important are the spirit and tone of the teaching, not only because they are reflected in the ultimate quality of the student's performance but also because they can influence the student to decide for or against a career in mathematics.

The student should be introduced to the language of mathematics in both its rigorous and idiomatic forms. He should learn to give clear explanations of some fundamental concepts, statements, and notations. He should develop facility with selected mathematical techniques, know proofs of a collection of basic theorems, and acquire experience in constructing proofs. He should appreciate the power of abstraction and of the axiomatic method. He should be aware of the applicability of mathematics and of the constructive interplay between mathematics and other disciplines. He should begin to read mathematical literature with understanding and enjoyment. He should learn from illustration and experience to cultivate curiosity and the habit of experimentation, to look beyond immediate objectives, and to make and test conjectures. In short, the student must be helped to mature mathematically as well as to acquire mathematical information.

There are many ways in which the student can be helped to mature mathematically. He can be taught in special "honors" classes for superior students. In the earlier stages he can be given independent reading assignments in textbooks, and later he can be assigned the more difficult task of reading papers in journals. He can be taught through "reading courses." He can make reports in seminars and colloquia. He can prepare an undergraduate thesis containing work original for him although not necessarily original in the stricter sense. He can be taught through the "developmental course," in which he is led to develop a body of mathematical material under the guidance of the professor. In general, the Panel feels very strongly that every pregraduate curriculum should include work to develop mathematical self-reliance, initiative, and confidence.

Identifying students. Far too few college students successfully complete a graduate program in mathematics. The Panel recommends strongly that every effort should be made to identify pregraduate mathematicians as early as possible, preferably when they enter college (a task often complicated by the students' own incorrect notions about their mathematical capabilities).

Lower-division courses. As we have already said, the Panel regards the basic sequence Mathematics 1, 2, 3, 4, 5 of the GCMC recommendations as essential for the pregraduate student. We comment briefly on this program.

For the first semester of college-level mathematics, the GCMC presents a course dealing with the integral and differential calculus of the elementary functions, together with the associated analytic geometry. To meet the needs of the future graduate student most effectively, this course should be designed with three specific objectives in mind. First, the course should build a strong intuitive concept of limits based on concrete examples. These examples can be drawn from geometry, physics, biology, etc. This may be followed by setting down a strong enough axiom system about limits to encompass their elementary properties obtained intuitively. The student should be told which of his current axioms will be future theorems. The formal definition of a limit is too difficult to be swallowed whole by the student at this point; our greatest service to the student would be to give him a firm intuitive grasp of the concept.

The second objective of this course should be to improve the student's ability to handle mathematical rigor. This can be done, for example, by using the axioms about limits in a rigorous development of the calculus.

The third and final purpose of this course should be to teach the student to calculate--for, certainly, it is the ability to calculate with the calculus that makes the calculus the powerful tool that it is.

In the next calculus courses, Mathematics 2 and 4, the GCMC is concerned, in part, with the possible introduction of a fair amount of multivariable calculus much earlier than is customary. This is less important for pregraduate mathematics students than for some other students, since pregraduate students will take all the courses.

The attitude in presenting the material is more important. After courses 1 or 3 there should be a gradual but considerable increase of mathematical maturity. Course 11-12 treats continuity, differentiation, and integration at the level of sophistication required in the theory of "real variables." Consequently, the study of these concepts in courses 2 and 4 should bring the student to an insight which makes the transition easier. There will be little in Mathematics 5 to help in this direction. Thus, the student must be led in Mathematics 2 and 4 to a considerable appreciation of rigor and to the effective personal use of mathematical language.

The 1965 GCMC recommendations included an introductory course in probability, Mathematics 2P, for all students in their first two years. The present Panel feels that the student preparing for graduate work in mathematics might better be making faster progress toward upper-division courses, deferring his work in probability.

Mathematics 3 is a short course in linear algebra. The Pregraduate Panel concurs with the GCMC committee in recommending that this course should come no later than the beginning of the second year.

Upper-division courses. Mathematics 5 contains material frequently presented as the second half of a course called "advanced calculus." Certainly the pregraduate student, whatever his branch of mathematical study, needs to acquire skill in the techniques and understanding of the concepts of mappings between Euclidean spaces of dimension at least 2 (i.e., systems of several functions of several variables).

After Mathematics 1, 2, 3, 4, 5 the Panel recommends for pregraduate students, for reasons discussed earlier, a year course in abstract algebra instead of GCMC Mathematics 6M and a year course in real analysis instead of GCMC Mathematics 11. Possible outlines for such an algebra course are presented below to indicate the flavor and scope that the Panel considers desirable for the pregraduate student; an outline for the analysis course, which is the same as GCMC Mathematics 11-12, can be found on page 93.

We repeat that any department which can offer more than these two courses should turn to the 1963 recommendations (Pregraduate Preparation of Research Mathematicians, page 369) for further suggestions.

COURSE OUTLINES

Abstract Algebra

The purpose of this year course is to introduce the student to the basic structures of abstract algebra and also to deepen and strengthen his knowledge of linear algebra. It provides an introduction to the applications of these concepts to various branches of mathematics. [Prerequisite: Mathematics 3]

OUTLINE A

1. Groups. (10 lessons) Definition. Examples: vector spaces, linear groups, additive group of reals, symmetric groups, cyclic groups, etc. Subgroups. Order of an element. Theorem: Every subgroup of a cyclic group is cyclic. Coset decomposition.

Lagrange theorem on the order of a subgroup. Normal subgroups. Homomorphism and isomorphism. Linear transformations as examples. Determinant as homomorphism of GL(n) to the nonzero reals. Quotient groups. The first two isomorphism theorems. Linear algebra provides examples throughout this unit.

2. Further group theory. (10 lessons) The third isomorphism theorem. Definition of simple groups and composition series for finite groups. The Jordan-Hölder theorem. Definition of solvable groups. Simplicity of the alternating group for $n > 4$. Elements of theory of p-groups. Theorems: A p-group has nontrivial center; a p-group is solvable. Sylow theory. Sylow theorem on the existence of p-Sylow subgroups. Theorems: Every p-subgroup is contained in a p-Sylow subgroup; all p-Sylow subgroups are conjugate and their number is congruent to 1 modulo p.

3. Rings. (10 lessons) Definition. Examples: the integers, polynomials over the reals, the rationals, the Gaussian integers, all linear transformations of a vector space, continuous functions on spaces. Zero divisors and inverses. Division rings and fields. Domains and their quotient fields. Examples: construction of field of four elements, embedding of complex numbers in 2×2 real matrices, quaternions. Homomorphism and isomorphism of rings. Ideals. Congruences in the ring of integers. Tests for divisibility by 3, 11, etc., leading up to Fermat's little theorem, $a^{p-1} \equiv 1 \pmod{p}$, and such problems as showing that $2^{32} + 1 \equiv 0 \pmod{641}$. Residue class rings. The homomorphism theorems for rings.

4. Further linear algebra (continuing Mathematics 3). (12 lessons) Definition of vector space over an arbitrary field. (Point out that the first part of Mathematics 3 carries over verbatim and use the opportunity for some review of Mathematics 3.) Review of spectral theorem from Mathematics 3 stated in a more sophisticated form. Dual-space adjoint of a linear transformation, dual bases, transpose of a matrix. Theorem: Finite-dimensional vector spaces are reflexive. Equivalence of bilinear forms and homomorphism of a space into its dual. General theory of quadratic and skew-symmetric forms over fields of characteristic different from 2. The canonical forms. (Emphasize the connections with corresponding material in

Mathematics 3.) The exterior algebra defined in terms of a basis--2- and 3-dimensional cases first. The transformation of the p-vectors induced by a linear transformation of the vector space. Determinants redone this way.

5. Unique factorization domains. (12 lessons) Primes in a commutative ring. Examples where unique factorization fails, e.g., in $Z[\sqrt{-5}]$. Definition of Euclidean ring, regarded as a device to unify the discussion for Z and $F[x]$, F a field. Division algorithm and Euclidean algorithm in a Euclidean ring; greatest common divisor. Theorem: If a prime divides a product, then it divides at least one factor; unique factorization in a Euclidean ring. Theorem: A Euclidean ring is a principal ideal domain. Theorem: A principal ideal domain is a unique factorization domain. Gauss' lemma on the product of twò primitive polynomials over a unique factorization domain. Theorem: If R is a unique factorization domain, then $R[x]$ is a unique factorization domain.

6. Modules over Euclidean rings. (14 lessons) Definition of module over an arbitrary ring viewed as a generalization of vector space. Example: vector space as a module over $F[x]$ with x acting like a linear transformation. Module homomorphism. Cyclic and free modules. Theorem: Any module is a homomorphic image of a free module. Theorem: If R is Euclidean, A an $n \times n$ matrix over R, then by elementary row and column transformations A can be diagonalized so that diagonal elements divide properly. Theorem: Every finitely generated module over a Euclidean ring is the direct sum of cyclic modules. Uniqueness of this decomposition, decomposition into primary components, invariant factors, and elementary divisors. Application to the module of a linear transformation, leading to the rational and Jordan canonical forms of the matrix. Several examples worked in detail. Similarity invariants of matrices. Characteristic and minimal polynomials. Hamilton-Cayley theorem: A square matrix satisfies its characteristic equation. Application of module theorem to the integers to obtain the fundamental theorem of finitely generated abelian groups.

7. Fields. (10 lessons) Prime fields and characteristic. Extension fields. Algebraic extensions. Structure of $F(a)$, F a

field, a an algebraic element of some extension field. Direct proof that if a has degree n, then the set of polynomials of degree $n-1$ in a is a field; demonstration that $F(a) \cong F[x]/(f(x))$, where f is the minimum polynomial of a. Definition of $(K:F)$, where K is an extension field of F. Theorem: If $F \subset K \subset L$ and $(L:F)$ is finite, then $(L:F) = (L:K)(K:F)$. Ruler-and-compass constructions. Impossibility of trisecting the angle, duplicating the cube, squaring the circle (assuming π transcendental). Existence and uniqueness of splitting fields for equations. Theory of finite fields.

OUTLINE B (including Galois theory)

A course culminating in and climaxed by Galois theory can be constructed by compressing the topics in Outline A into somewhat less time and adding material at the end. Outline B suggests such a course. The appeal of Galois theory as a part of a year course in algebra is obvious. The material ties together practically all the algebraic concepts studied earlier and establishes a clear connection between modern abstraction and a very concrete classical problem. The cost is equally obvious; the depth of much of the earlier material must be reduced, or several topics eliminated. Whether the advantages justify the cost is debatable.

Recognizing that there is merit on both sides, the Panel offers Outline B as an alternative to Outline A with the following words of caution and explanation:

(a) For most pregraduate students today, Outline A probably represents the better balance between coverage and pace.

(b) Outline B contains all the material in Outline A plus Galois theory. Thus Outline B should be attempted only when more time is available or the students are clearly capable of an accelerated pace. Accordingly, each unit in this outline is assigned a range of suggested times, the extremes representing these two alternatives.

(c) In order to break up the rather substantial concentration on group theory at the beginning of Outline A, some of this material has been moved to a position near the end of Outline B, where it fits naturally with Galois theory. This same shift may be made in Outline A by taking the units in the order 1, 3, 4, 5, 6, 7, 2.

1. Groups. (6-10 lessons) Outline A, unit 1.
2. Rings. (7-10 lessons) Outline A, unit 3.

3. Further linear algebra. (10-12 lessons) Outline A, unit 4.

4. Unique factorization domains. (10-12 lessons) Outline A, unit 5.

5. Modules over Euclidean rings. (12-14 lessons) Outline A, unit 6.

6. Fields. (7-10 lessons) Outline A, unit 7.

7. Galois theory. (8-10 lessons) Automorphisms of fields. Fixed fields. Definition of Galois group. Definition of Galois extension. Fundamental Theorem of Galois Theory. Separability. Equivalence of Galois extension and normal separable extension. Computation of Galois groups of equations. Existence of Galois extensions with the symmetric group as Galois group. Theorem on the primitive element: If $(L:F)$ is finite and there exist only a finite number of intermediate fields, then $L = f(a)$ for some $a \in F$.

8. Further group theory. (9-10 lessons) Outline A, unit 2.

9. Galois theory continued. (8-10 lessons) Hilbert's theorem 90: If L is finite and cyclic over F, g is a generator of the Galois group, and x is an element of L of norm 1 over F, then $x = y(yg)^{-1}$ for some $y \in L$. Also the additive form of Hilbert's theorem 90. Galois groups of $x^n - a$. Definition of solvability by radicals. Theorem: An equation is solvable by radicals if and only if its Galois group is solvable. The unsolvability of the general equation of degree n, $n \geqq 5$. Other examples. Roots of unity and cyclotomic fields.

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