# 2023 Session A

**A1.** For a positive integer $n$, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest $n$ such that $|f_n''(0)| > 2023$.

**Answer:** 18

**Solution 1:** The Taylor series is

$$\begin{aligned} f_n(x) &= \left(1 - \frac{x^2}{2} + \cdots\right)\left(1 - \frac{(2x^2)}{2} + \cdots\right)\cdots\left(1 - \frac{(nx)^2}{2} + \cdots\right) \\ &= 1 - \frac{x^2}{2}\left(1^2 + 2^2 + \cdots + n^2\right) + \cdots . \end{aligned}$$

Therefore (using the well-known summation formula for sums of squares)

$$f_n''(0) = -(1^2 + 2^2 + \cdots + n^2) = -\frac{n(n+1)(2n+1)}{6}.$$

The question is then to find the minimum $n$ such that $\frac{n(n+1)(2n+1)}{6} > 2023$. One can calculate that this occurs at $n = 18$, where the sum is $3 \cdot 19 \cdot 37 = 2109$ (and at $n = 17$ it is 1785).

**Solution 2:** By the product rule,

$$\begin{aligned} f_n'(x) &= -\sin(x)\cos(2x)\cos(3x)\cdots\cos(nx) - 2\cos(x)\sin(2x)\cos(3x)\cdots\cos(nx) \\ &\quad - \cdots - n\cos(x)\cos(2x)\cdots\cos((n-1)x)\sin(nx) \\ &= -f_n(x)\left(\tan(x) + 2\tan(2x) + \cdots + n\tan(nx)\right) \end{aligned}$$

for $x$ sufficiently small that all the tangents are well-defined. Applying the product rule again and substituting $x = 0$,

$$\begin{aligned} f_n''(0) &= -f_n'(0)\left(\tan(0) + 2\tan(2\cdot 0) + \cdots + n\tan(n\cdot 0)\right) \\ &\quad - f_n(0)\left(\sec^2(0) + 4\sec^2(2\cdot 0) + \cdots + n^2\sec^2(n\cdot 0)\right) \\ &= -(1 + 4 + \cdots + n^2). \end{aligned}$$

One can compute directly (or using the formula in Solution 1) that $|f_{17}''(0)| = 1+4+\cdots+17^2 = 1785$ and $|f_{18}''(0)| = 1 + 4 + \cdots + 18^2 = 1785 + 324 = 2109$, so the answer is 18.

**A2.** Let $n$ be an even positive integer. Let $p$ be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers $k$ such that $1 \leq |k| \leq n$. Find all other real numbers $x$ for which $p(1/x) = x^2$.

**Answer:** $\pm 1/n!$

**Solution 1:** The given condition can be equivalently written as $p(x) = \frac{1}{x^2}$ for $x = \pm\frac{1}{k}, k = 1, \ldots, n$. Now define $g(x) := x^2 p(x) - 1$, and note that $p(1/x) = x^2$ is equivalent to $g(1/x) = 0$. Notice that $g$ is a monic polynomial of degree $2n+2$, and by the preceding observation, it has roots at all $x = \pm\frac{1}{k}$. Unique factorization of polynomials (and/or the Fundamental Theorem of Algebra) then implies that

$$\begin{aligned} g(x) &= (x-1)(x+1)\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right)\cdots\left(x - \frac{1}{n}\right)\left(x + \frac{1}{n}\right)\cdot(x^2 + ax + b) \\ &= \left(x^2 - 1\right)\left(x^2 - \frac{1}{4}\right)\cdots\left(x^2 - \frac{1}{n^2}\right)\cdot(x^2 + ax + b), \end{aligned}$$

where $a$ and $b$ are real numbers.

In order to determine these final coefficients, first note that by definition of $g(x)$, the coefficient of $x$ is zero. But on the other hand, this coefficient is $\dfrac{(-1)^n}{n!^2}a$, so $a = 0$. Now consider the value at $x = 0$, which gives

$$g(0) = -1 = \frac{(-1)^n}{n!^2}b.$$

We therefore conclude (using that $n$ is even) that $b = -n!^2$. In all,

$$g(x) = \left(x^2 - 1\right)\left(x^2 - \frac{1}{4}\right)\cdots\left(x^2 - \frac{1}{n^2}\right)\cdot\left(x^2 - n!^2\right).$$

Finally, we see that $g(1/x)$ has the additional roots $x = \pm\frac{1}{n!}$.

**Solution 2:** We first show that $p$ is even.

**Claim.** *Suppose that $q$ is a monic, degree $2n$ polynomial. If there exists a sequence of distinct positive values $x_1, \ldots, x_n$ such that $q(x_j) = q(-x_j)$ for $1 \leq j \leq n$, then $q$ is even.*

*Proof.* The polynomial $q(x) - q(-x)$ has degree at most $2n-1$, but has (at least) $2n$ roots $\pm x_1, \ldots, \pm x_n$. Therefore, $q(x) - q(-x)$ is identically zero, so $q$ is even. □

Thus $p(x) = s(x^2)$, where $s$ is a monic, degree $n$ polynomial such that $s(1/k^2) = k^2$ for $1 \leq k \leq n$. Let $h(x) := x \cdot s(x) - 1$. Then $h$ is a monic, degree $n+1$ polynomial with roots at $\frac{1}{k^2}$, so

$$h(x) = (x-1)\left(x - \frac{1}{4}\right)\cdots\left(x - \frac{1}{n^2}\right)(x+b)$$

for some real number $b$. Plugging in $x = 0$ gives

$$h(0) = -1 = \frac{(-1)^n}{n!^2}b.$$

Finally, substituting $x^2$ for $x$ gives $h(x^2) = x^2 p(x) - 1$, and the remainder of the proof proceeds as in Solution 1.

*Remark.* Although this is not needed in the proof of the claim, one can use Lagrange interpolation to say slightly more. For $1 \leq a_j \leq n$, let $a_j := q(\pm x_j)$. Define $L(x)$ as the unique polynomial of degree strictly less than $n$ with the values $L(x_j^2) = a_j$ for all $j$ (Lagrange interpolation gives a formula for $L$). Then

$$q(x) = \left(x^2 - x_1^2\right) \cdots \left(x^2 - x_n^2\right) + L(x^2).$$

**A3.** Determine the smallest positive real number $r$ such that there exist differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,

(b) $g(0) = 0$,

(c) $|f'(x)| \le |g(x)|$ for all $x$,

(d) $|g'(x)| \le |f(x)|$ for all $x$, and

(e) $f(r) = 0$.

**Answer:** $\pi/2$

**Solution 1:** Let $h(x) := f(x)^2 + g(x)^2$. Then, using the AM-GM inequality,

$$\begin{aligned} |h'(x)| &= \left|2(f(x)f'(x) + g(x)g'(x))\right| \le 2\left(|f(x)||f'(x)| + |g(x)||g'(x)|\right) \\ &\le 4|f(x)||g(x)| \le 2\left(f(x)^2 + g(x)^2\right) = 2h(x). \end{aligned}$$

Thus, $h(x)e^{2x}$ has a nonnegative derivative. For $x \ge 0$, it follows that $h(x)e^{2x} \ge h(0)e^0 = f(0)^2$, so $h(x) \ge f(0)^2 e^{-2x} > 0$.

Now define $\theta(x) := \tan^{-1} \frac{g(x)}{f(x)}$ on the largest interval around $x = 0$ on which $f(x) > 0$ (this includes a neighborhood of 0 by continuity of $f$). Then

$$\theta'(x) = \frac{1}{1 + \frac{g(x)^2}{f(x)^2}} \cdot \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2} = \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2 + g(x)^2},$$

which implies that $|\theta'(x)| \le 1$, and (since $\theta(0) = 0$) therefore $|\theta(x)| \le x$ for $x \ge 0$ such that $\theta$ is defined.

Finally, observe that $f(x)^2 = h(x)\cos^2\theta(x)$, and therefore $f(x)^2 \ge f(0)^2 e^{-2x}\cos^2 x > 0$ for $x \in (0, \pi/2)$. Thus, $r \ge \pi/2$. Letting $f(x) = \cos x$ and $g(x) = \sin x$ shows that $r = \pi/2$ is possible.

**Solution 2:** Notice that $f(x) = \cos x$ and $g(x) = \sin x$ satisfy all the conditions of the problem with $r = \pi/2$.

To see that no smaller value of $r$ is possible, we claim that $f(x) \ge f(0)\cos x$ for $0 \le x < \pi/2$. If not, then there is some $z \in (0, \pi/2)$ such that $f(z)/\cos z < f(0)$. Since $f(x)/\cos x$ is continuous for $0 \le x < \pi/2$ and $f(0)/\cos 0 = f(0) > 0$, we can choose $z$ such that $f(x)/\cos x > 0$ [whence $f(x) > 0$] for $0 \le x \le z$. Since $f(x)/\cos x$ is differentiable for $0 < x < z$, the Mean Value Theorem implies that its derivative must be negative at some $y \in (0, z)$. Thus, $f'(y)\cos y + f(y)\sin y < 0$. Since $|f'(y)| \le |g(y)|$ and $\cos y > 0$, it follows that $f(y)\sin y - |g(y)|\cos y < 0$.

Let $h(x) = f(x)\sin x - |g(x)|\cos x$. Since $h(0) = 0$, $h(y) < 0$, and $h$ is continuous, there is a greatest value $w \in [0, y)$ such that $h(w) \ge 0$. Then for $w < x < y$, we have $h(x) < 0$ and $f(x)\sin x \ge 0$ [since $f(x) > 0$ for $0 \le x \le z$ and $y < z$], so $|g(x)|\cos x > 0$, and thus $|g(x)| > 0$, for all such $x$. In particular, $h$ is differentiable on the interval $(w, y)$, so by the Mean Value Theorem $h'(x) < 0$ for some $x \in (w, y)$. Further, since $g$ is continuous and nonzero on $(w, y)$, it does not change sign on this interval; without loss of generality, assume that $g$ is positive on $(w, y)$ [otherwise, replace $g$ with $-g$]. Then $h'(x) = (f'(x) + g(x)) \sin x + (f(x) - g'(x)) \cos x$. Since $h'(x) < 0$, $\sin x > 0$, and $\cos x > 0$, we must have $f'(x) + g(x) < 0$ or $f(x) - g'(x) < 0$. But since $f(x) > 0$ and $g(x) > 0$, this would require either $|f'(x)| > g(x) = |g(x)|$ or $|g'(x)| > f(x) = |f(x)|$, a contradiction.

**A4.** Let $v_1, \ldots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \ldots, a_{12}$ such that $\|a_1 v_1 + \cdots + a_{12} v_{12} - v\| < \varepsilon$.

**Solution 1:** We first claim that the vertices of a regular pentagon centered at the origin in $\mathbb{R}^2$ generate a dense additive subgroup. Identify $\mathbb{R}^2 \cong \mathbb{C}$, and assume without loss of generality that the vertices are the fifth roots of unity $\zeta_5^n = e^{\frac{2\pi i n}{5}}$. Define

$$r := \zeta_5 + \zeta_5^4 = 2\cos\frac{2\pi}{5} < 2\cos\frac{\pi}{3} = 1$$

(in fact, $r = \frac{\sqrt{5}-1}{2}$, which follows from observing that $r^2 + r = 1$). Then the positive powers of $r$ accumulate at 0, and are all contained in the subring $\mathbb{Z}[\zeta_5]$. Therefore this subring is dense in $\mathbb{R}$, since it contains all $\{mr^n \mid m \in \mathbb{Z}, n \geq 0\}$. Furthermore, it similarly contains a dense subset of $\zeta_5\mathbb{R}$, and thus a dense subset of $\mathbb{R} + \zeta_5\mathbb{R} = \mathbb{C}$.

Now suppose that $v_1, \ldots, v_5$ are the neighbors of some fixed vertex $v$ in the icosahedron. Then $v_1 - v_2, v_2 - v_3, \ldots, v_5 - v_1$ are the vertices of a regular pentagon in the plane perpendicular to the line through 0 and $v$. Therefore the set of vertices generates a dense set in this plane, and similarly, in the plane perpendicular to (say) the line through 0 and $v_1$. These two planes span $\mathbb{R}^3$.

**Solution 2:** Write the vertex-vectors as $\pm v_1, \ldots, \pm v_6$ where $v_2, \ldots, v_6$ are each adjacent to $v_1$, and are adjacent to each other in the pairs $(v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_2)$. Then $v_2$ is also adjacent to $-v_4$ and $-v_5$, etc. Since all sides of the icosahedron have the same length and $v_j \cdot v_j = 1$ for all $j$, the value of $v_j \cdot v_k$ is the same for all pairs of adjacent vertices $(v_j, v_k)$. Thus,

$$\begin{aligned} v_2 \cdot v_3 &= v_3 \cdot v_4 = v_4 \cdots v_5 = v_5 \cdot v_6 = v_6 \cdot v_2 \\ &= -v_2 \cdot v_4 = -v_2 \cdot v_5 = -v_3 \cdot v_5 = -v_3 \cdot v_6 = -v_4 \cdot v_6. \end{aligned}$$

Then the cross terms $2v_j \cdot v_k$ cancel each other in the following calculation:

$$(v_2 + \cdots + v_6) \cdot (v_2 + \cdots + v_6) = v_2 \cdot v_2 + \cdots + v_6 \cdot v_6 = 5.$$

By symmetry and adjacency, $v_2 + \cdots v_6$ is a positive multiple of $v_1$, so $v_2 + \cdots v_6 = \sqrt{5} v_1$. By Kronecker's Theorem on Diophantine approximation, the integer linear combinations of 1 and $\sqrt{5}$ are dense in the reals, so the integer linear combinations of $v_1$ and $v_2 + \cdots + v_6$ are dense in the line spanned by $v_1$. By the analogous argument, appropriate integer linear combinations of $v_1, \ldots, v_6$ are dense in the line spanned by $v_2$ and in the line spanned by $v_3$. Since $v_1, v_2, v_3$ are not coplanar, they span three-space. Thus, every vector $v$ in three-space can be written $c_1 v_1 + c_2 v_2 + c_3 v_3$, and since each term in this sum can be approximated arbitrarily closely by an integer linear combination of $v_1, \ldots, v_6$, so can $v$.

**A5.** For a nonnegative integer $k$, let $f(k)$ be the number of ones in the base 3 representation of $k$. Find all complex numbers $z$ such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)}(z+k)^{2023} = 0.$$

**Answer:** $\dfrac{1-3^{1010}}{2}$, $\dfrac{1-3^{1010}}{2} \pm \dfrac{\sqrt{3^{2020}-1}}{4}i$

**Solution 1:** If $n$ is an integer, a *quasi-base-3* representation of $n$ is $n = a_N 3^N + a_{N-1}3^{N-1} + \cdots + a_1 \cdot 3 + a_0$, where all $a_j \in \{-1, 0, 1\}$. This can also be written in the shorthand $(a_N a_{N-1} \cdots a_0)_3$, with parentheses used as needed for clarity. For example, $7 = 1(-1)1_3$. If the leading digit is required to be $\pm 1$, then it is a standard fact that this representation is unique (both existence and uniqueness are easily proven by induction). However, here we will also be interested in representations with some number of leading zeros. Let $A_N := \dfrac{3^N-1}{2} = \underbrace{11\cdots 1_3}_{N \text{ digits}}$. It can similarly be shown that if $|n| \leq A_N$, then $n$ has a unique quasi-base-3 representation consisting of exactly $N$ digits.

Define $f_{0,N}(\ell)$ to be the number of zeros when $\ell$ is written in its $N$-digit quasi-base-3 representation. For example, $f_{0,3}(8) = 1$ and $f_{0,5}(8) = 3$, as the left-extended quasi-base-3 expansion is $8 = \ldots 0010(-1)_3$.

**Claim.** *Let $u = z + A_N$. The sum in the problem statement is equivalent to (with $N = 1010$)*

$$S_N(u) := \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)}(u+\ell)^{2N+3}. \tag{1}$$

This follows from the summation shift $k = \ell + A_N$, as well as the straightforward observation that $f(\ell + A_N) = f_{0,N}(\ell)$ (note that $\ell \mapsto \ell + A_N$ is a bijection from $N$-digit quasi-base-3 representations to $N$-digit base-3 representations for the range $\ell = -A_N, \ldots, A_N$, while the definition of $f$ is unchanged by the presence of leading zeros).

If $d$ is an integer, define the centered, $d$-shifted second-difference operator by

$$\Delta_d^2(h(x)) := h(x+d) - 2h(x) + h(x-d).$$

This operator satisfies some simple properties that will be helpful later for reducing longer expressions.

**Lemma 1.** *1. $\Delta_d^2$ is a parity-preserving operator on functions (i.e. $\Delta_d^2(h)$ is an even/odd function as $h$ is).*

*2. $\Delta_d^2$ acts on monomials as*

$$\Delta_d^2(x^n) = 2\sum_{1\leq j\leq \frac{n}{2}} \binom{n}{2j} d^{2j} x^{n-2j}.$$

*3. If $p(x)$ is a polynomial of degree $n$, then $\Delta_d^2(p(x))$ is a polynomial of degree $n-2$ (or is identically zero if $n \leq 1$).*

*Proof.* 1. Suppose that $h$ is an even or odd function. By definition, this means that $h(-x) = \operatorname{sgn}(h) \cdot h(x)$, where $\operatorname{sgn}(h) = \pm 1$ denotes the parity of $h$. Then

$$\begin{aligned} \Delta_d^2(h(-x)) &= h(-x+d) - 2h(-x) + h(-x-d) \\ &= h(-(x+d)) - 2h(-x) + h(-(x-d)) \\ &= \operatorname{sgn}(h) \cdot \Delta_d^2(h(x)). \end{aligned}$$

2. By the Binomial Theorem,

$$\Delta_d^2(x^n) = \sum_{j=1}^{n} \binom{n}{j} d^j x^{n-j} + \sum_{j=1}^{n} \binom{n}{j} (-d)^j x^{n-j} = 2 \sum_{1 \le j \le \frac{n}{2}} \binom{n}{2j} d^{2j} x^{n-2j}.$$

3. This follows by considering only the highest degree term in the sum from part 2.

$\square$

We next show that sums of the form (1) can be written in terms of the second-difference operator.

**Lemma 2.** *If $N$ is a positive integer, then*

$$\Delta_{3^{N-1}}^2 \cdots \Delta_3^2 \Delta_1^2 h(x) = \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell).$$

*Proof.* The base case of $N = 0$ simply states that $h(x) = h(x)$. For the inductive step,

$$\begin{aligned} &\Delta_{3^N}^2 \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell) \\ &\quad = \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)} \left(h\left(x+3^N+\ell\right) - 2h(x+\ell) + h\left(x-3^N+\ell\right)\right) \\ &\quad = \sum_{\ell=3^N-A_N}^{3^N+A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell) + \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)+1} h(x+\ell) \\ &\qquad\qquad + \sum_{\ell=-3^N-A_N}^{-3^N+A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell) \\ &\quad = \sum_{\ell=-A_{N+1}}^{A_{N+1}} (-2)^{f_{0,N+1}(\ell)} h(x+\ell). \end{aligned}$$

The final line follows because in the first and third sum $\ell$ has leading coefficient $\pm 1$, so $f_{0,N+1}(\ell) = f_{0,N}(\ell)$, whereas in the middle sum the leading coefficient is 0, so $f_{0,N+1}(\ell) = f_{0,N}(\ell) + 1$. Furthermore, the three summation ranges combine to one because $A_{N+1} = 3^N + A_N$ by definition. $\square$

In particular, Lemma 2 implies that

$$S_N(u) = \Delta^2_{3^{N-1}} \cdots \Delta^2_3 \Delta^2_1 u^{2N+3}, \tag{2}$$

with the implicit notational assumption that the $\Delta^2$ operators now act on the variable $u$. Lemma 1 parts 1 and 3 then imply that $S_N(u)$ reduces to an odd, cubic polynomial (i.e. of the form $au^3 + bu$). The exact coefficients can now be determined using Lemma 1 part 2.

In particular, only the two highest-order terms are relevant. For $n \geq 4$, the Lemma states that

$$\Delta^2_d(x^n) = 2\binom{n}{2} d^2 x^{n-2} + 2\binom{n}{4} d^4 x^{n-4} + O\left(x^{n-6}\right).$$

For a nonnegative integer $m$ and real number $x$, the *falling factorial* is defined by $(x)_m := x(x-1)(x-2)\cdots(x-m+1)$.

**Lemma 3.** *If $N$, $k$ and $d$ are positive integers, where $k \geq 4$, then*

$$\begin{aligned}
&\Delta^2_{d^{N-1}} \cdots \Delta^2_d \Delta^2_1 x^{2N+k} \\
&\quad = (2N+k)_{2N} \cdot d^{N(N-1)} \left[ x^k + \frac{k(k-1)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)}\right) x^{k-2} + O\left(x^{k-4}\right) \right].
\end{aligned}$$

*The statement is also true for $k \leq 3$, except that any monomials in $x$ with negative exponents do not appear.*

*Proof.* We induct on $N$. The base case is $N = 1$, and by Lemma 1,

$$\Delta^2_1 x^{2+k} = (2+k)_2 \cdot 1^2 x^k + \frac{(2+k)_4}{3 \cdot 4} 1^4 x^{k-2}.$$

For the inductive step, suppose that the statement is true for $N$. Then we have (replacing $k$ by $2+k$ when applying the statement for $N$)

$$\begin{aligned}
&\Delta^2_{d^N} \Delta^2_{d^{N-1}} \cdots \Delta^2_1 x^{2(N+1)+k} \\
&= \Delta^2_{d^N} \Bigg( (2(N+1)+k)_{2N} \cdot d^{N(N-1)} \Bigg[ x^{2+k} \\
&\qquad\qquad + \frac{(2+k)(1+k)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)}\right) x^k + O\left(x^{k-2}\right) \Bigg] \Bigg) \\
&= (2(N+1)+k)_{2N} \cdot d^{N(N-1)} \Bigg[ \left( (2+k)(1+k) d^{2N} x^k + \frac{(2+k)_4}{3 \cdot 4} d^{4N} x^{k-2} \right) \\
&\qquad + \frac{(2+k)(1+k)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)}\right) \cdot k(k-1) d^{2N} x^{k-2} + O\left(x^{k-4}\right) \Bigg] \\
&= (2(N+1)+k)_{2N+2} \cdot d^{N(N+1)} \Bigg[ x^k + \frac{k(k-1)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)} + d^{2N}\right) x^{k-2} \\
&\qquad\qquad\qquad\qquad + O\left(x^{k-4}\right) \Bigg].
\end{aligned}$$

□

Applying Lemma 3 (with $d = 3$ and $k = 3$) to (2) gives

$$S_N(u) = (2N+3)_{2N} \cdot 3^{N(N-1)} \left( u^3 + \frac{3 \cdot 2}{3 \cdot 4} \frac{3^{2N} - 1}{3^2 - 1} u \right).$$

The cubic $u^3 + \frac{3^{2N}-1}{16}u$ has the roots $u \in \left\{0, \pm\dfrac{\sqrt{3^{2N}-1}}{4}i\right\}$. Finally, these are translated to roots of the original expression in $z$ using (1).

**Solution 2:** For nonnegative integers $n$, let

$$g_n(z) = \frac{1}{(2n+5)!}\sum_{k=0}^{3^n-1}(-2)^{f(k)}(z+k)^{2n+5},$$

and notice that the equation to be solved is $(2023!)g''_{1010}(z) = 0$. Let $c_n = (3^n-1)/2$. We will prove by induction on $n$ that

$$g''_n(z) = 3^{n^2-n}\left(\frac{(z+c_n)^3}{6} + \frac{(3^{2n}-1)(z+c_n)}{96}\right).$$

Since $g_0(z) = z^5/5!$, we have $g''_0(z) = z^3/3! = (z+c_0)^3/6$, verifying the base case. Assume now the formula for $g''_n$ above holds for a particular $n \geq 0$. Observe that $f(j+3^n) = f(j)+1$ and $f(j+2\cdot 3^n) = f(j)$ for $0 \leq j < 3^n$. Then, with the substitutions $\ell = k - 3^n$ and $m = k - 2\cdot 3^n$ in the corresponding sums below,

$$\begin{aligned}
g''_{n+1}(z) &= \frac{(2n+7)(2n+6)}{(2n+7)!}\sum_{k=0}^{3^{n+1}-1}(-2)^{f(k)}(z+k)^{2n+5} \\
&= \frac{1}{(2n+5)!}\sum_{k=0}^{3^n-1}(-2)^{f(k)}(z+k)^{2n+5} + \frac{1}{(2n+5)!}\sum_{\ell=0}^{3^n-1}(-2)^{f(\ell)+1}(z+\ell+3^n)^{2n+5} \\
&\quad + \frac{1}{(2n+5)!}\sum_{m=0}^{3^n-1}(-2)^{f(m)}(z+m+2\cdot 3^n)^{2n+5} \\
&= g_n(z) - 2g_n(z+3^n) + g_n(z+2\cdot 3^n) \\
&= \int_0^{3^n} g'_n(z+3^n+t)dt - \int_0^{3^n} g'_n(z+t)dt = \int_0^{3^n}\int_t^{3^n+t} g''_n(z+s)ds\,dt \\
&= \int_0^{2\cdot 3^n}\int_{\max(0,s-3^n)}^{\min(3^n,s)} g''_n(z+s)dt\,ds \\
&= \int_0^{2\cdot 3^n}(3^n - |s-3^n|)g''_n(z+s)ds = \int_{-3^n}^{3^n}(3^n-|u|)g''_n(z+u+3^n)du \\
&= 3^{n^2-n}\int_{-3^n}^{3^n}(3^n-|u|)\left(\frac{(z+u+3^n+c_n)^3}{6} + \frac{(3^{2n}-1)(z+u+3^n+c_n)}{96}\right)dt.
\end{aligned}$$

The integrand is $3^n - |u|$ times a cubic polynomial of $u$. Since $3^n - |u|$ is an even function of $u$, its integral from $-3^n$ to $3^n$ times an odd power of $u$ is zero, so we can eliminate the cubic and linear terms from the cubic polynomial of $u$. Having done so, the integrand becomes an even function of $u$, which we can integrate instead from 0 to $3^n$ and double the result. Using also the fact that $3^n + c_n = c_{n+1}$, we have

$$\begin{aligned}
g''_{n+1}(z) &= 2 \cdot 3^{n^2-n} \int_0^{3^n} (3^n - u) \left( 3u^2 \frac{(z + c_{n+1})}{6} + \frac{(z + c_{n+1})^3}{6} + \frac{(3^{2n} - 1)(z + c_{n+1})}{96} \right) du \\
&= 2 \cdot 3^{n^2-n} \left( \frac{3^{4n}}{4} \frac{(z + c_{n+1})}{6} + \frac{3^{2n}}{2} \left( \frac{(z + c_{n+1})^3}{6} + \frac{(3^{2n} - 1)(z + c_{n+1})}{96} \right) \right) \\
&= 3^{2n} \cdot 3^{n^2-n} \left( \frac{(z + c_{n+1})^3}{6} + \left( \frac{3^{2n}}{12} + \frac{(3^{2n} - 1)}{96} \right) (z + c_{n+1}) \right) \\
&= 3^{(n+1)^2-(n+1)} \left( \frac{(z + c_{n+1})^3}{6} + \frac{3^{2n+2} - 1}{96} (z + c_{n+1}) \right),
\end{aligned}$$

completing the induction.

Recall that the answers to the problem are the roots of $g''_{1010}$, which are those $z$ for which $z + c_{1010} = 0$ or $(z + c_{1010})^2 = -(3^{2020} - 1)/16$, yielding the answers given above.

**Solution 3:** For a sequence of integers $b$, set $p(b) = \#\{i : b_i = 0\}$. For nonnegative integers $n, m$, define

$$h_{n,m}(y) = \sum_{b \in \{-1,0,1\}^n} (-2)^{p(b)} \left( y + \sum_{i=0}^{n-1} b_i 3^i \right)^m$$

As described in Solution 1, the polynomial in the problem is $h_{n,m}(y)$ for $n = 1010, m = 2n+3$ and $y = z + 1 + 3 + \cdots + 3^{n-1} = z + \frac{3^n - 1}{2}$.

We have that $p(b) = n - \sum_i |b_i|$, and we can expand the polynomial by the Binomial and Multinomial Formulas as follows:

$$\begin{aligned}
h_{n,m}(y) &= \sum_{b \in \{-1,0,1\}^n} (-2)^{n - \sum |b_i|} \sum_{k=0}^{m} \binom{m}{k} y^{m-k} \left( \sum_{i=0}^{n-1} b_i 3^i \right)^k \\
&= \sum_{k=0}^{m} \binom{m}{k} y^{m-k} (-2)^n \sum_{b \in \{-1,0,1\}^n} (-2)^{-\sum_i |b_i|} \sum_{\substack{a_0 + \cdots + a_{n-1} = k \\ a_j \geq 0}} \binom{k}{a_0, a_1, \ldots} \prod_{i=0}^{n-1} (b_i 3^i)^{a_i} \\
&= \sum_{k=0}^{m} \binom{m}{k} y^{m-k} (-2)^n \sum_{\substack{a_0 + \cdots + a_{n-1} = k \\ a_j \geq 0}} \binom{k}{a_0, a_1, \ldots} \sum_{b_0 \in \{-1,0,1\}} \cdots \sum_{b_{n-1} \in \{-1,0,1\}} \prod_{i=0}^{n-1} (-2)^{-|b_i|} (b_i 3^i)^{a_i} \\
&= \sum_{k=0}^{m} \binom{m}{k} y^{m-k} (-2)^n \sum_{\substack{a_0 + \cdots + a_{n-1} = k \\ a_j \geq 0}} \binom{k}{a_0, a_1, \ldots} \prod_{i=0}^{n-1} \left( \sum_{b_i \in \{-1,0,1\}} (-2)^{-|b_i|} (b_i 3^i)^{a_i} \right)
\end{aligned}$$

Notice that the terms in the final parentheses evaluate as (recall that the correct convention for powers of 0 in multinomial expansions is $0^0 = 1$, and $0^\ell = 0$ for $\ell \geq 1$)

$$\sum_{b_i \in \{-1,0,1\}} (-2)^{-|b_i|} (b_i 3^i)^{a_i} = (-2)^{-1} (-3^i)^{a_i} + 0^{a_i} + (-2)^{-1} (3^i)^{a_i} = \begin{cases} 0 & \text{, if } a_i = 0 \text{ or odd} \\ -3^{a_i i} & \text{, if } a_i \geq 2 \text{ is even} \end{cases}$$

Thus the only nonzero summands can occur only when all $a_i \geq 2$ and are even and so $k = \sum a_i \geq 2n$ and is even. Since $k \leq m = 2n+3$, we have either $k = 2n$ and then $a_i = 2$ for all $i$, or $k = 2m+2$ and then $a_j = 4$ for some $j$ and $a_i = 2$ for all $i \neq j$. The summation simplifies as

$$h_{n,2n+3}(y) = (-2)^n \left( y^3 \binom{2n+3}{2n} \binom{2n}{2, 2, \ldots} \prod_{i=0}^{n-1} (-3^{2i}) + y \binom{2n+3}{2n+2} \binom{2n+2}{4, 2, 2, \cdots} \sum_{j=0}^{n-1} (-3^{4j}) \prod_{\substack{0 \leq i \leq n-1 \\ i \neq j}} (-3^{2i}) \right)$$

$$= (-2)^n (-1)^n \prod_{i=0}^{n-1} 3^{2i} \left( \binom{2n+3}{3} \frac{(2n)!}{2^n} y^3 + y(2n+3) \frac{(2n+2)!}{12 \cdot 2^n} \sum_{j=0}^{n-1} 3^{2j} \right)$$

Factoring out $\frac{(2n+3)!}{6 \cdot 2^n}$ and simplifying the geometric sum over $j$ we are left with looking for the solutions of

$$y^3 + \frac{3^{2n} - 1}{16} y = 0,$$

whose roots are $\{0, \pm i \frac{\sqrt{3^{2n}-1}}{4}\}$. Finally, recalling that $z = y - \frac{3^n - 1}{2}$, the roots of the original polynomial are then $\{-\frac{3^n-1}{2}, -\frac{3^n-1}{2} \pm i \frac{\sqrt{3^{2n}-1}}{4}\}$ with $n = 2020$.

**A6.** Alice and Bob play a game in which they take turns choosing integers from 1 to $n$. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the $n$ integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the $n$th turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon \text{the number } k \text{ was chosen on the } k\text{th turn}\}$ matches his goal. For which values of $n$ does Bob have a winning strategy?

**Answer:** Bob can always win by choosing the goal that matches the parity of $n$.

**Solution 1:** We say that $k$ is a "fixed point" if $k$ is chosen on the $k$th turn.

If $n$ is even, then Bob can win by following a simple "mirror" strategy. Divide the available numbers into adjacent pairs $(1,2),(3,4),\ldots,(n-1,n)$. Whenever Alice chooses a number from some pair, Bob chooses the other number from the pair on his turn. If on turn $2j+1$ Alice chooses $2j+1$, then Bob also creates a fixed point on turn $2j+2$, thereby adding two to the total number of fixed points. Otherwise, Alice and Bob add zero fixed points on turns $2j+1$ and $2j+2$. There are therefore an even number of fixed points after each of Bob's turns, and since the game ends after the $n$th turn, which is Bob's, he wins the game.

Now suppose that $n = 2m+1$ is odd. For the remainder of the proof, denote Alice's choices by $A_1, A_3, \ldots, A_{2m+1}$, and Bob's by $B_2, \ldots, B_{2m}$. In particular, on the first turn of the game, Alice chooses $A_1$, followed by Bob choosing $B_2$, and so on, and all of the $A$s and $B$s must be distinct integers from 1 to $2m+1$. Let $F_k$ be the number of fixed points after $k$ turns, reduced modulo 2. We have $F_0 = 0$, and Bob wins if $F_n = 1$.

If $n = 1$, then Bob clearly wins. Otherwise, for $m \geq 1$ we claim that Bob wins by playing according to the following rules. Throughout $2j$ will denote Bob's current turn in the game, beginning by applying Rule R1 on turn 2, followed by whichever of Rule R2 or R3 applies on turn $2j$ for $j \geq 2$. The rule statements include several assumed properties that will be justified inductively later, most importantly that $F_{2j-2} = 1$ for $j \geq 2$.

(R1) (a) If $A_1$ is in $\{1,2\}$, then Bob chooses the other integer in this pair as $B_2$. This results in either 0 or 2 fixed points, so $F_2 = 0$. Now rename the remaining integers $3,\ldots,n$ to $1,\ldots,n-2$, and inductively restart the game.

(b) If $A_1 = a \geq 3$, then Bob chooses $B_2 = 2$, so that $F_2 = 1$. If $n = 3$, then $a = 3$ and the game ends with the forced value $A_3 = 1$, so $F_3 = 1$. Otherwise, proceed to the following rules.

(R2) If $j \geq 2$ and $A_{2j-1} = 2j-1$, then (we will show later that $2j \leq a$ in this case):

(a) If $2j < a$, Bob chooses $B_{2j} = 2j$. Then $F_{2j} = F_{2j-2} + 2 \mod 2 = 1$.

(b) If $2j = a$, then Bob chooses $B_{2j} = 1$. Then $F_{2j} = F_{2j-2} + 1 \mod 2 = 0$. Rename the remaining integers $a+1,\ldots,n$ to $1,\ldots,n-a$ and inductively restart the game.

(R3) If $j \geq 2$ and $A_{2j-1} \neq 2j-1$, then Bob chooses $B_{2j} = 2j+1$ if it has not been previously chosen; otherwise, Bob chooses $B_{2j}$ to be an arbitrary unchosen integer not equal to $2j$. Then $F_{2j} = F_{2j-2} = 1$.

Since Rules R1b, R2a, and R3 all end with $F_{2j} = 1$, and the other rules result in a restart that resets $j$ to 1, the assumption that $F_{2j-2} = 1$ for $j \geq 2$ is true by induction. Bob's last turn occurs when $j = m$. If this turn results in a restart (from Rule R1a or R2b), then $F_{2m} = 0$, and Alice is forced to choose a fixed point on her final turn, so $F_n = 1$ and Bob wins. Otherwise, Rule R1b or R2a or R3 applies on Bob's last turn, and in each case $F_{2m} = 1$. Rule R1b explains why $F_n = 1$ in that case. If Rule R2a applies on turn $2m$, then $a = n = 2m + 1$, so Alice cannot choose $A_n = n$, and $F_n = 1$. If Rule R3 applies on turn $2m$, then either Bob chooses $B_{2m} = 2m + 1$, or $2m + 1$ has already been chosen; again, Alice cannot choose $A_n = n$, and $F_n = 1$.

To complete the proof, we need to verify that the strategy above respects the rules of the game, that the case $2j > a$ never occurs when Rule R2 applies, and that the inductive restart in Rule R2b is valid. In the arguments below, we assume that Bob has been able to apply the rules on all turns before the turn in question.

**Claim.** *If Rule R3 applies, then Bob is able to follow its instructions. Further, Rule R3 will apply on all of Bob's remaining turns.*

*Proof.* If $2j + 1$ has already been chosen, then since there are $n - (2j - 1) = 2m - 2j + 2 \geq 2$ remaining integers, there is at least one choice for $B_{2j}$ other than $2j$. After Bob's turn, Alice cannot choose $A_{2j+1} = 2j + 1$. Thus, Rule R3 applies on turn $2j + 2$, and by induction it applies on all of Bob's subsequent turns. $\square$

It follows that if Rule R2 applies, then Rule R3 could not have previously been applied. Thus, after the most recent restart (if any), Rule R1b must have been applied on turn 2, and Rule R2a must have been applied on turn $2i$ for $2 \leq i < j$.

**Claim.** *If Rule R2 applies, then $2j \leq a$. Further, if Rule R2a applies, then Bob is able to choose $B_{2j} = 2j$. If Rule R2b applies, then $a$ is even, and all integers from 1 to $a$ are chosen on turns 1 to $a$.*

*Proof.* After Rule R1b was applied, 2 and $a$ had been chosen. Since Rule R2 has applied ever since, Alice has chosen all of the odd numbers from 3 to $2j - 1$, and since Rule R2a was applied on the previous turns, Bob has chosen all of the even numbers from 4 to $2j - 2$. Since $a$ is different from the other chosen numbers, and $a \neq 1$, we must have $a \geq 2j$. If Rule R2a applies, $2j \neq a$, so Bob can choose $B_{2j} = 2j$. If Rule R2b applies, then Alice chose $A_1 = a = 2j$ on her first turn, and Bob chooses $B_{2j} = 1$. Then all of the numbers from 1 to $a$ (and only those numbers) have been chosen. $\square$

**Solution 2:** Let $S = \{k : \text{the number } k \text{ was chosen on the } k \text{ th turn}\}$; at the beginning of the game, $S$ is empty, and each turn either adds an element to $S$ or keeps $S$ the same. Call a number "available" if it hasn't been chosen yet.

If $n$ is even and Bob chooses the "even" goal, then Bob can win by following the rules below on the $k$th turn (where $k$ is even).

(1) If all numbers greater than $k$ are available, then Bob chooses the one remaining available number in $\{1, \ldots, k\}$. This rule always applies when $k = n$.

(2) If $k \leq n - 2$ and Rule 1 doesn't apply, then Bob chooses $k + 1$ if available; if not, he chooses an available number greater than $k + 1$ if possible; otherwise, Bob can choose any remaining value other than $k$ (since $k < n$, there is more than one available number for Bob to choose).

**Claim.** *If $k$ is even, then after $k$ turns, $S$ has an even number of elements, and either* 0 *or at least* 2 *of the numbers greater than $k$ have been chosen.*

The proof is by induction on even values of $k$ from 0 to $n$, the base case $k = 0$ being the beginning of the game with 0 elements in $S$ and 0 numbers chosen. Assume now that the claim hold for some even $k < n$.

If $k = 0$ or if Bob applies Rule 1 on the $k$th turn, then all numbers from 1 to $k$ are chosen before Alice makes the $(k + 1)$st turn. If Alice chooses $k + 1$ (adding an element to $S$) or $k + 2$, then Bob applies Rule 1 on the $(k + 2)$nd turn to choose $k + 2$ (adding an element to $S$) or $k + 1$, keeping an even number of elements in $S$ and leaving 0 chosen numbers greater than $k + 2$. If, on the other hand, Alice chooses a number greater than $k + 2$, then exactly 1 number greater than $k + 2$ has been chosen before Bob makes the $(k + 2)$nd turn. Bob must then apply Rule 2, and since $k + 2 \leq n - 2$ in that case, there is still an available number greater than $k + 2$ for him to choose. Thus, Alice's and Bob's turns add no elements to $S$, and result in 2 chosen numbers greater than $k + 2$.

If $k > 0$ and Bob applies Rule 2 on the $k$th turn, then either Bob chooses $k + 1$, or $k + 1$ was previously chosen; either way, Alice can't add an element to $S$ on the $(k + 1)$st turn. Since Rule 1 didn't apply on the $k$th turn, by the inductive hypothesis there were already at least 2 chosen numbers greater than $k$, and thus at least 1 chosen number greater than $k + 1$. If before Bob makes the $(k + 2)$nd turn, the only chosen number greater than $k + 1$ is $k + 2$, then Bob applies Rule 1 and chooses a number strictly less than $k + 2$; this adds no new element to $S$, and leaves 0 chosen numbers greater than $k$. Otherwise, there is at least 1 chosen number greater than $k + 2$ before Bob makes the $(k + 2)$nd turn, so Bob applies Rule 2, which never adds an element to $S$. Either he chooses a number greater than $k + 2$, making at least 2 such numbers chosen, or else all of the numbers (and in particular, at least 2 numbers) greater than $k + 2$ were already chosen before Bob's turn.

This completes the induction that proves the claim. Applying the claim with $k = n$ shows that Bob wins when $n$ is even.

If $n$ is odd and Bob chooses the "odd" goal, then Bob can win by following the rules below on the $k$th turn; since $k$ is even, $k < n$.

(1) If $S$ has an odd number of elements, then Bob chooses $k + 1$ if available; otherwise, Bob can choose any number other than $k$ (since $k < n$, Bob must have an option other than $k$).

(2) If $S$ has an even number of elements, then Bob chooses $k$ if available; otherwise, Bob chooses a number less than $k$ (since only $k - 1$ numbers are chosen before Bob's turn, there must be an available number less than or equal to $k$).

If Bob applies Rule 1, then he doesn't add an element to $S$, and Alice can't add an element to $S$ on the $(k + 1)$st turn, so the number of elements in $S$ remains odd after the $(k + 1)st$ turn. By induction, Bob applies Rule 1 on all future turns, and the number of elements in $S$ remains the same for the rest of the game, so Bob wins.

Assume hereafter that Bob is never able to apply Rule 1 for the entire game.

**Claim.** *If $k$ is even, then after $k$ turns, either $S$ has an even number of elements and no numbers greater than $k$ have been chosen, or $S$ has an odd number of elements and exactly one number greater than $k$ has been chosen.*

In the base case $k = 0$, there are 0 elements in $S$ and no elements at all have been chosen. Proceeding by induction, assume that the claim holds for some even $k < n - 1$.

If after $k$ turns $S$ has an even number of elements, then by the inductive hypothesis all numbers less than or equal to $k$ have been chosen so far, and only those numbers. To prevent Bob from applying Rule 1, Alice must not choose $k+1$ on the $(k+1)$st turn. If Alice chooses $k + 2$, Bob chooses $k + 1$ on the $(k + 2)nd$ turn; then $S$ still has an even number of elements and no numbers greater than $k+2$ have been chosen. If Alice chooses a number greater than $k + 2$, Bob chooses $k + 2$, adding an element to $S$; then $S$ has an odd number of elements, and exactly one number greater than $k + 2$ has been chosen.

If after $k$ turns $S$ has an odd number of elements, then Alice must choose $k + 1$ on the $(k+1)$st turn, adding an element to $S$ to prevent Bob from applying Rule 1. By the inductive hypothesis, exactly one other number greater than $k$ was chosen before Alice's turn, and we now know that number couldn't have been $k+1$. If that number is $k+2$, then all but one of the numbers from 1 to $k + 2$ have been chosen prior to the $(k + 2)$nd turn, and Bob chooses the remaining avalaible number in that range; then $S$ has an even number of elements, and no numbers greater than $k + 2$ have been chosen. If, on the other hand, a number greater than $k + 2$ was previously chosen, Bob chooses $k + 2$, adding another element to $S$; then $S$ has an odd number of elements, and exactly on number greater than $k + 2$ has been chosen.

This completes the induction that proves the claim. Applying the claim with $k = n - 1$, either $S$ has an even number of elements and Alice is forced to choose $n$ on the $n$th turn, adding an element to $S$, or $S$ has an odd number of elements and $n$ is already chosen, so Alice is unable to add an element to $S$ on the $n$th turn; either way, Bob wins.

# 2023 Session B

**B1.** Consider an $m$-by-$n$ grid of unit squares, indexed by $(i,j)$ with $1 \le i \le m$ and $1 \le j \le n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares $(i,j)$ with $1 \le i \le m-1$ and $1 \le j \le n-1$. If a coin occupies the square $(i,j)$ with $i \le m-1$ and $j \le n-1$ and the squares $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$ are unoccupied, then a legal move is to slide the coin from $(i,j)$ to $(i+1,j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

**Answer:** $\displaystyle\binom{m+n-2}{m-1}$

**Solution 1:** Think of $(1,n)$ as the northwest corner of the grid and $(m,1)$ as the southeast corner. Consider the unoccupied squares. Initially, they consist of all the squares in the north row ($j=n$) and/or the east column ($i=m$). We think of these squares as forming a lattice path from $(1,n)$ to $(m,1)$, and we denote this path by $\underbrace{E\ldots E}_{m-1}\underbrace{S\ldots S}_{n-1}$, representing the sequence of eastward and southward steps that traverse the path from northwest corner to southeast corner.

**Claim.** *The unoccupied squares always form a lattice path from $(1,n)$ to $(m,1)$, with a total of $m-1$ eastward steps and $n-1$ southward steps, and all such lattice paths can be achieved by a sequence of legal moves.*

*Proof.* The squares $(1,n)$ and $(m,1)$ always remain unoccupied, because no legal move can slide a coin to either square. A legal move changes an $ES$ portion (consisting of three unoccupied squares that make the move legal) of an unoccupied lattice path to $SE$, so by induction, after every move the unoccupied squares continue to form a lattice path consisting of the same number of $E$ and $S$ steps.

On the other hand, given a lattice path of $E$ and $S$ steps from $(1,n)$ to $(m,1)$, which must consist of $m+n-1$ squares, consider the configuration with coins on every square not on the path. If the sequence of steps does not contain an $SE$ portion, then we are in the initial configuration. Otherwise, choose any $SE$ portion; the three squares connected by the $SE$ steps must have coordinates $(i,j+1)$, $(i,j)$, and $(i+1,j)$, and there must be a coin in square $(i+1,j+1)$. Sliding that coin to square $(i,j)$ is the reverse of a legal move, and changes $SE$ to $ES$ in the path. Continue to change instances of $SE$ to $ES$ in this way until the initial configuration is reached. Reversing the sequence of coin slides made yields a sequence of legal moves from the initial configuration to the given configuration. $\square$

Thus, the total number of configurations is the number of possible lattice paths, which is the number of different sequences of $m-1$ $E$'s and $n-1$ $S$'s; this number is $\binom{m+n-2}{m-1}$.

**B2.** For each positive integer $n$, let $k(n)$ be the number of ones in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?

**Answer:** 3

**Solution 1:** Clearly $k(n)$ must be more than one, since no power of 2 is a multiple of $2023 = 7 \cdot 17^2$. Furthermore, powers of 2 are all congruent to $1, 2,$ or $4 \pmod 7$, and no sum of two of these residues can be zero. So $k(n)$ must be at least three. We now show that this can be achieved.

Note that the binary representations of both prime power divisors of 2023 have exactly three ones, as $7 = 2^2+2^1+2^0$ and $17^2 = 289 = 2^8+2^5+2^0$. It is now possible to piece together three powers of 2 whose sum is simultaneously a multiple of 7 and 289 using the Chinese Remainder Theorem, since the multiplicative order of 2 modulo 7 is 3, which is coprime to $16 \cdot 17 = 272$, the order of the group of multiplication modulo 289 (from which it follows that $2^{272} \equiv 1 \pmod{289}$, by Euler/Lagrange's Theorem).

In particular, the triple $(0, 277, 8)$ is congruent to $(0, 1, 2) \pmod 3$ and $(0, 5, 8) \pmod{272}$, so $2^{277} + 2^8 + 2^0$ is a multiple of 2023. (Another natural solution here is $2^{280} + 2^5 + 2^0$.)

**Solution 2:** As in Solution 1, $k(n) \geq 3$ for all $n$. This solution gives an alternative simple approach for constructing $n$ that achieve $k(n) = 3$.

Since $2^4 = 17 - 1$, raising both sides to the 17th power and using the binomial theorem yields $2^{68} \equiv -1 \pmod{17^2}$. Also, squaring both sides yields $2^{136} \equiv 1 \pmod{17^2}$. Since $2^{68} = 2^{67}+2^{67}$, we therefore have $2^{67}+2^{67}+2^0 \equiv 0 \pmod{17^2}$. As in Solution 1, $2^i+2^j+2^k \equiv 0 \pmod 7$ if and only if $\{i, j, k\} \equiv \{0, 1, 2\} \pmod 3$. Since $136 \equiv 1 \pmod 3$, this can be achieved by $2^{203} + 2^{67} + 2^0$, which is a multiple of 2023.

As a minor variant, we similarly have $2^{69} + 2^0 + 2^0 \equiv -2 + 1 + 1 \equiv 0 \pmod{17^2}$, which naturally leads to the following solutions: $2^{69}+2^{136}+2^{272}$, $2^{205}+2^{272}+2^0$, and $2^{341}+2^{136}+2^0$.

**Solution 3:** This solution provides an exhaustive description of all possible triples of non-negative integers $(i, j, k)$ such that $2^i + 2^j + 2^k \equiv 0 \pmod{2023}$. All solutions can be reduced to a canonical form by factoring out (and removing) common powers of 2, and without loss of generality the terms may be ordered as $2^i+2^j+2^0$, where $i \geq j$. The Chinese Remainder Theorem implies that the exponents $i$ and $j$ may be taken as residues modulo $\operatorname{lcm}[3, 136] = 408$ (note that the calculations in Solution 2 imply that the multiplicative order of 2 modulo $17^2$ is 136; this is not necessary for the construction of single examples as in that solution, but it is needed here in order to define the shape of canonical solutions). With these restrictions, we will show that there are exactly 51 unique solutions, which are listed at the end.

In fact, all solutions lie in two simple families. Indeed, any $(i, j)$ such that $2^i + 2^j \equiv -1 \pmod{2023}$ also projects to a solution modulo 17. The order of 2 modulo 17 is 8, and there are two solutions: $(3, 3)$ and $(5, 0)$. We now must check whether these lift to solutions modulo $17^2$.

In the first case, such a lift has the form

$$2^{8\alpha+3} + 2^{8\beta+3} \equiv -1 \pmod{17^2}$$

for some integers $0 \leq \alpha, \beta \leq 16$. The Binomial Theorem implies that

$$2^{8\alpha} \equiv (15 \cdot 17 + 1)^\alpha \equiv 1 + \alpha \cdot 15 \cdot 17 \pmod{17^2}.$$

Plugging this in, the above equation reduces to

$$(8(\alpha+\beta)\cdot 15+1)\cdot 17 \equiv 0 \pmod{17^2} \iff 8(\alpha+\beta)\cdot 15+1 \equiv 0 \pmod{17},$$

which reduces to $\alpha+\beta+1 \equiv 0 \pmod{17}$. Since $\alpha, \beta$ play a symmetric role, the possible sets of solutions modulo $17^2$ are spanned by choosing $0 \le \alpha \le 8$ and $\beta = 16-\alpha$. To recover solutions modulo 2023, multiples of 136 must be added to $(8\alpha+3, 8\beta+3)$ until the values are congruent to $\{1,2\} \pmod 3$. For $\alpha < 8$ there are two distinct ways of doing this, but for $\alpha = 8$ there is only one, making for 17 total solutions. (In fact, when $\alpha = 8$ we recover the congruence from Solution 2, namely $2^{67}+2^{67}+2^0 \equiv 0 \pmod{17^2}$, which lifts uniquely to $(203, 67)$).

For the case $(5, 0)$, the calculations are similar. We find that $2^{8\alpha+5}+2^{8\beta} \equiv -1 \pmod{17^2}$ has solutions parameterized by $\beta = 1+2\alpha \mod 17$. Now each such pair lifts to two solutions modulo 2023, since $\alpha$ and $\beta$ are no longer symmetric, so this gives a total of 34 solutions. For example, $\alpha = 0, \beta = 1$ corresponds to $2^5+2^8+2^0 \equiv 1 \pmod{17^2}$ from Solution 1, and $\alpha = 8, \beta = 0$ corresponds to $2^{69}+2^0+2^0$ from Solution 2.

The set of all reduced $(i, j)$ is as follows:

$$\begin{aligned}
&(77,16),(85,32),(101,64),(109,80),(125,112),(133,128),(139,131),(155,115),\\
&(160,149),(163,107),(176,157),(179,91),(187,83),(203,67),(208,173),(211,59),\\
&(224,181),(227,43),(235,35),(251,19),(256,197),(259,11),(272,205),(277,8),\\
&(280,5),(293,40),(296,13),(301,56),(304,221),(317,88),(320,229),(325,104),\\
&(328,29),(341,136),(344,37),(347,331),(349,152),(352,245),(355,323),(365,184),\\
&(368,253),(371,307),(373,200),(376,53),(379,299),(389,232),(392,61),(395,283),\\
&(397,248),(400,269),(403,275).
\end{aligned}$$

**B3.** A sequence $y_1, y_2, \ldots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \ldots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \ldots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \ldots, X_n)$ be the largest value of $k$ for which there exists an increasing sequence of integers $i_1, i_2, \ldots, i_k$ such that $X_{i_1}, X_{i_2}, \ldots X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \ldots, X_n)$ for $n \geq 2$.

**Answer:** $(2n + 2)/3$ for $n \geq 2$.

**Solution 1:** We begin by noting that with probability one, the $X_i$ are all distinct. Indeed, the event in which there are at least two identical values defines a finite collection of hyperplanes in $[0, 1]^n$, which has measure zero. Furthermore, since we consider only relative order , we can translate the problem to computing $a(w)$, where $w = w_1 \ldots w_n$ is a permutation of $[1, \ldots, n]$ chosen uniformly at random.

Let $u(w)$ be the length of maximal zigzag sequence starting with a descent, and let $d(w)$ be the maximal zigzag sequence starting with an ascent. Then $a(w) = \max\{u(w), d(w)\}$ and moreover, for $n \geq 2$ we see that $u(w) = d(w) \pm 1$, as we can obtain one from the other by adding/removing an initial element. By symmetry we see that $u(w)$ and $d(w)$ are identically distributed and exactly half of the time $u(w)$ will be the larger. Thus, by linearity of expectation, we have

$$\mathbf{E}[a(w)] = \mathbf{E}[u(w)] + \frac{1}{2}\mathbf{E}[1] = \mathbf{E}[u(w)] + \frac{1}{2} = \mathbf{E}[d(w)] + \frac{1}{2}.$$

We claim that a maximal zigzag subsequence can always be chosen containing the value $n$. If a zigzag subsequence has $w_r = n$ where $i_j < r < i_{j+1}$, then we can reassign $i_j = r$ if $w_{i_j} > w_{i_{j+1}}$ or $i_{j+1} = r$ if $w_{i_j} < w_{i_{j+1}}$, resulting either way in a zigzag subsequence of the same length containing $n$. If $r < i_1$, we can reassign $i_1 = r$ if $w_{i_1} > w_{i_2}$ or prepend $w_r$ to $w_{i_1}$ if $w_{i_1} < w_{i_2}$, resulting an a zigzag subsequence of the same length or longer. The case $r > i_k$, where $k$ is the length of the subsequence, is similar.

Let $S_n$ denote the permutations of $[1, \ldots, n]$, and define

$$f_n := \mathbf{E}[d(w)|w \in S_n], \qquad g_n := \mathbf{E}[a(w)|w \in S_n].$$

By the above reasoning we have $g_n = f_n + \frac{1}{2}$.

Let $w \in S_{n+1}$. By the above reasoning, we can assume that the element $n + 1$ is part of a maximal zigzag sequence $w_{i_1} \cdots w_{i_k}$. Suppose that $w_{j+1} = n + 1$ and $i_r = j + 1$. Then $w_{i_{r-1}} < w_{i_{r-2}} > \cdots$ is a maximal UD alternating subsequence (i.e. starting with an ascent) in the reverse permutation $w_j \ldots w_1$, and $w_{i_{r+1}} < w_{i_{r+2}} > \cdots$ is a maximal UD alternating subsequence in $w_{j+2} \ldots w_{n+1}$. Thus we see that, choosing $j$ with probability $\frac{1}{n+1}$, we have

$$g_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} (f_j + f_{n-j} + 1).$$

We have the following boundary values, which are verified directly: $f_0 = 0$, $f_1 = 1$, $f_2 = \frac{3}{2}$ ($d(12) = 2$ and $d(21) = 1$).

Then

$$g_{n+1} = f_{n+1} + \frac{1}{2} = \frac{1}{n+1} \left( 2f_0 + 2f_1 + 2\sum_{j=2}^{n} f_j \right) + 1$$

Rewriting this recursion for $f$ we get

$$f_{n+1} = \frac{2}{n+1}\left(1 + \sum_{j=2}^{n} f_j\right) + \frac{1}{2}.$$

Note that this implies a relation of $f_2 + \cdots + f_n$ via $f_{n+1}$, which can be then reiterated for $n-1$ and substituted

$$\frac{n+1}{2}\left(f_{n+1} - \frac{1}{2}\right) = 1 + f_2 + \cdots + f_{n-1} + f_n \Longrightarrow \frac{n+1}{2}\left(f_{n+1} - \frac{1}{2}\right) = \frac{n}{2}\left(f_n - \frac{1}{2}\right) + f_n$$

Then

$$(n+1)f_{n+1} = (n+2)f_n + \frac{1}{2} \Longrightarrow (n+1)\left(g_{n+1} - \frac{1}{2}\right) = (n+2)\left(g_n - \frac{1}{2}\right) + \frac{1}{2},$$

so

$$(n+1)g_{n+1} = (n+2)g_n \Longrightarrow \frac{g_{n+1}}{n+2} = \frac{g_n}{n+1} = \cdots = \frac{g_2}{3} = \frac{2}{3},$$

which completes the proof.

**Solution 2:** The event that two of the $X_j$'s are equal has probability zero, so in the argument below we assume that no two are equal. For $1 < j < n$, call $j$ a "turning point" if $X_j - X_{j-1}$ and $X_{j+1} - X_j$ have opposite signs, and let $T$ be the total number of turning points.

We claim that $a(X_1, \ldots, X_n) = T + 2$. For $1 \le j < n$, let $D_j = X_{j+1} - X_j$. Suppose that $X_{i_1}, \ldots, X_{i_k}$ is zigzag. For each $1 \le m < k$, we have $X_{i_{m+1}} - X_{i_m} = D_{i_m} + D_{i_m+1} + \cdots + D_{i_{m+1}-1}$. Choose $\ell_m$ with $i_m \le \ell_m < i_{m+1}$ so that $D_{\ell_m}$ has the same sign as $X_{i_{m+1}} - X_{i_m}$. Then the sequence $D_{\ell_1}, D_{\ell_2}, \ldots, D_{\ell_{k-1}}$ alternates sign. Thus, there are at least $k-2$ changes of sign in the sequence $D_1, D_2, \ldots, D_{n-1}$. Each such change of sign is a turning point, so $k - 2 \le T$, and $k \le T + 2$. To see that $k = T + 2$ is possible, let $j_1, j_2, \ldots, j_T$ be the turning points. Then $D_1, D_2, \ldots, D_{n-1}$ changes sign only between $D_{j_m}$ and $D_{j_m - 1}$ for $m = 1, 2, \ldots, T$. It follows that the sequence

$$\begin{aligned} &X_{j_1} - X_1, X_{j_2} - X_{j_1}, \ldots, X_{j_T} - X_{j_{T-1}}, X_n - X_{j_T} \\ &\quad = D_1 + \cdots + D_{j_1 - 1}, Dj_1 + \cdots + D_{j_2 - 1}, \ldots, D_{j_{T-1}} + \cdots + D_{j_T - 1}, D_{j_T} + \cdots + D_{n-1} \end{aligned}$$

alternates sign, so $X_1, X_{j_1}, \ldots, X_{j_T}, X_n$ is a zigzag subsequence with length $T + 2$. This verifies our claim.

For $1 < j < n$, the probability that $j$ is a turning point is $2/3$, since 4 of the 6 equally likely orderings of $X_{j-1}, X_j, X_{j+1}$ yield a turning point. Therefore, $\mathbf{E}[a(X)] = (n-2)2/3 + 2 = (2n+2)/3$.

**Solution 3:** Let $Z_j = a(X_1, \ldots, X_j) - a(X_1, \ldots, X_{j-1})$ for $j \ge 3$. Then for $n \ge 2$, we can write $a(X_1, \ldots, X_n) = a(X_1, X_2) + Z_3 + \cdots + Z_n$, so it suffices to determine

$$\mathbf{E}[a(X_1, X_2) + Z_3 + \cdots + Z_n] = \mathbf{E}[a(X_1, X_2)] + \mathbf{E}[Z_3] + \cdots \mathbf{E}[Z_n].$$

Notice that $\mathbf{E}[a(X_1, X_2)] = 2$ since with probability one either $X_1 > X_2$ or $X_1 < X_2$, and in either case $X_1, X_2$ is zigzag. We claim that $\mathbf{E}[Z_j] = 2/3$ for $j \ge 3$, from which the answer $2 + 2(n-2)/3 = (2n+2)/3$ follows immediately.

To verify the claim, we first excluded the possibility that two of the $X_i$'s are equal, which has probability zero. Next, notice that if $X_{i_1}, \ldots, X_{i_k}$ is zigzag and $i > i_k$, then either $X_{i_1}, \ldots, X_{i_{k-1}}, X_i$ or $X_{i_1}, \ldots, X_{i_k}, X_i$ is zigzag, according (if $k \geq 2$) to whether or not $X_{i_k}$ is between $X_{i_{k-1}}$ and $X_i$. Then if $k = a(X_1, \ldots, X_{j-1})$ and $X_{i_1}, \ldots, X_{i_k}$ is zigzag, $X_{i_1}, \ldots, X_{i_{k-1}}, X_{j-1}$ must also be zigzag (since otherwise there would be a longer zigzag subsequence). Further, $X_{j-1} - X_{j-2}$ must have the same sign as $X_{j-1} - X_{i_{k-1}}$; otherwise, $X_{j-2} - X_{i_{k-1}} = (X_{j-2} - X_{j-1}) + (X_{j-1} - X_{i_{k-1}})$ would have the same sign as $X_{j-2} - X_{j-1}$ and $X_{j-1} - X_{i_{k-1}}$, hence the opposite sign as $X_{j-1} - X_{j-2}$, making a longer zigzag subsequence $X_{i_1}, \ldots, X_{i_{k-1}}, X_{j-2}, X_{j-1}$. Thus, $Z_j = 0$ if $X_j - X_{j-1}$ has the same sign as $X_{j-1} - X_{j-2}$, and $Z_j = 1$ if they have opposite signs. In other words, $Z_j = 0$ if $X_{j-1}$ is between $X_{j-2}$ and $X_j$, and $Z_j = 1$ otherwise. Since $X_{j-1}$ is between $X_{j-2}$ and $X_j$ for 2 of the 6 equally likely orderings of $X_{j-2}, X_{j-1}, X_j$, the probability that $Z_j = 1$ is $4/6 = 2/3$, and $\mathbf{E}[Z_j] = 2/3$ as claimed.

**B4.** For a nonnegative integer $n$ and a strictly increasing sequence of real numbers $t_0, t_1, \ldots, t_n$, let $f(t)$ be the corresponding real-valued function defined for $t \geq t_0$ by the following properties:

(a) $f(t)$ is continuous for $t \geq t_0$, and is twice differentiable for all $t > t_0$ other than $t_1, \ldots, t_n$;

(b) $f(t_0) = 1/2$;

(c) $\lim_{t \to t_k^+} f'(t) = 0$ for $0 \leq k \leq n$;

(d) For $0 \leq k \leq n-1$, we have $f''(t) = k+1$ when $t_k < t < t_{k+1}$, and $f''(t) = n+1$ when $t > t_n$.

Considering all choices of $n$ and $t_0, t_1, \ldots, t_n$ such that $t_k \geq t_{k-1} + 1$ for $1 \leq k \leq n$, what is the least possible value of $T$ for which $f(t_0 + T) = 2023$?

**Answer:** 29

**Solution 1:** Let $T$ be the value for which $f(t_0 + T) = 2023$, and assume without loss of generality that $t_n < t_0 + T$, since greater values do not affect $f(t_0 + T)$. Let $t_{n+1} = t_0 + T$. Notice that for each $1 \leq k \leq n+1$ we have $f(t) = f(t_{k-1}) + k(t - t_{k-1})^2/2$ when $t_{k-1} \leq t \leq t_k$. Let $\tau_k = t_k - t_{k-1}$; then $\tau_k \geq 1$ for $1 \leq k \leq n$ and $\tau_{n+1} \geq 0$. Let $m = n+1$. The goal is to minimize $T = \tau_1 + \tau_2 + \cdots + \tau_m$ subject to the constraint

$$C(\tau) := \sum_{k=1}^{m} k\tau_k^2 = 2(f(t_0 + T) - f(t_0)) = 4045.$$

The space to be minimized over consists of all $m \geq 1$ and all real $m$-tuples $(\tau_1, \ldots, \tau_m)$ with $\tau_k \geq 1 - \delta_{mk}$, where $\delta_{mk}$ is the Kronecker delta. This space can be made topologically connected with the identification $(\tau_1, \ldots, \tau_{m-1}, 0) = (\tau_1, \ldots, \tau_{m-1})$ for $m \geq 2$. The subset of this space that satisfies the constraint is bounded and hence compact, because the constraint excludes values of $m$ for which $m(m-1)/2 > 4045$ and values of $\tau_k$ greater than $\sqrt{4045}$.

Let $(\tau_1, \tau_2, \cdots, \tau_m)$ lie in the constrained space, so that $C(\tau) = 4045$ and $\tau_k \geq 1 - \delta_{mk}$. If there is more than one value of $k$ for which $\tau_k > 1 - \delta_{mk}$, let two of these values be $k_1$ and $k_2$, and assume without loss of generality that $k_1\tau_{k_1} \geq k_2\tau_{k_2}$. Choose $\varepsilon > 0$ sufficiently small that $\tau_{k_2} - \varepsilon \geq 1 - \delta_{mk_2}$, and consider the $m$-tuple for which $(\tau_{k_1}, \tau_{k_2})$ is replaced by $(\tau_{k_1} + \varepsilon, \tau_{k_2} - \varepsilon)$, and the other $\tau_k$ are unchanged. Then $T$ is unchanged, while the sum $\sum_{k=1}^{m} k\tau_k^2$ in the constraint changes by

$$2(k_1\tau_{k_1} - k_2\tau_{k_2})\varepsilon + (k_1 + k_2)\varepsilon^2,$$

which is (strictly) positive by our earlier assumption. Thus, the sum is now greater than 4045. Next, reduce $\tau_{k_1} + \varepsilon$ until the constraint is satisfied again, and notice that the value that meets this condition is strictly between $\tau_{k_1}$ and $\tau_{k_1} + \varepsilon$ (since $C(\tau)$ is increasing in each $\tau_k$), so it satisfies the same lower bound as $\tau_{k_1}$. The resulting $m$-tuple has a strictly lower value of $T$, so the original $m$-tuple could not have minimized $T$. Thus, the (constrained) minimum of $T$ cannot be achieved with more than one value of $k$ for which $\tau_k > 1 - \delta_{mk}$.

Next, suppose that $T$ is minimized with $\tau_k > 1$ for some $k < m$; then by the argument above, $\tau_m = 0$. In that case, reduce $m$ by 1, and since $\tau_{m-1} > 0$, the above argument implies that $T$ is not at a minimum for the new value of $m$, and hence not a global minimum. By the compactness described above, there must be a global minimum value for $T$, and by what we have argued so far, this minimum must satisfy $\tau_1 = \tau_2 = \cdots = \tau_{m-1} = 1$. Thus, the problem is reduced to minimizing

$$T(m) = m - 1 + \sqrt{\left(4045 - \sum_{k=1}^{m-1} k\right) / m} = m - 1 + \sqrt{4045/m - (m-1)/2}$$

over all $1 \le m \le M$, where $M$ is the greatest integer for which the square root is well-defined. The fact that $M \ge 20$ suffices for the arguments below.

If $m < M$, then $T(m+1) > T(m)$ is equivalent to

$$1 + \sqrt{4045/(m+1) - m/2} > \sqrt{4045/m - (m-1)/2}.$$

Squaring both sides (which we observe are nonnegative) and simplifying, this is equivalent to

$$2\sqrt{4045/(m+1) - m/2} > 4045/[m(m+1)] - 1/2.$$

Notice that the right side is nonnegative when the left side is well-defined (when $m < M$). We square and simplify again to conclude that $T(m+1) > T(m)$ is equivalent to

$$4m > 4045/[m(m+1)] - 1/2 \iff m(m+1)(m+1/8) > 1011.25.$$

Thus, $T(m+1) > T(m)$ for $10 \le m < M$.

Similarly, $T(m+1) < T(m)$ is equivalent to $m(m+1)(m+1/8) < 1011.25$, which is true for $1 \le m \le 9$. We conclude that $T(10) = 9 + \sqrt{404.5 - 4.5} = 29$ is the minimum possible value of $T$.

**Solution 2:** Following the same notation and initial steps as the previous solution, the goal is to minimize $T = \tau_1 + \tau_2 + \cdots + \tau_{n+1}$ subject to the constraint

$$C(\tau) := \sum_{k=1}^{n+1} k\tau_k^2 = 4045,$$

where $n \ge 0$ and $\tau_k \ge 1$ for $k = 1, \dots, n$ and $\tau_{n+1} \ge 0$.

We now make the following substitution: Let $x_k = k\tau_k^2 - k \ge 0$ for $k = 1, \dots, n$ and $x_{n+1} = (n+1)\tau_{n+1}^2$. We have $x_i \ge 0$ and

$$x_{n+1} = 4045 - \binom{n+1}{2} - x_1 - \cdots - x_n \ge 0.$$

Then the goal is to minimize

$$F_n(x_1, \dots, x_n) := \sum_{k=1}^{n} \sqrt{\frac{x_k + k}{k}} + \sqrt{\frac{4045 - \binom{n+1}{2} - x_1 - \cdots - x_n}{n+1}}$$

over the simplex $x_i \ge 0$ and $x_1 + \cdots + x_n \le 4045 - \binom{n+1}{2}$, and over $n \ge 0$ for which this simplex is non-empty. (We include $n = 0$, for which the simplex is undefined but the original constraint requires $\tau_1 = 4045$; in this case, $T = \sqrt{4045}$, so we let $F_0 = \sqrt{4045}$.)

We now analyze the function $g_k(x_k) = F_n(x_1, \ldots, x_n)$ for fixed values of $x_j$, $j \neq k$. We have

$$g_k'(x_k) = \frac{1}{2\sqrt{k(x_k+k)}} - \frac{1}{2\sqrt{(n+1)\left(4045 - \binom{n+1}{2} - x_1 - \cdots - x_n\right)}}$$

$$g_k''(x_k) = -\frac{1}{4\sqrt{k}(x+k)^{3/2}} - \frac{1}{4\sqrt{n+1}\left(4045 - \binom{n+1}{2} - x_1 - \cdots - x_n\right)^{3/2}} < 0,$$

so $g_k$ is a concave function and the minimum is achieved at the boundary. Thus we either have $x_k = 0$ or $x_1 + \cdots + x_n = 4045 - \binom{n+1}{2}$ for every $k$. So the minimum is achieved for some $(x_1, \ldots, x_n)$ either on the hyperplane $H_n := \{x_1 + \cdots + x_n = 4045 - \binom{n+1}{2}\}$ or else we must have $x_k = 0$ for all $k$ so $(x_1, \ldots, x_n) = (0, \ldots, 0)$.

In the first case we have $x_n = 4045 - \binom{n+1}{2} - x_1 - \cdots - x_{n-1}$, so

$$F_n(x_1, \ldots, x_n) = \sum_{k=1}^{n-1} \sqrt{\frac{x_k+k}{k}} + \sqrt{\frac{4045 - \binom{n+1}{2} - x_1 - \cdots - x_{n-1} + n}{n}} = F_{n-1}(x_1, \ldots, x_{n-1}).$$

If $(x_1, \ldots, x_{n-1})$ minimizes $F_{n-1}$ and $(x_1, \ldots, x_{n-1}) \in H_{n-1}$, then we can again reduce to the case of one fewer variable. Recursively, we can eventually reach a minimum in the second case where $(x_1, \ldots, x_N) = (0, \ldots, 0)$, which is vacuously true for $N = 0$ in case we get that far.

Having reduced to the second case, we now need only minimize the function

$$t(N) := f_N(0, \ldots, 0) = N + \sqrt{\frac{4045 - \binom{N+1}{2}}{N+1}} = N + \sqrt{\frac{4045}{N+1} - \frac{N}{2}}$$

over the integers $0 \leq N \leq 89$ (since the square root is undefined for $N \geq 90$). To motivate the next steps, crude estimates show that $t(1) > 40$, $t(3) < 40$, $t(10) < 40$, and $t(30) > 40$. This suggests that the minimum occurs when $\frac{4045}{N+1}$ is considerably larger than $\frac{N}{2}$. To get a rough idea where the minimum occurs, replace $N$ for the moment with a real variable $y$, and consider the approximation $t(y) \approx y + \sqrt{\frac{4045}{y+1}}$. From its derivative, the latter function is minimized when $y = -1 + (4045/4)^{1/3} \approx 9$.

Consider then $t(y) - t(9) = t(y) - 29$; we will find values of $y \geq 0$ for which this is nonpositive. Thus, we are solving for

$$\sqrt{\frac{4045}{y+1} - \frac{y}{2}} \leq 29 - y \iff y \leq 29 \text{ and } \frac{4045}{y+1} - \frac{y}{2} \leq (29-y)^2.$$

The last inequality is equivalent to (using the fact that we have equality for $y = 9$ to factor)

$$\begin{aligned}(y-29)^2(y+1) - 4045 + \frac{y(y+1)}{2} &= 1/2(y-9)(2y^2 - 95y + 712) \\ &= \frac{1}{4}(y-9)\left(y - \frac{95+\sqrt{3329}}{4}\right)\left(y - \frac{95-\sqrt{3329}}{4}\right) \geq 0.\end{aligned}$$

This inequality is true only when $y \geq \frac{95+\sqrt{3329}}{4} > 29$ or when $y$ is (not strictly) between 9 and $\frac{95-\sqrt{3329}}{4}$. The latter number is strictly between 9 and 10 because $55^2 = 3025 < 3329 < 3481 = 59^2$. Thus, $t(y) - t(9) \leq 0$ only when $9 \leq y \leq \frac{95-\sqrt{3329}}{4} < 10$. Then $t(N) \geq t(9)$ for integers $N$, and $t(9) = 29$ is the desired minimum.

**B5.** Determine which positive integers $n$ have the following property: For all integers $m$ that are relatively prime to $n$, there exists a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $\pi(\pi(k)) \equiv mk \pmod{n}$ for all $k \in \{1, 2, \ldots, n\}$.

**Answer:** Those $n$ of the form $4j + 2$ where $j$ is a nonnegative integer, and $n = 1$.

**Solution:** Suppose that $\tau \in S_n$, where $S_n$ is the group of permutations on $n$ elements. We say that a permutation $\pi \in S_n$ is a *square root* of $\tau$ if $\pi^2 = \tau$ (we will shortly see that in general there is not a unique square root, if one exists). If $m$ is an integer coprime to $n$, let $\tau_m \in S_n$ denote the permutation defined by multiplication by $m$ modulo $n$, in other words, $\tau_m(k) = m \cdot k \mod n$ for $1 \leq k \leq n$. The problem statement asks for a classification of the $n$ such that *all* $\tau_m$ have a square root in $S_n$.

Now consider the standard decomposition of $\tau$ into disjoint cycles. For a positive integer $i$ such that $i \leq n$, let $f_i(\tau)$ denote the number of distinct cycles of length $i$.

**Lemma.** *Suppose that $\tau \in S_n$. Then $\tau$ has a square root if and only if $f_{2i}(\tau)$ is even for all $i \leq \frac{n}{2}$.*

*Proof.* If $\ell$ is a positive integer, consider the permutation in $S_n$ defined by a cycle of length $\ell$, say $\gamma = (x_1 x_2 \cdots x_\ell)$. When squared, the resulting permutation either preserves the cycle length, or splits it into two cycles of half the length, depending on the parity of $\ell$:

$$\gamma^2 = \begin{cases} (x_1 x_3 \cdots x_{2k+1} x_2 x_4 \cdots x_{2\lambda}) & \text{if } \ell = 2\lambda + 1, \\ (x_1 x_3 \cdots x_{2\lambda-1})(x_2 x_4 \cdots x_{2\lambda}) & \text{if } \ell = 2\lambda. \end{cases} \tag{3}$$

Applying this fact to the cycle decomposition of a permutation $\pi \in S_n$ implies that (noting that all of the resulting cycles remain disjoint)

$$\begin{aligned} f_{2\lambda+1}(\pi^2) &= f_{2\lambda+1}(\pi) + 2f_{4\lambda+2}(\pi), \\ f_{2\lambda}(\pi^2) &= 2f_{4\lambda}(\pi). \end{aligned}$$

Here we set $f_i(\pi) = 0$ if $i > n$. The forward direction of the claim follows, as $f_i(\pi^2)$ is even for all even $i$.

For the reverse direction, suppose that $\tau$ is a permutation such that $f_{2i}(\tau)$ is even for all $i$. We now construct a square root $\pi$ as a list of cycles, essentially using (3) in reverse. For each cycle in $\tau$ of odd length, say $\gamma = (x_1 x_2 \cdots x_{2\lambda+1})$, append the following length $2\lambda + 1$ cycle to $\pi$:

$$(x_1 x_{\lambda+2} x_2 x_{\lambda+3} x_3 \cdots x_{2\lambda+1} x_{\lambda+1}).$$

By assumption, there are an even number of cycles of even length in $\tau$, so they may be grouped (arbitrarily) in pairs. For each pair of the form $(x_1 x_2 \cdots x_{2\lambda}), (y_1 \cdots y_{2\lambda})$, append the following cycle of length $4\lambda$ to $\pi$:

$$(x_1 y_1 x_2 y_2 \cdots x_{2\lambda} y_{2\lambda}).$$

Now it is immediate that $\pi^2 = \tau$, since by construction they share the same cycle decomposition.

□

This result also shows that even parity of $\tau$ is a necessary condition for the existence of a square root (though not sufficient, as for example, $\tau = (12)(3456)$ is an even permutation without a square root).

**Corollary.** *If $\tau$ is an odd permutation, then it has no square root.*

Returning to the problem at hand, note that the cycles of $\tau_m \in S_n$ are of the form

$$\langle m \rangle \cdot x := (x \; mx \; \cdots m^{\ell-1}x),$$

where $x$ is a positive integer, and $\ell$ is the minimum positive exponent such that $m^\ell x \equiv x \pmod{n}$. This is equivalent to the minimal $\ell$ such that $m^\ell \equiv 1 \pmod{\frac{n}{\gcd(x,n)}}$.

If $n = 4k+2$, then consider an arbitrary value of $m$, which must be odd. In this case, we will show that $f_i(\tau_m)$ is even for all $i$, and therefore $\tau_m$ has a square root by the Lemma. Specifically, we claim that $f_i(\tau_m) = 2o_i(\tau_m)$, where $o_i(\tau_m)$ denotes the number of cycles of length $i$ such that all elements are odd.

To verify this claim, suppose that $x$ is odd. Then $\langle m \rangle \cdot x$ consists of only odd integers (and any such cycle is generated by some odd $x$), and maps bijectively to $\langle m \rangle \cdot 2x$ (a cycle with only even integers), since $x \mapsto 2x \pmod{n}$ bijectively maps the odd residues modulo $n$ to the even residues (noting that $n$ is divisible by 2 but no higher power of 2).

If $n = 1$, then the property trivially holds because all integers are congruent to each other modulo 1, and because there does exist a (trivial) permutation $\pi\colon \{1\} \to \{1\}$.

For all other $n$, we will provide a value $m = m_n$ such that $\tau_{m_n}$ does not have a square root in $S_n$. If $n = 4k$, let $m_n = -1$. Then $f_1(\tau_m) = 2$ (the fixed points being $x = 0$ and $2k$) and $f_2(\tau_m) = 2k-1$, so the conclusion follows by the Lemma. A similar argument also shows that $\tau_{-1}$ does not have a square root if $n = 4k+3$. However, this does not work for $n = 4k+1$, and the case that $n$ is odd can be treated in a unified manner as follows.

Suppose that $n > 1$ is odd, with prime factorization $n = p_1^{e_1} \cdots p_r^{e_r}$, where the $p_i$ are distinct odd primes, and $e_i$ are positive integers. Let $m_1$ be a primitive root of the multiplicative group modulo $p_1^{e_1}$ (cf. Euler, Lagrange, Legendre, Gauss for the existence of such a root). Then $m_1$ is also a primitive root modulo $p_1^e$ for $1 \le e \le e_1 - 1$. Now (using the Chinese Remainder Theorem) set $m$ to be the residue modulo $n$ that satisfies

$$m \equiv m_1 \pmod{p_1^{e_1}}, \quad m \equiv 1 \pmod{p_j^{e_j}} \text{ for } 2 \le j \le r.$$

Then all non-fixed-point cycles of $\tau_m$ have lengths of the form $p_1^{e-1}(p_1 - 1)$ for some $1 \le e \le e_1$. Specifically, if $x$ is a multiple of $p_1^{e_1}$, then $\langle a \rangle \cdot x$ has length 1, and otherwise if $p_1^s$ is the largest power of $p_1$ dividing $x$ (where $0 \le s \le e_1 - 1$), then the length is $p_1^{e_1-s-1}(p_1 - 1)$.

Thus by considering the cycles formed by all $x$'s that are not multiples of $p_1$ (this is the case $s = 0$, but in fact any $s < e_1$ works similarly), we find that

$$f_{p_1^{e_1-1}(p-1)}(\tau_m) = \frac{n\left(1 - \frac{1}{p_1}\right)}{p_1^{e_1-1}(p_1 - 1)} = p_2^{e_2} \cdots p_r^{e_r}.$$

Since this is odd, the Lemma implies that $\tau_m$ is not a square.

*Remark.* The Corollary can also be proven directly (almost immediately using the fact that the sign map is a homomorphism). This gives an alternative criterion for showing that $\tau_{-1}$ does not have a square root when $n \equiv 0, 3 \pmod 4$.

There are many possible constructions for permutation square roots in the case $n = 4k+2$. The proof above implicitly defines a square root of $\tau_m$ by "zipping" together all cycles of the form $(x, mx, m^2x, \dots)$ and $(2x, m \cdot 2x, m^2 \cdot 2x, \dots)$. One alternative is to instead pair cycles of the form $(x, mx, m^2x, \dots)$ and $((x+n'), m(x+n'), m^2(x+n'), \dots)$, where $n' = 2k+1 = \frac{n}{2}$. To be more precise, it is straightforward to show that the following permutation is also a square root of $\tau_m$:

$$\pi(x) := \begin{cases} x + n' \pmod n & \text{if } x \text{ is odd,} \\ mx + n' \pmod n & \text{if } x \text{ is even.} \end{cases}$$

**B6.** Let $n$ be a positive integer. For $i$ and $j$ in $\{1, 2, \ldots, n\}$, let $s(i, j)$ be the number of pairs $(a, b)$ of nonnegative integers satisfying $ai + bj = n$. Let $S$ be the $n$-by-$n$ matrix whose $(i, j)$-entry is $s(i, j)$.

For example, when $n = 5$, we have $S = \begin{bmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$.

Compute the determinant of $S$.

**Answer:** $(-1)^{m+1}2m$ where $m$ is the least integer greater than or equal to $n/2$

**Solution 1:** Let $d(i, j) = 1$ if $i \mid j$ and 0 otherwise, and let $D = (d(i, j))_{(i,j)=(1,0)}^{(n,n)}$ be the corresponding $n \times (n+1)$ matrix whose rows are indexed by $1, \ldots, m$ and whose columns are indexed by $0, 1, \ldots, n$. For example, for $n = 8$, we have

$$D = \begin{pmatrix} & j=0 & j=1 & j=2 & j=3 & j=4 & j=5 & j=6 & j=7 & j=8 \\ i=1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ i=2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ i=3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ i=4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ i=5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ i=6 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ i=7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ i=8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

All possible solutions $(a, b)$ to $ai+bj = n$ in nonnegative integers can be indexed by those $0 \le k \le n$ such that $i \mid k$ and $j \mid n - k$, so

$$s(i, j) = \sum_{k=0}^{n} d(i, k)d(j, n - k).$$

Thus $S = D(D^r)^T$, where $D^r$ is the matrix $D$ with the rows reversed, i.e., $D^r_{i,j} = D_{i,n-j}$.

The Cauchy–Binet formula implies that if $M$ is an $n\times(n + 1)$-matrix and $N$ is an $(n + 1)\times n$-matrix, then

$$\det (MN) = \sum_{k=0}^{n} \det \left(M_{\widehat{k}}\right) \cdot \det \left(N^{\widehat{k}}\right),$$

where $M_{\widehat{k}}$ denotes the matrix $M$ with the $k$-th column removed, and $N^{\widehat{k}}$ denotes the matrix $N$ with the $k$-th row removed.

In the present case we have $M = D$ and $N = (D^r)^T$, and can further reduce

$$\det \left(\left(D^r\right)^T\right)^{\widehat{k}} = \det \left((D^r)_{\widehat{k}}\right) = (-1)^{\lfloor n/2\rfloor} \det D_{\widehat{n-k}}.$$

The first equality is due to the fact that determinants are preserved by transposition, and the sign arises from writing the row reversals as $\lfloor n/2\rfloor$ column swaps. In all, we therefore have

$$\det S = \sum_{k=0}^{n} \det D_{\widehat{k}}(-1)^{\lfloor n/2\rfloor} \det D_{\widehat{n-k}}. \tag{4}$$

This sum can be further simplified by evaluating $\det D_{\widehat{k}}$ for large $k$.

**Lemma.** *(a) If $k > n/2$, then* $\det D_{\widehat{k}} = (-1)^{k-1}$.

*(b) If $n$ is even, then* $\det D_{\widehat{n/2}} = 0$.

*Proof.* For $k > n/2$ we can compute $\det D_{\widehat{k}}$ by cofactor expansion along row $k$. For such a $k$ we have $d(k, j) = 0$ unless $j = 0$ or $k$, and on removing column $k$, row $k$ of $D_{\widehat{k}}$ is $[1, 0, \dots, 0]$. Thus $\det D_{\widehat{k}} = (-1)^{k-1}$, since after removing row $k$ and columns 0 and $k$ of $D$, the remaining matrix is upper triangular with 1s on the diagonal.

If $n$ is even, say $n = 2m$, then row $m$ of $D_{\widehat{m}}$ is $[1, 0, \dots, 0, 1]$ (since $d(m, j) = 1$ only when $j = 0, m, 2m$, and $j = m$ has been removed). Thus $d(m, j) = d(2m, j)$ for $j \neq m$, and since row $m$ and row $n = 2m$ are then identical, $\det D_{\widehat{m}} = 0$. $\square$

Plugging in to (4), we can hence in all cases set $m := \lceil n/2 \rceil$ and write

$$\begin{aligned}\det S = 2(-1)^{\lfloor n/2 \rfloor} \sum_{k=0}^{m-1} \det D_{\widehat{k}} \det D_{\widehat{n-k}} &= 2(-1)^{\lfloor n/2 \rfloor} \sum_{k=0}^{m-1} (-1)^{n-k-1} \det D_{\widehat{k}} \\ &= 2(-1)^{m-1} \sum_{k=0}^{m-1} (-1)^k \det D_{\widehat{k}},\end{aligned}$$

where we have also used that $n - m = \lfloor n/2 \rfloor$. The proof is complete on evaluating this sum.

**Claim.** *We have*

$$\sum_{k=0}^{m-1} (-1)^k \det\left(D_{\widehat{k}}\right) = m.$$

*Proof.* Let $D'$ be the $(n+1) \times (n+1)$-matrix obtained from $D$ by attaching an extra row $(1, 1, \dots, 1)$ at the top of the matrix (i.e., at row index $i = 0$). Then, the topmost two rows of the matrix $D'$ are equal, so that $\det(D') = 0$. On the other hand, expanding the determinant of $D'$ along the topmost row, and using part (b) of the Lemma to eliminate the middle term of the sum when $n$ is even, we obtain

$$\begin{aligned}\det\left(D'\right) &= \sum_{k=0}^{n} (-1)^k \det\left(D_{\widehat{k}}\right) = \sum_{k=0}^{m-1} (-1)^k \det\left(D_{\widehat{k}}\right) + \sum_{k=n-m+1}^{n} (-1)^k \det\left(D_{\widehat{k}}\right) \\ &= \sum_{k=0}^{m-1} (-1)^k \det\left(D_{\widehat{k}}\right) + \sum_{k=n-m+1}^{n} (-1)^k \cdot (-1)^{k-1} = \sum_{k=0}^{m-1} (-1)^k \det\left(D_{\widehat{k}}\right) - m.\end{aligned}$$

The second line follows by part (a) of the Lemma. Recalling that $\det(D') = 0$, the proof is complete. $\square$

**Solution 2:** As in Solution 1, the $(i, j)-$th entry of $S$ is the convolution of indicator series for multiples of $i$ and $j$. In particular, this gives the factorization $S = B \cdot \mathrm{Rev}(B)^T$, where $\mathrm{Rev}(B)$ denotes the matrix formed by reversing each row of $B$, and $B$ consists of the rows $R_1, \dots, R_n$, where

$$R_i = (1 \underbrace{0 \cdots 0}_{i-1} 1 \underbrace{0 \cdots 0}_{i-1} 1 \cdots) = (\mathbb{1}(i \mid j))_{j=0}^{n}.$$

The key idea in this proof is to apply row operations directly on $B$. In particular, if $M$ is an $n \times n$ matrix with $\det(M) = 1$, then

$$\det\left(MB \cdot \mathrm{Rev}(B)^T M^T\right) = \det(S).$$

Furthermore, $\mathrm{Rev}(B)^T M^T = \mathrm{Rev}(MB)^T$, since under left-multiplication $M$ acts as row operations on $B$, which are commutative with row reversal.

The preceding claims can also be justified more explicitly by noting that $\mathrm{Rev}B = M \cdot J$, where

$$J := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

For example, commutativity follows since $M \cdot \mathrm{Rev}(B) = M \cdot B \cdot J = \mathrm{Rev}(MB)$.

The rows $R_i$ form a full rank system, since $B$ excluding the first column is upper triangular. We can therefore reduce $B$ to the canonical form

$$MB = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ a_{n-1} & 0 & 0 & 0 & \cdots & 1 & 0 \\ a_n & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

by a sequence of row operations that acts on the rows from bottom to top, with each operation subtracting a row from below the row being acted on. It follows that $a_i$ can be defined recursively for $i = n, n-1, \ldots, 1$ by the formula

$$a_i = 1 - a_{2i} - a_{3i} - \cdots,$$

where for convenience we set the initial conditions $a_i = 0$ for $i > n$. Equivalently, we can set additional initial conditions $a_{n-m+1}, \ldots, a_n = 1$, where $m = \lceil \frac{n}{2} \rceil$, and define $a_i$ recursively for $i = n-m, \ldots, 1$.

Assume now that $n$ is odd, so that $n = 2m-1$; it will turn out that $n = 2m$ reduces to this case almost immediately. Then $n - m + 1 = m$, and

$$
\begin{aligned}
MSM^T &= MB \cdot \mathrm{Rev}(B)^T M^T \\
&= \begin{pmatrix}
a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
a_2 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & \vdots \\
a_{m-1} & 0 & \cdots & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & & & & & & \vdots \\
1 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
0 & 0 & \cdots & & & 1 & 0 \\
\vdots & & & & & & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & & & & & & \vdots \\
0 & 1 & \cdots & & & & 0 \\
1 & 0 & \cdots & & & & 0 \\
a_1 & a_2 & \cdots & a_{m-1} & 1 & \cdots & 1
\end{pmatrix} \\
&= \begin{pmatrix}
 & & & & & 1 & a_1 \\
 & & & & \iddots & & \vdots \\
 & & & 1 & & & a_{m-1} \\
 & & 1 & & & & 1 \\
 & \iddots & & & & & \vdots \\
1 & & & & & & 1 \\
a_1 & \cdots & a_{m-1} & 1 & \cdots & 1 & 2
\end{pmatrix}
\end{aligned}
$$

The determinant of this matrix can now be evaluated by reducing the final row to zeroes in all except the final column, and then expanding along that row. This gives

$$\det(S) = \det(MSM^T) = (-1)^{m-1} \left(2 - 2(a_1 + \cdots + a_{m-1})\right).$$

**Lemma 4.** *With $a_j$ defined as above,*

$$a_1 + \cdots + a_{m-1} = -(m-1).$$

*Proof.* Plugging in the definition of $a_1$ and initial values for $a_m, \ldots, a_n$,

$$
\begin{aligned}
a_1 + \cdots + a_{m-1} &= (1 - a_2 - a_3 - \cdots - a_n) + a_2 + \cdots + a_{m-1} \\
&= 1 - a_m - \cdots - a_n = 1 - m.
\end{aligned}
$$

□

This completes the proof for $n = 2m - 1$.

Finally, in the case that $n = 2m$, the recursion gives $a_m = 0$. Thus, the $m$th row of $MB$ has only one nonzero entry, a 1 in the first column, and is orthogonal to all other rows of $MB$. It follows that the $m$th row in $MSM^T$ has a 1 in the $m$th column, and is 0 elsewhere, so this row has no effect on its determinant. Further, the conclusion in Lemma 4 also still holds, since now $a_m = 0$ implies that $a_m + a_{m+1} + \cdots + a_n = m$.

**Solution 3:** As in Solution 2, let $m$ be the least integer greater than or equal to $n/2$, so that either $n = 2m$ or $n = 2m - 1$. We will define $(n+1) \times (n+1)$ matrices $C$ and $D$ and show that

$$CD = \begin{bmatrix} -2m & O \\ O^T & S \end{bmatrix}$$

where $O$ denotes a row of $n$ zeros. It will then follow that $\det S = (\det C)(\det D)/(-2m)$. We will write the entries of $C$ and $D$ as $c_{ij}$ and $d_{ij}$ where $0 \le i \le n$ and $0 \le j \le n$.

For $1 \le i \le n$ and $0 \le k \le n$, let $c_{ik} = 1$ if $k$ is a multiple of $i$, and $c_{ij} = 0$ otherwise. For $0 \le k \le n$ and $1 \le j \le n$, let $d_{kj} = 1$ if $n-k$ is a multiple of $j$, and $d_{kj} = 0$ otherwise. Then each allowed solution of $ai + bj = n$ corresponds to a case where $c_{ik} = d_{kj} = 1$, with $k = ai$. Thus, $s(i,j) = \sum_{k=0}^{n} c_{ik} d_{kj}$, verifying the $S$ block in the equation above for $CD$.

Next, we will choose column 0 of $D$ (corresponding to $j = 0$) to be orthogonal to rows 1 to $n$ of $C$, which will satisfy the $O^T$ block of the equation for $CD$. Such a column must exist because these $n$ rows can't span $(n+1)$-dimensional space. Notice that the $n \times n$ matrix $\{c_{ik}\}_{1 \le i,k \le n}$ is upper triangular, with all ones on its diagonal, so the rows of this reduced matrix are linearly independent. Thus, for the desired orthogonality, $d_{00}$ must be nonzero, and since rows 1 to $n$ of $C$ span an $n$-dimensional space, choosing $d_{00} = 1$ uniquely determines column 0 of $D$. For $n - m + 1 \le i \le n$, we have $2i > n$, so $c_{ik} = 1$ if and only if $k = 0$ or $k = i$. Thus, the desired orthogonality requires that $d_{i0} = -1$ for these values of $i$. Also, if $n$ is even, then row $m$ of $C$ is the same as row $n$ of $C$ except that $c_{mm} = 1$ while $c_{nm} = 0$. Thus, the desired orthogonality requires that $d_{m0} = 0$ when $n$ is even. Then, whether $n$ is even or odd, we have $\sum_{k=m}^{n} d_{k0} = -m$. For column 0 of $D$ to be orthogonal to row 1 of $C$, all of whose entries are 1, we then must have $\sum_{k=0}^{m-1} d_{k0} = m$. We will not need to determine the individual values of $d_{0k}$ for $1 \le k \le m-1$.

Now let $c_{0k} = d_{n-k,0}$ for $0 \le k \le n$. Notice that this makes row 0 of $C$ (which is the reverse of column 0 of $D$) orthogonal to columns 1 to $n$ of $D$ (which are the reverses of rows 1 to $n$ of $C$), satisfying the $O$ block of the equation for $CD$. To complete the verification of this equation, the upper left entry of $CD$ is

$$\sum_{k=0}^{n} c_{0k} d_{k0} = \sum_{k=0}^{n} d_{n-k,0} d_{k0} = \sum_{k=0}^{m-1} (-1) d_{k0} + \sum_{k=n-m+1}^{n} d_{n-k,0}(-1) = -m - m = -2m.$$

Next, to determine $\det C$, replace column 0 of $C$ with column 0 of $CD$. Since the latter column is the sum over $0 \le k \le n$ of $d_{k0}$ times column $k$ of $C$, and $d_{00} = 1$, this replacement is a column operation on $C$ that this does not change its determinant. The resulting matrix is upper triangular, and its diagonal consists of $-2m$ followed by $n$ ones. Thus, $\det C = -2m$.

Finally, since $D$ can be obtained by reversing the columns of $C^T$, which amounts to swapping $m$ pairs of columns, $\det D = (-1)^m \det C^T = (-1)^m \det C = (-1)^{m+1} 2m$. Therefore, $\det S = -2m(-1)^{m+1} 2m/(-2m) = (-1)^{m+1} 2m$.