



# The 84th William Lowell Putnam Mathematical Competition 2023

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**A1** For a positive integer  $n$ , let  $f_n(x) = \cos(x) \cos(2x) \cos(3x) \cdots \cos(nx)$ . Find the smallest  $n$  such that  $|f_n''(0)| > 2023$ .

**A2** Let  $n$  be an even positive integer. Let  $p$  be a monic, real polynomial of degree  $2n$ ; that is to say,  $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x + a_0$  for some real coefficients  $a_0, \dots, a_{2n-1}$ . Suppose that  $p(1/k) = k^2$  for all integers  $k$  such that  $1 \leq |k| \leq n$ . Find all other real numbers  $x$  for which  $p(1/x) = x^2$ .

**A3** Determine the smallest positive real number  $r$  such that there exist differentiable functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  satisfying

- (a)  $f(0) > 0$ ,
- (b)  $g(0) = 0$ ,
- (c)  $|f'(x)| \leq |g(x)|$  for all  $x$ ,
- (d)  $|g'(x)| \leq |f(x)|$  for all  $x$ , and
- (e)  $f(r) = 0$ .

**A4** Let  $v_1, \dots, v_{12}$  be unit vectors in  $\mathbb{R}^3$  from the origin to the vertices of a regular icosahedron. Show that for every vector  $v \in \mathbb{R}^3$  and every  $\varepsilon > 0$ , there exist integers  $a_1, \dots, a_{12}$  such that  $\|a_1v_1 + \cdots + a_{12}v_{12} - v\| < \varepsilon$ .

**A5** For a nonnegative integer  $k$ , let  $f(k)$  be the number of ones in the base 3 representation of  $k$ . Find all complex numbers  $z$  such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)} (z+k)^{2023} = 0.$$

**A6** Alice and Bob play a game in which they take turns choosing integers from 1 to  $n$ . Before any integers are chosen, Bob selects a goal of “odd” or “even”. On the first turn, Alice chooses one of the  $n$  integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the  $n$ th turn, which is forced and ends the game. Bob wins if the parity of  $\{k : \text{the number } k \text{ was chosen on the } k\text{th turn}\}$  matches his goal. For which values of  $n$  does Bob have a winning strategy?



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- B1** Consider an  $m$ -by- $n$  grid of unit squares, indexed by  $(i, j)$  with  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . There are  $(m-1)(n-1)$  coins, which are initially placed in the squares  $(i, j)$  with  $1 \leq i \leq m-1$  and  $1 \leq j \leq n-1$ . If a coin occupies the square  $(i, j)$  with  $i \leq m-1$  and  $j \leq n-1$  and the squares  $(i+1, j)$ ,  $(i, j+1)$ , and  $(i+1, j+1)$  are unoccupied, then a legal move is to slide the coin from  $(i, j)$  to  $(i+1, j+1)$ . How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?
- B2** For each positive integer  $n$ , let  $k(n)$  be the number of ones in the binary representation of  $2023 \cdot n$ . What is the minimum value of  $k(n)$ ?
- B3** A sequence  $y_1, y_2, \dots, y_k$  of real numbers is called *zigzag* if  $k = 1$ , or if  $y_2 - y_1, y_3 - y_2, \dots, y_k - y_{k-1}$  are nonzero and alternate in sign. Let  $X_1, X_2, \dots, X_n$  be chosen independently from the uniform distribution on  $[0, 1]$ . Let  $a(X_1, X_2, \dots, X_n)$  be the largest value of  $k$  for which there exists an increasing sequence of integers  $i_1, i_2, \dots, i_k$  such that  $X_{i_1}, X_{i_2}, \dots, X_{i_k}$  is zigzag. Find the expected value of  $a(X_1, X_2, \dots, X_n)$  for  $n \geq 2$ .
- B4** For a nonnegative integer  $n$  and a strictly increasing sequence of real numbers  $t_0, t_1, \dots, t_n$ , let  $f(t)$  be the corresponding real-valued function defined for  $t \geq t_0$  by the following properties:
- (a)  $f(t)$  is continuous for  $t \geq t_0$ , and is twice differentiable for all  $t > t_0$  other than  $t_1, \dots, t_n$ ;
  - (b)  $f(t_0) = 1/2$ ;
  - (c)  $\lim_{t \rightarrow t_k^+} f'(t) = 0$  for  $0 \leq k \leq n$ ;
  - (d) For  $0 \leq k \leq n-1$ , we have  $f''(t) = k+1$  when  $t_k < t < t_{k+1}$ , and  $f''(t) = n+1$  when  $t > t_n$ .
- Considering all choices of  $n$  and  $t_0, t_1, \dots, t_n$  such that  $t_k \geq t_{k-1} + 1$  for  $1 \leq k \leq n$ , what is the least possible value of  $T$  for which  $f(t_0 + T) = 2023$ ?
- B5** Determine which positive integers  $n$  have the following property: For all integers  $m$  that are relatively prime to  $n$ , there exists a permutation  $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  such that  $\pi(\pi(k)) \equiv mk \pmod{n}$  for all  $k \in \{1, 2, \dots, n\}$ .
- B6** Let  $n$  be a positive integer. For  $i$  and  $j$  in  $\{1, 2, \dots, n\}$ , let  $s(i, j)$  be the number of pairs  $(a, b)$  of nonnegative integers satisfying  $ai + bj = n$ . Let  $S$  be the  $n$ -by- $n$  matrix whose  $(i, j)$ -entry is  $s(i, j)$ .

For example, when  $n = 5$ , we have  $S = \begin{bmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$ .

Compute the determinant of  $S$ .