A1. Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

Answer. The possible values are the nonnegative integers that are either divisible by 9 or *not* divisible by 3.

Solution 1. Let $f(A, B, C) = A^3 + B^3 + C^3 - 3ABC$. By direct computation, for nonnegative integers A we have $f(A, A, A+1) = 3A+1$ and $f(A, A, A) = 0$, while for positive integers A we have $f(A, A, A-1) = 3A-1$ and $f(A, A+1, A-1) = 9A$. This shows that all the values listed in the answer can actually be obtained. To show that no other values are possible, first note that by the AM-GM inequality, for nonnegative A, B, C we have

$$\frac{1}{3}(A^3 + B^3 + C^3) \geq \sqrt[3]{A^3B^3C^3} = ABC, \text{ and therefore } f(A, B, C) \geq 0.$$

So when A, B, C are nonnegative integers, the value $f(A, B, C)$ must be a nonnegative integer; it remains to show that if $f(A, B, C)$ is divisible by 3, then it is also divisible by 9. Note that $f(A, B, C) \equiv A^3+B^3+C^3 \equiv A+B+C \pmod 3$, so we are concerned with the case that $A+B+C \equiv 0 \pmod 3$. In this case we have $C = 3k - A - B$ for some integer k, and then

$$\begin{aligned} f(A, B, C) &= A^3 + B^3 + (3k - A - B)^3 - 3AB(3k - A - B) \\ &= 9k(a^2 + ab + b^2 - 3k(a+b) + 3k^2) \end{aligned}$$

is divisible by 9, completing the proof.

Solution 2. Start by observing the factorization

$$\begin{aligned} A^3 + B^3 + C^3 - 3ABC &= (A+B+C)(A^2 + B^2 + C^2 - AB - BC - CA) \\ &= (A+B+C) \cdot \frac{(A-B)^2 + (B-C)^2 + (C-A)^2}{2}, \end{aligned}$$

from which it is clear that the only possible values are nonnegative integers. For $(A, B, C) = (k, k, k)$ we get 0; for $(A, B, C) = (k, k, k+1)$ we get $(3k+1) \cdot 1$; for $(A, B, C) = (k, k+1, k+1)$ we get $(3k+2) \cdot 1$; and for $(A, B, C) = (k, k+1, k+2)$ we get $(3k+3) \cdot 3 = 9(k+1)$, showing that all the values listed in the answer can actually be obtained. To show that no others are possible, we show that of the two factors $A+B+C$ and $\dfrac{(A-B)^2 + (B-C)^2 + (C-A)^2}{2}$, either both are divisible by 3 or neither is divisible by 3. If $A+B+C$ is divisible by 3, then either A, B, C are all equal mod 3, in which case the second factor is clearly divisible by 3, or A, B, C are all different mod 3, in which case $(A-B)^2 + (B-C)^2 + (C-A)^2$ is equal to $1+1+1$ modulo 3 and the second factor is again divisible by 3. If $A+B+C$ is not divisible by 3, then A, B, C take on precisely two different values mod 3 (one twice, the other once), so $(A-B)^2 + (B-C)^2 + (C-A)^2$ is equal to $0+1+1$ modulo 3 and the second factor is not divisible by 3.

A2. In the triangle ΔABC, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α.

Answer. $\alpha = \dfrac{\pi}{2}$.

Solution 1. Recall that I is the intersection point of the angle bisectors, and G is the intersection point of the medians, of the triangle. Let Z be the "foot" of the angle bisector at C (the intersection of that bisector and AB) and M be the midpoint of AB. Then because IG and AB are parallel, ΔCIG and ΔCZM are similar triangles, so

$$\frac{CI}{IZ} = \frac{CG}{GM} = 2.$$

On the other hand, AI is the angle bisector through A in ΔACZ, which divides the opposite side in the ratio of the sides adjacent to A, so $AC = 2\,AZ$. Similarly, $BC = 2\,BZ$, and adding these two equations we get $AC + BC = 2\,AB$. Now use the law of sines to express all the side lengths of ΔABC in terms of BC:

$$AC = BC\,\frac{\sin\beta}{\sin\alpha}\,,\ \ AB = BC\,\frac{\sin(\pi-\alpha-\beta)}{\sin\alpha} = BC\,\frac{\sin(\alpha+\beta)}{\sin\alpha},$$

and it follows that

$$\sin\beta + \sin\alpha = 2\,\sin(\alpha+\beta)\,.$$

Given that $\beta = 2\tan^{-1}(1/3)$, we have

$$\sin\beta = 2\sin(\tan^{-1}(1/3))\cos(\tan^{-1}(1/3)) = 2\cdot\frac{1}{\sqrt{10}}\cdot\frac{3}{\sqrt{10}} = \frac{3}{5}\,,$$

$$\cos\beta = 2\cos^2(\tan^{-1}(1/3)) - 1 = \frac{4}{5}\,,$$

and using the addition formula for sine we get

$$\begin{aligned}\frac{3}{5} + \sin\alpha &= 2\,\sin\alpha\,\cos\beta + 2\,\cos\alpha\,\sin\beta\\ &= \frac{8}{5}\sin\alpha + \frac{6}{5}\cos\alpha,\end{aligned}$$

which simplifies to

$$\sin\alpha + 2\cos\alpha = 1\,.$$

If we set $x = \cos\alpha,\ y = \sin\alpha$ we have $2x + y = 1,\ x^2 + y^2 = 1$. Eliminating y yields $5x^2 - 4x = 0$, so $x = 0$ or $x = 4/5$, but $x = 4/5$ would yield a negative value for y, which is impossible. So $\cos\alpha = 0$ and $\alpha = \pi/2\,$.

Solution 2. Let h be the length of the altitude from C in ΔABC, and let r be the radius of the inscribed circle. Then the area of ΔABC is equal to $h\cdot AB/2$ and also, as can be seen by dissecting ΔABC into three triangles with a common vertex at I, equal to $r\cdot(AB + BC + AC)/2$. On the other hand, because IG is parallel to AB, the distances from I and from G to AB are equal. The distance from I is r, and because the centroid is two-thirds of the way (along the median) from C to AB, the distance from G is $h/3$. So $r = h/3$, and comparing the expressions above for the area of the triangle, we see that

$$\frac{1}{3}(AB + BC + AC) = AB,\ \text{that is,}\ BC + AC = 2AB\,.$$

From here we can proceed as in the first solution, or we can use the law of cosines:

$$AC^2 = AB^2 + BC^2 - 2\,AB\cdot BC\,\cos\beta = AB^2 + BC^2 - \frac{8}{5}\,AB\cdot BC\,,\ \text{so}$$

$$(2AB - BC)^2 = AB^2 + BC^2 - \frac{8}{5}\,AB\cdot BC\,,\ \text{which yields}\ AB = \frac{4}{5}BC,\ AC = \frac{3}{5}BC\,.$$

So $AB^2 + AC^2 = BC^2$, and ΔABC is a right triangle with the right angle α at A.

A3. Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \ldots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019\,.$$

Answer. $M = \left(\dfrac{1}{2019}\right)^{\frac{1}{2019}} = \dfrac{1}{\sqrt[2019]{2019}}\,.$

Solution. Because the polynomial factors as

$$P(z) = b_{2019}(z - z_1)(z - z_2)\cdots(z - z_{2019}),$$

its constant term is

$$b_0 = -b_{2019} z_1 z_2 \cdots z_{2019}\,,$$

and therefore we have

$$|z_1||z_2|\cdots|z_{2019}| = \frac{b_0}{b_{2019}} \geq \frac{1}{2019}.$$

By the AM-GM inequality,

$$\mu = (|z_1| + \cdots + |z_{2019}|)/2019 \geq (|z_1||z_2|\cdots|z_{2019}|)^{\frac{1}{2019}} \geq \left(\frac{1}{2019}\right)^{\frac{1}{2019}}.$$

So to finish the proof, it is enough to exhibit a specific polynomial for which $\mu = M$, where $M = \left(\dfrac{1}{2019}\right)^{\frac{1}{2019}}$. Note that for such a polynomial, all the $|z_i|$ must be equal to M, and we must have $b_0 = 1, b_{2019} = 2019$. Specifically, let $\omega = \exp(2\pi i/2020)$, a primitive 2020th root of unity. Then the polynomial

$$\begin{aligned} P(z) &= 2019(z - M\omega)(z - M\omega^2)\cdots(z - M\omega^{2019}) \\ &= 2019 \cdot \frac{z^{2020} - M^{2020}}{z - M} \\ &= 2019(z^{2019} + Mz^{2018} + \cdots + M^{2018}z + M^{2019}) \end{aligned}$$

has coefficients

$$b_0 = 2019M^{2019} = 1 < b_1 = 2019M^{2018} < \cdots < b_{2018} = 2019M < b_{2019} = 2019$$

and roots $z_i = M\omega^i$ with $|z_i| = M$, so $\mu = M$ as desired.

A4. Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?

Answer. No.

Solution. We will show that any nonzero continuous function f that depends on just one of the three variables x, y, z and that is periodic with period 2 and average value 0 provides a counterexample. (One such function is $f(x,y,z) = \sin(\pi z)$.) Let f be such a function, and assume that f depends only on z. Then to find the surface integral of $f(z)$ over a sphere of radius 1, we may assume without loss of generality that the sphere is centered at $(0,0,c)$; it can then be parametrized by

$$x = \sin\phi\cos\theta,\ y = \sin\phi\sin\theta,\ z = c + \cos\phi,\ \ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

For this parametrization $\mathbf{R}(\phi,\theta) = <\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ c + \cos\phi>$ we have the surface area element

$$\begin{aligned} dS &= |\partial\mathbf{R}/\partial\phi \times \partial\mathbf{R}/\partial\theta|\, d\phi\, d\theta \\ &= |<\cos\phi\cos\theta,\ \cos\phi\sin\theta, -\sin\phi> \times <-\sin\phi\sin\theta, \sin\phi\cos\theta, 0>|\, d\phi\, d\theta \\ &= |<\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi>|\, d\phi\, d\theta = \sin\phi\, d\phi\, d\theta. \end{aligned}$$

Therefore, the surface integral is equal to

$$\begin{aligned} \int_0^{\pi}\int_0^{2\pi} f(c+\cos\phi)\sin\phi\, d\theta\, d\phi &= 2\pi\int_0^{\pi} f(c+\cos\phi)\sin\phi\, d\phi \\ &= 2\pi\int_{c-1}^{c+1} f(z)\, dz. \end{aligned}$$

Because the integral is over a full period of f and f has average value 0, the integral is always 0.

A5. Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k,$$

where

$$a_k = k^{(p-1)/2} \mod p\,.$$

Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Answer. $n = \dfrac{p-1}{2}$.

Solution. Let $m = \dfrac{p-1}{2}$. Then the polynomial $q(x)$ can be obtained from the polynomial $P(x) = \sum\limits_{k=0}^{p-1} x^k$ by m applications of the linear operator L on $\mathbb{F}_p[x]$ defined by $L(f(x)) = xf'(x)$, where the prime denotes taking the formal derivative. Note that if a polynomial $f(x) \in \mathbb{F}_p[x]$ is divisible by $(x-1)^r$, then $L(f(x))$ is divisible by $(x-1)^{r-1}$ (to see this, write $f(x) = (x-1)^r g(x)$ and differentiate both sides). We now observe that because the coefficients are in $\mathbb{F}_p$, we can write

$$P(x) = \sum_{k=0}^{p-1} x^k = 1 + x + \cdots + x^{p-1} = \frac{1-x^p}{1-x} = \frac{(1-x)^p}{1-x} = (1-x)^{p-1},$$

so $P(x)$ is divisible by exactly $p-1 = 2m$ factors $x-1$, and by the observations above, $q(x) = L^m(P(x))$ is divisible by at least $2m - m = m$ factors $x-1$. To show that $q(x)$ cannot have more than m factors $x-1$ in $\mathbb{F}_p[x]$, note that the result of applying L another m times to the polynomial $q(x)$ is

$$\begin{aligned} L^m(q(x)) &= \sum_{k=1}^{p-1} k^{p-1} x^k \\ &= \sum_{k=1}^{p-1} x^k \quad \text{(by Fermat's Little Theorem)} \\ &= P(x) - 1. \end{aligned}$$

Because $P(x)$ is divisible by $x-1$, $L^m(q(x))$ is not, so $q(x)$ cannot have more than m factors $x-1$, showing that $n = m$.

A6. Let g be a real-valued function that is continuous on the closed interval $[0,1]$ and twice differentiable on the open interval $(0,1)$. Suppose that for some real $r > 1$,

$$\lim_{x\to 0^+} \frac{g(x)}{x^r} = 0\,.$$

Prove that either

$$\lim_{x\to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x\to 0^+} x^r |g''(x)| = \infty\,.$$

Solution. Throughout this solution, we will abbreviate $\lim\limits_{x\to 0^+}$ by $\lim$, and similarly for $\limsup\limits_{x\to 0^+}$ and $\liminf\limits_{x\to 0^+}$. Suppose that under the assumptions of the problem, $\lim g'(x) \neq 0$. We will show that then $\limsup x^r |g''(x)| = \infty$. First of all, from the given limit of $g(x)/x^r$ we have $g(0) = 0$. We claim that

$$\liminf g'(x) \le 0 \le \limsup g'(x). \quad (*)$$

In fact, if the left-hand inequality would fail, there would be some positive constant c and some interval $(0,\delta)$ on which $g'(x) > c$. But that would imply $g(x) > cx$ on this interval, in contradiction with the given limit. The argument for the right-hand inequality is similar. Also, if both inequalities were equalities, then we would have $\lim g'(x) = 0$ after all, so at least one of the two inequalities in $(*)$ must be strict.

Now suppose that the inequality $\limsup g'(x) > 0$ is strict. (The other case is completely analogous.) Choose a positive constant $C < \limsup g'(x)$. Then because g' is continuous on $(0,1)$ and $\liminf g'(x) \le 0$, we can find a sequence (b_n) of real numbers tending to zero such that $g'(b_n) = C$, and for each b_n we can find a number a_n with $0 < a_n < b_n$ such that $g'(a_n) = C/2$ and such that on the interval $[a_n, b_n]$, $g'(x) \ge C/2$. By the Mean Value Theorem on that interval, we then have $g(b_n) - g(a_n) \ge C(b_n - a_n)/2$, so at least one of $|g(b_n)| \ge C(b_n - a_n)/4$ and

$|g(a_n)| \geq C(b_n - a_n)/4$ must hold. Let

$$t_n = \frac{b_n - a_n}{b_n^r}.$$

Then because we know $\lim g(x)/x^r = 0$, it follows that $t_n \to 0$ as $n \to \infty$, and in particular, a_n/b_n approaches 1 as $n \to \infty$.

Finally, applying the Mean Value Theorem to g' on the interval $[a_n, b_n]$ shows the existence of $u_n \in (a_n, b_n)$ such that

$$g''(u_n) = \frac{g'(b_n) - g'(a_n)}{b_n - a_n} = \frac{C - C/2}{b_n - a_n} = \frac{C}{2t_n b_n^r}.$$

If we take n sufficiently large so that $a_n/b_n \geq (2/3)^{1/r}$, we will have $3u_n^r > 3a_n^r \geq 2b_n^r$ and thus

$$g''(u_n) \geq \frac{C}{3t_n u_n^r}, \quad \text{that is, } u_n^r\, g''(u_n) \geq \frac{C}{3t_n}.$$

As t_n approaches 0 as $n \to \infty$, it follows that $\limsup x^r g''(x) = \infty$.

B1. Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

Answer. $5n + 1$.

Solution. Let S_k be the set of all points $(x, y) \in \mathbb{Z}^2$ such that $x^2 + y^2 = 2^k$, so that

$$P_n = \{(0, 0)\} \cup \bigcup_{k=0}^{n} S_k \,.$$

Then $S_0 = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$ and $S_1 = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. For $k \geq 2$ and any $(x, y) \in S_k$, we have $x^2 + y^2 \equiv 0 \pmod 4$, so because 0 and 1 are the only squares (mod 4), x and y must both be even. If we put $x = 2x_1, y = 2y_1$, then $4(x_1^2 + y_1^2) = 2^k$, so $x_1^2 + y_1^2 = 2^{k-2}$ and $(x_1, y_1) \in S_{k-2}$. It follows by induction on k that

$$\begin{aligned} S_k &= \{(2^q, 0), (-2^q, 0), (0, 2^q), (0, -2^q)\} && \text{for } k = 2q \text{ even} \quad \text{and} \\ S_k &= \{(2^t, 2^t), (2^t, -2^t), (-2^t, 2^t), (-2^t, -2^t)\} && \text{for } k = 2t + 1 \text{ odd.} \end{aligned}$$

Note that for any k, the four points of S_k form the vertices of a square; also, for any q there are four squares with one vertex at the origin, two vertices in S_{2q}, and one vertex in S_{2q+1} (the square with vertices $(0, 0), (2^q, 0), (2^q, 2^q), (0, 2^q)$ and the three squares obtained from it by rotation through $\pi/2, \pi, 3\pi/2$ around the origin), and for any t there are four squares with one vertex at the origin, two vertices in S_{2t+1}, and one vertex in S_{2t+2} (the square with vertices $(0, 0), (2^t, 2^t), (0, 2^{t+1}), (-2^t, 2^t)$ and the three squares obtained from it by rotation). Thus when we pass from P_n to P_{n+1} by including the points in S_n, we get at least five additional squares, whether n is even or odd. Because there is exactly one four-point subset of P_0 (namely S_0) that gives a square, there will be exactly $5n + 1$ such subsets of P_n , provided that the only squares of which all vertices are in the set

$$P_\infty = \bigcup_{n=0}^{\infty} P_n = \{(0, 0)\} \cup \bigcup_{k=0}^{\infty} S_k$$

are the ones we have mentioned so far.

To see that there are no additional such squares, first note that for all $k \geq 2$, all points in S_k have only even coordinates; if we have a square for which each vertex is in $\{(0, 0)\} \cup \bigcup_{k=2}^{\infty} S_k$, we can scale down all coordinates by a factor 2 and get another square of which all vertices are in P_∞ . Thus it is sufficient to consider squares for which all vertices are in P_∞ and at least one vertex is in $S_0 \cup S_1$.

It is impossible to have just *one* of the vertices of such a square be in $S_0 \cup S_1$, because the square of the side length from that vertex to any other vertex would be 1 or 2 mod 4, whereas the square of a side length not involving that vertex would be 0 mod 4. By the same argument, if exactly *two* of the vertices of such a square are in $S_0 \cup S_1$, those two must be opposite vertices of the square. And if three or four of the vertices of such a square are in $S_0 \cup S_1$, we can choose two such vertices that are opposite each other. Thus it is enough to analyze squares of which all vertices are in P_∞ and two opposite vertices are in $S_0 \cup S_1$.

If one of the two opposite vertices in $S_0 \cup S_1$ is in S_0, up to rotational symmetry we can assume it is $(1,0)$. Then it can be checked by a quick case analysis that the vertex of the square opposite it cannot be in S_1; if it is $(-1,0)$, then the vertices of the square are the four points of S_0, otherwise it is $(0,1)$ up to reflectional symmetry, and the vertices of the square are $(0,0),(1,0),(1,1),(0,1)$. The final possibility is that the two opposite vertices in $S_0 \cup S_1$ are both in S_1, in which case we can assume up to symmetry that they are $(1,1)$ and $(-1,-1)$ (and the vertices of the square are all the points of S_1) or $(1,1)$ and $(-1,1)$ (and the vertices of the square are $(0,0),(1,1),(0,2),(-1,1)$). We have now checked that the only possible squares whose vertices are all in P_n are the $5n+1$ squares found above.

B2. For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)} .$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3} .$$

Answer. $\dfrac{8}{\pi^3}$.

Solution. Let $\theta_n = \dfrac{\pi}{2n}$, and note that $\sin\theta_n \neq 0$. Then we have

$$\begin{aligned}
a_n &= \sum_{k=1}^{n-1} \frac{\sin((2k-1)\theta_n)}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \frac{\sin((2k-1)\theta_n)\sin\theta_n}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \frac{1}{2}\frac{\cos((2k-2)\theta_n) - \cos(2k\theta_n)}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \frac{1}{2}\frac{2\cos^2((k-1)\theta_n) - 1 - (2\cos^2(k\theta_n) - 1)}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \Big(\frac{1}{\cos^2(k\theta_n)} - \frac{1}{\cos^2((k-1)\theta_n)}\Big).
\end{aligned}$$

We now see that the sum telescopes, and we get

$$a_n = \frac{1}{\sin\theta_n}\Big(\frac{1}{\cos^2((n-1)\theta_n)} - 1\Big).$$

Because $n\theta_n = \dfrac{\pi}{2}$, we have $\cos((n-1)\theta_n) = \cos(\dfrac{\pi}{2} - \theta_n) = \sin\theta_n$, so

$$a_n = \frac{1}{\sin^3\theta_n} - \frac{1}{\sin\theta_n}.$$

Now let $n \to \infty$. Then $\theta_n \to 0$, so

$$\lim_{n\to\infty} n \sin\theta_n = \lim_{n\to\infty} n\theta_n = \frac{\pi}{2}.$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{n^3} = \lim_{n\to\infty} \frac{1}{(n\sin\theta_n)^3} - \lim_{n\to\infty} \frac{1}{n^2(n\sin\theta_n)}$$
$$= \frac{1}{(\pi/2)^3} - 0 = \frac{8}{\pi^3}\,.$$

B3. Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q , then 1 is an eigenvalue of PQ .

Solution. Note that $P(u) = u - 2uu^T u = u - 2u = -u$, while if $v \in \mathbb{R}^n$ is a vector orthogonal to u, that is, if $u^T v = 0$, we have $P(v) = v - 2uu^T v = v$. So P has a one-dimensional eigenspace for the eigenvalue $\lambda = -1$ and an $(n-1)$-dimensional eigenspace for the eigenvalue $\lambda = 1$, and thus $\det(P) = -1$. Also, P is an orthogonal matrix; this can be seen geometrically by noting that P is the matrix of the reflection in the hyperplane through the origin with normal vector u, or by direct computation:

$$P^T P = (I - 2(uu^T)^T)(1 - 2uu^T) = (1 - 2uu^T)(1 - 2uu^T)$$
$$= 1 - 4uu^T + 4u(u^T u)u^T = 1 - 4uu^T + 4uu^T = 1\,.$$

Now recall that any orthogonal matrix has determinant ± 1, and that the product of orthogonal matrices is orthogonal. Therefore, because $\det(P) = -1$, we know Q and PQ are orthogonal matrices of the same size that have opposite determinants ± 1. The desired result now follows immediately from the following.

Lemma. If A is an n-by-n real orthogonal matrix such that either (i) $\det(A) = 1$ and n is odd or (ii) $\det(A) = -1$ and n is even, then 1 is an eigenvalue of A.

To prove the lemma, first let $\lambda \in \mathbb{C}$ be any eigenvalue of A and $v \in \mathbb{C}^n$ be an associated eigenvector. Then, taking complex conjugates, $Av = \lambda v$ yields $A\overline{v} = \overline{\lambda}\overline{v}$, so

$$(A\overline{v})^T A v = \overline{\lambda}\lambda\, \overline{v}^T v = |\lambda|^2 |v|^2\,, \quad \text{while also}$$
$$(A\overline{v})^T A v = \overline{v}^T (A^T A) v = \overline{v}^T v = |v|^2\,.$$

Because $|v| \neq 0$, it follows that $|\lambda| = 1$. Thus the eigenvalues of A that are not 1 or -1 must occur in complex conjugate pairs for which $\lambda\overline{\lambda} = 1$. The product of all the eigenvalues (counting multiplicity) is $\det(A)$, and if we leave out the complex conjugate pairs, the product of the real eigenvalues ± 1 will still be $\det(A)$. If n is odd, the number of real eigenvalues is odd, but to get $\det(A) = 1$ the number of factors -1 must be even, so the eigenvalue 1 must occur at least once. Similarly, if n is even, the number of real eigenvalues is even (in general, possibly zero), but to get $\det(A) = -1$ the number of factors -1 must be odd, and again the eigenvalue 1 must occur.

B4. Let $\mathcal{F}$ be the set of functions $f(x, y)$ that are twice continuously differentiable for $x \geq 1$, $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

$$x f_x + y f_y = xy \ln(xy)\,,$$
$$x^2 f_{xx} + y^2 f_{yy} = xy\,.$$

For each $f \in \mathcal{F}$, let

$$m(f) = \min_{s\geq 1}\Big(f(s+1,s+1) - f(s+1,s) - f(s,s+1) + f(s,s)\Big)\,.$$

Determine $m(f)$, and show that it is independent of the choice of f.

Answer. $m(f) = 2\ln 2 - \dfrac{1}{2}$, independently of the choice of $f \in \mathcal{F}$.

Solution. First note that for any $f \in \mathcal{F}$,

$$\begin{aligned} f(s+1,s+1)-&f(s+1,s) - f(s,s+1) + f(s,s) = \\ &= (f(s+1,s+1) - f(s,s+1)) - ((f(s+1,s) - f(s,s)) \\ &= \int_s^{s+1} f_x(x,s+1)\,dx - \int_s^{s+1} f_x(x,s)\,dx \\ &= \int_s^{s+1} (f_x(x,s+1) - f_x(x,s))\,dx \\ &= \int_s^{s+1}\int_s^{s+1} f_{xy}(x,y)\,dy\,dx, \end{aligned}$$

so to find $m(f)$ we must minimize this double integral. We now use the given partial differential equations to find f_{xy}. Taking partial derivatives of both sides of $xf_x + yf_y = xy\ln(xy)$ with respect to each of x and y, we get the two equations

$$f_x + xf_{xx} + yf_{yx} = y\ln(xy) + y, \quad xf_{xy} + f_y + yf_{yy} = x\ln(xy) + x. \quad (*)$$

Note that because f is twice continuously differentiable, $f_{yx} = f_{xy}$. If we multiply the first equation in $(*)$ by x and the second equation by y and add the results, we obtain

$$(xf_x + yf_y) + (x^2 f_{xx} + y^2 f_{yy}) + 2xyf_{xy} = 2xy\ln(xy) + 2xy.$$

Using the two given equations to replace the bracketed expressions on the left and then dividing by $2xy$ leads to

$$f_{xy} = \frac{1}{2}(\ln(xy) + 1) = \frac{1}{2}(\ln x + \ln y + 1).$$

Therefore, we have

$$\begin{aligned} m(f) &= \min_{s\geq 1}\int_s^{s+1}\int_s^{s+1} \frac{1}{2}(\ln x + \ln y + 1)\,dy\,dx \\ &= \frac{1}{2}\min_{s\geq 1}\int_s^{s+1}\Big(\ln x + 1 + \int_s^{s+1}\ln y\,dy\Big)\,dx \\ &= \frac{1}{2}\min_{s\geq 1}\Big(\int_s^{s+1}\ln x\,dx + 1 + \int_s^{s+1}\ln y\,dy\Big) \\ &= \min_{s\geq 1}\Big(\int_s^{s+1}\ln t\,dt + \frac{1}{2}\Big). \end{aligned}$$

Because the function ln is increasing, the minimum occurs for $s = 1$, and so

$$m(f) = \frac{1}{2} + \int_1^2 \ln t\,dt = \frac{1}{2} + (t\ln t - t)\Big|_{t=1}^{2} = 2\ln 2 - \frac{1}{2}\,.$$

Comment. With some additional calculation it can be shown that the functions in $\mathcal{F}$ are exactly those of the form

$$f(x,y) = \frac{1}{2}xy\ln(xy) - \frac{1}{2}xy + C(\ln x - \ln y) + D,$$

where C and D are arbitrary constants.

B5. Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

Answer. $p(2019) = F_{2019} - F_{1010}$, so $j = 2019, k = 1010$.

Solution 1. More generally, let $p_N(x)$ be the polynomial of degree N such that $p_N(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, N$. We will show by induction on N that $p_N(2N+3) = F_{2N+3} - F_{N+2}$; setting $N = 1008$ then gives the desired answer. For the basis step, $p_1(x)$ is the linear polynomial with $p_1(1) = 1$, $p_1(3) = 2$, so $p_1(x) = (x+1)/2$ and $p_1(5) = 3 = F_5 - F_3$. To start the induction step, note that $p_N(x)$ and $p_{N-1}(x)$ have the same values for $x = 1, 3, 5, \ldots, 2N-1$, and therefore there is a constant c_N such that

$$p_N(x) = p_{N-1}(x) + c_N(x-1)(x-3)\cdots(x-(2N-1)).$$

We can find c_N by substituting $x = 2N+1$ and using the induction hypothesis $p_{N-1}(2N+1) = F_{2N+1} - F_{N+1}$, which yields

$$F_{2N+1} = F_{2N+1} - F_{N+1} + c_N(2N)(2N-2)\cdots 2 \quad \text{and thus} \quad c_N = \frac{F_{N+1}}{2^N N!}.$$

It follows that

$$\begin{aligned}
p_N(x) &= (x+1)/2 + c_2(x-1)(x-3) + \cdots + c_N(x-1)(x-3)\cdots(x-(2N-1)) \\
&= (x+1)/2 + \sum_{i=2}^{N} \frac{F_{i+1}}{2^i i!}(x-1)(x-3)\cdots(x-(2i-1)),
\end{aligned}$$

and in particular

$$\begin{aligned}
p_N(2N+3) &= N+2+\sum_{i=2}^{N} \frac{F_{i+1}}{2^i i!}(2N+2)(2N)\cdots(2N-2i+4) \\
&= N+2+\sum_{i=2}^{N} \frac{F_{i+1}(N+1)N\cdots(N-i+2)}{i!} \\
&= N+2+\sum_{i=2}^{N} F_{i+1}\binom{N+1}{i} \\
&= \sum_{i=0}^{N} F_{i+1}\binom{N+1}{i} = \sum_{i=0}^{N+1} F_{i+1}\binom{N+1}{i} - F_{N+2}.
\end{aligned}$$

Thus the induction on N will be complete if we can prove that $\sum_{i=0}^{K} F_{i+1}\binom{K}{i} = F_{2K+1}$ for any positive integer K. This in turn follows from the more general fact

$$\sum_{i=0}^{K} F_{i+m}\binom{K}{i} = F_{2K+m}\,,$$

which is true for all positive integers K and m and can be shown by a relatively straightforward induction on K (the generality helps because the induction step uses the induction hypothesis both for m and for $m+1$).

Solution 2. By Binet's formula, we have

$$\begin{aligned} F_{2n+1} &= \frac{1}{\sqrt{5}}\Big(\big(\frac{1+\sqrt{5}}{2}\big)^{2n+1} - \big(\frac{1-\sqrt{5}}{2}\big)^{2n+1}\Big) \\ &= \frac{1+\sqrt{5}}{2\sqrt{5}}\, r_1^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\, r_2^n\,, \end{aligned}$$

where r_1, r_2 are given by

$$r_1 = \frac{3+\sqrt{5}}{2} = \big(\frac{1+\sqrt{5}}{2}\big)^2, \quad r_2 = \frac{3-\sqrt{5}}{2} = \big(\frac{1-\sqrt{5}}{2}\big)^2.$$

Therefore, if we define $q_1(x), q_2(x)$ to be the polynomials of degree 1008 such that

$$q_1(2n+1) = r_1^n \quad \text{and} \quad q_2(2n+1) = r_2^n \quad \text{for} \quad n = 0, 1, 2, \ldots, 1008,$$

we will have

$$p(x) = \frac{1+\sqrt{5}}{2\sqrt{5}}\, q_1(x) \; - \; \frac{1-\sqrt{5}}{2\sqrt{5}}\, q_2(x).$$

Thus the following fact about interpolating polynomials will be useful.

Lemma. If $q(x)$ is the polynomial of degree N such that $q(n) = r^n$ for $n = 0, 1, 2, \ldots N$, where r is some fixed real number, then $q(N+1) = r^{N+1} - (r-1)^{N+1}$.

To prove the lemma, first define

$$T_{N,k}(x) = \prod_{\substack{j=0 \\ j\neq k}}^{N} (x-j) \quad \text{and} \quad Q_{N,k}(x) = \frac{T_{N,k}(x)}{T_{N,k}(k)}\,.$$

Then $Q_{N,k}(x)$ is the polynomial of degree N such that for integers n with $0 \leq n \leq N$, we have $Q_{N,k}(n) = \delta_{n,k}$, where δ is the Kronecker delta. Therefore, $q(x)$ is the linear combination

$$q(x) = \sum_{k=0}^{N} r^k Q_{N,k}(x)$$

of these "basic" interpolating polynomials. We then get

$$\begin{aligned}
q(N+1) &= \sum_{k=0}^{N} r^k\, Q_{N,k}(N+1) = \sum_{k=0}^{N} r^k\, \frac{T_{N,k}(N+1)}{T_{N,k}(k)} \\
&= \sum_{k=0}^{N} r^k\, \frac{(N+1)!/(N+1-k)}{k!\,(-1)^{N-k}\,(N-k)!} \\
&= \sum_{k=0}^{N} (-1)^{N-k} r^k\, \frac{(N+1)!}{k!\,(N+1-k)!} = \sum_{k=0}^{N} (-1)^{N-k} r^k \binom{N+1}{k} \\
&= -\sum_{k=0}^{N+1} (-1)^{N+1-k} r^k \binom{N+1}{k} + r^{N+1} = r^{N+1} - (r-1)^{N+1},
\end{aligned}$$

proving the lemma.

The lemma applies to the polynomials $q(x) = q_1(2x+1)$ and $q(x) = q_2(2x+1)$, so we can compute

$$\begin{aligned}
p(2019) &= \frac{1+\sqrt{5}}{2\sqrt{5}}\, q_1(2019) \;-\; \frac{1-\sqrt{5}}{2\sqrt{5}}\, q_2(2019) \\
&= \frac{1+\sqrt{5}}{2\sqrt{5}}\big(r_1^{1009} - (r_1-1)^{1009}\big) \;-\; \frac{1-\sqrt{5}}{2\sqrt{5}}\big(r_2^{1009} - (r_2-1)^{1009}\big) \\
&= \frac{1}{\sqrt{5}}\Big(\big(\frac{1+\sqrt{5}}{2}\big)^{2019} - \big(\frac{1-\sqrt{5}}{2}\big)^{2019}\Big) - \frac{1}{\sqrt{5}}\Big(\big(\frac{1+\sqrt{5}}{2}\big)^{1010} - \big(\frac{1-\sqrt{5}}{2}\big)^{1010}\Big) \\
&= F_{2019} - F_{1010}\,,
\end{aligned}$$

where we have used that $r_1 - 1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 - 1 = \dfrac{1-\sqrt{5}}{2}$.

B6. Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called *neighbors* if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers $n \geq 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?

(1) If p is in S, then none of the neighbors of p is in S.
(2) If $p \in \mathbb{Z}^n$ is not in S, then exactly one of the neighbors of p is in S.

Solution. We will show how to construct such a subset for every n. Because each point in $\mathbb{Z}^n$ has exactly $2n$ neighbors, for each point there is a set of size $2n+1$ (consisting of its neighbors and itself) of which exactly one element should be in S. This may suggest looking at congruences modulo $2n+1$. More specifically, for each integer k with $0 \leq k \leq 2n$ we can define a subset S_k of $\mathbb{Z}^n$ by

$$S_k = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \,|\, x_1 + 3x_2 + 5x_3 + \cdots + (2n-1)x_n \equiv k \mod (2n+1)\}.$$

It is immediate that these $2n+1$ subsets partition $\mathbb{Z}^n$; we claim that any of the subsets has the desired properties for S. To see this, let

$f(x_1, x_2, \ldots, x_n) = x_1 + 3x_2 + \cdots + (2n-1)x_n\,,$ so that

$S_k = \{(x_1, \ldots, x_n) \,|\, f(x_1, \ldots, x_n) \equiv k \mod (2n+1)\}$. Note that moving from a point $p = (x_1, \ldots, x_n)$ to one of its neighbors adds one of the numbers $\pm1, \pm3, \ldots, \pm(2n-1)$ to the value of $f(x_1, \ldots, x_n)$. Because these numbers represent all the nonzero

congruence classes mod $(2n+1)$
(specifically, $1 \equiv 1, 2 \equiv -(2n-1), 3 \equiv 3, 4 \equiv -(2n-3), \ldots, 2n-1 \equiv 2n-1, 2n \equiv -1$), for any k exactly one of the point p and its $2n$ neighbors is guaranteed to be in the set S_k, as desired.