2024 Session B

B1. Let n and k be positive integers. The ith row and jth column of an n-by-n grid of squares contains the number $i + j - k$. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly $1, 2, \ldots, n$?

Answer: It is possible if and only if $n = 2k - 1$.

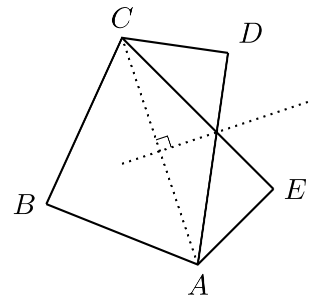
Solution: Suppose that it is possible to select such squares, and let their coordinates be $(i, w(i))$, where $w : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijection. We must have that the sum of the entries in those squares is $1 + 2 + \cdots + n = \binom{n+1}{2}$. On the other hand, the sum is equal to

$$\sum_{i=1}^{n}(i + w(i) - k) = \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} w(i)\right) - kn = 2\left(\sum_{i=1}^{n} i\right) - kn = 2\binom{n+1}{2} - kn,$$

Thus, we must have $kn = \binom{n+1}{2}$, so $k = (n + 1)/2$, and $n = 2k - 1$ has to be odd.

To exhibit a possible construction for these values, let $w(i) = k+i-1$ for $i = 1, \ldots, k$ and $w(i) = i - k$ for $i = k + 1, \ldots, n$, so that $w(1), w(2), \ldots, w(n) = k, k + 1, \ldots, n, 1, \ldots, k - 1$. The value in the square $(i, w(i))$ is $2i - 1$ for $i = 1, \ldots, k$ (the odd numbers $1, 3, \ldots, n$), and $2(i - k)$ for $i = k + 1, \ldots, n$ (the even numbers $2, 4, \ldots, n - 1$).

B2. Two convex quadrilaterals are called *partners* if they have three vertices in common and they can be labeled $ABCD$ and $ABCE$ so that E is the reflection of D across the perpendicular bisector of the diagonal $\overline{AC}$. Is there an infinite sequence of convex quadrilaterals such that each quadrilateral is a partner of its successor and no two elements of the sequence are congruent?



Answer: No.

Solution 1: Let $Q_0, Q_1, \ldots, Q_6$ be a sequence of convex quadrilaterals such that Q_n and Q_{n+1} are partners for $0 \le n \le 5$. We will prove that Q_0 and Q_6 are congruent, unless Q_{n-1} and Q_{n+1} are congruent for some $1 \le n \le 5$.

In the notation of the problem statement, we'll say that $ABCD$ and $ABCE$ are partners with respect to diagonal $\overline{AC}$. A quadrilateral can have two different partners with respect to the same diagonal, but these partners are congruent to each other, because one partner is the reflection of the other about the perpendicular bisector of the diagonal. Thus for $1 \le n \le 5$, either Q_{n-1} and Q_{n+1} are congruent or they are partners of Q_n with respect to different diagonals of Q_n. Hereafter, we assume that the latter is true for all $1 \le n \le 5$.

Label the vertices of Q_n as $W_nX_nY_nZ_n$ in such a way that the three vertices in common between Q_n and Q_{n+1} are assigned the same letters. Without loss of generality, we can assume for $0 \le n \le 5$ that Q_n and Q_{n+1} are partners with respect to $\overline{W_nY_n}$ if n is even and with respect to $\overline{X_nZ_n}$ if n is odd. For each such Q_n, this allows two possibilities for Q_{n+1} that are congruent to each other, so without loss of generality we can assume that $W_{n+1} = W_n$ and $Z_{n+1} = Z_n$.

Since Q_0 is convex, its diagonals intersect, and in particular they are not parallel, so neither are their perpendicular bisectors. Let P be the intersection of their perpendicular bisectors, and let P be the origin of a polar coordinate system. Then W_0 and Y_0 are equidistant from P, and so are X_0 and Z_0. Let the coordinates of these vertices be $W_0 = (r, \alpha)$, $X_0 = (s, \beta)$, $Y_0 = (r, \gamma)$, $Z_0 = (s, \delta)$.

To form Q_1, we reflect X_0 across the perpendicular bisector of $\overline{W_0Y_0}$ to get X_1. The angle that the bisector makes with respect to P is $\frac{\alpha+\gamma}{2}$, reflecting X_0 to X_1 then gives an angular coordinate for X_1 as $2(\frac{\alpha+\gamma}{2} - \beta) + \beta$. Since P is on this perpendicular bisector, it is equidistant from X_0 and X_1, so we have $X_1 = (s, \alpha + \gamma - \beta)$, while $Y_1 = Y_0 = (r, \gamma)$. Notice also that X_1 and $Z_1 = Z_0$ are equidistant from P, so P is on the perpendicular bisector of diagonal $\overline{X_1Z_1}$. Continuing in this manner, keeping in mind that $W_n = W_0$ and $Z_n = Z_0$, we calculate

$$\begin{aligned}
X_2 &= (s, \alpha + \gamma - \beta), & Y_2 &= (r, \alpha + \delta - \beta);\\
X_3 &= (s, \alpha + \delta - \gamma), & Y_3 &= (r, \alpha + \delta - \beta);\\
X_4 &= (s, \alpha + \delta - \gamma), & Y_4 &= (r, \beta + \delta - \gamma);\\
X_5 &= (s, \beta), & Y_5 &= (r, \beta + \delta - \gamma);\\
X_6 &= (s, \beta), & Y_6 &= (r, \gamma).
\end{aligned}$$

In particular, Q_0 and Q_6 are congruent (though they would not necessarily coincide with each other if we had chosen partners so that W_n or Z_n changed at some step).

Solution 2: We say that an ordered quintuple (w, x, y, z, θ) "represents" a convex quadrilateral $ABCD$ if $w = AB$, $x = BC$, $y = CD$ and $z = DA$, and θ is the sum of the interior angles at B and D. (By relabeling its vertices, a quadrilateral can be represented by more than one quintuple.) We claim that two convex quadrilaterals that can be represented by the same quintuple must be congruent. Before we prove this claim, we explain why it solves the problem.

For partners $ABCD$ and $ABCE$, the interior angles at D and E are the same, and $CD = AE$ and $DA = EC$. Thus, if (w, x, y, z, θ) represents $ABCD$ using the labeling of the previous paragraph, then (w, x, z, y, θ) represents $ABCE$. Furthermore, if (w, x, y, z, θ) represents $ABCD$ using a different labeling, then $ABCE$ can be represented by some permutation of w, x, y, z followed by θ. (Notice that θ might be the sum of the interior angles at A and C in this representation, but since the sum of all four interior angles of a quadrilateral is always 2π, the value of θ is the same for $ABCD$ and $ABCE$ in this case too.) By induction, for an infinite sequence of convex quadrilaterals in which each is a partner of its successor, if (w, x, y, z, θ) represents the first member of the sequence, then each member of the sequence can be represented by a quintuple that is some permutation of w, x, y, z followed by θ. Since there are only a finite number of such permutations, two members of the sequence can be represented by the same quintuple.

To prove the claim, we again use the labeling of the first paragraph. Let ϕ be the interior angle at B; then the interior angle at D is $\theta - \phi$. By the law of cosines, both of the following expressions equal AC^2:

$$w^2 + x^2 - 2wx \cos \phi = y^2 + z^2 - 2yz \cos(\theta - \phi).$$

Since $ABCD$ is convex, both ϕ and $\theta - \phi$ lie between 0 and π, so the left side of the equation above is a strictly increasing function of ϕ, and the right side is a strictly decreasing function of ϕ. Thus, there can be only one value of ϕ that achieves equality. The values of w, x, y, z, θ, ϕ determine triangles ABC and CDA up to congruence, and therefore they determine $ABCD$ up to congruence.

Solution 3: We prove that the number of noncongruent quadrilaterals in such a sequence of convex quadrilaterals cannot exceed 12.

Observe that $\triangle ACD \cong \triangle CAE$. Thus, two convex partners have the same set of four side-lengths, the same area, and the same sums for the two pairs of opposite angles.

In quadrilateral $ABCD$, let $\rho = \angle A$ and $\sigma = \angle A + \angle C$, $w = AB$, $x = BC$, $y = CD$, $z = DA$. The areas of the sequence of quadrilaterals all equal

$$\frac{1}{2} wz \sin \rho + \frac{1}{2} xy \sin(\sigma - \rho) = \frac{wz - xy \cos \sigma}{2} \sin \rho + \frac{xy \sin \sigma}{2} \cos \rho.$$

Since the area is positive, the coefficients of $\sin \rho$ and $\cos \rho$ cannot both be zero. By convexity, $0 < \rho < \pi$, so $\sin \rho > 0$. Thus, for the derivative of the area with respect to ρ to be 0, we must have

$$\cot \rho = \frac{xy \sin \sigma}{wz - xy \cos \sigma},$$

which holds for as most one of the possible values of ρ. Then for given w, x, y, z, σ, the area takes on any particular value for at most two values of ρ. Knowing ρ determines BD; hence determines $\angle ABD$, $\angle ADB$, $\angle CBD$, and $\angle CDB$; hence determines $\angle ABC$ and $\angle ADC$.

More generally, after a sequence of convex partnerships, there is always an angle adjacent to the side of length w that is one of the opposite pair of angles that sum to σ. Let the side lengths be, in order, w, s_1, s_2, s_3, where s_1, s_2, s_3 is a permutation of x, y, z, and σ is the sum of the angle ρ between w and s_3 and the angle between s_1 and s_2. As in the previous paragraph, the values of $w, s_1, s_2, s_3, \sigma, \rho$ determine the quadrilateral up to congruence, and for given w, s_1, s_2, s_3, σ, there are at most two values of ρ that make the area of the quadrilateral equal to the area of $ABCD$. This yields at most $3! \cdot 2 = 12$ noncongruent quadrilaterals.

Remark. Solution 3 yields twice as many possibilities as Solution 1 because, in fact, there is only one possible value of ρ for given w, x, y, z, σ. This follows from an argument similar to the last paragraph of Solution 2, requiring that angles ρ and $\sigma - \rho$ yield the same value for the length of diagonal BD.

B3. Let r_n be the nth smallest positive solution to $\tan x = x$, where the argument of tangent is in radians. Prove that

$$0 < r_{n+1} - r_n - \pi < \frac{1}{(n^2+n)\pi}$$

for $n \geq 1$.

Solution: Set $d_n = r_{n+1} - r_n - \pi$. Because $\frac{d}{dx}(\tan x - x) = \sec^2 x - 1 \geq 0$ where the derivative exists, with equality only at integer multiples of π, for each period of $\tan x$ the function $\tan(x) - x$ is increasing and has a unique root. Thus, $n\pi < r_n < (n+1/2)\pi$ and $d_n < \pi/2$. Since $\tan(r_n - n\pi) = \tan r_n = r_n < r_{n+1} = \tan r_{n+1} = \tan(r_{n+1} - (n+1)\pi)$, we have $r_n - n\pi < r_{n+1} - (n+1)\pi$, and hence $d_n = r_{n+1} - (n+1)\pi - (r_n - n\pi) > 0$. Then $0 < d_n < \pi/2$, and in particular $d_n < \tan d_n$. By the formula for the tangent of a difference, we have

$$\begin{aligned} d_n < \tan(d_n) = \tan(r_{n+1} - (r_n + \pi)) &= \frac{\tan(r_{n+1}) - \tan(r_n + \pi)}{1 + \tan(r_{n+1})\tan(r_n + \pi)} \\ &= \frac{r_{n+1} - r_n}{1 + r_{n+1}r_n} = \frac{\pi + d_n}{1 + r_{n+1}r_n} \end{aligned}$$

Isolating d_n, we find

$$d_n < \frac{\pi}{r_{n+1}r_n} < \frac{\pi}{(n+1)\pi \cdot n\pi} = \frac{1}{(n^2+n)\pi}.$$

B4. Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from the set $\{1, \ldots, n\}$, and let

$$a_{n,k+1} = \begin{cases} a_{n,k} + 1, & \text{if } m_{n,k} > a_{n,k}; \\ a_{n,k}, & \text{if } m_{n,k} = a_{n,k}; \\ a_{n,k} - 1, & \text{if } m_{n,k} < a_{n,k}. \end{cases}$$

Let $E(n)$ be the expected value of $a_{n,n}$. Determine $\lim_{n\to\infty} E(n)/n$.

Answer: $\dfrac{1 - e^{-2}}{2}$.

Solution 1: Let $p_{n,k}(j)$ denote the probability that $a_{n,k} = j$ and let $E(n,k)$ denote the expected value of $a_{n,k}$. When $a_{n,k} = j$, the expected value of $a_{n,k+1} - a_{n,k}$ is

$$1 \cdot \frac{n-j}{n} + 0 \cdot \frac{1}{n} - \frac{j-1}{n} = \frac{n+1-2j}{n}.$$

Therefore,

$$\begin{aligned} E(n,k+1) &= E(n,k) + \sum_{j=1}^{n} \frac{n+1-2j}{n}\, p_{n,k}(j) = E(n,k) + \frac{n+1}{n} - \frac{2}{n}\, E(n,k) \\ &= \frac{n+1}{n} + \frac{n-2}{n}\, E(n,k). \end{aligned}$$

Iterating from $E(n,0) = 1$, we find

$$E(n,n) = \left(\frac{n-2}{n}\right)^n + \frac{n+1}{n} \sum_{k=0}^{n-1} \left(\frac{n-2}{n}\right)^k = \left(\frac{n-2}{n}\right)^n + \frac{n+1}{n} \cdot \frac{1 - \left(\frac{n-2}{n}\right)^n}{2/n}.$$

Observing that

$$\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = e^{-2},$$

we conclude

$$\lim_{n\to\infty} \frac{E(n)}{n} = \lim_{n\to\infty} \frac{E(n,n)}{n} = \frac{1-e^{-2}}{2}.$$

Solution 2: Let $E_k(d)$ be the expected value of $a_{n,n}$ given that $a_{n,k} = d$. Note that $E_n(d) = d$. We seek $E(n) = E_0(1)$. We have the recursion

$$E_k(d) = \frac{d-1}{n} E_{k+1}(d-1) + \frac{1}{n} E_{k+1}(d) + \frac{n-d}{n} E_{k+1}(d+1).$$

We can prove two lemmas by (downward) induction on k:

Lemma 1: For $k \leq n$, $E_k(d+1) - E_k(d)$ is independent of d for $d \leq n-1$.

Proof: This is true for $k = n$. Now suppose it is true for $k+1$; let $c_{k+1} = E_{k+1}(d+1) - E_{k+1}(d)$. Then subtracting two terms of the recursion gives

$$\begin{aligned}
E_k(d+1) - E_k(d) &= \frac{n-d-1}{n} E_{k+1}(d+2) - \frac{n-d-1}{n} E_{k+1}(d+1) \\
&\quad + \frac{d-1}{n} E_{k+1}(d) - \frac{d-1}{n} E_{k+1}(d-1) \\
&= \frac{n-d-1}{n} c_{k+1} + \frac{d-1}{n} c_{k+1} \\
&= \left(1 - \frac{2}{n}\right) c_{k+1},
\end{aligned}$$

which is independent of d.

Corollary 1: For $k \le n$, $E_k(d+1) - E_k(d) = \left(1 - \frac{2}{n}\right)^{n-k}$.

Proof: This follows directly from the relationship between c_k and c_{k+1} from the above proof.

Lemma 2: For $k \le n$, $E_k(d) + E_k(n+1-d) = n+1$.

Proof: This is true for $k = n$. Now suppose it is true for $k+1$. Then

$$\begin{aligned}
E_k(d) + E_k(n+1-d) &= \frac{d-1}{n} E_{k+1}(d-1) + \frac{1}{n} E_{k+1}(d) + \frac{n-d}{n} E_{k+1}(d+1) \\
&\quad + \frac{n-d}{n} E_{k+1}(n-d) + \frac{1}{n} E_{k+1}(n+1-d) + \frac{d-1}{n} E_{k+1}(n+2-d) \\
&= \frac{d-1}{n}(n+1) + \frac{1}{n}(n+1) + \frac{n-d}{n}(n+1) \\
&= n+1,
\end{aligned}$$

by the inductive hypothesis (and pairing the first/sixth, second/fifth, and third/fourth terms of the expansion).

Corollary 2: $\sum_{d=1}^{n} E_k(d) = \frac{n(n+1)}{2}$.

Proof: $2 \sum_{d=1}^{n} E_k(d) = \sum_{d=1}^{n} E_k(d) + \sum_{d=1}^{n} E_k(n+1-d) = n(n+1)$, so the result follows.

The rest of the proof is algebra. We have

$$\frac{n(n+1)}{2} = \sum_{d=1}^{n} E_0(d) = \sum_{d=1}^{n} \left(E_0(1) + \left(1 - \frac{2}{n}\right)^n (d-1) \right) = nE_0(1) + \left(1 - \frac{2}{n}\right)^n \frac{n(n-1)}{2},$$

so we get

$$\frac{E(n)}{n} = \frac{E_0(1)}{n} = \frac{n+1}{2n} - \frac{n-1}{2n} \left(1 - \frac{2}{n}\right)^n.$$

The limit of this expression is $\frac{1}{2} - \frac{1}{2e^2}$.

B5. Let k and m be positive integers. For a positive integer n, let $f(n)$ be the number of integer sequences $x_1, \ldots, x_k, y_1, \ldots, y_m, z$ satisfying $1 \le x_1 \le \cdots \le x_k \le z \le n$ and $1 \le y_1 \le \cdots \le y_m \le z \le n$. Show that $f(n)$ can be expressed as a polynomial in n with nonnegative coefficients.

Solution 1: For a given z, the number of sequences is the number of ways to put k balls in z (labeled) boxes, i.e. $\binom{k+z-1}{k}$, and m balls in z boxes. Summing over z, the number of sequences is

$$p_{k,m}(n) = \sum_{z=1}^{n} \binom{k+z-1}{k}\binom{m+z-1}{m} = \sum_{i=0}^{n-1} \binom{i+k}{k}\binom{i+m}{m}.$$

We may assume $k \ge m$ and proceed by induction on m, beginning with $m = 0$. First, by the hockey-stick identity,

$$p_{k,0}(n) = \sum_{i=0}^{n-1} \binom{i+k}{k} = \binom{n+k}{k+1} = \frac{(n+k)(n+k-1)\cdots n}{(k+1)!},$$

the latter expression showing this is a polynomial in n of degree $k+1$ and the coefficients are nonnegative.

Next, we find a recursion, again using the hockey-stick identity,

$$\begin{aligned}
p_{k,m+1}(n) &= \sum_{i=0}^{n-1} \binom{i+k}{k}\binom{i+m+1}{m+1} = \sum_{i=0}^{n-1} \binom{i+k}{k} \sum_{j=0}^{i} \binom{j+m}{m} \\
&= \sum_{j=0}^{n-1} \binom{j+m}{m} \sum_{i=j}^{n-1} \binom{i+k}{k} = \sum_{j=0}^{n-1} \binom{j+m}{m}\left(\binom{n+k}{k+1} - \binom{j+k}{k+1}\right) \\
&= \binom{n+k}{k+1}\binom{n+m}{m+1} - \sum_{j=0}^{n-1} \binom{j+m}{m}\left(\binom{j+k+1}{k+1} - \binom{j+k}{k}\right) \\
&= \binom{n+k}{k+1}\binom{n+m}{m+1} - p_{k+1,m}(n) + p_{k,m}(n). \qquad (1)
\end{aligned}$$

This shows, by induction on m, that $p_{k,m}(n)$ is a polynomial in n. From

$$\frac{i+k+1}{k+1} = \frac{m+1}{k+1} \cdot \frac{i+m+1}{m+1} + \frac{k-m}{k+1},$$

for $k \ge m$, we deduce

$$\begin{aligned}
p_{k+1,m}(n) &= \sum_{i=0}^{n-1} \frac{i+k+1}{k+1} \binom{i+k}{k}\binom{i+m}{m} \\
&= \frac{m+1}{k+1} \sum_{i=0}^{n-1} \frac{i+m+1}{m+1} \binom{i+k}{k}\binom{i+m}{m} + \frac{k-m}{k+1} \sum_{i=0}^{n-1} \binom{i+k}{k}\binom{i+m}{m} \\
&= \frac{m+1}{k+1}\, p_{k,m+1}(n) + \frac{k-m}{k+1}\, p_{k,m}(n).
\end{aligned}$$

Substituting for $p_{k+1,m}(n)$ in (1) yields the weighted average

$$p_{k,m+1}(n) = \frac{k+1}{k+m+2}\binom{n+k}{k+1}\binom{n+m}{m+1} + \frac{m+1}{k+m+2}\, p_{k,m}(n),$$

completing the induction.

Solution 2: Without loss of generality, assume that $k \le m$. For a given value of z, there are $\binom{z+k-1}{k}$ sequences $x_1, x_2, \ldots, x_k$ that meet the condition, because the corresponding sequences $x_1+1, x_2+1, \ldots, x_k+k$ are in one-to-one correspondence with subsets of k elements in $\{2, 3, \ldots, z+k\}$. Similarly, there are $\binom{z+m-1}{m}$ possibilities for $y_1, \ldots, y_m$. Thus,

$$f(n) = \sum_{z=1}^{n} \binom{z+k-1}{k}\binom{z+m-1}{m} = \frac{1}{k!m!}\sum_{z=1}^{n}\prod_{j=0}^{k-1}(z+j)^2 \prod_{j=k}^{m-1}(z+j).$$

Here and below, an empty product (for example, the second product above if $k = m$) should be interpreted as the number 1.

In the summation above, each term is a degree $k+m$ polynomial in z. Thus, $f(n)$ can be expressed as a linear combination of sums of the form $1^\ell + 2^\ell + \cdots + n^\ell$, where ℓ goes from 0 to $k+m$. For each ℓ, this sum can be expressed as a degree $\ell+1$ polynomial in n (with rational coefficients), which is a well-known fact. Thus, $f(n)$ can be expressed as a degree $k+m+1$ polynomial $p(n)$. It remains to show that the coefficients of this polynomial are nonnegative.

For all real t, we have the identity

$$k!m!(p(t+1) - p(t)) = \prod_{i=1}^{k}(t+i)^2 \prod_{i=k+1}^{m}(t+i),$$

since both sides are polynomials, and the identity is true for all positive integers t. Applying the identity for $t = 0$ yields $p(0) = 0$. Then, applying the identity for $x = -1, -2, \ldots, -m$ yields $0 = p(0) = p(-1) = \cdots = p(-m)$ (we call this equation Property I). Also, differentiating the identity and substituting $x = -1, -2, \ldots, -k$ yields $p'(0) = p'(-1) = \cdots = p'(-k)$ (we call this equation Property II).

We claim that Properties I and II, and the fact that p has degree less than $k+m+2$, uniquely determine p up to a multiplicative constant. Indeed, if polynomials p and q with degree less than $k+m+2$ both satisfy Properties I and II, then so does each linear combination $ap+bq$. Choose a and b such that $ap'(0) + bq'(0) = 0$. Then $ap+bq$ has double roots at $0, -1, \ldots, -k$ and single roots at $-k-1, \ldots, -m$, for a total of $k+m+2$ roots. Since the degree of $ap+bq$ is less than its number of roots, it must be identically zero.

Next, we will construct a polynomial of degree less than $k+m+2$ with Properties I and II that is not identically zero, and conclude that p is a constant multiple of it. The approach is similar to Lagrange interpolation. For $j = 0, 1, \ldots, k$, let

$$q_j(t) = (t+j)\prod_{\substack{i=0\\ i\ne j}}^{k}\frac{(t+i)^2}{(-j+i)^2}\prod_{i=k+1}^{m}\frac{t+i}{-j+i}.$$

Notice that q_j has degree $k+m+1$, and that $0 = q_j(0) = q_j(-1) = \cdots = q_j(-m)$. Notice also that by the product rule,

$$q_j'(t) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{(t+i)^2}{(-j+i)^2} \prod_{i=k+1}^{m} \frac{t+i}{-j+i} + (t+j) \prod_{\substack{i=0 \\ i\neq j}}^{k} (t+i) r_j(t)$$

for some polynomial r_j. Thus, $q_j'(-i) = 0$ for $i = 0, 1, \ldots, k$ and $i \neq j$, while $q_j'(-j) = 1$. It follows that $q_0 + q_1 + \cdots + q_k$ has Properties I and II, and hence that $p = c(q_0 + q_1 + \cdots + q_k)$ for some real number c. Notice that $c > 0$ because the t^{k+m+1} coefficient of p must be positive. Each q_j is a polynomial with nonnegative coefficients; thus, so is p.

Remark: This problem was inspired by Corollary 3.4 in [Luis Ferroni, "On the Ehrhart Polynomial of Minimal Matroids", Discrete & Computational Geometry 68 (2022), 255–273, `https://doi.org/10.1007/s00454-021-00313-4`].

B6. For a real number a, let $F_a(x) = \sum_{n\ge 1} n^a e^{2n} x^{n^2}$ for $0 \le x < 1$. Find a real number c such that

$$\lim_{x\to 1^-} F_a(x)e^{-1/(1-x)} = 0 \quad \text{for all } a < c, \text{ and}$$
$$\lim_{x\to 1^-} F_a(x)e^{-1/(1-x)} = \infty \quad \text{for all } a > c.$$

Answer: $c = -1/2$.

Solution: Recall that $\ln x \le x - 1$ for all positive x, and thus also $\ln(1/x) \le 1/x - 1 = (1-x)/x$. For $0 < x < 1$, it follows that

$$\frac{1}{\ln x} \ge \frac{1}{x-1} = -\frac{1}{1-x} = -\frac{x}{1-x} - 1 \ge -\frac{1}{\ln(1/x)} - 1 = \frac{1}{\ln x} - 1.$$

Thus, for $0 < x < 1$,

$$e^{1/(\ln x)} \ge e^{-1/(1-x)} \ge \frac{e^{1/(\ln x)}}{e}.$$

Thus, replacing $e^{-1/(1-x)}$ with $e^{1/(\ln x)}$ does not affect whether the limit in question is 0, or whether it is ∞.

Let $z = -1/(\ln x)$, so that $x = e^{-1/z}$. Then

$$F_a(x)e^{1/(\ln x)} = \sum_{n=1}^{\infty} n^a e^{-z+2n-\frac{n^2}{z}} = \sum_{n=1}^{\infty} n^a e^{-\frac{(n-z)^2}{z}}.$$

Notice that $z \to \infty$ as $x \to 1^-$. For $z \ge 4$, so that $\sqrt{z} \le z/2$, consider the portion of the sum for which $z - \sqrt{z} \le n \le z + \sqrt{z}$, which is equivalent to $(n-z)^2/z \le 1$. There are at least $z + \sqrt{z} - (z - \sqrt{z}) - 1 = 2\sqrt{z} - 1$ terms in this portion, and since $z/2 \le n < 2z$, we obtain the bound $n^a \ge 2^{-|a|} z^a$. Thus,

$$F_a(x)e^{1/(\ln x)} \ge \sum_{z-\sqrt{z}\le n\le z+\sqrt{z}} n^a e^{-\frac{(n-z)^2}{z}} \ge (2\sqrt{z} - 1)2^{-|a|} z^a e^{-1}.$$

If $a > -1/2$, this lower bound approaches ∞ as $z \to \infty$, so $F_a(x)e^{1/(\ln x)} \to \infty$ as $x \to 1^-$.

For $a < -1/2$, write $F_a(x)e^{1/(\ln x)} = S_1(z) + S_2(z)$ where

$$S_1(z) = \sum_{1\le n\le z/2} n^a e^{-\frac{(n-z)^2}{z}}; \quad S_2(z) = \sum_{n>z/2} n^a e^{-\frac{(n-z)^2}{z}}.$$

Since both sums are nonnegative, it suffices to show that each has an upper bound that approaches 0 as $z \to \infty$

We bound $S_1(z)$ above by the number of terms in the sum (which is as most $z/2$) times an upper bound on each term. Since $a < 0$, we have $n^a \le 1$, and since $(n-z)^2/z \ge z/4$ for $n \le z/2$, we have $S_1(z) \le (z/2)e^{-z/4}$. Thus, $S_1(z) \to 0$ as $z \to \infty$.

Since $a < 0$, in $S_2(z)$ we can bound n^a above by $(z/2)^a$. We write $S_2(z) = s_0 + s_1 + s_2 + \cdots$ where s_k includes the terms in $S_2(z)$ for which $k \le (n-z)^2/z \le k+1$. Then each term in

s_k is at most $(z/2)^a e^{-k}$, and since $z - \sqrt{(k+1)z} \le n \le z + \sqrt{(k+1)z}$, there are at most $z + \sqrt{(k+1)z} - (z - \sqrt{(k+1)z}) + 1 = 2\sqrt{k+1}\sqrt{z} + 1$ terms in s_k. Thus,

$$S_2(z) \le \sum_{k=0}^{\infty} (2\sqrt{k+1}\sqrt{z} + 1)(z/2)^a e^{-k}.$$

Since $\sum_{k\ge 0} \sqrt{k+1} e^{-k} < \infty$, this upper bound approaches 0 as $z \to \infty$ if $a < -1/2$. Therefore, $F(x)e^{1/(\ln x)} \to 0$ as $x \to 1^-$, completing the proof.

Remark. This problem was inspired by Proposition 3.2 in [K. Bringmann, C. Jennings-Shaffer, K. Mahlburg , "On a Tauberian theorem of Ingham and Euler–Maclaurin summation", The Ramanujan Journal 61 (2023), 55–86, `https://doi.org/10.1007/s11139-020-00377-5`]. A similar argument shows that the limit in this problem exists for $a = -1/2$, and is equal to $\sqrt{\pi/e}$. An outline of the proof, using the notation of the solution above, follows. First, by multiple applications of L'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 1^-} \left(-\frac{1}{\ln x} - \frac{1}{1-x} \right) &= -\lim_{x\to 1^-} \frac{1 + (\ln x)/(1-x)}{\ln x} = -\lim_{x\to 1^-} \frac{1/(x(1-x)) + (\ln x)/(1-x)^2}{1/x} \\ &= -\lim_{x\to 1^-} \frac{1 - x + x\ln x}{(1-x)^2} = -\lim_{x\to 1^-} \frac{\ln x}{-2(1-x)} = -\lim_{x\to 1^-} \frac{1/x}{2} = -\frac{1}{2}.\end{aligned}$$

Thus, the limit of the exponential of the expression above is $e^{-1/2}$.

Next, make the change of variables $u = (n - z)/\sqrt{z}$ and $n = z + \sqrt{z}u$ to write

$$F_{-1/2}(x)e^{1/(\ln x)} = \sum_{n=1}^{\infty} \left(z + \sqrt{z}u\right)^{-1/2} e^{-u^2}.$$

This is a Riemann sum, using intervals of length $1/\sqrt{z}$, for

$$\int_{(1-z)/\sqrt{z}}^{\infty} \sqrt{z} \left(z + \sqrt{z}u\right)^{-1/2} e^{-u^2}\, du.$$

Notice that for fixed u, the integrand approaches e^{-u^2} as $z \to \infty$. The remainder of the proof is to justify that the integral approaches $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$, and that the limit of the Riemann sums is the limit of the integrals.