Carl B. Allendoerfer Awards

John Chase and Matthew Wright

"Bacterial Growth: Not So Simple," *Mathematics Magazine,* 96:4, pp. 433–441.
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If a bacterium divides once per hour, what will the population size be after t hours?

Exponential models of bacterial growth are among the first mathematical models students see, often when first encountering logarithms. Due to these models' ubiquity and reliance on techniques from high school algebra, it is easy to discount them as "simple" and assume they hold little interest beyond being a starting point for more sophisticated population models. In "Bacterial Growth: Not So Simple," authors John Chase and Matthew Wright demonstrate that this seemingly straightforward problem holds unexpected depths.

The authors start by encouraging us to question our assumptions via a deterministic model, in which every bacterium divides after exactly the same amount of time. This produces an "exponential step function" that is equal to the familiar population size function 2t at integer values of t. However, this model highlights the absurdity of supposing that a large population of bacteria would double in size instantaneously, and then remain a constant size for exactly an hour. It is more reasonable to suppose that the time to division varies within the bacteria culture; in other words, time to division is a random variable, not a constant. With that in mind, we might expect that if we replace a constant 1-hour time to division with a 1-hour mean time to division, then 2t would still be a reasonable model for the number of bacteria after t hours. And indeed, this is how we often talk informally about the situation with our students. But as the authors of this article describe, things are much more interesting than this approach suggests: the bacteria population depends on the distribution of the time to division, not just the mean.

After the introduction to the deterministic model, the authors explore a stochastic model in which the time to division has an exponential distribution with mean a = 1, since exponential distributions are often appropriate for lifespans or time to failure. Using well-known results for this distribution, they reveal the surprising result that in this case, the population size is not approximated by 2t, but by et instead! Because the median of the exponential distribution is less than the mean, the classic deterministic model substantially underestimates the population growth function.

The splitting of bacteria, however, is not a single biological process but a sequence of interrelated subprocesses. If each subprocess is modeled by an exponential distribution, then the splitting process might be better modeled by a sum of exponentials, which is a Gamma distribution. This is the next case the authors consider. Using simulations, the authors determine that the population size is still modeled well by an exponential function whose base depends on the variance of the underlying Gamma distribution. One might consider other time-to-division distributions, so the authors conclude by describing a result that under mild assumptions, every distribution will lead to an exponential model for the population size.

This paper is a delight to read, with an enjoyable mix of theory and simulation. The authors take a topic many readers will feel they know well and reveal hidden aspects of the underlying models. That these new insights can be seen playing out in the real world just adds to the fun.

Responses

John Chase: I am honored to receive the Carl B. Allendoerfer Award, together with my coauthor Matthew Wright. Our paper was inspired by conversations with one of my high school math teacher colleagues, Will Rose, who questioned the underlying premises of bacterial growth. I did work in stochastic processes in my graduate program, and I thought this would be a perfect time to put that knowledge to use. When Matthew and I first uncovered the results in our paper, we found them surprising and delightful. Using bacterial growth as a first example of exponential growth is so commonplace it seemed unlikely that there would be anything new to say. We are pleased that others found the paper surprising and delightful as well. Our results are not groundbreaking and are likely well-known by those who have a deep knowledge of stochastic processes, but we were glad for the opportunity this paper gave us to popularize these results. We hope that this expository treatment of the topic will open conversations among educators and students in both undergraduate and secondary settings. I hope that any recognition the award brings will broaden the reach of our paper and highlight the delightful mathematics, not just the authors.

Matthew Wright: It is a surprise and an honor to receive the Allendoerfer Award. Ever since I was an undergraduate student, I have sought out and enjoyed reading well-written mathematical exposition. While it has been my desire to write mathematical papers that others would enjoy reading, I never imagined winning the Allendoerfer Award. It is especially an honor to win this award together with John Chase. I met John in college, where we were friends, classmates, and roommates. In the years since, we've co-written two mathematical papers. Discussions that led to this award paper started in late 2016. As I recall, John asked me about whether I used cell division as an example of exponential growth in my teaching, and whether I had ever thought about how the individual splitting times affected the growth rate. Since I regularly teach differential equations and probability theory, I was intrigued to explore this question. We did some calculations and simulations, but we only thought about this question sporadically until 2019. The results we found were surprising to us, and we thought they deserved to be more widely known, so we decided to bring our thoughts together into a coherent paper. It took us a few more years to finish the paper, and we are grateful to *Mathematics Magazine* for publishing it. We hope others find this paper as enjoyable to read as it was for us to write!

Biographical Sketches

John Chase is the head of the math department at Walter Johnson High School in Bethesda, Maryland. He earned a BA in Math Education from Messiah University and an MS in Applied and Computational Mathematics from Johns Hopkins University. John is a National Board-certified teacher. He has spoken at the national NCTM conference, the Association of Christians in the Mathematical Sciences annual conference, and the Association of Teacher Educators annual meeting. He performs every year at the New York City Math Festival and has been a guest presenter at the Museum of Mathematics in New York City. John maintains a math education blog at mrchasemath.com. Outside of mathematics, he enjoys spending time with his wife and three daughters and pursuing hobbies such as juggling, Lego, and magic. Most important of all is John's faith in Jesus—the axiom on which the rest of his life is built.

Matthew Wright is an associate professor at St. Olaf College in Northfield, MN, where he teaches applied and computational math courses. He earned an undergraduate degree in mathematics and computer science from Messiah University and a PhD in mathematics from the University of Pennsylvania. He was a postdoctoral fellow at the Institute for Mathematics and its Applications. His research is in topological data analysis and computational mathematics, and he is an author of the RIVET software for topological data analysis. Matthew lives in Minnesota with his wife and two children. In his spare time, he enjoys reading, juggling, and anything constructive. Find him online at mlwright.org.