

S E N O T E S

BEYOND LECTURE:

RESOURCES AND PEDAGOGICAL TECHNIQUES
FOR ENHANCING THE TEACHING OF
PROOF-WRITING ACROSS THE CURRICULUM

Rachel Schwell,
Aliza Steurer, and
Jennifer F. Vasquez
Editors



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Beyond Lecture

**Resources and Pedagogical Techniques for Enhancing the
Teaching of Proof-Writing Across the Curriculum**

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Beyond Lecture

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Teaching of Proof-Writing Across the Curriculum**

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The MAA Notes Series, started in 1982, addresses a broad range of topics and themes of interest to all who are involved with undergraduate mathematics. The volumes in this series are readable, informative, and useful, and help the mathematical community keep up with developments of importance to mathematics.

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Foreword

What sort of volume is this? It is a compendium of phronesis; that is, it contains much practical wisdom for those wondering how one teaches undergraduate students to write proofs that are acceptable to the mathematical community. There are approximately 40 short pieces of usable advice, written by college and university mathematics teachers who have tried out the ideas themselves and found them worthy of presenting to others. In short, it's a handbook of practical suggestions—some large, some small—of “what works” for helping students develop proof-writing skills.

Why is learning to write proofs so hard? Many mathematics education research studies have been devoted to this question. In our chapter, “Overcoming Students’ Difficulties in Learning to Understand and Construct Proofs,” in the MAA *Notes* volume, *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, we detailed some of the reasons reported in the mathematics education research literature of the time. These included difficulties with understanding and using mathematical definitions and theorems, understanding the structure of a proof and how it is written, reading and checking proofs, bringing appropriate knowledge to mind, and dealing with various symbolic representations. At the time, there were only a few suggestions in the mathematics education research literature on how to alleviate these difficulties. While there are now a few more tested methods, there need to be more. Meanwhile, a volume of practical wisdom, such as this, can fill the gap for college and university teachers who need help now.

Much of this volume is devoted to activities to use in the classroom. Indeed, the mathematician Hans Freudenthal, turned mathematics educator par excellence, had a strong belief that mathematics is a human activity, and hence, learning mathematics should be an activity. He disliked what he referred to as the “anti-didactical inversion;” that is, that in most traditional mathematics instruction, the results of the mathematical activity of others, mainly mathematicians, is often taken as a starting point, rather than the facilitation of that activity itself with students. Freudenthal emphasized the importance of having students engage in group work and reflection to reach new levels of insight; of having students reinvent mathematics under guidance; and of using research, particularly observations of individual students’ thinking processes, to inform curriculum development. Many of the chapters in this volume suggest activities, including group work and/or reflection, for helping students with proving—clearly something of which Freudenthal would have approved.

This volume is not a book one reads cover-to-cover. Instead, readers may pick and choose what they find useful. As such, the volume contains a host of good practical advice, organized in a way that lets one find it.

Annie and John Selden
New Mexico State University

Preface

The three editors of this book met through Project NExT, a fellowship program of the MAA for new faculty. Being young instructors thrust into upper-level and proof-based courses, new to teaching proofs, we found it challenging to find the time to research, adapt, and implement innovations in our pedagogy. Hoping to learn about effective strategies for teaching proof-writing in our own classes, we organized a contributed paper session at MathFest 2009 entitled “Getting Students Involved in Writing Proofs,” which then ran two more times at MathFest 2010 and the Joint Math Meetings 2011. These sessions were very well attended, and inspired a great deal of discussion.

We believed it would be helpful to have, collected in one place, detailed and practical descriptions of a variety of the methods presented there. The ideas discussed and further explored within this volume include those from the contributed paper sessions, along with others that are new to us since that time. This resource is meant to serve as a handbook for faculty members seeking techniques to insert into any course containing proof-writing in its curriculum, even including those that may not typically emphasize proofs.

We have enjoyed reading the many innovative ideas put forth by our authors, and are grateful to them for their creativity, motivation, and willingness to share their innovations with the mathematical community. In addition, we appreciate the thoughtful feedback from the reviewers on the MAA *Notes* Board. Lastly, we would like to thank Stephen Maurer, Mike May, and Mike Axtell (who at different times were the editors of the *Notes* series) for their insightful comments and helpful advice during the editing process.

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Articles are assigned codes that categorize them by difficulty of implementation and course level.

Difficulty level:

- Low (L): little preparation (beyond the usual) required of the instructor and students, and little or no change in current course structure; or, can be adapted to require minimal preparation
- Medium (M): a moderate amount of preparation and time required of the instructor and students, and possibly some significant change in course structure; or, can be adapted to require less preparation
- High (H): sustained extra preparation (at least the first time the method is employed) and time required of both the instructor and students, and comprehensive change in course structure

Course level:

- Transitional (T): transition-to-proof or mid-level course such as discrete math or linear algebra
- Advanced (A): advanced classes such as abstract algebra or real analysis
- Non-traditional (N): courses that are not traditionally considered proofs courses, such as calculus or math for liberal arts

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Introduction

Our vision for this volume is for it to be a practical handbook containing “nuts-and-bolts” descriptions of pedagogical techniques to insert into courses containing proof-writing. The articles address a variety of courses, including mathematics for liberal arts, calculus, linear algebra, modern geometry, and advanced courses. Methods range from simple, easy-to-implement activities, such as using codes to grade proofs, to ones requiring a moderate amount of preparation, such as using peer grading of exams, to those needing a high level of preparation and experience, such as teaching via inquiry-based learning. Our contributors come from a wide range of institutions, including private and public schools, small liberal arts colleges and large universities, and religious and secular schools. The authors have different levels of experience, some having even written or edited their own books [7, 8, 11, 25, 27]. We hope that the broad variety of techniques, contexts, and viewpoints provided here will be a springboard for discussion and exploration in our readers’ own classrooms.

Motivation

The purpose and benefits of learning to write proofs are vast; in fact, an entire *Notes* volume alone could be dedicated to discussing them. This is part of the reason a great deal of attention is given to the teaching of proof-writing in the undergraduate mathematics curriculum. Indeed, many colleges and universities have semester-long courses devoted at least partially to this topic, and there are numerous textbooks dedicated to it. However, students can still find the transition from computational mathematics to proof-writing both challenging and overwhelming [21], and instructors can find it challenging to help them overcome this.

To understand why students face such difficulties, consider the analogy of learning to become a skilled chef. Would a person who can successfully follow step-by-step recipes (and prepare meals only that way) be considered a skilled chef or even a chef at all? How difficult is it to develop the intuition of a chef if all one has ever done is follow recipes? This can be likened to the difficulties our students face in learning to read and write proofs: their experience has predominantly been comprised of following computational algorithms [19], i.e., dutifully following recipes, which they can sometimes even do very well. Developing a proof contains many steps that are not necessarily algorithmic: understanding the question, deciding what mathematical concepts to use and fitting them together into an argument, achieving the right level of detail, presenting the argument in a clear, coherent way, and assessing the correctness of the proof. Creating a meal involves similar steps that are not necessarily explicit: choosing the dishes to make, selecting in the correct proportions complementary ingredients for the dishes, presenting the courses in an appealing way, and having confidence in the meal one has made. Our students have been asked only to make one dish at a time following a step-by-step recipe with a cookbook, and when they face proof-writing, not only are they being asked to create a full meal without really knowing what one is, we expect them to become skilled chefs by working without a recipe and with little hands-on training or guidance.

With all the details and subtleties involved, becoming a chef is an extremely difficult task. It thus seems unreasonable to assume that merely observing a master chef cook a delicious meal is sufficient for students to become chefs themselves (except possibly for those who would have been master chefs anyway), even if the observation is an interactive one. However, this is what we typically ask our beginning proof students

to do. Furthermore, leading up to this point, in introductory/computational courses, we have largely been paying lip service to the communication of mathematics [6]. Now, they are being asked to not only know how to communicate well, but to understand *why* they should know how to do it well, and even further, in a context of non-procedural tasks, to which they are not accustomed. This is a vast leap.

Students encounter a variety of challenges in making this jump, including those related to both the language and also the reasoning and logic of mathematical proofs. More specifically, these include the following (references following each item point the reader to further information about that difficulty):

- Understanding the problem [22]
- Conception of what a proof is [13, 23]
- Knowing where to begin [21]
- Recognizing the role of false starts [2]
- Understanding conditional statements, specifically the difference between assumptions and conclusions [24]
- Writing for the proper audience [18]
- Differentiating between solving the problem and writing the proof [20]
- Using correct notation, symbols, and vocabulary [21, 24]
- Using good grammar, sentence structure, and flow [18]

These challenges lead us to believe that to successfully teach proof-writing, pedagogical techniques that go beyond just lecture are crucial. Some such approaches are discussed in current literature; for example, articles in *Problems, Resources, and Issues in Mathematics Undergraduate Studies (PRIMUS)* devoted to this topic include, among others, Campbell [1], Greer and Reed [12], Goff [10], Johnson and Green [15], Kasman [16], and Zerr and Zerr [28]. The MAA *Notes* series has relevant volumes such as Coppin, Mahavier, May, and Parker [4]; Hibbard and Maycock [14]; Knuth, Larrabee, and Roberts [17]; and Sterrett [26]. There are also many sources online.

This *Notes* volume unites many new resources in one easy-to-access document, which can make it easier for the busy instructor to thoroughly consider the variety of available techniques. In addition, the volume also contains many other methods that are not currently discussed in the literature. Our purpose is to share, in practical terms, a variety of methods for teaching proof-writing that have inspired our authors and that they have found useful for their students.

Structure of the Volume

The volume is organized by type of approach: Jump-Starters and Other Activities; Small Group Activities and Presentations; Whole-Class Activities; Portfolios, Journals, and Peer-Review; Inquiry-Based Learning and Flipped Classrooms; Ideas Borrowed From English and Other Composition Courses; Teaching Proofs in Non-Traditional Courses; and Long-Term Activities. Included in the chapters are techniques that involve:

- Small activities to ease students into proof-writing, such as letter-writing and the use of two-column proofs
- Critiques of incorrect proofs
- Group work, such as group exams and proof circles

- Presentation models
- Use of technology, such as wireless capabilities and wikis
- Portfolios
- Class journals
- Variations of flipped classrooms
- Inquiry-based learning
- Structuring the math major to build proof-writing skills across multiple courses

A natural question that might arise is how employing the techniques could affect content coverage. Our authors have taken care to either explain how cutting material can be avoided, or how the benefits of the improved proof-writing stemming from their methods compensate for the reduction in content. In fact, the techniques in this volume align well with the trend in secondary education to emphasize depth over breadth in mathematical learning [3, 9].

Some of the techniques can be used with virtually no change in current course structure, while others require a complete overhaul of one's teaching style or an investment of a significant amount of time. Therefore, noted next to the title of each article is one of the following difficulty levels:

- Low (L): little preparation (beyond the usual) required of the instructor and students, and little or no change in current course structure; or, can be adapted to require minimal preparation
- Medium (M): a moderate amount of preparation and time required of the instructor and students, and possibly some significant change in course structure; or, can be adapted to require less preparation
- High (H): sustained extra preparation (at least the first time the method is employed) and time required of both the instructor and students, and comprehensive change in course structure

In addition, we indicate the course level(s) in which the implementation mainly took place, with an (A) for advanced classes such as abstract algebra or real analysis, a (T) for a transitional or mid-level course such as discrete math or linear algebra, and an (N) for courses that are not traditionally considered proofs courses, such as calculus or math for liberal arts. However, we want to emphasize that the setting does not imply that the technique is specific to that subject or level; rather, most can be used in any course where proof has a role. Lastly, we have placed a † next to each article that is technology based.

Each article has the same general structure, with four main sections:

1. Background and Context: describes the type of institution and course(s) in which the technique was used and the author's motivation for using the technique, possibly including his or her personal views on the nature and/or importance of proof
2. Description and Implementation: describes in practical terms how to implement the method, including the amount of preparation required for both student and instructor, as well as how the technique fits into a required or standard curriculum
3. Outcomes: summarizes evidence of success of the method, and possible shortcomings or pitfalls
4. Extending the Method: describes adaptations or suggestions for implementation in other settings

In addition to the adaptations in the “Extending the Method” section, some approaches are described in multiple articles, from different perspectives; internal cross-references within the articles direct readers accordingly. By providing varied points of references, modes of implementation, and levels of intensity for a technique or idea, we hope that readers will see that the methods can be adapted to each instructor’s course goals and teaching style.

Instructors with a variety of backgrounds, experience levels, and teaching goals should find this book useful. Perhaps you are new to teaching proofs and are looking for ways to reach the students beyond the text you’ve been assigned. Or, you may be a seasoned instructor who is looking for new ways to target areas of weakness among your students. Regardless of your experience, this volume presents ideas that are likely to be useful for your teaching of proofs. Additionally, the reader who is interested in conducting scholarly studies of student learning may find this volume useful in conjunction with the *Notes* volume *Doing the Scholarship of Teaching and Learning in Mathematics* [5].

We hope that you will enjoy learning about these new techniques as much as we have, and will be inspired to implement them in your classes. Feel free to use as much or as little as you like, and to mix and match among methods. Our aim is that you will view this volume as a helpful resource to which you can frequently refer for ideas to help improve your students’ proof skills.

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A decorative border consisting of multiple parallel lines forming a square frame. The corners are embellished with a complex, interlocking geometric pattern of overlapping lines.

1

Jump-Starters and Other Activities

Introductory Discussions of “What Constitutes a Proof?”

Jane M. Keiser

Abstract

A handout is given to junior/senior students in an upper-level geometry course at the beginning of the semester. It contains five arguments that the sum of the interior angles of any triangle is 180° , and students are asked to distinguish which are simply good arguments and which are mathematical proofs. The handout, student deliberations, and the discussion that develops makes a nice introductory activity for any proof-based course by informing instructors about their students' thinking and helping students evaluate the criteria for establishing mathematical proofs.

Difficulty Level: Low; Course Level: Advanced

1 Background and Context

The teaching of proof is much like teaching problem-solving. You can model good strategies and outline the structure of its various forms, but the “apprenticeship of observation” is often not enough to enrich most students' conceptions of proof. Why? Because the pathway to the result is rarely formulaic or self-evident at first glance, and the tools to choose vary with each problem. Besides this, the preparation for this type of learning varies greatly among our students. Some are ripe for the learning, while others carry with them false notions from past experiences that may hinder what they believe constitutes a proof.

When I am the instructor of proof-based courses, rather than diving right into modeling the methods, I first attempt to discover what my students believe about the proof concept. In the past, when my students were asked, “What is a proof?” they overwhelmingly responded with, “a sound argument,” but I wanted to learn more about what they believe makes an argument sound.

In order to explore this question more deeply, I designed a handout (described below) that I now use as an introductory activity for any proof-based course that I teach. The first time I used it, I collected some data from a class of twenty-two Miami University juniors and seniors a few weeks into the semester of their college geometry course and again at the end of the semester. The course is required for secondary mathematics education majors; however, there are often other majors who choose to take it as well. They take it after completing the calculus sequence and a discrete mathematics course that teaches proof-writing techniques, and some students take abstract algebra concurrently.

2 Description and Implementation

2.1 Designing the handout

The handout begins with a statement so elementary and evident that all my students would already be convinced of its truth—that the sum of the interior angles of a triangle equals 180 degrees. I did this so that they could focus on the arguments rather than the statement. The instrument contains five different “proofs” of this statement. The word “proof” is in quotation marks because some are clearly not proofs; some are; and some might get different responses from different mathematicians. The goal of the instrument is not to

establish beyond a shadow of a doubt which are valid arguments and which are not; it is to generate good discussions of what constitutes a proof.

The boundary line that divides the set of all arguments into two categories—mathematical proofs and non-mathematical proofs—is not well-defined. However, by forcing the students to decide and to share to which of the two categories each situation belongs, I am given access to what they believe are the distinguishing features: “A proof must have. . .”, “A non-proof is one that. . .” Though some of the questions posed in the handout have no clear-cut answers, the classroom discussion reveals many of the criteria my students believe necessary to comprise a mathematical proof. Vinner calls the composition of mental images associated with any concept the “concept image,” and he suggests that it is much more rich and complete with connections than a simple concept definition of an idea [4]. Since many of my students have limited experiences with proof (i.e., their one geometry course in high school), enriching their proof concept images with new and different ideas is vital in an introductory proof course. The directions of the handout were

On this sheet, five situations are provided that suggest that the sum of the interior angles of any triangle is 180° . Read through them and answer the following questions for each situation.

1. Is this a mathematical proof?
2. If not, why is this not a proof?
3. Regardless of what you wrote for #1 or #2, what are the underlying assumptions that one must accept in order to accept the arguments of this “proof?”

Before I share the responses given by my students, I will describe the five proofs and why I chose them (see Figure 1). The first is a demonstration that I am certain has been seen by many of us in lower grade levels where you tear off the three corners of a triangle, line them up, and see that a straight line is formed. I certainly don’t believe Situation 1 is a proof; however, I wanted to see if my students knew that 1) a finite number of examples is never enough for a proof, and 2) there is no way of knowing from the activity that the angle is actually 180° (it could be 179° or 181° , etc.).

Situations 2 and 5 are more traditional, and 3 and 4 involved motion. I wanted a variety of styles of arguments and specifically ones that were questionable. Situations 2 and 3 are based on the assumption that the sum of the angles is a constant, s ; however, this is stated for Situation 2 and left out for Situation 3. Situation 5 relies upon the parallel postulate which is also not stated but is assumed when we state that alternate interior angles are congruent. In fact, the statement that the sum of the angles is 180° is equivalent to the parallel postulate, so each would rely solely upon the other in order to be proved. Therefore, it was intended that some of the situations presented have hidden assumptions that are not clearly stated. I hoped this issue would be raised in class since it is often a problem in a proof-based course to determine how much justification is needed. Students often ask, “How far do we need to go back with justifications? Do we need to start at the very beginning with each proof? Or, once we have established a postulate or theorem as a class, can we assume the postulate and quickly state the theorem?”

I also purposely chose not to have any of the situations written in the two-column format that is sometimes used in high school geometry classes. I did this so I could see how many students believed that a proof must have this format. Finally, I wanted to have a few arguments that were different from what they may have heard before, so I included Situations 3 and 4. Both can be demonstrated with any interactive geometry software, and I was curious to see if my students would accept a proof that relied upon dynamic rather than static objects. Some of my students have been correctly taught that the use of interactive software to illustrate theorems is not a proof. However, the arguments of Situations 3 and 4 are true of any triangle, not just the few you have the time to test with software. Both arguments also have the weakness that each step of reasoning is valid in hyperbolic geometry as well as Euclidean, but in hyperbolic geometry, the sum of angles in a triangle is less than 180° . These situations raised some interesting comments from my students.

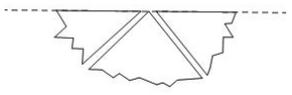
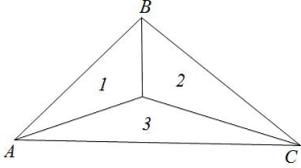
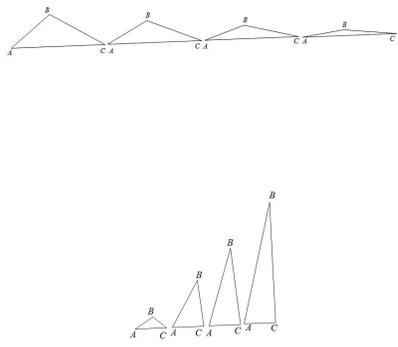
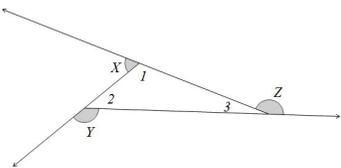
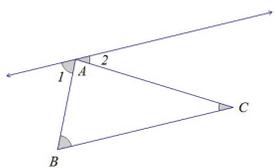
Situation with Picture	Summary
<p style="text-align: center;">Situation 1</p> 	<p>A teacher asks her students to cut out a unique triangle from construction paper (<i>unique</i> meaning different angles and different side measures from the person next to them). She has the students color in the corners of their triangles and then rip off each corner and align the three vertices as shown at left. As it turns out, all the students in the class determined that the three angles formed a straight line on one side. Therefore, the sum of the interior angles of any triangle is 180°.</p>
<p style="text-align: center;">Situation 2</p> 	<p>In this situation, we are calling the sum of the angles of any triangle, s. Looking at the diagram, you can see that a point was placed anywhere near the center of the triangle and line segments were drawn from this point to the vertices of $\triangle ABC$. We know that the sums of the angles of each of $\triangle 1$, $\triangle 2$, and $\triangle 3$ are s. Therefore, $3s$ minus 360° (the sum of the angles around the central point) must equal s (the sum of the interior angles of the large $\triangle ABC$). If you solve the equation $3s - 360 = s$, $2s = 360$, it follows that $s = 180$.</p>
<p style="text-align: center;">Situation 3</p> 	<p>As you can see in the top picture, as you let the vertex B be dragged closer and closer to side AC, angle B gets larger and larger as angles A and C get smaller and smaller. If you continued to do this, the limit of the measure of angle B would be 180° since its limit would be a straight line. The limit of angles A and C would both be 0°. Therefore the sum of the limits of angles A, B, and C would be $0^\circ + 180^\circ + 0^\circ = 180^\circ$.</p> <p>Alternatively, if you begin with any triangle and pull the vertex B upward, the angle B would begin to decrease in measure while the other two angles would increase. So the limit of angle B is 0°, and the limit of each of the angles A and C is 90°. So the sum of the limits is $90^\circ + 0^\circ + 90^\circ = 180^\circ$. You could use both arguments for any triangle since you can always find one vertex that is opposite one side.</p>
<p style="text-align: center;">Situation 4</p> 	<p>In the picture, the exterior angles of the triangle are drawn in and labeled angles X, Y, and Z. The interior angles are labeled 1, 2, and 3. Angles 1 and X, 2 and Y, and 3 and Z, are supplementary since they each form a straight line, and therefore $m\angle 1 + m\angle X = 180^\circ$, $m\angle 2 + m\angle Y = 180^\circ$, and $m\angle 3 + m\angle Z = 180^\circ$. The sum of angles X, Y, and Z is 360° since, if you walked around the outside and pivoted each time you needed to turn to trace the triangle, you would have turned an entire revolution, which is 360°. If you add together the first three equations, you get $m\angle 1 + m\angle 2 + m\angle 3 + m\angle X + m\angle Y + m\angle Z = 540^\circ$ and if you substitute 360° for $m\angle X + m\angle Y + m\angle Z$, you will solve and get that $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$.</p>
<p style="text-align: center;">Situation 5</p> 	<p>Given $\triangle ABC$, construct a line through A parallel to the line segment BC. We know that if two parallel lines are cut by a transversal, then alternate interior angles are congruent. Therefore, $\angle 1 \cong \angle B$, and $\angle 2 \cong \angle C$. We also know that $m\angle 1 + m\angle 2 + m\angle BAC = 180^\circ$ since the three angles together form a line on one side. So, by substitution, $m\angle B + m\angle C + m\angle BAC = 180^\circ$.</p>

Figure 1: Handout given to students to answer all three questions for each.

Having amassed five decent arguments, I placed them in no particular order and wrote each argument as briefly and clearly as I could. The goal was to learn more of my students' thinking and to gain an opportunity to discuss my expectations for written proofs with the class.

2.2 Conducting the activity

The activity can be conducted many times throughout the semester, but the first time I used this handout, I used it only during the first week of class. Introductory activities are a normal part of our curriculum so no content needed to be missed or cut in order to find time for it.

I asked the students to thoughtfully examine each situation and answer the three questions above for each, devoting the entire 50-minute class to the exercise. (We met three times a week.) Between classes, I read their responses and tallied their votes. The next class day, I asked some students to clarify their responses to hear their verbal explanations about what they had written. A few students were confused by the question, "Is this a mathematical proof?" because they felt that, as in the case of a "mathematical problem," what is a proof to a fourth grader may not be a proof for our level. Therefore, some of my students believed that Situation 1 *is* a proof for children but not a proof for the college level. I clarified that I wanted them to use the term "mathematical proof" if they thought most mathematicians would identify it as a proof. This changed a few responses, and their final votes are reported in Table 1.

Is it a mathematical proof?	Yes	No	Maybe
Situation 1	23%	77%	0%
Situation 2	82%	13.5%	4.5%
Situation 3	32%	63.5%	4.5%
Situation 4	82%	18%	0%
Situation 5	100%	0%	0%

Table 1: Tally of "Which are Proofs?" handout.

Their written responses were then categorized by similarity and I looked for patterns. Below I have summarized what stood out to me and what then led to some great discussions with that class and those that followed. Because the idea proved to be successful in my college geometry course, I have since designed a similar handout for my middle childhood arithmetic/algebra course that provides six "proofs" that $0.\bar{9} = 1$. It handout can be found in the Appendix.

2.3 Conducting Post-Discussions

The handout has led to varied discussions. But often there are common themes that are raised and issues that are argued. I will summarize some of these to provide instructors with ideas for questions or prompts to help generate discussion:

- Most students agree that Situation 1 is not a proof, but it is important to address those who think it is and make sure they come away from the experience clearly understanding why it is not a mathematical proof. Situation 1 is a sound and convincing argument and it certainly is suitable for convincing children; however, our students need to understand why it is not sufficient for a mathematical proof.
- Students need to understand that good proofs explain each step thoroughly and leave no statements unjustified. However all proofs begin with at least some assumptions (called postulates), and it's impossible to justify these. You will find that Situation 2 and 3 both rely upon the assumption that all triangles' interior angles have a constant sum. However, many of my students voted for Situation

2 and against Situation 3, but when they inspected their thinking, the assumption was the reason for their rejection of Situation 3. One of my students said, “I don’t see how it’s telling you that the sum of the angles in the first picture is 180 since they only said [that in] the last picture it was, and those are totally different pictures.” Yet this student accepted this premise easily for Situation 2.

Also of interest is that a few of the students claimed that Situation 5 was a mathematical proof because “no assumptions” were made. These comments were made even after the very first day of the course when it had already been discussed that nothing can be proven without “some” beginning agreed upon assumptions or postulates. This led to a nice discussion reiterating that we should eliminate as many assumptions as possible, but eliminating all is impossible.

- What was mentioned often by students about Situation 3 was that they didn’t believe that “at its limit,” you would still have a triangle, and therefore you could not use information about a straight line to inform you about a triangle. One student commented, “To me, a limit is a sort of asymptote, so really what this says to me is that the angles get really close to 180, but never really 180. To accept this, I would have to accept the fact that the limit would be 180 inclusive.” Many of my students have trouble with the limit arguments on both handouts (see Appendix A for the second handout). This is something that we must address, because students truly don’t believe that a limit equals something; rather, they believe it approaches something. We had a great discussion in our class concerning this.
- Finally, the issue of structure and/or formality is often raised when discussing what most students liked about their identified “proofs” and didn’t like about the “non-proofs.” Because the proof structure prevalent in most high schools’ mathematics curricula is presented in geometry class using a two-column approach, it may be the case that students’ entire concept image is based upon it. This activity, while still being geometric, allows the instructor the opportunity to challenge some of these images with examples and non-examples of proofs. In the end, what is most vital in a proof is the quality of its argument so in this context of teaching of proof, structure should be of less importance. Gila Hanna writes,

Once having embarked upon an examination of a proof, a mathematician is still much more interested in the message embodied in the proof than in its formal codification and syntax. The mechanics of proof are seen as a necessary but ultimately less significant aspect of mathematics. [1, p. 72]

3 Outcomes

There are three major positive outcomes that result from using this activity.

At the beginning of the semester, when your students are unknown quantities—sitting there passively taking notes—the activity reveals to you their thinking and their level of maturity. It does this well because students are motivated by the simplicity of the arguments and each can serve as exemplars in class discussion. “This one IS a proof because..., however this one IS NOT because...” The activity is useful since usually students don’t contribute readily in mathematics classes, especially at the beginning of the year. The activity draws them in, and they really all have opinions that they want to share.

Besides providing information to the instructor, the activity helps to strengthen and build up what might be an insufficient concept image for the concept of proof. Moore established that most of his students had “inadequate concept images,” and determined that many students progress through the sequence, “Images → Definitions → Usage.” He states:

[this] sequence illustrates that the students’ ability to use the definitions in proofs depended on their knowledge of the formal definitions, which in turn depended on their informal concept

images. The students often needed to develop their concept images through examples, diagrams, graphs, and other means before they could understand the formal verbal or symbolic definitions. It seems that this reliance on concept images for understanding definitions and notation may diminish as the students move beyond this transition point in their learning of mathematics and become more comfortable with standard notation, mathematical grammar and syntax, and the logical structure of proofs.[2, p. 262]

This finding provides evidence that using example arguments and discussing their features or lack of features may help improve informal concept images, which is necessary for success in actual proof-writing.

The activity also provides a forum to establish some guidelines as a class for future proof-writing. Students learn what level of formality is expected, what needs to be stated up front and what can be excluded given an axiomatic system. When I finished the first semester where I had used the activity just once during the first week, I asked a 4-point question (out of 100) on the final exam, “What is the difference (if there is one) between a convincing argument and a mathematical proof?” Here are excerpts from two students’ papers:

Student 1: ...“In my mind,...a mathematical proof is something that demonstrates the validity of a conjecture beyond the shadow of a doubt. A mathematical proof is complete and proves something to be true for all possible cases. It is something that is able to “withstand” critique...”

Student 2: “This process of following a clear path from the known to the unknown is the first of three criteria I feel proofs must meet. Nothing must be assumed beyond the most basic, elemental definitions and axioms.

The second is structure. Although not everyone agrees on the strict set of guidelines to be used in mathematical proofs, most would grant that there must be a rigorous logical structure present.

...

The third point on which arguments differ from proofs is in their inclusiveness. For example, the first triangle “proof” with the torn paper [angles from] triangles cannot account for all possible triangles, and for this reason is not a mathematical proof...”

The fact that students still used examples from the handout from the first week to help describe their thinking during the last week is evidence that the activity was worthwhile and they learned something from it. Both instruments shared in this chapter serve as useful tools for learning about students’ thinking prior to any other discussions about proof. They also often generate heated discussions, bringing to the forefront issues of hidden assumptions and levels of formality and structure which helps students to be more critical about what counts and what does not count as valid reasoning.

4 Extending the Method

The activity could be extended by revisiting the handout periodically throughout the semester to see if any new questions are raised as the result of becoming more proficient in proof-writing. As I wrote above, when I first used it in a college geometry course at the beginning of the year, I returned to it only at the end of the semester when on the final I asked students to refer back to it. If it had been revisited several times throughout the semester, I think that I would have had richer reflections for that question on the final. (Editors’ note: For another activity in this volume verifying the validity of a proof, see Pfeiffer and Quinlan [3].)

The activity also lends itself to a large lecture course since the five arguments are fairly simple to read and interpret and could be discussed in small groups first rather than in a full-class discussion. After spending time sharing their thinking with each other, the small groups could report their opinions to the entire class.

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B Resources for Other Questionable Proofs

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30 “Classical” Statements and What We’ve Learned

Luke Bennett

Abstract

I have developed a list of 30 “classical” mathematical statements that I distribute on the first day of my Introduction to Mathematical Reasoning class. The students are to break these statements down into categories that determine their confidence level of knowing whether the statement is true or false and also whether they are able to give justification that is conclusive. Their final project for the class is then to do the activity again. This allows the students to realize how much they have grown throughout the course, and allows them to see what it means to truly have a conclusive justification.

Difficulty Level: Low; Course Level: Transitional

1 Background and Context

Grand View University is a small liberal arts institution with about 2300 students, 800 of whom live on campus. We have between five and fifteen graduates each year in applied mathematics. Roughly half of them go on to become secondary education teachers. Our majors are required to take a course titled “Introduction to Mathematical Reasoning”, which is the first time students see proofs. The course has a pre-requisite of first-semester calculus, so most of the students are either freshmen or sophomores. The enrollment in the course is capped at twenty students, and usually has between ten and fifteen. All theoretical courses later in the curriculum have this one as a prerequisite. Historically, our students have struggled with the idea of mathematical proof and its relevance to their career aspirations. This inspired me to look into ways to help students realize what the goal of the class was early on and also to show the importance of proofs for all students, whether they plan on being secondary educators, work in industry, or continue on to graduate school.

In the past, I have tried both lecturing and inquiry-based learning to teach the course. Neither style seemed to fit the needs of my students, so I felt I needed a happy medium of the two of them. While lecturing, I noticed that the students were able to do proofs in the short time after the lecture, but they struggled with the final exam, and, more importantly, future courses. When I tried inquiry-based learning, the class didn’t flow quite like I wanted it to. The combination of my inexperience with the method and the lack of confidence that our students have with the material made the class struggle with the basics, which hindered the class as a whole. Currently, I do a mix between lecture and inquiry-based learning. Sometimes, I will go through a whole proof with the class. Other times, I will have the students work on problems and discover the solutions on their own.

2 Description and Implementation

To help students realize why proofs are important and where they are used, I start the first day of class with “30 ‘Classical Statements’”. I have identified them as “classical” because they are ones I feel all math majors should know how to prove or identify as false. While creating the list, I made sure there was diversity in the types of statements. I included statements that are familiar to them but are hard to prove, easy counterexample problems, statements that include terminology the students probably have never seen before, and

statements I feel all math students should know, such as “ $\sqrt{2}$ is irrational.” I also included a number of false statements to show that they are something we, as mathematicians, encounter and should be able to identify. A few examples are listed below. (For a complete list, see Appendix A.)

- A number is divisible by 4 if and only if its last two digits are divisible by 4.
- If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
- There are infinitely many prime numbers.
- Every positive integer is the sum of distinct powers of two.

During the first week, the students think about the statements, determine whether they are true or false, and then give a justification if possible. The students put each into a category to determine their confidence. The categories are adopted from Carol Schumacher [2], with a little editing and an addition of a category. They are

- a. I am confident I answered the T/F correctly, and the justification I gave is conclusive.
- b. I am confident I answered the T/F correctly but am not confident that the justification I gave is conclusive.
- c. I am confident I answered the T/F correctly and am confident that the justification I gave is *not* conclusive. (If you have no justification at all, your answer falls into this category.)
- d. I could not decide whether the statement is true or false.
- e. I do not understand some of the terminology so am unable to answer the question.

I feel this is a good activity for the first week of class to get students thinking about these ideas before they are formally introduced. The students are allowed to work together to think about all of the statements, but are required to hand in their own work. Since I use the first day of class to start the project, this allows the students to work together from the very beginning and sets the precedent that they should find a classmate with whom to discuss math. Their answers are due by the end of the week. I make it clear that I don't expect justifications for all the problems, but encourage them to try to come up with justifications for some of them if they are able.

In the past, I didn't allow students to think about these statements on their own. I initially set the ground rules, and then started proving and disproving statements. There were several issues that I encountered. First, I realized students didn't necessarily know definitions at the end of the semester. For instance, they struggled with the definitions of even and odd integers. After I implemented this project on day one, students were forced to think about what it means to be even or odd on their own before seeing the formal definition. This makes the definition much easier for them to work with. Also, I realized that they were not familiar with some common notations, like summation. Now, I give them statements with summation symbols to see if they know this. Another issue is algebra skills. I give statements that test these skills from the beginning, such as whether $(a + b)^2$ equals $a^2 + b^2$. This gives me a good base for identifying any student inadequacies.

Most importantly, starting the students with the statements at the beginning gets them to really think about why something works from the very beginning of class. It also creates curiosity, as I tell them I will answer the questions they have about this activity throughout the semester. One thing that boggles their minds from the beginning is the statement “ $0.\overline{9} = 1$.” To me, this shows that the students are not used to working with different representations of the same thing. All these issues make for good discussions at the beginning of the semester.

The final piece of this activity is for the students to do a project at the end of the semester. They are given the same 30 questions and they do the same thing they did the first week of class. This time, the expectation

is for all of their answers to fall into category (a) from the categories above. They are given a few weeks to complete the project. Many of the questions I end up going over throughout the semester, but I leave five or six that they never do as a class and never discuss after the first week. This allows students to see if they can apply the material they learn throughout the semester to statements we never examined formally. This serves as a great reminder to them how much they have grown throughout the semester, and it is also great assessment data for our department as we collect information about their progress.

The main work for the instructor is determining the 30 statements to use and grading the students’ work. The creation of the list is a lot of work the first time it is used but little work to maintain. I see myself making adjustments to it from year to year, but only a statement or two at a time. The changes to the list that I see necessary are ones to make sure the students work through many different types of problems. As I analyze the problems, I’m sure I will find areas that need to be addressed. The amount of time grading the students’ work will not change drastically from year to year. The initial grading of the first-week activity is quick as the students attempt very few proofs. The end project takes time to grade as there are 30 proofs or counterexamples, with varying lengths. This takes a lot of time, especially as the number of students in the class increases. The length of the list could easily be changed, however, to cut down on the time. Overall, I feel the time is very well spent and useful in helping students with their proof-writing ability and critical thinking.

The activity has not diminished the content of the course. But, if an instructor is not willing to give the entire first day of class to it, s/he can spend 10-15 minutes introducing it in class and leave the rest for homework. I am hopeful that the activity will strengthen the students’ abilities long-term and give them a better background for our theoretical courses, thus gaining more time to spend on the content in them. The reader may be interested in a similar method involving assessing the truth value of a statement; for details, see Ensley’s article [1] in this volume for a technique involving finding counterexamples.

3 Outcomes

My main goals when creating the activity were to improve students’ ability to write proofs, give them concrete evidence that they had learned a lot, and to use the data for our assessment report.

3.1 Strengths

The way the activity drove the semester really helped our students get excited about proving statements. They were much more active in class and inquisitive. I had secondary education majors discussing how they felt it was relevant to their understanding of algebraic ideas and would help them with their teaching. There were many cases where we would be discussing a topic of proof throughout the class, and a student would bring up the idea that this would work for one of the problems given to them on the first day of class.

3.2 Numerical Outcomes

One can see the growth of the students’ ability to write proofs through the data in tables 1 and 2. The columns are taken from the categories discussed previously. The percentages are for total number of statements through all students. The percentages in table 1 represent 13 students with 30 statements each giving a total of 390 statements. Two students dropped the class, so the percentages in table 2 represent 11 students for a total of 330 statements. The data convey the students’ confidence regarding the correctness of the statements. One can see that there were still 10.61% in category (d) in the project at the end of the semester (Table 2). This seems rather high, but a lot of them were ones students left blank, so that is the category where they were placed. It is hard to determine if they didn’t know the truth value of the statement or if they didn’t have time to work on it. The data show only 1.54% of statements received conclusive justification during

the first week, while 70.30% were conclusive for the final project. Although the data are solely based on students' perception, my analysis of their correctness backs this up. Students were able to learn what makes a conclusive proof throughout the course.

% of (a)	% of (b)	% of (c)	% of (d)	% of (e)
1.54	13.59	45.90	23.59	15.38

Table 1: First week data

% of (a)	% of (b)	% of (c)	% of (d)	% of (e)
70.30	7.27	11.82	10.61	0.00

Table 2: Project data

Overall, I feel the activity has helped our students grow in their mathematical reasoning. A lot of our students struggle with confidence, and the ability to show them concrete evidence of their growth allows them to build self-belief in their ability to write proofs. The activity had our students thinking about proofs from day one, which wasn't always the case since our material starts with basic logic. I see this as a very useful tool for both our students and our department's assessment to show students are achieving our goal of having the ability to apply techniques of mathematical reasoning to construct and analyze arguments and hypotheses.

3.3 Things to Consider

During the first part of the semester, many students wanted to know the truth values of the statements to make sure they were thinking correctly, so I went over about a third of them in class informally. I realized that this may hinder the effectiveness of this activity throughout the semester, as I want students to come to an understanding of the ideas themselves. In the future, I plan on having students discover the truth values of all of the statements on their own. I will still make comments on students' initial work, but more general ones. Another thing to address is that many students did not answer all thirty statements for their project. I aim to find ways to motivate students to finish all the problems. I may give the project out earlier and give multiple due dates for a certain number of proofs to be completed. For instance, I will have them hand in five proofs every week for the last six weeks of class. The lack of completeness does skew the data a little, but overall, one can see growth from the students.

4 Extending the Method

This method can be used in many types of classes. For larger class sizes, the instructor may want to decrease the number of statements used to manage the time spent on grading. For smaller class sizes, there should be no issues with it. Another dimension that could be added to it would be for the students to critique each other's proofs to have more practice with reading and understanding them.

References

- [1] Douglas E. Ensley, "Counterexamples and Correspondence: Establishing Two Roles in a Proof", in this volume.
- [2] Carol Schumacher, *Chapter Zero: Fundamentals of Abstract Mathematics*, Addison Wesley, Boston, 2001.

Appendix

A 30 “Classical” Statements

1. A number is divisible by 4 if and only if its last two digits are divisible by 4. (True)
2. A number is divisible by 8 if and only if its last three digits are divisible by 8. (True)
3. A number is divisible by 3 if and only if the sum of its digits are divisible by 3. (True)
4. If x is an integer, then $x^2 \geq x$. (True)
5. If x is an integer, then $x^3 \geq x$. (False)
6. There exists a real number x such that $x^3 = x$. (True)
7. If 3 divides ab , then 3 divides a or 3 divides b . (True)
8. If x divides ab , then x divides a or x divides b . (False)
9. $\sqrt{2}$ is an irrational number. (True)
10. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd. (False)
11. If $x + y$ is even and $y + z$ is even, then $x + z$ is even. (True)
12. If x is an even integer, then x^2 is an even integer. (True)
13. If x^2 is an even integer, then x is an even integer. (True)
14. Every positive integer is the sum of distinct powers of two. (True)
15. Every positive integer is the sum of distinct powers of three. (False)
16. If x is an integer, then x is even or x is odd. (True)
17. If x is an integer, then x cannot be both even and odd. (True)
18. There are infinitely many prime numbers. (True)
19. $\sum_{i=1}^n i = \frac{n(n+1)}{2}$. (True)
20. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$. (True)
21. $0.\overline{9} = 1$. (True)
22. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. (True)
23. $(a + b)^2 = a^2 + b^2$. (False)
24. If $ab = 0$, then $a = 0$ or $b = 0$. (True)
25. If n is an odd integer, then 8 divides $n^2 - 1$. (True)
26. If x is a positive real number, then $x + \frac{1}{x} \geq 2$. (True)
27. The product of a rational number and an irrational number is irrational. (False)
28. Given that A , B , and C are sets, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (True)
29. Given that A and B are sets, $(A \cup B)^c = A^c \cap B^c$. (True)
30. Given that A , B , C , and D are sets, if $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$. (True)

Using a Proof-Scrambling Activity to Enhance Student Understanding of Mathematical Proof*

Russell E. Goodman

Abstract

The author shares a proof-scrambling activity he created for his introduction-to-proof course. Students are given the sentences, in random order, from a proof and are asked to arrange them in order to recreate the proof. Devised as an in-class activity, in-class quiz, or take-home quiz, proof-scrambling helps students improve their understanding of the language, conventions, and flow of mathematical proofs in an active and holistic way.

Difficulty Level: Low; Course Level: Transitional

1 Background and Context

I want to share an interesting activity I have done with my students a few times that, while not involving *writing* mathematical proofs, helps students better understand the language, conventions, and flow of mathematical writing by organizing the scrambled sentences of existing, logically correct, proofs. Although the activity does not address all relevant skills for effective proof-writing, it helps students improve their writing and produce better proofs.

1.1 My Institution

I teach at Central College, a four-year, residential, liberal arts college of around 1,400 students in the rural midwestern town of Pella, Iowa. I teach in a combined mathematics and computer science department with eight full-time faculty consisting of five mathematicians and three computer scientists. We currently have 27 mathematics majors, 28 computer science majors, 18 actuarial science majors, and a fair number of mathematics minors. The majority of our mathematics graduates go on to teach at the middle-school or high-school level, quite a few go into industry or engineering, and a few enter graduate school for mathematics or an associated discipline like statistics.

1.2 Our Introduction-to-Proofs Course

I have implemented the following activity in our Foundations of Mathematics (“Foundations”) course, which I have taught four times in the past eight years. This is our intro-to-proofs course, which a typical mathematics major (or minor) would take in the fall of her second year of study. The typical Foundations student has completed Calculus 2, despite Foundations requiring only Calculus 1 as a prerequisite. It is also common for Foundations students to have already taken either Differential Equations or Linear Algebra and so I have grown to expect a bit of mathematical maturity from them. In my experience, the course enrolls 15-20 students and the grade distribution is approximately normal but slightly skewed towards higher grades.

*Parts of this chapter were originally published by MAA’s Mathematical Sciences Digital Library (MathDL). That material can be found at <http://mathcomm.org/folder/proof-scrambling-activity-2/>.

This is because I make my expectations clear to my students and, anecdotally, students are more motivated and perform better when they understand what is expected of them. Thus, my students have a strong sense of what they need to do to earn better grades. Students moving on in our mathematics program must have taken Foundations as a prerequisite for upper-level courses, including Abstract Algebra, Geometry, and Real Analysis.

1.3 My Inspiration

Through my experiences teaching Foundations, I learned that I need to help my students develop their logical thinking and proof-writing skills. As we know, mathematicians are concise in their writing with every sentence precisely stated, logically following preceding sentences, and logically implying succeeding sentences. My students, and likely students at many institutions, fared generally well with learning logic, but they struggled to demonstrate an understanding of the flow of a proof. They needed an activity that helped them to see the act of proof-writing broken down into simple steps. I already had many examples of proofs for us to discuss in class, thus I knew I had proofs I could slice-and-dice into pieces for the students to reassemble. Thus was born my proof-scrambling activity.

2 Description and Implementation

2.1 Description

My proof-scrambling activity proceeds as follows: students are given the sentences and calculations from a correct mathematical proof in one-sentence or one-calculation chunks. The students' task is to arrange them in an order producing the original mathematical proof.

The roles for the activity are clear in that students are expected to work individually or in a group (both are effective) to arrange the existing scrambled sentences of a proof, and the instructor's role is to observe and speak with students about the meaning of the words and phrases mathematicians use in proof-writing. These are opportunities to help students learn, for example, that words like "therefore," "next," etc., indicate the flow of a proof. The word "therefore" is a backwards-looking word as it stresses to the reader that statements preceding the sentence where it appears lead to what the author is about to say. On the other hand, a word like "next" is forward-looking. Using "next" helps the reader see the logical direction the author is about to take in the proof. Additionally, the students will see how mathematicians put formal definitions of mathematical structures to use in constructing proofs. In order to reassemble the scrambled sentences of a proof, the students have to consider the structures in play and understand how the proof's author utilized their definitions to logically move forward to create a correct proof. As a result, students appreciate the value of using a structure's formal definition. Furthermore, reassembling the scrambled proofs helps students learn how mathematicians (and thus they!) must rely on formal definitions of terms rather than constructing informal descriptions of mathematical constructs.

In my experience using this activity in my Foundations classes, I have not had to look far to find reasonable examples for the students to attempt. If one were to spend even a few minutes in a number theory, abstract algebra, or any other moderately high-level mathematics textbook, they would likely find a proof of reasonable length for students to be able to unscramble in, say, ten minutes of a class period. For an in-class or take-home quiz, one would look for a slightly longer or more complex proof for the students to unscramble. I will introduce a few examples I have used with some success.

The first proof an instructor might use for an in-class activity includes the straightforward steps in a direct proof of the theorem

Theorem 1. *Let $x, y \in \mathbb{Z}$ and assume x is even and y is odd. Then $x + y$ is odd and $x \cdot y$ is even.*

For an in-class quiz, the instructor might use a proof by contradiction of the theorem

Theorem 2. *Every integer greater than 1 is expressible as a product of primes.*

And finally, depending on the instructor's students, he could offer a proof of the following theorem as a challenging in-class group activity, or as a take-home quiz:

Theorem 3. *Let \mathcal{H} be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let T be a bounded linear operator on \mathcal{H} . Then there exists a bounded linear operator T^* on \mathcal{H} such that, for all x and y in \mathcal{H} , $\langle Tx, y \rangle = \langle x, T^*y \rangle$.*

The fun thing about Theorems 2 and 3, whose proofs are provided in the appendix, is that the instructor's students do not need to have had any formal lessons on prime numbers or know anything about Hilbert spaces, inner products, or bounded linear operators to be able to reassemble the sentences of their proofs. Moreover, Theorem 3 is most useful due to its complexity. The typical student will not be familiar with the structures in it and can be guided to focus on the accompanying language and how it indicates connections between phrases. The focus of the activity is to understand the logic of a proof and how its pieces come together, so I believe the instructor would enjoy using Theorems 2 or 3. While the content may be foreign and intimidating, students will likely still successfully assemble the correct proof.

I encourage the reader to view a sampling of my activities on the Mathematical Sciences Digital Library website [1].

2.2 Implementation

I have implemented this activity during class time (as a group activity or a quiz) and as out-of-class assignments (take-home quizzes). It is a relatively short group activity that requires only a bit of preparatory work from the instructor. The instructor would introduce the activity, set the students to work, and then summarize and provide closure at the end of the activity period, possibly calling on a group of students to share their results.

My recommendation is to sequence the use and frequency of the activity in the following order in one's unit or course on proof-writing:

1. one or two small group in-class activities (10 minutes each)
2. one in-class individual activity (12 minutes)
3. one in-class individual quiz (12 minutes)
4. one take-home individual quiz.

This sequencing allows the instructor to gradually give students proofs of increasing length or complexity while leading them to independently demonstrate their understanding of the language, conventions, and flow of a mathematical proof. There is obvious flexibility in one's use of this activity and one should consider that as she plans to implement this activity in her course. Finally, I should point out that the enterprising student might search for a correct proof for his take-home quiz, but he is either unlikely to find the exact language and notation for a proof of Theorem 2, or even the theorem itself for one like Theorem 3.

For the instructor, the preparation is primarily to make copies of the assignment and to bring scissors and tape to class. Many students are visual or tactile learners who prefer to have their hands on the individual sentences and to see where they might go on a piece of paper. As a result, I have written the activity handout in a form where students can cut the already-mixed sentences into slips of paper, then arrange and tape them onto a blank sheet as their output. Having students cut out the scrambled sentences also saves the instructor a great deal of time. Preparing students for the activity is necessary by way of discussing how mathematicians use words in their writing (as described in Subsection 2.1) and how important concision is for mathematicians. In my experience, I have not had to cut any other course content while using the activity, as it is aligned with what I already cover in my course.

3 Outcomes

I have implemented the activity with my last three classes of Foundations students. It is a lively activity on which the students have always enjoyed working. It is challenging enough to make them think while being straightforward enough that they are generally successful in correctly completing it. A shortcoming is that, while one might already have a large quantity of proofs to use, one would need to take the time to produce classroom-ready materials for them. This shortcoming underscores the need for the instructor to put in some work ahead of time to prepare a variety of scrambled proofs, encompassing a variety of proof techniques.

Also, while I have student attitudinal data on the usefulness of my Foundations course (quite positive!), I unfortunately have no data on student attitudes towards this specific activity. Anecdotally, however, it has been quite successful in my classes, and others have reported the same, as I have received quite a few requests for the materials from my network of mathematical colleagues.

3.1 Primary Outcomes

For the purposes of this volume, “primary outcomes” are outcomes most germane to the proof-writing process and those that have helped students show improvement in that area. I feel confident stating that my students showed significant improvement in their ability to write proofs. In particular, I noticed their proofs began to have better logical consistency from sentence to sentence as the semester went on. I attribute this partially, but not solely, to my activity, as its purpose was to force students to pay attention to how one sentence in a proof should be implied by previous sentences while implying subsequent sentences. Furthermore, students learning from this activity began to show more intentionality with their word choices in proofs. As expected, they learned to use words like “therefore,” “next,” “consequently,” and others to effectively indicate the flow and direction of a proof. A colleague at a nearby university shared the following with me regarding her classroom conversations related to this activity:

Usually [to wrap up the activity] I discuss structure with the students. For example, which statement sounded like it was the “introduction” of the proof? Which statements sounded like they were justifying other statements, and therefore must be a follow up to another statement? How do most proofs conclude, i.e., which statement sounds like it would wrap things up?

This colleague went on further to share, “I agree that they [scrambled proofs] are a great way to ease students into getting a feel for the mechanics of a certain type of proof (direct, indirect, even induction).” As a result of this type of feedback, I feel confident reiterating that the activity is successful at helping students improve the level of their mathematical proof-writing.

3.2 Secondary Outcomes

For the purposes of this volume, “secondary outcomes” are outcomes that are benefits of doing the activity, but are more general measures of success. I can report from my experience that the two biggest secondary outcomes are that students definitely enjoyed participating in the activity, and doing so improved their confidence in reading and understanding proofs. This confidence boost was especially timely, as one colleague shared with me, because her students participated in the activity at a time when they felt that proof-writing seemed to be “way over their heads.” One final indication that this activity has been viewed as successful is in my departmental colleagues’ willingness to adopt a scrambled proof as part of our assessment of student learning. We collect scores from students’ work on reassembling a scrambled proof to assess how well we are reaching our goal of helping students learn to “employ logical reasoning skills and produce mathematical arguments.”

4 Extending the Method

I believe the activity would be a reasonable one to attempt in a larger class of, say, 40–50 students, perhaps in groups of three to four. One could also consider converting it into an electronic one to be performed on computers or mobile devices, rather than with paper, scissors, and tape.

References

- [1] “Proof-scrambling activities: Mathematical Communication pages of MathDL,” MAA Mathematical Sciences Digital Library, mathcomm.org/folder/proof-scrambling-activity-2/, accessed January 2015.

Appendix

A Proofs of Theorems 2 and 3

Theorem 2. *Every integer greater than 1 is expressible as a product of primes.*

Proof. We proceed using proof by contradiction. Suppose there are some positive integers greater than 1 that are not expressible as products of primes. By the Well-Ordering Principle, there must be a smallest such integer n . Then n must be a composite number (non-prime). As a result, there must be an integer a such that $1 < a < n$ and such that a divides n . Using the definition of divisibility, there is an integer b with $a \cdot b = n$. It also must be the case that $1 < b < n$. But a and b must then be expressible as products of primes due to the minimality of n . Thus, $a = p_1 \cdot p_2 \cdot \dots \cdot p_s$ and $b = q_1 \cdot q_2 \cdot \dots \cdot q_t$ where all p_i and q_j are primes and $s, t \geq 1$. Hence, we see that $n = ((p_1 \cdot p_2 \cdot \dots \cdot p_s) \cdot (q_1 \cdot q_2 \cdot \dots \cdot q_t))$. This is contrary to our assumption about n . Therefore, it must be the case that every integer greater than 1 is expressible as a product of primes. \square

Theorem 3. *Let \mathcal{H} be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let T be a bounded linear operator on \mathcal{H} . Then there exists a bounded linear operator T^* on \mathcal{H} such that, for all x and y in \mathcal{H} , $\langle Tx, y \rangle = \langle x, T^*y \rangle$.*

Proof. Let \mathcal{H} and T be as in the hypothesis. We will construct the required T^* vector by vector. Let y be a fixed vector in \mathcal{H} and consider the function f_y defined on \mathcal{H} by $f_y(x) = \langle Tx, y \rangle$ for all values of x . Observe that this function f_y is linear in x by standard properties of the inner product and that it maps \mathcal{H} to the complex numbers. Further, f_y is bounded, since for any x , $|f_y(x)| = |\langle Tx, y \rangle| \leq \|T\| \|x\| \|y\|$. Therefore, f_y is a bounded linear functional on \mathcal{H} and since \mathcal{H} is self-dual, there exists a unique vector y^* in \mathcal{H} such that $f_y(x) = \langle x, y^* \rangle$ for all x .

Our candidate for T^* is to set, for each y in \mathcal{H} , $T^*(y) = y^*$ where y^* is as constructed above for y . Next, observe that by construction, T^* satisfies $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all x and y in \mathcal{H} . First, clearly T^* so constructed maps \mathcal{H} to \mathcal{H} . Further, to show T^* is bounded, note $\|T^*y\| = \sup \{ |\langle x, T^*y \rangle| : x \in \mathcal{H}, \|x\| \leq 1 \} = \sup \{ |\langle Tx, y \rangle| : x \in \mathcal{H}, \|x\| \leq 1 \} \leq \sup \{ \|T\| \|x\| \|y\| : x \in \mathcal{H}, \|x\| \leq 1 \} = \|T\| \|y\|$. Finally, to show that T^* is linear, a computation shows that for any complex numbers α and β and y_1 and y_2 in \mathcal{H} , the vector $\alpha T^*(y_1) + \beta T^*(y_2)$ satisfies $\langle Tx, \alpha y_1 + \beta y_2 \rangle = \langle x, \alpha T^*(y_1) + \beta T^*(y_2) \rangle$ for all x , and using uniqueness, therefore, $T^*(\alpha y_1 + \beta y_2) = \alpha T^*(y_1) + \beta T^*(y_2)$ as required. Thus, the candidate for T^* is satisfactory, and the proof is complete. \square

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Dependency Trees

Nazarré Merchant

Abstract

A theorem encountered by a student often relies on numerous previous works. Understanding this reliance is a necessary step in comprehending the logic of the proof. Here I describe a “dependency tree” assignment in which a student represents in tree form the dependencies on theorems, statements, and lemmas encountered earlier in a course for its proof. The tree structure serves to visually reinforce the logical construction of the proof, thereby facilitating comprehension.

Difficulty Level: Low; Course Level: Advanced

1 Background and Context

While teaching a mathematical logic course (a heavily proof-reliant course), I encountered a situation familiar to many mathematics instructors: my students, even my best, got lost in the details of proofs (both proofs they were reading and those they were producing), failing to see how many proofs built directly on previously encountered work. It was the desire to emphasize the interconnectedness of the material in the course, besides the immediate goal of understanding, that led me to assign what I call the “dependency tree” assignment.

As mentioned, the course in which I first used this assignment was Mathematical Logic, offered at Eckerd College, a small liberal arts school located in Saint Petersburg, Florida. The school has a student population of approximate 1850 and is residential, with over 80% of the students living on campus. In the Mathematical Logic class, there were 22 students, most of whom were junior and senior math and physics majors who had taken a number of previous mathematics courses, some of them proof based. The students were highly motivated and competent and the grade distribution reflected this: there was about an even split between As and Bs with one or two Cs. A majority (60%) of our math majors go on to graduate school in mathematics or a closely related field, and a smaller percentage, though still large, of our physics majors go on to graduate school in physics or a related subject.

The course was a senior-level mathematics course that counted as one of the eight upper-level math courses that mathematics majors must take to graduate. It also counted as a course for the math minor. Even though it counted towards the major and minor, the course was not required; majors and minors could have avoided it and taken other upper-level courses. The course fulfilled no requirements outside of the math department. The prerequisite was merely a single-variable integral calculus class. At Eckerd there is no transition class from lower-level math to upper, so for some students this was their first exposure to formal proofs.

While originating in my math logic course, I have since used this technique in other upper-level undergraduate math courses including Topology and Abstract Algebra. The students in them shared the same demographics as the math logic course: highly motivated and interested math and physics majors, some of whom were being exposed to rigorous proof-writing and reading for the first time.

2 Description and Implementation

The assignment works as follows: the instructor chooses one theorem recently encountered. It can either be a theorem the instructor has presented and proved in class or a theorem the students have proved themselves. From it the student must produce a dependency tree that consists of all theorems, propositions, and lemmas that the theorem requires, organized as a tree.¹ This is done so that if the assigned theorem relies on Theorem B and Theorem B relies on C the relationship is produced in tree structure. In addition, the student provides a brief summary of how the theorem uses each of the given statements, often only one or two sentences. One can require that the tree have any depth appropriate for the course. I found that depth two was often sufficient to reveal the structure of the proof and its relationship to previous theorems, and also, trees of greater depth can eclipse the capacity of students' comprehension. The assignment serves the useful purpose of visually representing the interrelated and dependent nature of the material.

While I initially employed the technique in my Mathematical Logic course, below is an example from the likely more familiar domain of abstract algebra. In it, I ask students to construct a dependency tree of depth two for the second isomorphism theorem for rings (material taken from *The Basics of Abstract Algebra* by Paul Bland; it is the third isomorphism theorem in some texts).

Second Isomorphism Theorem for Rings *If I_1 and I_2 are ideals of a ring R and $I_2 \subseteq I_1$, then*

1. I_1/I_2 is an ideal of R/I_2 .
2. The factor rings $(R/I_2)/(I_1/I_2)$ and R/I_1 are isomorphic.

Proof. It is easy to show that I_1/I_2 is an ideal of R/I_2 . For the second part, define $\phi : R/I_2 \rightarrow R/I_1$ by $a + I_2 \mapsto a + I_1$. It is straightforward to show that this is well-defined, a ring homomorphism, and surjective. An element $a + I_2$ is in the kernel of ϕ iff $a + I_2 = I_1$. But this is true exactly when $a \in I_1$. Therefore the kernel of ϕ is I_1/I_2 . The proof is completed by invoking the first isomorphism theorem for rings. ■

Instructions to Students for Dependency Tree Assignment *Write out a dependency tree for the second isomorphism theorem for rings of depth 2. This requires of you two things. (1) Write out a tree, having depth 2, that organizes all theorems, propositions, corollaries, and lemmas that this corollary uses in its proof, and all thms., props., cors., and lemmas that those statements use in their proofs. (2) For each node in the tree (thm., prop., etc), write a short statement explaining how the above node on the tree depends on the lower node. These statements should be brief; a sentence or two is all that is required.*

The correct result of the assignment will depend on the presentation and organization of the theorems in the text. Even so, many theorems do not vary significantly by text. Below I give a typical set of theorems that might appear immediately before the presentation of the second isomorphism theorem. I then give what might be a typical student's final product for the assignment, including the dependency tree and the brief summaries the student might produce explaining the dependencies.

First Isomorphism Theorem for Rings *If $f : R \rightarrow S$ is a ring epimorphism with kernel K , then R/K and S are isomorphic rings.*

Theorem A *If $f : R \rightarrow S$ is a ring homomorphism, then the kernel of f is an ideal of R .*

Theorem B *If I is an ideal of ring R , then R/I is a ring.*

Theorem C *If $f : R \rightarrow S$ is a ring homomorphism, then $f(0_R) = 0_S$, where 0_R and 0_S are the additive identities of R and S respectively.*

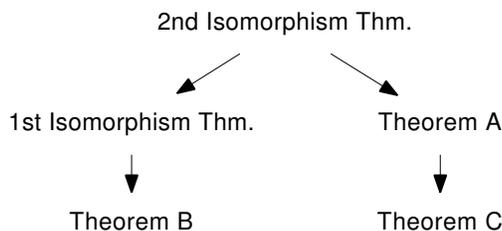


Figure 1: Dependency tree for the second isomorphism theorem for rings

Possible Student Solution *Explanations of dependencies given in the tree in Figure 1.*

1. *Iso. 2 uses Iso. 1: The kernel of $f : R/I_2 \rightarrow R/I_1$ is I_1/I_2 , so the first isomorphism theorem applies, proving part 2.*
2. *Iso. 2 uses Thm. A: The kernel of the homomorphism $f : R/I_2 \rightarrow R/I_1$ is an ideal because of Theorem A.*
3. *Iso. 1 uses Thm. B: For R/K and S to be isomorphic, R/K must be a ring, which it is by Theorem B.*
4. *Thm. A uses Thm. C: For the kernel to be an ideal it must not be empty, which Theorem C guarantees.*

As one can see, the student's explanation of the dependencies need not be extensive. Often a one-sentence response is sufficient to ensure that the student understands the logic of the proof. (I've omitted some of the theorems from the dependency tree represented here for clarity.)

The direction of the arrows is down, and not up. I have found that this direction better aligns with how students think of the assignment, since the top-positioned theorem depends on the theorems below it and so on, and so the arrow represents the dependency relation: Theorem A depends on Theorem B, etc.

For successful execution of this assignment both the students and the instructor must take active roles in ensuring that a number of roadblocks to understanding are avoided. The instructor must choose appropriate theorems for the students to tackle. A useful heuristic for the first time one gives the assignment is to choose a theorem whose proof is exceptionally explicit in its reliance on previously established results. This helps ease the students into the task. Later instances could provide terser and more opaque theorems for the students. The students' role here is as in a typical homework assignment. I allow them a week to finish the problem, and assign it as part of their weekly homework.

In my courses I will typically employ this assignment every other week. Initial construction of the assignment can be moderately lengthy, requiring identification of candidate theorems sufficiently dependent on material covered in some depth in the course. Subsequent constructions though are quite brief; it is straightforward to produce assignments after training myself to be on the lookout for requisitely structured proofs. As the theorems encountered in a course usually do not vary from year to year, once one has used this technique, producing the problems should require very little work.

The first instance of the assignment in my class replaced a typical week's worth of problem sets, a choice made from extreme conservatism, done out of concern for the length of time the students would spend disassembling and reassembling the proofs, and simply understanding the assignment. This was unnecessary, as students immediately grasped the thrust of the assignment and reported that its completion took at most a couple of hours. In the second round, the assignment replaced a single (challenging) problem from their weekly problem set. This seems an appropriate scope for a tree problem.

¹For various theorems, a more accurate name for the assignment is a dependency directed graph, though this suffers from being less evocative for the student.

I do find that initially students tend to over-articulate the tree by putting in everything remotely relevant to the theorem. This is done enthusiastically, as on more than one occasion it has been reported to me that just listing previously seen theorems and propositions helps them understand the structure of a proof. On the second and subsequent iterations students become sharper at discerning which are the central dependent theorems.

Given that the tree assignment is roughly equivalent to a single problem from students' weekly assignments, this type of problem does not require an instructor to cut content that would otherwise be covered.

3 Outcomes

I have used this technique in a number of courses for math majors including Abstract Algebra, Topology, and Mathematical Logic. In each of them there have been ample theorems that are appropriate for the assignment.

The primary strength of the method is the visual representation, in tree form, of the logical structure of a proof. Dependency trees often consist of only a few nodes, allowing students to recall the full tree for a theorem. This helps in cementing in the content of theorems (from the student's point of view, a theorem becomes simply a node on a tree, more easily recalled than something tucked away in last week's lecture notes). This helps with the production of proofs: typically when students are asked to produce a proof, they have as models only proofs they vaguely understand. By having a set of proofs readily available, ones in which they have pulled apart the logical structure, students can turn to them when attempting their own.

Overall, the assignment seems quite successful. While no perfect control exists, in one class, I attempted to measure the efficacy of the assignment by conducting a small experiment. I assigned two problems each requiring that the student prove a given statement. For one of the problems I also required they produce a dependency tree. A week after they completed the assignment, I gave them a quiz requiring them to prove both statements again. For the dependency tree question, the average score was 95%. For the other, it was 91%. This could suggest that some benefit was gained from the assignment.

Also, my impression was that students' comprehension of the assigned proofs and ones similar in structure was significantly boosted. Completion of proof-based homework assignments also seemed improved. This was exemplified in a number of ways. During lectures, students would inquire about the relation of a theorem to previous material, more so than in other similar classes, and they would regularly ask for a dependency tree of a theorem being presented. On more than one occasion during lecture, students proffered up their own dependency trees, demonstrating to me that they understood the theorem presented and its relation to previous material. Besides lecture-related comprehension, I saw the impact of these assignments in the students' homework. In all work, students became much better at distinguishing work they themselves were providing and work they were citing in their proofs. The dependency tree assignment emphasizes that proofs are constructed using both novel logical deductions and previously established work, and students understood this distinction and incorporated it into their work.

Even though the classes that I have used this assignment in are typically senior-level math classes, regularly a sizeable minority of the students have not taken a proof-oriented math class before. They often struggle with reading proofs in the text, understanding those I present in class, and producing those assigned in homework. The dependency tree assignment seems to turn, for many of them, the typical opaque proof into something manageable. At each step of the assignment only one logical structure needs to be understood and explained. This breaks the proof into smaller pieces, and many students have reported that they feel the assignment has greatly increased their understanding of the material. As one student said in an evaluation, "The dependency trees were really illuminating in understanding the proof." The assignment, from the students' perspective, makes producing a proof more concrete and manageable.

An issue to watch out for is repetition of the dependency tree assignments. If one selects the first isomorphism theorem, for example, as a dependency tree assignment every year, solutions may be shared among the students. This is slightly ameliorated by the portion of the assignment in which students have to write

a sentence or two explaining how a theorem relies on another. One can also vary the theorems assigned to avoid repetition from year to year. At my college, upper-level courses like Abstract Algebra are offered every other year, providing enough distance between student cohorts.

Perhaps the largest shortcoming of the method is the initial time for setup. Each proof needs to be carefully chosen so that the students will not feel overwhelmed with determining or explaining dependencies. In choosing theorems for students to work on, I found it useful to keep in mind that often the difficulty in the assignment stems not from producing the tree, but in providing the justifications for the dependencies. Authors of undergraduate texts are often exceptionally explicit of reliance on previous work, being acutely aware of students' needs. Because of this, the tree-production portion of the assignment can be utterly straightforward. Even so, care must be taken in choosing logically straightforward proofs for the students, since the assignment relies on the production of the tree and the students' explanation of the dependencies. A benefit of the assignment is that proofs covered rarely change from year to year, and so once a collection of assignments has been built up, the work has been done, though, as mentioned above, care must be taken not to allow students to know of the repeated problems.

4 Extending the Method

The method could be extended in a number of useful ways. For a larger class, one could assign the students to groups, and assign a single dependency tree to each group. The groups could then present their results to the class, or could be required to type up their results for dissemination to the other groups, providing a mini-study guide for the proofs covered.

In most proofs, there are numerous types of dependencies. The above assignment focuses the student on explicit dependencies, though of course implicit ones abound. My sharper students regularly found implicit ones and inquired as to their status in the assignment. A useful addition to the assignment could be to find the implicit dependencies. For example, in a real analysis class, before a rigorous definition of the reals has been given, numerous proofs implicitly rely on the existence of irrational numbers. Ferreting out such assumptions could be a challenging problem for the more ambitious students.

Further modifications of this approach abound:

1. Students could produce a definitional dependency tree, showing what definitions a theorem relies upon.
2. They could produce a tree for a theorem they had to prove, so that an assignment would consist of two parts: proving a theorem, and producing a dependency tree of the proof.
3. They could analyze other students' proofs instead of one from the text.
4. They could build up a tree for all the theorems presented in a semester, showing the interrelatedness of the material in the course.

Focusing on the "definitional dependency tree" assignment for a moment, the proof dependency tree could profitably be coupled with a definitional dependency tree. Extrication of definitional dependencies often constitutes the entirety of proofs at the introductory level (regularly proofs often aren't the product of wholly novel mathematical constructions, but rather the dogged pursuit of definitional consequences). The parallel definitional assignment emphasizes this reliance. Asked to produce a definitional dependency tree to the second isomorphism theorem, a student might give the tree in Figure 2.

This assignment drives home the fact that virtually all the material is built upon the definition of a ring. Also apparent is that one produces a directed graph structure, and not a tree.

The techniques of proof dependency trees and of definitional dependency trees, I believe, can be useful in many courses. Interestingly, some typical undergraduate material is more suited to the proof dependency

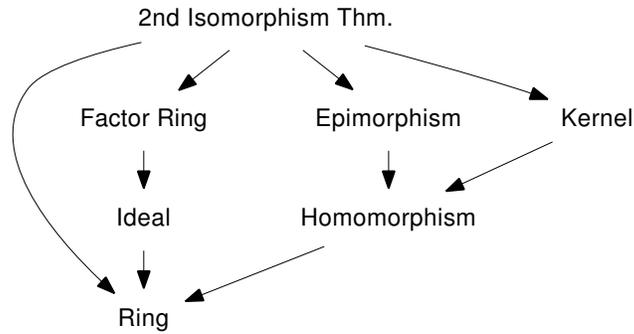


Figure 2: Definitional dependencies of second isomorphism theorem

tree approach than the definitional and vice versa. In preparing problems for my upcoming topology course, I found that deep proof dependency trees were fairly uncommon, though definitional trees were quite extensive. This was the opposite of what I saw in my Mathematical Logic course where deep proof trees were prevalent and definitional ones were shallower.

Getting students to understand and remember the cumulative nature of mathematics is a continuing and challenging problem. The dependency tree assignment visually reinforces it, along with articulating the logical structure of a theorem. It can be useful alongside a number of proof-instruction techniques including typical classroom lecture, peer-review, and the usual problem-set approach.

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Counterexamples and Correspondence: Establishing Two Roles in a Proof

Douglas E. Ensley

Abstract

To make mathematical proofs more concrete, we emphasize the two roles in a mathematical argument: the proof's author and the proof's reader. We use an activity focused on counterexamples to establish the role of the proof's reader, and then we use this perspective to motivate writing a direct proof in the form of a letter to convince the skeptical reader of the hopelessness of finding a counterexample. The letter-writing activity extends very well to other proof forms, including proofs with cases and proof by contradiction. The counterexample activity can be used in every mathematics course, even ones with no proof expectations from the students, and doing so contributes to students' ability to write a coherent argument in subsequent courses.

Difficulty Level: Low; Course Level: Transitional

1 Background and Context

Shippensburg University is a comprehensive public, regional institution with 7000 undergraduate students. We teach introductory proof in the first half of a one-semester, freshman-level discrete mathematics course, where topics in the second half come from combinatorics and graph theory. The course runs in six sections per year with an average class size of 28 students. The only prerequisite for Discrete Mathematics is the completion of a college-level algebra course or equivalent placement. The course is required of students majoring in mathematics, computer science, and middle-level education, and it was designed specifically in response to professional association recommendations [4, 5, 6]. All mathematics majors in their fourth or fifth semester will subsequently take Introduction to Abstract Algebra, a course intended to provide a transition to abstract mathematics, effectively spreading out the introduction to proof over two semesters. The mathematics majors then take a course in real analysis or complex analysis, which we characterize as a capstone experience with mathematical proofs. Because of the mixed audience and the desire for the discrete mathematics course to be accessible in the first year, we created a textbook [3] for this course at Shippensburg University.

The approach to mathematical proof in it is based on three primary tenets:

- Just as students learn new mathematical definitions by working from concrete examples to abstract descriptions, we attempt to lead students to *construct* their own, individual understanding of mathematical proof.
- All the students in the course are in a major for which they are expected to *communicate* mathematics precisely, and so we maintain a high standard for mathematical writing.
- Students will learn how to *write* mathematics only if they first understand how to *read* mathematics, so we intentionally work on the transition from reading to writing mathematical proofs.

The first two tenets comprise the focus of this article, in which we will share some classroom activities and an explanation of how students are led through the curriculum. Though used in the author's textbook, the ideas described can be applied to any introductory discussion of mathematical proof.

2 Description and Implementation

2.1 Constructing the Logical Meaning of Implication

The first activity involves asking students to find counterexamples to well-formed, universally quantified implicational statements. Because the integers are familiar objects with which students have years of experience, we initially use the set of integers as the domain for these statements. The act of finding a counterexample feels concrete to students, and most students can complete the activity with no prior instruction on vocabulary or logic. See Bennett [1] for a similar activity with an even longer list of nice statements that build critical-thinking skills.

The statements below come from a worksheet that we use at the beginning of the discussion of implicational statements. Our goal is to have the students tell us the circumstances under which an implicational statement is true by first asking them to describe the circumstances under which such a statement is false. This is the final section of a more general conversation about formal logic and language, and it is followed immediately by our unit on mathematical proof.

Counterexample Worksheet: *For each of the following, state whether or not you believe the given statement is “universally true” of whole numbers. If you claim the statement is not universally true, please provide a specific example to support your claim.*

1. If n ends in the digit 2, then n is divisible by 2.
2. If n ends in the digit 3, then n is divisible by 3.
3. If n is divisible by 5, then n ends in the digit 5.
4. If n ends in the digit 5, then n is divisible by 5.
5. If n is a prime number greater than 3, then $n^2 - 1$ is divisible by 12.
6. If n is prime, then so is $4n - 1$.
7. If $2^n - 1$ is prime, then so is n .
8. If n is a prime number greater than 3, then $n + 1$ is not a perfect square.

While the worksheet is completed by groups in class using paper and pencil, we follow up by assigning ungraded, outside-of-class exercises that use a web-based applet that can verify for the student whether he or she has correctly identified counterexamples to a wide range of statements. To see these applets, look under the Section 2.1 heading at the course website [2].

Notes on Using the Counterexample Worksheet

A common result of the activity is that students independently and without instructor intercession decide that the statement, “If p , then q ,” is not universally true when they can find an example that makes p true while q is false. Immediately following completion of the worksheet, a ten-minute class discussion is led by the instructor to ensure that everyone realizes they have come to this same conclusion about what constitutes a counterexample. Another important point during this discussion is the process subconsciously used by the students in searching for a counterexample. It defines the role of the Skeptic for the letter-writing activity that follows. Specifically, when considering a universally quantified statement of the form “ $p \rightarrow q$,” a Skeptic

searches for a counterexample by choosing examples satisfying p , and then checking whether q is satisfied. In the next section of this article we will discuss how this point of view is carried over to writing direct proofs as a correspondence between the proof's author and the Skeptic.

The follow-up is crucially important as later we hope students will reflect on this process as they construct their understanding of mathematical proof. Another valuable result of this activity is establishing that not every mathematical-sounding statement in this course will necessarily be true. We will see that a healthy skepticism is the most important ingredient to reading proofs critically.

2.2 Constructing the Structure of Direct Proofs

The transition from “the logic of implications” to “mathematical proof-writing” is achieved through an intentional emphasis on communication, and, of course, central to communication is the participation of more than one party. A mathematical proof is not a form of communication where we impart knowledge or ideas—rather, we deliberately use the term “argument” when we refer to a proof, because we envision one party attempting to persuade another.

In our discrete mathematics course, we focus on both parties in an argument – we call them the Author and the Skeptic, separating the roles of writer and reader of the mathematical proof. In the context of the counterexample activity, the students have already played the role of the Skeptic, doggedly searching for a counterexample to a statement. The post-activity discussion for the counterexample activity has also established the process by which the Skeptic pursues this quest: choosing examples that satisfy the hypothesis and checking to see if they make the conclusion false.

Now we introduce the Author, whose goal is to convince the Skeptic that no counterexample could possibly exist. Since the Author understands how the Skeptic is searching for counterexamples, the Author will write the argument to address this process. Specifically, the Skeptic is picking examples satisfying the hypothesis and then checking the conclusion. We imagine the Skeptic stubbornly repeating this process forever unless someone convinces him otherwise. The Author intercedes in this process by explaining algebraically why the very fact that an example satisfies the hypothesis precludes its being a counterexample. We encourage the early proofs to take the form of a letter from the proof Author to the Skeptic.

To illustrate this idea, here is the first proof we write in this course. It is written by the instructor at the board, reminding the students at each point how we are addressing the process the Skeptic is using.

CLAIM: If n is a perfect square greater than 4, then $n - 1$ is not prime.

Proof

Dear Skeptic,

I know you are busy choosing perfect squares beyond 4 and checking the number before each for primeness, but I think I can save you some time. Consider the latest example you picked. Since I don't know what it is, I will refer to your number as n . Since your n is a perfect square, this means $n = m^2$ for some integer m . So when you subtract 1 from n , you are really computing $m^2 - 1$, which from high school algebra we know to factor as $(m + 1)(m - 1)$. And since n is greater than 4, neither your $m + 1$ nor your $m - 1$ is 1, so you can see that you have actually written $n - 1$ as the product of two smaller numbers. So even without knowing which perfect square you picked, I know perfectly well that this example cannot be a counterexample to the original statement. Feel free to apply this reasoning to *any* perfect square you pick, and you will see why the original statement can have no possible counterexample.

Hugs and kisses,

Author

It is easy for an experienced mathematician to see how the traditional structure of direct proof derives from this verbose writing style. After students have experience writing a proof as a form of correspondence, we encourage them to edit for brevity, as

Proof. Let a perfect square $n > 4$ be given. By definition, we know that $n = m^2$ for some integer $m \geq 3$. This means that $n - 1 = m^2 - 1 = (m - 1)(m + 1)$. Since each factor is greater than 1, this shows that $n - 1$ is not prime.

This initial example is chosen because it's not clear at first glance whether the statement is even true, so it is entirely plausible that the Skeptic would be hard at work with his process when he receives this letter. For the first proofs the students write on their own, we opt for simpler statements such as “If n is odd, then $n^2 - 1$ is divisible by 4,” or “If a and b are odd, then $a + 3b$ is even.” The identification of the two roles in a mathematical proof allows us to put the role of “definition” in its proper context: the important thing about a definition is that it is agreed upon by the person writing the proof and the person reading it.

During the first week of working with direct proofs, we encourage students to write the verbose letter form of their proofs. Throughout the five weeks of working with all types of proof, we bring struggling students back to the letter-writing model before allowing them to move forward with the more condensed form. The idea of emphasizing the communication between two parties greatly improves student writing of mathematical proofs, since we strongly convey the notion that a proof is written to be read.

While writing a proof as a correspondence between two people creates a certain critical point of view, no one really wants to write this much and certainly no one wants to read this much for every argument. However, it is a mistake to call one of these arguments an “informal proof,” since that term is typically reserved for a rough argument that leaves out details, and a “proof as letter” has a lot of details. As we allow students to transition to traditional proofs, we emphasize that they are making the argument *more concise*, not more formal.

Both the counterexample activity and letter-writing convention address the first tenet's emphasis on constructing one's own understanding of mathematical proof. In addition, letter-writing sets a standard of communication, which nicely addresses the second tenet.

3 Outcomes

The counterexample problems have been assessed in two different courses as part of a multi-semester study conducted several years ago. On the first day of the semester the counterexample activity was given with identical statements to students in Discrete Mathematics and abstract algebra, the course mathematics majors take a year later. Predictably, the percentage of Discrete Mathematics students able to give a correct counterexample to a false statement was largely dependent on the logical complexity of the statement—throw in a few “not”s and “or”s and students have a hard time parsing the statement. However, we consistently saw marked improvements with these same statements in the abstract algebra course, a sign that our students are dealing with logical complexity better as a result of work completed in earlier courses.

The letter-writing method requires the instructor to adopt a point of view that may feel odd at first, but the method does not require much new preparation. After the experience of writing out long, descriptive sentences for one or two class periods, most students quickly prefer the shorter, terse proofs found in most textbooks. Because students come to this preference on their own, the quantity and quality of writing remains generally high. A common comment by instructors is that it is the rare Discrete Mathematics student who turns in “random math gibberish.” In fact, we see the results of this in upper-level courses, where the general level of writing (i.e., the use of complete sentences and whole words) is noticeably high even as the mathematical constructs become more abstract.

Overall we have been pleased with the way the discrete mathematics course prepares mathematics majors for the logical rigors of our advanced courses, and we believe the emphasis on concrete counterexamples and the role of the Skeptic has played an important role.

4 Extending the Method

The counterexample activity is used throughout the discrete mathematics course, especially when new definitions have been introduced. For example, in our unit on sets and operations, students consider universally quantified statements such as, “If $|A| = |B|$, then $|A \cap C| = |B \cap C|$.” In our unit on functions and relations, students consider statements such as, “If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$,” for various relations R .

The counterexample activity can be adapted easily for any mathematics course, providing the opportunity to build critical-thinking skills across the curriculum, even in courses where we may not expect students to write formal proofs. Here are examples from two such courses. In each, the instructions are, “Give a counterexample or state that you believe there is no counterexample.”

Precalculus

1. If $r < 3$, then $r^2 < 9$.
2. If f and g are functions with period π , then $f + g$ has period π .

Calculus I

1. If f has a local minimum at $x = 4$, then $f'(4) = 0$.
2. If $f(x) < g(x)$ for all x , then $f'(x) < g'(x)$ for all x .

Other proof techniques such as proof by cases or proof by contradiction fit the letter-writing paradigm very well. Proof by contradiction is particularly easy to understand—each proof starts with a sentence like, “Imagine, dear Skeptic what catastrophes will unfold if you ever do find the counterexample you’re looking for.” Early in a proof by cases, the Author may include a sentence like, “I have no way of knowing if your number is even or odd, but I do know it has to be one of them; so I will tell you what I know about each case separately.”

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Two Techniques for Sense-Making: Proof-Tracing and Induction Tables

Douglas E. Ensley

Abstract

This article describes two techniques to help students better understand proof structures such as direct proof, proof by cases, and proof by induction. The proof-tracing activity makes students active readers of mathematical arguments, encouraging them to analyze an argument line by line in a critical way. The approach presented for mathematical induction establishes the role of the proof's reader as an active participant in the argument, thereby assuaging the feeling among students that an induction proof uses circular reasoning.

Difficulty Level: Low; Course Level: Transitional

1 Background and Context

Shippensburg University is a comprehensive public, regional institution with 7000 undergraduate students. We teach introductory proof in the first half of a one-semester, freshman-level discrete mathematics course, where topics in the second half come from combinatorics and graph theory. The course runs in six sections per year with an average class size of 28 students. The only prerequisite for Discrete Mathematics is the completion of a college-level algebra course or equivalent placement. The course is required of students majoring in mathematics, computer science, and middle-level education, and it was designed in response to professional association recommendations [3, 4, 5]. All mathematics majors in their fourth or fifth semester will subsequently take Introduction to Abstract Algebra, a course intended to provide a transition to abstract mathematics, effectively spreading out the introduction to proof over two semesters. The mathematics majors then take a course in real analysis or complex analysis, which we characterize as a capstone experience with mathematical proofs. Because of the mixed audience and the desire for the discrete mathematics course to be accessible in the first year, we created a textbook [2] for this course at Shippensburg University.

The approach to mathematical proof is based on three primary tenets:

- Just as students learn new mathematical definitions by working from concrete examples to abstract descriptions, we attempt to lead students to *construct* their own, individual understanding of mathematical proof.
- All the students in the course are in a major for which they are expected to *communicate* mathematics precisely, and so we maintain a high standard for mathematical writing.
- Students will learn how to *write* mathematics only if they first understand how to *read* mathematics, so we intentionally work on the transition from reading to writing mathematical proofs.

The first and third tenets comprise the focus of this article, in which we will share some classroom activities and an explanation of how students are led through the curriculum. Though used in the author's textbook, the ideas described in this paper can be applied to any introductory discussion of mathematical proof.

2 Description and Implementation

2.1 Proof-Tracing

As described in the companion article [6], we introduce direct proofs as a form of correspondence between the proof's Author and a Skeptic for whom the proof forms an intervention into a process of tirelessly searching for a counterexample. To establish this point of view, the Discrete Mathematics students play the role of both Author and Skeptic at different points in the introduction. In the present article we further develop this role-playing with another in-class activity focused on the Skeptic; see companion article [6] for a more detailed description of these roles.

The act of "proof-tracing" is akin to the way a computer programmer might test a piece of code by calling a function with different inputs to see if the correct output is generated. A programmer would call this "tracing" a piece of code, so we use the same terminology with proofs. In a nutshell, to trace a proof of a statement of the form $\forall x \in S, P(x)$, the Skeptic chooses a value of x in S and reads through the argument to check that the proof clearly and correctly addresses the chosen value. It is important to emphasize that this process does not validate the proof, but rather the goal of the activity is to make the student an active reader of the finished argument.

The worksheet described below exemplifies the structure of the exercises. It is a paper-and-pencil activity that students complete first individually and then discuss within a small group. If an error or ambiguity is discovered in a proof, this is shared by the group with the whole class during a discussion period at the end of the activity. For a fifty-minute class, we typically allocate half of the class to the activity and discussion, including three or four proofs to be traced. The number of proofs, of course, determines the amount of time to allow, but it is good to have examples of four types of proofs: (a) correct arguments written well, (a) correct arguments written ambiguously, (c) incorrect arguments for true statements, and (d) incorrect arguments for false statements. The sample exercises below fall into categories (a) and (c).

Proof Reading Worksheet: *Refer to the given proof to write responses to each set of exercises. After you finish, we will discuss your responses and any observations you make as a class.*

Claim 1: If n is divisible by 6, then n^2 is divisible by 12.

Proof. Let an integer n that is divisible by 6 be given. This means that $n = 6L$ for some integer L . But then $n^2 = (6L)(6L) = 36L^2 = 12(3L^2)$. Since $3L^2$ is an integer, this shows that n^2 is divisible by 12. \square

Tracing the Proof of Claim 1:

1. Respond to the first sentence. That is, choose a specific integer that is divisible by 6: $n = \underline{\hspace{2cm}}$
2. For your choice of n , respond to the second sentence. That is, what is the value of L that comes from your choice of n ? $L = \underline{\hspace{2cm}}$
3. Check the third sentence. That is, for your values of n and L , compute $n^2 = \underline{\hspace{2cm}}$ and $12(3L^2) = \underline{\hspace{2cm}}$ to verify these are equal.
4. Were you able to complete all three steps? (Yes or No) If yes, you have verified that this value of n is handled correctly by the proof, and you should feel free to repeat this exercise with other choices of n to gain more understanding of the structure of this proof. If no, either the proof is broken or it is not written clearly enough. Discuss the problem with your group.

Claim 2: If n is odd, then $n^2 - 1$ is divisible by 4.

Proof. Let an odd integer n be given. This means that $n = 2k + 1$ for some integer k . But then simple algebra tells us that $n^2 - 1 = (2k + 1)^2 - 1 = (4k^2 + 1) - 1 = 4k^2$. Since k^2 is an integer, this demonstrates that $n^2 - 1$ is divisible by 4. \square

Tracing the Proof of Claim 2:

1. Respond to the first sentence. That is, choose a specific odd integer: $n = \underline{\hspace{2cm}}$
2. For your choice of n , respond to the second sentence. That is, what is the value of k that comes from your choice of n ? $k = \underline{\hspace{2cm}}$
3. Check the third sentence. That is, for your values of n and k , compute $n^2 - 1 = \underline{\hspace{2cm}}$ and $4k^2 = \underline{\hspace{2cm}}$ to verify these are equal.
4. Were you able to complete all three steps? (Yes or No) If so, you have verified that this value of n is handled correctly by the proof, and you should feel free to repeat this exercise with other choices of n to gain more understanding of the structure of this proof. If not, either the proof is broken or it is not written clearly enough. Discuss the problem with your group.

Since the language used in a proof should sound like steps to be followed, having students literally follow the steps as instructions supports our goal that when the student becomes the proof author, the proof will be written in clear, coherent English. The proof-tracing activity is particularly valuable for explaining proof by cases, since the proof's reader must follow one of the parallel branches in the proof depending on which case is satisfied by the example initially chosen. On the other hand, this activity does not fit well with proof by contradiction.

For follow-up there are web-based activities under Section 2.1 of Ensley [1] for students to work on outside of class. They provide ungraded practice tracing through various correct and incorrect proofs. In addition, we create custom, online homework problems within our campus course management system (D2L) since short-answer problems with formulas are particularly easy to code. They give a short proof and a question of the form "If you choose 7 for n in line 1, what is the value of k in line 3?"

2.2 Constructing an Understanding of Mathematical Induction

Mathematical induction is a core topic in discrete mathematics, and it is one of the more difficult proof structures for students to understand. Once again we want students to have a less abstract experience with mathematical proof, this time with a focus on constructing an understanding of a proof by induction. And once again we have students begin by exploring concrete examples before moving on to the abstract proof structure.

Our course begins with a focus on sequences of numbers. Before we discuss logic and proof, we have students work with number sequences defined with recursive rules such as $a_n = a_{n-1} + (2n - 1)$ (with $a_1 = 1$) and algebraic rules such as $b_n = n^2$. This work sets the stage for our first mathematical induction proofs, in which our goal is to show that two given rules describe the same sequence.

We explain to the students the reason that mathematical induction can be applied to statements of the form $\forall n \in \mathbb{N}, P(n)$. That is, we explain that this type of statement is the same as

$$P(1) \text{ and } P(2) \text{ and } P(3) \text{ and } P(4) \text{ and } \dots$$

When the Author declares that a proof argument will use mathematical induction, that declaration tells the Skeptic that the statements above should be checked in order – $P(1)$, then $P(2)$, then $P(3)$, etc. If all statements check out, then this establishes $\forall n \in \mathbb{N}, P(n)$.

For the purpose of this article we will describe the initial presentation as a worksheet completed in stages, but in reality the initial presentation is given by the instructor and then students complete group worksheets similar in structure. In all the initial examples, the student is establishing that a recursively defined sequence, such as c_n in the worksheet that follows, is the same as a sequence defined by an algebraic formula, such as $d_n = 2^n - 1$ in the worksheet.

Induction Worksheet – Part A: Consider the sequence recursively defined by $c_1 = 1$ and $c_n = 2 \cdot c_{n-1} + 1$ for all $n \geq 2$. For each value of n , use the recursive definition of c_n to fill in the empty remaining cells in the first column, and use the algebraic formula $2^n - 1$ to fill in the second column.

n	$c_n = 2 \cdot c_{n-1} + 1$	$2^n - 1$	Equal?
1	$c_1 = 1$ (by definition)	$2^1 - 1 = 1$	Yes
2			
3			
4			
5			

The completed worksheet should look like

n	$c_n = 2 \cdot c_{n-1} + 1$	$2^n - 1$	Equal?
1	$c_1 = 1$ (by definition)	$2^1 - 1 = 1$	Yes
2	$c_2 = 2 \cdot c_1 + 1 = 2 \cdot 1 + 1 = 3$	$2^2 - 1 = 3$	Yep
3	$c_3 = 2 \cdot c_2 + 1 = 2 \cdot 3 + 1 = 7$	$2^3 - 1 = 7$	You bet
4	$c_4 = 2 \cdot c_3 + 1 = 2 \cdot 7 + 1 = 15$	$2^4 - 1 = 15$	Yeah
5	$c_5 = 2 \cdot c_4 + 1 = 2 \cdot 15 + 1 = 31$	$2^5 - 1 = 31$	Yup

Notes on Using the Induction Worksheet – Part A After the first part of the worksheet has been completed, the instructor should make a few points before continuing to Part B.

Students should show every step in the computation of c_n from the recursive formula. The larger the value of n , the less arbitrary a request this will seem. That is, to make an important point about the necessity of showing every step, it is helpful to ask students to compute c_n for some larger values of n , where the mental arithmetic is not as easy as in the typical base case(s).

To prepare for Part B of the worksheet, the final row of the table should be broken down by the instructor into more detail as follows:

$$\begin{aligned}
 c_5 &= 2 \cdot c_4 + 1 && \text{by the recurrence relation for } c_n \\
 &= 2 \cdot 15 + 1 && \text{since } c_4 = 15 \text{ from the previous row} \\
 &= 31 && \text{by arithmetic.}
 \end{aligned}$$

Before Part B, students should extend the table a few more rows; the number of additional rows is up to each student. This is an important detail, because it will help us give inherent meaning to the important variable m in the induction step.

Induction Worksheet – Part B Recall that we are working with the sequence recursively defined by $c_1 = 1$ and $c_n = 2 \cdot c_{n-1} + 1$ for all $n \geq 2$. You have filled in the table completely down to some point, but the number of rows you have completed is unknown to the proof author.

There is a first row you have not yet checked. Let's use m to denote that row number. So your table looks like this, with all rows completed prior to row m :

n	$c_n = 2 \cdot c_{n-1} + 1$	$2^n - 1$	Equal?
1	$c_1 = 1$ (by definition)	$2^1 - 1 = 1$	Yes
2	$c_2 = 2 \cdot c_1 + 1 = 2 \cdot 1 + 1 = 3$	$2^2 - 1 = 3$	Yes
3	$c_3 = 2 \cdot c_2 + 1 = 2 \cdot 3 + 1 = 7$	$2^3 - 1 = 7$	Yes
4	$c_4 = 2 \cdot c_3 + 1 = 2 \cdot 7 + 1 = 15$	$2^4 - 1 = 15$	Yes
5	$c_5 = 2 \cdot c_4 + 1 = 2 \cdot 15 + 1 = 31$	$2^5 - 1 = 31$	Yes
\vdots	\vdots	\vdots	\vdots
m			

Since all previous rows have been completed, we know what row $m - 1$ looks like, more or less, so we can add that information as well:

n	$c_n = 2 \cdot c_{n-1} + 1$	$2^n - 1$	Equal?
1	$c_1 = 1$ (by definition)	$2^1 - 1 = 1$	Yes
2	$c_2 = 2 \cdot c_1 + 1 = 2 \cdot 1 + 1 = 3$	$2^2 - 1 = 3$	Yes
3	$c_3 = 2 \cdot c_2 + 1 = 2 \cdot 3 + 1 = 7$	$2^3 - 1 = 7$	Yes
4	$c_4 = 2 \cdot c_3 + 1 = 2 \cdot 7 + 1 = 15$	$2^4 - 1 = 15$	Yes
5	$c_5 = 2 \cdot c_4 + 1 = 2 \cdot 15 + 1 = 31$	$2^5 - 1 = 31$	Yes
\vdots	\vdots	\vdots	\vdots
$m - 1$	$c_{m-1} = \text{some calculations}$	$2^{m-1} - 1$	Yes
m			

Now fill in row m using the same steps you used in Part A of this worksheet. In particular, you should be able to fill in the blanks below. (For the purposes of the article, the blanks have already been completed.)

$$\begin{aligned}
 c_m &= \frac{2 \cdot c_{m-1} + 1}{} && \text{by the recurrence relation for } c_n \\
 &= \frac{2 \cdot (2^{m-1} - 1) + 1}{} && \text{since } c_{m-1} = 2^{m-1} - 1 \text{ from the previous row} \\
 &= \underline{\underline{2^m - 2 + 1 = 2^m - 1}} && \text{by algebra.}
 \end{aligned}$$

For follow-up there are web-based activities under Section 2.3 of Ensley [1] for students to work on outside of class. They provide ungraded practice that further develop the temporal aspect of thinking about an induction proof.

Notes on Using Induction Tables A common reaction to mathematical induction, even among high-achieving students, is that the argument structure is circular, and the approach we use is designed to alleviate that concern. Giving meaning to the variable m in the induction step forms the key. If we imagine the Skeptic checking the statements in order, then it is clear we need a variable to stand for an arbitrary “first row not yet checked.” Hence, the role of the variable m in the induction proof becomes a central point of emphasis.

In the discrete mathematics course the first induction proofs are written about statements like the one given here and statements about closed formulas for sums such as the formulas for the sums of arithmetic and geometric progression. Clearly, induction proofs on non-numerical objects such as graphs do not fit this model well. For this reason, we try to get students to write their proofs in traditional, concise paragraph form. The transition to a concise form is the natural opportunity to address the so-called weak form of induction, if one is inclined to do so. Our choice to let m always indicate the first row not yet checked is tantamount to focusing only on the so-called strong form of induction. Since the strong and weak forms of induction are equivalent, we do not teach students that there are two separate forms.

3 Outcomes

The proof-tracing activity is part of an effort to get students to understand that mathematical proofs are written to be read, though we have undertaken no studies to separate the effect of this activity from the overall approach in the course. Anecdotal evidence suggests that tracing through a proof is effective in one-on-one tutoring in which a student is confused about the notion of proofs, even in higher level classes such as abstract algebra or analysis. In fact, the activity is born out of these effective tutoring situations.

Most of our instructors for Discrete Mathematics adopt the induction table approach, primarily because the structure takes the mystery out of this form of reasoning. We have been pleased that Discrete Mathematics instructors very rarely have students ask about the circularity of the induction proof structure. Some instructors require students to write concise, paragraph form induction proofs on exams in order to begin weaning from the table structure. Without this nudge, many students cling to the use of the table as an organizational structure, and instructors in subsequent courses have had to reinforce the concise form of induction proofs to break the habit.

4 Extending the Method

The proof-tracing activity can be adapted to any course where a proof is presented, regardless of any intent to ask students to write proofs for themselves. For example, the following calculus homework problem will build facility with proof: “For the proof in Example 3 in Section 2.8, if $\epsilon = 0.01$ is chosen in response to the first sentence, what is the value of δ described in the third sentence?” In this way we can emphasize in every course we teach that proofs are written to be read.

The use of induction tables feels fairly limited, but it extends well to other problems in which we prove a persistent property of a sequence of numbers. For example, proving that $8^n - 1$ is divisible by 7 for all $n \geq 1$ works well with the induction table method. The method also works well in Discrete Mathematics where algebraic formulas for sums, such as $\sum_{k=1}^n 2k = n^2 + n$ for all $n \geq 1$, are prevalent. The table method does not work well when induction is applied to graph-theoretical statements, primarily because there is no single, natural conclusion to draw from statement $P(n - 1)$ before addressing statement $P(n)$. However, by building a strong understanding of the logic behind inductive reasoning early on, we believe students more easily understand more complex arguments later in the course.

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Proof-Evaluation as a Step towards Proof Authorship

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Abstract

This article presents a case for the use of proof-evaluation exercises throughout the undergraduate mathematics curriculum, as a means of promoting genuine engagement with the mechanisms and purposes of mathematical proof, and as a means of affording students an accessible opportunity to develop and exercise a sense of responsible personal mathematical authority. What is involved is easily described and also easily adapted: students are presented with a selection of proposed “proofs” and/or counterexamples to a mathematical statement, and asked to critically assess each one and perhaps to rank them in order of preference. An example from an elementary university course in linear algebra is described.

Difficulty Level: Low; Course Level: Transitional

1 Background and Context

Proof is a difficult mathematical concept for students. Research shows that most university students experience difficulties not only in proof-writing [1, 2, 4, 7, 9] but also in knowing what constitutes a proof [2, 6]. Research by Selden and Selden [8] demonstrates that students have difficulties in determining whether a deductive argument is valid. They see the “lack of validation skills as linked to beginning university students’ well-documented inadequate conceptions of proof”.

Our own accumulated experience has left us unconvinced of the value of assigning direct proof-writing tasks in the early stages of our students’ university education. Many of our students appear to be unequipped at this point to take the independent creative and/or technical steps that they demand. This article posits that the task of assessing and comparing different proposed proofs of the same statement provides natural and relatively easy access for students to essential considerations about the purposes and mechanisms of mathematical proof, and facilitates the development of a sense of, and responsibility for, individual mathematical agency. In support of this position we report on a task of this nature that was used in an elementary university course on linear algebra.

The recent doctoral thesis of the first author [5] involves a comprehensive exploratory study of the approaches taken by novice students to the validation and evaluation of simple mathematical proofs. The outcomes of the study, some of which are discussed in Section 3 below, motivated us to include a proof-evaluation activity on a higher mathematical level in a linear algebra module for our first-year students at the National University of Ireland, Galway. This institution has a student population of about 17,000 and offers a full range of undergraduate and graduate degree programs. The module is a specialist one for students of the sciences or humanities who have declared an interest in mathematics, and many (but not all) of the participants continue to degree level in the subject. Most of the students in the class have just completed their second-level (i.e., high-school) education and are registered as full-time students of the university. Upon graduation they are likely to further their careers in a number of ways including graduate school, employment in industry, and training for professions such as teaching. The enrollment in the course is typically around 100 students.

Students in Ireland spend eight years in primary education and five or six years in second-level education. University entrants are usually aged 18 or 19. Second-level education concludes with a national

school-leaving examination, results of which determine access to most university courses. Most students in our first-year class would have recently completed this examination in seven subjects, including mathematics at one of two levels (ordinary and higher). At both levels, the mathematics syllabus includes material on coordinate and Euclidean geometry (including theorems with proofs), probability and statistics, number systems, and basic differential calculus. The higher level syllabus is more extensive and includes, for example, an introduction to integral calculus. Mathematical education at the second level tends to be dominated by procedural rather than conceptual activities. Although the syllabus proposes emphasis at higher level on “the idea of rigorous proof”, proof-writing does not feature among the objectives or key skills. Beginning university students’ prior experience with mathematical proof is typically limited to the study of some standard proofs of well-known statements, confined mostly to the area of Euclidean geometry.

The proof-evaluation exercise described in this article took place in the context of the transition to university education, where we expect students to engage in deductive reasoning, testing of conjectures and writing of proofs as well as in more procedural mathematical activities. The students had limited prior experience with proof, and had not taken part in any transition-to-proof courses. Such courses do not typically form part of the university mathematics curriculum in Ireland. The task that is discussed here was part of the students’ first assignment, which was completed approximately in their fourth week at the university. They were familiar, from lectures, with the definition of a linear transformation (of \mathbb{R}^2) as a function that respects addition and multiplication by scalars, and with the matrix representation of a linear transformation.

2 Description and Implementation

2.1 The Activity

Students were presented with the following text.

Amani, Bob, Charlie, Deirdre, and Eduardo are thinking about proving the following statement.
If the function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation, then T fixes the origin, i.e., $T(0, 0) = (0, 0)$.

Amani’s Proof

Suppose that $T(1, 1) = (a, b)$. Then

$$T[(1, 1) + (0, 0)] = T(1 + 0, 1 + 0) = T(1, 1) = (a, b).$$

But on the other hand since T respects addition

$$\begin{aligned} T[(1, 1) + (0, 0)] &= T(1, 1) + T(0, 0) \\ &= (a, b) + T(0, 0) = (a, b) \text{ from above.} \end{aligned}$$

So $T(0, 0) = (a, b) - (a, b) = (0, 0)$.

Bob’s Proof

We know that for any element u of \mathbb{R}^2 and for any real number k we have

$$T(ku) = kT(u).$$

Then applying T to $(0, 0)$ and multiplying the result by any real number k must give the same result as multiplying $(0, 0)$ by k first and then applying T . But multiplying $(0, 0)$ by k always results in $(0, 0)$ no matter what the value of k is. So it must be that the image under T of $(0, 0)$ is a point in \mathbb{R}^2 that does not change when it is multiplied by a scalar. The only such point is $(0, 0)$. So it must be that $T(0, 0) = (0, 0)$.

Charlie's Proof

Think of T as the function that moves every point one unit to the right. So T moves the point $(0, 0)$ to the point $(1, 0)$. Then T is a linear transformation but T does not fix the origin. This example proves that the statement is not true.

Deirdre's Proof

Suppose that (a, b) is a point in \mathbb{R}^2 for which $T(a, b) = (0, 0)$. Then

$$T[2(a, b)] = T(2a, 2b) = 2T(a, b) = 2(0, 0) = (0, 0).$$

Thus $T(2a, 2b) = T(a, b)$, so $(2a, 2b) = (a, b)$, so $2a = a$ and $2b = b$. Thus $a = 0$, $b = 0$ and $T(0, 0) = (0, 0)$.

Eduardo's Proof

Since T is a linear transformation it can be represented by a matrix. Suppose that the matrix of T is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then the image of $(0, 0)$ under T can be calculated as

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a0 + c0 \\ b0 + d0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $T(0, 0) = (0, 0)$.

The students were asked the following questions:

- Does Amani's answer prove that the statement is true? If not, why not?
- Does Bob's answer prove that the statement is true? If not, why not?
- Does Charlie's answer prove that the statement is *not* true? If not, why not?
- Does Deirdre's answer prove that the statement is true? If not, why not?
- Does Eduardo's answer prove that the statement is true? If not, why not?
- Please rank the five answers in order of your preference (according to your own opinion). Include some comments to explain your ranking.

The following advisory note was included.

There is not necessarily a "right answer" to this question, certainly not to part (f). What you are expected to do is read each of the proposed proofs and think about the content. Is the argument correct? Does it do what it claims? There may be more than one correct proof here, and part of the point of this question is the fact that there can be many different approaches to proving a mathematical statement. You might prefer one argument to another, for example if you find it easier to follow or if you find it particularly convincing. If you are in doubt, discuss the problem with classmates or post a message on the Discussion Board.

In our linear algebra course, the exercise was included as a homework problem. Students worked on it independently (although possibly cooperatively) and submitted their written responses. Feedback on some recurrent features of these responses was provided at a subsequent lecture, in a discussion that was notable for the students' interested attention. We have used similar exercises in assigned homework (and not only in

linear algebra) and in workshops and tutorial sessions, where they have proved extremely useful in stimulating discussion about what is required in a proof and what is inadequate. In our experience the use of this technique has not impacted significantly syllabi or necessitated the cutting or reorganising of material.

The role of the student in this activity is to carefully consider each of the proposed proofs, in terms of its content and internal consistency, its logical structure, and its relationship to the statement that it asserts to prove. In doing so, the student must cast herself or himself in the role of a knowledgeable critical reader, and must be prepared to take a position on the correctness of each proof and to defend any objection with a coherent statement of what is inadequate or incorrect. The inclusion of the request to rank the proofs in order of preference invites the student to consider their subjective appeal (or at least of those that are accepted as correct) and acknowledges the existence of different mathematical tastes.

The instructor has multiple roles. One is to supply students with a selection of proposed proofs that is varied and appropriately challenging. They may feature errors of different types, for example minor omissions, slightly imprecise statements, flawed attempts at constructing counterexamples, and arguments whose logical structure is entirely mismatched to the statement whose proof is purported. If the statement admits numerous correct proofs that are essentially different, they may be included, perhaps written in different styles. Composing a suitable collection of “proofs” can be an absorbing challenge for an instructor. There is complete freedom in terms of what can be included, but our usual tendency has been to include (at least) one “proof” that is written almost entirely in text, one that is written in algebraic formalism, one that (correctly or incorrectly) proposes a counterexample to the statement, and one whose logical structure is mismatched to the statement that it claims to prove. Most of our examples involve four or five “proofs” in all, of which one or two are fully or substantially correct. However, there are few rules in this game, and instructors may for example choose to use a proof-evaluation task to draw attention to a misconception that has been observed in class. Instructors also have freedom to decide what questions to ask and how much explanation to request.

Another role for the instructor is to facilitate a discussion without fully directing it, if the task is to be considered during a classroom session. Another is to assess students’ work and to consider useful feedback to provide to the group or to individuals, and other responses to what is learned about the students’ understanding of and competence with mathematical proof. If proof-evaluation tasks are included in formal assessment, the instructor needs to consider how grades are to be assigned. This can require some care, since (for example) erroneous conclusions on the correctness of a proof are sometimes accompanied by comments that reveal appropriate thinking on the part of the student. The possibility of partial credit, even for short parts of a question, needs to be considered. The question of how to grade a task involving ranking of proofs presents a particular challenge. In the example discussed here we awarded full marks for parts (a), (b), and (e) to any student who stated that these proofs were correct. In some cases we awarded partial credit for these parts to students who rejected the proof on the basis of a clearly stated perceived deficiency (even if the deficiency did not warrant rejection of the proof in our view). In parts (c) and (d), full marks were awarded to students who rejected the proof and gave a precise and mathematically sound reason for doing so. Partial credit was given for rejections of the arguments that were unexplained or explained only in vague terms. We awarded full marks for part (f) for any full ranking with explanatory comments; however we recognize that the feasibility of this depends on the nature and significance of the particular assessment.

2.2 Student Responses

Our students’ written responses to this proof-evaluation task revealed some recurrent themes that are relevant to all aspects of their thinking on the subject of mathematical proof and certainly to their immediate potential to develop reliable competence in proof-writing.

The following observations are based on a sample of 28 students who submitted complete responses to the task in Section 2.1. We begin by mentioning a striking feature of our students’ work on Deirdre’s “proof”.

Only eight students considered it to be incorrect, and not one rejected it on the basis of the inappropriate logical structure of its argument. None of our students noted the mismatch between the statement to be proved and Deirdre's (erroneous) attempt to show that a hypothetical point whose image under T is the origin must itself be the origin. Two students rejected Deirdre's argument because of the incorrect assertion that

$$T(2a, 2b) = T(a, b) \implies (2a, 2b) = (a, b).$$

Other students who rejected Deirdre's proof did so for more spurious reasons, for example objecting to the use of the scalar 2 instead of a "more general" k .

Similarly, seven students objected to both Amani's and Bob's proofs on the grounds that their arguments use only part of the definition of linear transformation. This objection (as well as objections to Deirdre's choice of the scalar 2 and Amani's choice of the point $(1, 1)$) seem to be based on a superficial inspection of the "ingredients" of these proofs rather than on a careful validation of the logical structure of the arguments in relation to the statement to be proved. No student noted that either Amani or Bob actually proves a more general statement than the one they claim.

In general, there was little evidence in the work of our students of careful attention to the mechanisms of the deductive reasoning in the arguments. This is most strikingly apparent in the case of the responses to Deirdre's proof, whose misdirected argument seems to have gone unnoticed. While this may be a disconcerting outcome for an instructor, it led to an excellent opportunity for a classroom discussion of the importance of attention to the overall logical structure of a proof and to its internal details, in the context of a task with which the students had genuinely engaged.

Eduardo's proof was by far the most popular with our students, with all 28 accepting it and 23 ranking it first or second. Given that the students were occupied at the time with many other exercises involving matrix representations of linear transformations, and given their inexperience with proof, we were not surprised by their preference for Eduardo's translation of the problem to an easy matrix calculation over Amani's and Bob's reasoning from the defining properties of a linear transformation.

3 Outcomes

The proof-evaluation task allowed our students an opportunity to explore the nature and purposes of mathematical proof in a manner that was at least partly self-directed and required reliance on their own emerging mathematical judgement. Moreover, their responses to the task revealed to us a great deal of information about what aspects of the "proofs" attracted their attention and what escaped comment. For example, two striking features of the students' responses to the linear algebra task are the apparent inattention to the logical structure of Deirdre's proof and the universal approval and high ranking of Eduardo's proof. The latter suggests that proofs involving routine technical demonstrations may be more accessible to inexperienced students than those involving chains of deductive reasoning. While this is not surprising it may suggest a route into the complexities of mathematical argumentation.

Expecting a student to write a mathematical proof if s/he does not have a sound awareness of the essential purpose(s) of proof and of the appropriate logical relationship between an assertion and a claimed proof of that assertion seems unreasonable. The ability to use such awareness to critically assess a proposed proof and arrive at a position on its correctness (at least in a context where the concepts and terminology are familiar) is a prerequisite for successful proof-writing. Proof-evaluation exercises such as the one discussed here provide instructors with a means of checking these requisite skills, and our classroom experience and Pfeiffer's study suggest that they also provide students with a means of developing them. Proof-evaluation tasks can be used in parallel with tasks involving proof-writing, as they can be effectively used by instructors to draw the attention of students to key features and purposes of mathematical proofs.

Another positive effect of proof-evaluation activities is that they encourage students to engage collaboratively with proof. We recently used proof-evaluation tasks in an introductory course in group theory for students at a more advanced stage of their university education. It included tutorial sessions at which there were opportunities for discussion in small groups. We observed that the proof-evaluation tasks stimulated lively and thoughtful debate, and afforded meaningful engagement with the intricacies of proofs. We remark also that this sort of activity reflects the typical collaborative behavior of experienced mathematicians, and may be regarded as authentic participation in mathematical practice. The group theory course also involved numerous proof-writing exercises at varying levels of complexity. At the tutorial sessions we were able to draw explicit parallels between the mental processes required for the proof-evaluation and proof-writing exercises. On occasion we were also able to use recurrent errors in the students' written work to inform our composition of flawed proofs for the proof-evaluation exercises. As the course progressed, we noticed a marked improvement in the quality of the students' written proof and in the care taken over their writing. We cannot attribute this solely to the proof-evaluation exercises, but they were a factor.

Findings from Pfeiffer's study also indicate positive effects of proof-evaluation exercises on students' understanding of the purposes and mechanisms of mathematical proof. In particular, the invitation (in an interview) to compare and rank different proposed proofs encouraged reflection about what is important in a deductive argument. For instance, some students would have accepted a simple check of one or more examples until they saw a general argument correctly used to prove the statement. Only then they realized that correctness of a statement for one example is not sufficient to prove it. Some students acknowledged this learning effect through comparison, as the following quotation indicates. "That's a good answer. It's general again. I knew I'd change my mind about the first [answer]." (The first answer was a verification of the statement for one example).

A significant obstacle for students is their fear of proof. Many do not feel confident about their ability to write a proof and often do not even attempt proof-writing tasks. This does not seem to play a role in the case of proof-evaluation tasks. Participants in Pfeiffer's interviews indicated that they enjoyed tasks of this kind; many saw themselves in the role of teachers and felt able to comment on proposed proofs and to make suggestions for improvements, despite their own limited capacities in proof-writing. One of the interviewees mentioned his/her lack of confidence in the area of proving: "It's a better answer than what I have given. I'm not very good at proving things to be honest." Comments like this suggest that the interviewees did not feel confident in their abilities to prove even very simple statements themselves.

We conclude this section by remarking that proof-evaluation is an integral feature of the practice of mathematicians and that it is inextricably intertwined in this practice with proof production. The mental processes that the activity involves may thus be worthy of more explicit attention in curricula than they generally receive in our experience, if we view mathematical education as a process of becoming competent in the specialized practices of an expert community.

4 Extending the Method

This method is easily and readily adaptable. It is appropriate for use with classes of any size and in both group and individual work. Proof-evaluation tasks are suitable for students at all stages, from beginning undergraduate to research-level. (Editors' note: For another approach to this in this volume, see Keiser [3].) Errors or inaccuracies in proposed "proofs" can be as flagrant or as subtle as we wish. As well as challenging students to exercise sound judgement about correctness of a proof, evaluation tasks may challenge them to consider their own preferences in terms of style and presentation, and to develop their awareness of the choices that are available to us in mathematical writing.

One potential drawback of proof-evaluation tasks is that they are very time-consuming to compose. A shared repository of such tasks, that could be adapted locally by individual users, would be a useful resource for the community of mathematics instructors with interests in promoting engagement with proof.

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Teaching with Two-Column Proofs

Mindy Capaldi

Abstract

Teaching and learning proof-writing are well known for being difficult tasks. Often, students struggle with learning to think through the justifications of their proofs, as well as knowing how to organize them and how much detail to include. In an abstract algebra course, a resolution to this problem was attempted by partially requiring a two-column style of proof. Students followed a strict format, with one numbered column listing the steps of the proof, including any given assumptions and goal statements, and a parallel column defending each step. Rigor was demanded, so that every logical step was included in the proof, even those that seemed obvious to the student. Proving with two columns can help ease the transition to learning to prove in paragraph style.

Difficulty Level: Low; Course Level: Advanced

1 Background and Context

More than **whether** a conjecture is correct, mathematicians want to know **why** it is correct. We want to understand the proof, not just be told it exists. [2, p. 390]

Abstract algebra students at Valparaiso University are juniors, seniors, or, more rarely, sophomores. They are largely traditional students who live on campus. All the students are math majors, and many have double majors, with secondary education being the most common additional major. Few students plan to go on to graduate school for mathematics. The university as a whole is a liberal arts school with around 4,000 students.

Abstract algebra is the beginning of one of three upper-level sequences that are offered, all of which include proof-based classes. The prerequisite is linear algebra, which is where students are exposed to the techniques of proving. When I first taught abstract algebra, I quickly realized that my students were confused about proofs. They were not sure how to write, read, or create them. They may have only experienced proving in linear algebra, or they may have taken additional proof-based classes; previous exposure to proving did not seem to affect their ability level or comfort with it once they got to my course. Though I tried to explain how I approached proofs, from how to start them to how detailed they should be, the students continued to struggle. I knew that the next time I taught a proof-based class, I would need to try something different.

As an undergraduate student, I was taught proofs using two columns in discrete mathematics and abstract algebra. The left column was a list of numbered lines detailing the logical progression of the proof. The right side justified each step, by referring to previous lines or definitions and theorems. Proofs were started with an “assume” line and a “show” line, and the last line of the proof must connect to that “show”. It seemed easier, compared to reading textbook proofs, to notice the different structures of a proof. When writing, I learned to force myself to not use a step in the proof unless I was able to defend it.

This article details how I taught abstract algebra using the two-column method. While the style of proving was similar to the way it was taught to me, I also spent time in class emphasizing the transition to and from paragraph style and my expectations in terms of rigor. In this paper, I explain the benefits, as I see them,

as well as some of the drawbacks of teaching with two columns. While my class was abstract algebra at a liberal arts university, this teaching method could be employed in any proof-based class at any level. Using two columns provides students with a beneficial step along the path to becoming proficient proof-writers, even though this style is not how professional mathematicians write their proofs.

2 Description and Implementation

In the first week of class I laid out my expectations for the students. They were required to use this method of proof-writing at least through the first half of the course, until told otherwise.

The following guidelines summarize what my students were instructed to do.

- **Write an “Assume” statement.** This tells you what information was given, and what you can use to progress to the next step of the proof.
- **Write a “Show” statement.** This tells you what your end goal is. Once you have shown it, you are finished with the proof.
- **Use two columns.** On the left, number each logical step of your proof. On the right, defend your logic. Your defense can be previous lines of the proof or definitions and theorems covered in class.
- **Include all details.** Any step, even the obvious ones, should be listed in the proof. If you aren’t sure whether to include it, do.

Figure 1 in Section 3.1.2 is an example of a student’s two-column proof.

2.1 Why Require This Style?

At the beginning, and later in the class, it helps to explain some of the benefits of requiring such a strict structure. I explain many of the following points to my students.

2.1.1 Assume and Show

The “assume” statement helps with getting started. If students know what information they have to work with, they can look for theorems or definitions associated with it.

The “show” statement gives students a specific goal and helps them to know when they are done with the proof. Sometimes they might need nested “show” statements. For example, if they are trying to prove that $f : A \mapsto B$ is one-to-one, then their first “show” statement is “Show f is 1-1.” Then below that they could write “Show for every $x, y \in A$, if $f(x) = f(y)$, then $x = y$.” This leads to a new “assume”, as well, since they would now assume that x and y are in A and that $f(x) = f(y)$. Just writing out these “assume”s and “show”s gets them through the first several lines of the proof, and makes it very clear what the end goal is: “ $x = y$.”

2.1.2 Two Columns

Based on previous semesters of teaching abstract algebra, I do not believe that encouraging students to write assume and show statements is enough when teaching them proof-writing strategies. Selden and Selden discussed the importance of validation, which is a term that describes “... the process an individual carries out to determine whether a proof is correct and actually proves the particular theorem it claims to prove. This process involves much more than just passive reading—it is often quite complicated and includes making affirming assertions, asking and answering numerous questions of oneself, and perhaps even constructing

subproofs” [6, p. 127]. Two-column proofs basically force validation to occur. As the Seldens also pointed out, it is difficult to observe the validation occurring [7]. Another study discussed the ways that two-column proofs, when implemented correctly, can allow for more authentic mathematics [8]. Validation is a technique that mathematicians need to master, and I believe that it is easier to tell whether a student is correctly validating, or validating at all, with two-column proofs.

There is often a difference between convincing oneself that a proof is true, and understanding the logic of a true proof. Even a Fields medalist, Pierre Deligne, who was convinced that his proof was true, wished that someone else could explain it to him [5]. Segal remarks that internationally renowned mathematicians sometimes find formal proofs, including their own, unconvincing, even when the proofs are acceptable mathematically. She also points out that students are in even more of a confused state than mathematicians when reading community-accepted proofs [5]. Lamport, who advocates a structure of proof related to the two-column method, argues that proofs written in prose are difficult to understand and suggests, based on anecdotal evidence, that a third of all published papers have incorrect proofs and theorems [3]. One way to ensure that students are convinced by a proof, whether in a book, article, or one of their own making, is to train them to validate it. The second column aids in this. Once they are used to considering the justification of each step, they will not necessarily need to write the validation down, but will be able to validate mentally.

There are many benefits to using the two-column style of proof in addition to paragraph style. The most important benefit is that it makes students think about why they can proceed from one step to the next. They must defend their logic. They cannot get by on just hoping that what they are doing is “legal,” mathematically.

Requiring justification demonstrates the importance of definitions and theorems. When students are repeatedly using such information in the second column of their proof, it is easier for them to realize that it is essential. One of the struggles that students have once they get to upper-level proof courses is a naive belief that math definitions are like regular English vocabulary, and they only need a basic understanding of the concept. They were able to get by in their other classes with surface-level understanding. For example, when a continuous function is defined in calculus, it is often described with “you can’t lift your pencil off the paper.” The mathematical definition is glossed over. We have to retrain them to believe that definitions are critical.

2.1.3 The Details

In a class like abstract algebra, students often want to leave out steps in a proof that cover ideas they learned in grade school, like multiplication or division. Those operations are not necessarily allowed in different groups, rings, etc., so they cannot always assume them. By requiring every detail to be justified in the second column with definitions, theorems, or lemmas, students cannot use even a simple or obvious concept unless they know it is allowed. Students should be showing similar justification in a paragraph proof, of course. Nonetheless, it seemed to help my students to think through not just what definition or theorem they were using, but which previous parts of the proof, if they had to cite those references in the second column. Segal states “... the learner is left wondering: what knowledge may be safely assumed and left out? And, on the other hand, what knowledge is critical and must be explicitly addressed?” [5, p. 195]. Any logical step that connects to a definition or theorem learned in the course had to be defended, at least until otherwise agreed by the class and teacher.

2.2 Transitioning to Two-Column Proving

Most of my students had never written a two-column proof, or not since high school geometry. In the first week of class, I presented a theorem stating that a greatest common divisor of two nonzero integers can be written as a linear combination of the integers. They were given the theorem statement and a proof from the textbook [1] and asked, in groups of four, to write it in the two-column style. I expected this task to

take 10-15 minutes. It took 75 minutes. They really struggled to understand the justification behind each line of what most mathematicians, including me, would consider a very nice proof. The activity confirmed my belief that they needed practice with the validation step that should have been occurring when they read proofs. One way to train them to validate is through the creation of the second column. The first column can to some degree take the original proof and split it up into separate lines. The parts of the proof that do not end up in the first column should go into the second, usually with some additional information, like numbers referring to previous lines.

After practicing converting given proofs to two-columns, the class then practiced writing proofs of their own in the two-column style. They did this first during class in groups, and then on homework individually. The first several proofs that were completed in class ultimately ended up on the board. Thus, we were able to compare each group's logic and justification and decide if anything needed to be added or removed. This led to wonderful discussions of "what is a proof?" and why we should care about rigor and details. After turning in their individual efforts through homework, they were given feedback on their two-column proofs and given the chance to turn them in again.

2.3 Transitioning Away From Two-Columns

Requiring students to use the two-column method could lead to worries that they would not be able to write a paragraph style proof later on. I stopped requiring the two-column style about two-thirds of the way through the semester. At that point we practiced converting from two-columns to paragraph. They tried the task in groups, and similarly to when different groups' two-column proofs were compared at the board, the different paragraph proofs were written up to compare. This again allowed for class discussion of what is necessary or wanted in a proof. In the future, I plan to have students practice both styles earlier in the semester. One way to implement this would be to have students write a two-column draft of their proofs, and then a paragraph final version. Another good option is to pick a homework problem and ask students to write it up using both styles.

3 Outcomes

3.1 Benefits

3.1.1 Getting the Process Down

For decades, math teachers have discussed the difficulty students have with starting proofs [4]. In a previous semester, I did not require "assume" and "show" statements, though I recommended them. At the end of the semester, I asked the students about their strategies when starting or working through a proof. Five out of seven listed writing the "assume" and "show" as the first step. Several additionally wrote "write a more specific show step." The next time I taught the class these statements were required within the two-column format. When they came to office hours about a proof, every student at least had it started. Also, because they had to write at least one "show" statement, they learned to think through the goal of the proof and what getting to it entailed.

3.1.2 Validating

In my previous abstract algebra classes, it seemed as though the students were so happy when they finished a proof that they did not look back to consider whether or not their conclusion was justified. I also had problems with those students wanting to leave out "obvious" points. By telling my students from the start that they should include every detail that they could think of, until the class concluded it was obvious, they rarely missed significant points for lack of rigor. Also, when reading a proof that they knew to be correct,

Prove that $\phi : (\mathbb{R}^+, \cdot) \rightarrow (\mathbb{R}, +)$, where $\phi(x) = \log_{10}(x)$, is an isomorphism.	
<i>Proof:</i>	
1. Assume $\phi : (\mathbb{R}^+, \cdot) \rightarrow (\mathbb{R}, +)$, where $\phi(x) = \log_{10}(x)$. (Show ϕ is a bijection)	
2. Assume $a, b \in (\mathbb{R}^+, \cdot)$ where $\phi(a) = \phi(b)$. (Show ϕ is 1-1: $a = b$)	
3. $\log_{10}(a) = \log_{10}(b)$	1,2
4. $10^{\log_{10}(a)} = 10^{\log_{10}(b)}$	3, Alg.
5. $a = b$	4, Def: log
6. Assume $b \in (\mathbb{R}, +)$. (Show ϕ is onto: $\exists a \in (\mathbb{R}^+, \cdot)$ where $\phi(a) = b$)	
7. Let $a \in (\mathbb{R}^+, \cdot)$ where $a = 10^b$.	1,6
8. $\phi(a) = \phi(10^b) = \log_{10}(10^b) = b$	6,7,1, Def: logs
9. Therefore, ϕ is a bijection.	5,8
10. Assume $a, b \in (\mathbb{R}^+, \cdot)$. (Show $\phi(ab) = \phi(a)\phi(b)$)	
11. $\phi(ab) = \log_{10}(a \cdot b)$	10,1
12. $\phi(a)\phi(b) = \log_{10}(a) + \log_{10}(b) = \log_{10}(a \cdot b)$	10,1, Multl Prop. of logs
13. Therefore, $\phi(ab) = \phi(a)\phi(b)$.	11,12

Figure 1: Two-column example

they assumed it should easily make sense. However, they did not think through why it made sense. Their struggle with converting proofs to two-columns indicated to me that they needed to be trained to validate proofs, both those in books and their own. By the end of the semester, my students were able to do this.

Figure 1 is a representative example of student work from the two-column class. This student laid out what he can assume and what he needs to show, helping to eliminate possible confusion. I can tell that he knows the definition of an isomorphism, and what proving it entails. In the justification column, although I have to look back at previous lines to understand the references, I can clearly see this student's understanding of the logic behind each new line. In general, if students are able to justify their new line of argument, then I believe that they understand why it holds. I am measuring understanding by how correct the justification is. For example, did the student miss citing a previous line or theorem, or include a citation that was not really part of the justification for the new line? Consideration of such criteria is a way to evaluate student understanding.

3.1.3 Proving Correctly

Requiring two columns significantly increased the level of organization and correctness in my students' proofs. This became especially clear by the end of the semester. I gave similar final exams in the different abstract algebra classes, although the earlier class was allowed to use their notes and book on the exam, and the later class was not. The proofs were about the same level of difficulty. The two-column class got more points for correctness.

In the two-column class, I gave an anonymous, voluntary survey at the end of the semester that asked various questions about the two-column style. Most responses indicated that the students felt that proving with two-columns benefited them. For example, "this was my first time really writing formal proofs and it kept me organized. It helped me make sure that every line was cited from somewhere and it had a purpose." Another student commented that "the column proofs . . . are helpful to those who are new to writing proofs. It is an easy format to follow and helps keep a clear line of logic."

3.1.4 Secondary Benefits

An additional benefit to teaching with two-column proofs was that the students enjoyed it. They felt more confident when reading and writing two-column proofs, and indicated they understood more of what was going on. Many responders also commented on how much more organized they felt their work was.

4 Extending the Method

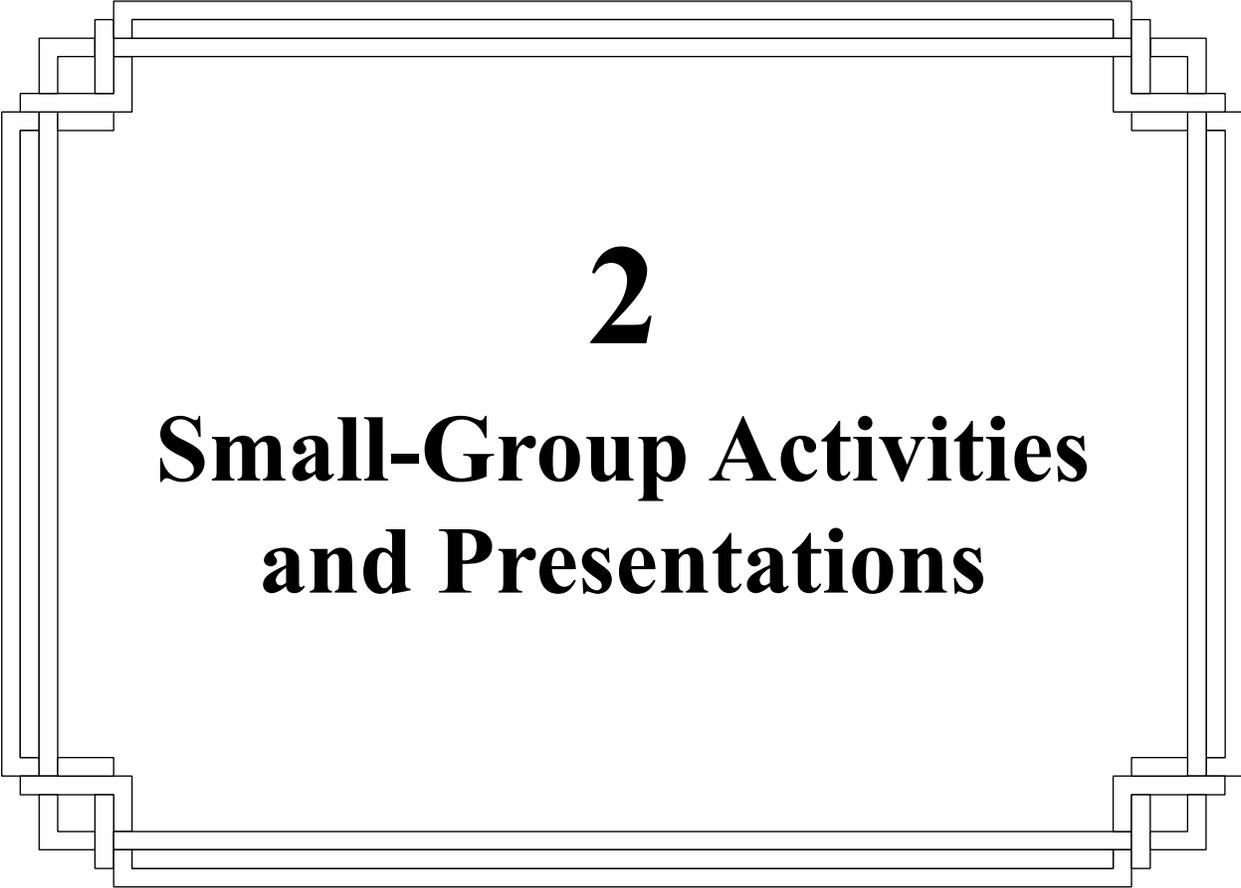
While I taught the two-column method in abstract algebra, it could be used in any proof-based course. It could benefit high-schoolers to college seniors. In fact, my abstract algebra class included freshmen to senior-level students. Teaching through two-column proofs was especially useful for me since my university did not have a specific transitions-to-proofs course; I was able to more quickly get students on the same page with a basic understanding of what a proof entailed. However, writing two-column proofs, in conjunction with paragraph-style training, could be used in an introduction-to-proofs course. By the time students finish that class they should be ready to write in paragraph style for the rest of their proof-based courses. Of course, since many instructors of those latter classes do not have the opportunity to teach the introduction course, the use of two-columns could still be beneficial.

Thinking through two-column proofs can be part of training students to think like a mathematician, just as training wheels are used to help kids learn to ride bikes. Lamport advocates a structure similar to two columns, but with the second column not separated from the first [3]. He motivated the need for what he called “structured proofs” by referencing the use of structured formulas in mathematics instead of sentence descriptions of them. Similarly, reading and understanding structured proofs can be easier compared to prose proofs [3]. Structuring puts proofs, and the logic involved in them, at a level that is more approachable and clearer to students. We want students to think and write like mathematicians. Two-column proofs help them think that way.

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2

Small-Group Activities and Presentations

Proving in the Right Circles: A Collaborative Learning Activity to Develop and Improve Proof-Writing Skills

Linda McGuire

Abstract

A *proof circle* is a pedagogical framework through which proof-writing techniques are understood, critically examined, explored, and refined. It involves assigning students specific roles to play and tasks to complete during organized group-work activities. This paper will detail the interdisciplinary foundations for this framework, the structure and implementation of proof circles in class, and the outcomes evident from using this technique over several semesters. Examples of student work and feedback will also be provided.

Difficulty Level: Medium; Course Level: Transitional

1 Background and Context

This article will detail the development, implementation, and assessment of a pedagogical structure called a *proof circle*. A *proof circle* is a group-work model similar to a literature circle or *junto* in the humanities, but with a decidedly mathematical focus and format. The technique is designed to help students acquire, develop, and deepen critical-reading, problem-solving, oral-communication, and proof-writing skills. As a result of implementing and honing the method over several semesters, assessment evidence suggests that proof circles are effective at supporting student skill development.

Proof circles are implemented in courses at Muhlenberg College, a four-year liberal arts college whose mission stresses a dedication to the development of effective oral and written communication skills within and across disciplines. The technique provides a creative way to facilitate the advancement of the mission in the mathematics classroom. Muhlenberg College has a residential campus of approximately 2,200 undergraduates, and class sizes are normally capped at twenty-four. After graduation, approximately one-half of mathematics majors enter graduate school, one-third take jobs in business and industry, and one-sixth begin work as secondary-education mathematics teachers.

Although first developed for use in a sophomore-level bridge course designed to teach basic proof techniques, I have modified proof circles for use in introductory-level calculus classes and upper-division mathematics classes that are sub-discipline specific, such as Mathematical Statistics and Combinatorics & Graph Theory. From a pedagogical perspective, the method is designed so that it (1) encourages small groups of students to discuss mathematics in depth, (2) scaffolds this process for students by providing a structure to guide discussions and written responses, (3) provides an appropriate context in which to develop technical reading and writing skills, and (4) promotes student responsibility for and ownership of their own learning.

1.1 Literature Circles and Juntos

Commonly used in humanities classes, literature circles and juntos are methods that inspired and informed the creation of proof circles. Both techniques are designed to encourage thoughtful and informed discussion and debate while nurturing a love of reading. Student collaboration is at the heart of these.

Literature circles have been a successful learning strategy in K-12 education for decades. They are intended to provide a way for students to engage in critical thinking and reflection as they read, discuss, and respond to articles or books [1, pp. 2-4]. In literature circles, small groups of students are guided by prompts and responses to what they have read to discuss a piece of literature in depth. Often, a student in a literature circle has a specific, unique role to play in a group discussion. Roles are configured depending on the application, but the general structure has proven to be beneficial for student learning [7, pp. 18-19]. Students explore, edit, and refine their understanding of a text as they construct meaning with other readers. Typically, literature circles lead students to deeper understanding of what they read through structured discussion and extended written responses.

Similarly, a *junto* is a discussion platform that can be adapted for pedagogical purposes. Historically, a *junto* was a club established by Benjamin Franklin in 1727. Also known as the *Leather Apron Club*, the primary purpose of Franklin's group was to engage in discussions on societal and academic issues. Franklin described the formation and goals of the *junto* in his autobiography:

I should have mentioned before, that, in the autumn of the preceding year, I had form'd most of my ingenious acquaintance into a club of mutual improvement, which we called the JUNTO; we met on Friday evenings. The rules that I drew up required that every member, in his turn, should produce one or more queries on any point of Morals, Politics, or Natural Philosophy, to be discuss'd by the company; and once in three months produce and read an essay of his own writing, on any subject he pleased [3, p. 1].

Students are often surprised to find out that, as a testament to Franklin's interest in life-long learning, a *junto*-inspired academy of learning was formed in 1751 that went on to become the University of Pennsylvania [9].

The goals and outcomes associated with these formats complement many of the core aims of mathematics education. So, there appeared to be an opportunity to adapt them for use in the mathematics classroom.

2 Description and Implementation

In proof circles, students are organized into working groups in which each person is assigned a role. Groups work both outside and during class to complete specified tasks. Normally, a group is given a reading assignment and a related set of problems to solve. Each student assumes his/her role in the group and a student must approach the discussions and problems from the perspective that assigned part dictates. Each role is characterized by specific types of information to gather, ideas to formulate, and tasks to complete prior to the next class meeting. During in-class work sessions, group members discuss their findings with each other, begin to collectively develop solutions based upon their pooled information, and ultimately, present their solutions to the rest of the class. Students rotate roles throughout the semester so that each student has the opportunity to play each role at least once.

2.1 Proof-Circle Roles and Responsibilities

The roles students assume are categorized and defined as

- *Discussion Coordinator*: presents an outline of the assigned reading focusing on the main points. This person is also responsible for facilitating the entire group process, from organization to discussion and assignment completion.
- *Passage Master*: locates passages in the reading where key information is provided and summarizes the main points of the reading assignment.

- *Concept Collector*: develops a detailed list of terms, definitions, theorems, strategies, and concepts that are important to understanding the reading.
- *Visualizer*: makes illustrations or schematics related to the reading. Visualizations may include graphs, charts, concept maps, drawings, collages, or other forms of visual representation.
- *Content Connector*: leads the group in standing back and making connections to previous material discussed in class and may also invite connections to content from other classes, mathematical or otherwise.
- *Devil's Advocate*: challenges the ideas under consideration by raising questions, examples, or arguments that could be put forward by critics of the argument being constructed.

Depending on the number of students in a class, the roles can be successfully adapted to groups of sizes four through six. For groups of size four, one can combine the roles of Passage Master and Concept Collector into one entity and disperse the Visualizer role by having it “played” by all group members. Similarly, groups of size five can distribute the Visualizer role among group members if physical representations of concepts are important to understanding the assignment. Additional suggestions for adapting the model to other classroom configurations will be addressed in section 4.

A general schematic for how the process is systematically implemented begins with students’ being assigned a reading to complete before a class meeting. The instructor prepares for class by writing and assembling the problem set and officially assigning roles to group members. Each student knows ahead of time what classmates he/she will work with and what role each will be asked to play during the session. A student writes up his/her own personal preparation notes for class in accordance with the role assigned. In class, students work in groups to analyze and discuss the reading, while the instructor circulates to listen to the conversations. As needed, class time is dedicated to a question-and-answer session involving the professor and the class. Groups are given a common set of problems to work on based on the reading for that day. Student groups usually go to the board to write out and present completed proofs or solutions. Normally, each group presents a different problem from the common problem set so that, by the end of class, the entire problem set has been presented and discussed. Group presentations are assessed on a 10-point scale focusing on mathematical accuracy, clarity of explanations, and balance of contributions among group members. Presentation grades and comments are sent to groups via email after class.

A student group “completes the circle” when it submits for assessment a single write-up of problem solutions and each group member’s preparation notes. Each student receives a grade per proof-circle assignment determined by his/her preparation for and participation during class exercises ($\approx 35\%$), the group’s performance during in-class presentations ($\approx 25\%$), and the quality of the group’s written work ($\approx 40\%$). One formal write-up is submitted per group and the authorship of group papers must consistently rotate among group members. Students write a brief self-assessment of their contribution to the proof circle as part of their assignment. The multiple layers of assessment offer the instructor a window into the individual progress and the collaborative efforts of his/her students. Not surprisingly, there is a strong positive correlation between how well students prepare for class and how well they do in proof circle activities.

Proof-circle sessions are easily adapted to align with the amount of class time an instructor is able to devote to such activities. The in-class time commitment can be anywhere from ten minutes for groups to work on a single problem to 50-75 minutes for an extended problem-solving session. For students to learn and benefit from using the technique, it is advisable that proof circles become a regular classroom activity. Given the flexibility the instructor has with regard to the length of exercises, proof circles can be used at least once or twice a week. It has been my experience that the coverage of course content is not significantly compromised when using the technique. Three observed trends contribute to explaining this: students are required to do a significant amount of pre-class preparation to effectively use proof circles, students are asking fewer homework questions during class, and the instructor has significantly reduced the use of other

types of in-class labs and worksheet activities in favor of proof circles. In addition, the amount of reading comprehension required of students leads them to ask more probing, informed questions in class. Questions rarely take the form “I don’t understand what this means,” but instead evolve to “On page 10, I am not clear as to the logic that leads from line three to line four of the argument and I am not seeing how this connects to the definition on page 8.” The specificity in the latter question allows the instructor to quickly identify and address points of student confusion or misunderstanding.

2.2 Examples

To give more detail regarding how this method is applied in practice, we will consider examples from two different sophomore-level courses: introductory proof-writing and a first course in linear algebra. The examples are chosen to emphasize that an instructor may adapt proof circles for use on a wide range of assignments, from those where the material is relatively straightforward to tasks where the content is more demanding and complex.

In the early stages of a proof-writing course, when the basic techniques of direct proof, indirect proof, proof by contradiction, and case-by-case analyses are being introduced, proof circles provide an effective means to encourage students to “speak” mathematics with each other.

Example 1: *Assume $a, b, c,$ and d are integers. Prove that if a divides b and c divides d , then ac divides bd [8, p. 38].*

In a four-member group problem-solving setting, the Discussion Coordinator might begin by highlighting the possible proof techniques that the group should consider employing. To assist the group members in making their decision as to what technique to apply, the Passage Master/Concept Collector would explicitly review the definition of divisibility and specify what the group needs to show mathematically to solve the problem. The Content Connector can explain what would be assumed and what would have to be demonstrated if the proof were constructed directly, indirectly, using cases, or by contradiction. The Devil’s Advocate can raise questions and concerns at this phase, as well as after a proof is constructed and is being edited. Notice that the role of Visualizer is not used in this situation. Recently when this problem was assigned, a student in Devil’s Advocate role noticed that the group had failed to specify that $a, b, c,$ and d were integers and questioned his classmates in a way that led them to find the error without his explicitly identifying it. From start to finish this exercise takes approximately ten to fifteen minutes of class time and is well worth it.

Example 2: *Let $C[a, b]$ be the set of all continuous real-valued functions defined on a closed interval $[a, b]$ in \mathbb{R} . Define $T : C[0, 1] \rightarrow C[0, 1]$ as follows: for f in $C[0, 1]$, let $T(f)$ be the antiderivative F of f such that $F(0) = 0$. Show that T is a linear transformation, and describe the kernel of T [6, p. 207].*

Example 2, taken from a linear algebra course, demonstrates that complex material often provides the opportunity for students to work in proof circles for an entire class period. Once group roles are assigned, students are asked to read the textbook section that introduces null spaces, column spaces, and linear transformations [6, pp. 198-205]. Class begins with each group’s meeting to review the section, clarify definitions, and formulate and ask any questions they have on the reading.

The Discussion Coordinator would provide an overview to the group that highlights the introduction of definitions and properties such as the null space and column space of an $m \times n$ matrix A , the difference between them, the introduction of the idea of a linear transformation between vector spaces, as well as its kernel and range. More specifically, the Discussion Coordinator might begin by asking group members to confirm the definitions of $C[0, 1]$, T , linear transformation, and kernel. The role requires a student to outline the big picture and begin the conversation. The Passage Master directs the group to locations in the text where concepts are defined or explained and examples are given, while the Concept Collector provides a summary list of definitions and theorems that capture these ideas. The Content Connector tries to contextualize this material in the larger conversation about vector spaces and to ask why the concepts are useful or

important. While the Passage Master and Concept Collector play important roles with regard to organizing and understanding information, the Content Connector serves a critical purpose in situations where the material is more theoretical. The Visualizer often provides drawings to help explain, for example, that $\text{Nul}(A)$ is the set of all vectors in \mathbb{R}^n that are mapped into the zero vector via the linear transformation $T(\vec{x}) = A \cdot \vec{x}$, or how the kernel and range of the linear transformation are interpreted. The Devil's Advocate is the great questioner who may ask the group to work through detailed examples to confirm definitions or pose questions to test whether an assumption is correct. The Devil's Advocate not only helps refine the argument by challenging decisions and assertions made, but often poses questions such as "why is this important?" and "why is this useful?"

During the discussions, the instructor circulates, listens, probes, and determines which questions or issues raised require full-class discussion or comment, and which are specific to a particular group. After questions are addressed and any necessary instruction is completed, groups begin working on a problem set.

Whether used as a single, short-term group exercise in class, or as a framework for a longer group problem-solving session on more complicated material, the proof circle can be readily adapted to many classroom scenarios.

3 Outcomes

To date, I have modified proof circles for use in six different classes over the last four years. In each situation, the approach has proven to be successful at increasing student engagement and focus, promoting an *esprit de corps* in class, and developing mathematical (as well as general) oral and written communication skills.

When comparing outcomes in courses taught without using proof circles to classes in subsequent semesters where they were used, results on traditional evaluative devices (exams, quizzes, and problem sets) have improved for me over time with concentrated use of the proof circle method. Exam grades reflect gains in both factual and definition recall questions (15-20% increase) and problem-solving questions (8-15% increase). Group quizzes, which this instructor has used in a limited way, have shown a 15-20% increase in grades. Significant improvement has been seen in the quality of student writing, both narrative and technical, as well as in the focus and complexity of class conversations.

Positive aspects of the proof-circle process are that it

- encourages close, careful reading of the text
- brings about higher-quality in-class questions and discussion
- identifies common student difficulties, misunderstandings, and errors more quickly
- allows the instructor to begin to experiment with "flipped classroom" pedagogies
- requires students to prepare for class ahead of time
- makes theoretical concepts more concrete and easier to understand
- allows students to see mathematics as a collegial, rather than solitary, effort.

Assessment information that affirms the efficacy of the proof-circle method includes comparison data between course outcomes over time, comments from students on anonymous course evaluations, and noticeable qualitative improvements in the level of student commentary and work.

Anonymous student feedback provided on course evaluations has been generally positive. Sample student comments include

- It was interesting to see what jobs were difficult for me. I realize I am not the best content connector and that this is something I really need to work on in all of my courses. I need to stop and think more about the big picture.
- Now when I study I try to play all of the roles in my head to make sure I am not missing anything.
- The proof circle group discussions were one of the most helpful parts of the course. We could talk through sticking points in the reading and then we knew what we could figure out on our own and what we really had to ask you about.
- Proof circles changed the way I read everything, not just math. I used to just read for the gist of the chapter, but now I read like a math-biologist, always dissecting every little sentence or example.

The method succeeds in the small-college setting precisely because it aligns with the goals of both the College and the mathematics curriculum. Students across the range of mathematical skill levels are challenged by the exercises. While the technique is demanding, especially for mathematically weaker or less organized students, proof circles require students to be responsible for their work. Observed patterns over time suggest that student desire not to let classmates down leads them to seek counsel and assistance more regularly during office hours. From the instructor's perspective, a key advantage is that it allows students to compartmentalize tasks and thus work in a more directed and purposeful manner. Many students report that they value having a specific role to play and a unique contribution to make to their group. At the early stages of learning to write mathematical arguments, this aspect of proof circles appears to make the tasks of learning and applying mathematical concepts less daunting.

4 Extending the Method

While the method has already been adapted for use in different mathematics courses, there are challenges with regard to its implementation depending upon class size. This article already includes suggestions in section 2 as to how some roles can be combined or redistributed to fit groups of different sizes. There are also, however, ways in which to modify this idea to work in high-enrollment classes. A colleague in the natural sciences at the University of Colorado has adapted the idea for use in a class of 180-200 students. Smaller recitation sections are often dedicated to discussion circles and the roles presented here are used to facilitate class discussion about journal articles.

Technology that facilitates student interaction is also showing great potential for incorporation into this method. Students this past semester used a wiki to capture, discuss, and refine their group work. It proved to be a useful and manageable way for the instructor to provide commentary and guidance outside of class. It may also provide another way for students in large lectures to adapt proof circles for use outside of class, capturing and contributing to interactions with each other and the instructor. The use of online formats represent a new addition to the proof circle process, but outcomes thus far are encouraging.

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Group Examinations in Introduction-to-Proof Courses

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Abstract

We discuss the use of in-class group exams as a method of instruction, peer-review, and assessment. We describe our experience implementing them in an introduction-to-proof course designed primarily for math majors and minors. In our model, each of the group members has a proposition he or she is responsible for proving. Group members review each other's work, and a grade is assigned to each student based on the quality of the student's proof as well as that of his or her group members. This process gives students valuable experience actively creating and analyzing proofs, while introducing a second level of critical review and feedback. Because students are responsible for their own proofs as well as determining the validity of the work of their peers, the exams provide assessment of content knowledge, proof-writing skills, and reading comprehension of proofs. In this article, we discuss what makes a good group exam and summarize student reaction. We provide specific examples of exam questions and comments on how to write a group exam.

Difficulty Level: Medium; Course Level: Transitional

1 Background and Context

In mathematics, the act of writing a proof involves both figuring out a logical argument, and effectively communicating it to an audience. The goal of an introduction-to-proof course is to teach students both facets of the proof-writing process. But how do you assess student progress on them? In our experience, homework, though valuable, has several disadvantages as an assessment tool. When students work together, there is a likelihood of over-dependence on peers. On the other hand, while working alone, they do not receive immediate feedback, and misconceptions fester. At our institution, individual, in-class midterms are limited by time constraints, making it difficult for students to figure out and craft proofs of more than two or three unfamiliar statements during the exam period. Thus, if we ask them to prove challenging statements, or statements that are unlike those they have proved before, we run the risk of students failing the entire exam because they became stuck on one problem. This high-stakes testing can lead to anxiety and loss of confidence. However, if we don't ask students to prove difficult or unfamiliar statements, how much of the *figuring out* process are we assessing?

The group exam format described herein uses peer discussion and solution-vetting to resolve these issues. Given the ability to brainstorm ideas, students are able to tackle challenging and novel proofs in the exam setting. Cooperative-learning activities such as this have been shown to increase student learning and retention of course content [7, 3]; students self-report reduced test anxiety, elevated confidence, benefits from seeing the problem-solving techniques of others, deeper critical thinking, and increased enjoyment of the course [6, 5]. The group exam model we describe is unique in that it includes the benefits of collaboration, while assessing individual achievement and minimizing social loafing¹.

We have successfully implemented group exams at a large state school, a private research 1 university, and our current school, Willamette University, a selective liberal arts college in Salem, Oregon. Willamette

¹*Social loafing* is a common phrase in social psychology. It refers to when individuals exhibit less motivation while working in a group setting than they would on their own [4].

enrolls approximately 2000 undergraduate students and graduates approximately 12 math majors and 9 math minors per year. In this article we discuss our experiences using group exams in Willamette's introduction-to-proof course, Foundations of Advanced Mathematics. Approximately four sections of this course are offered annually, with enrollments nominally capped at 15 students per section. In recent years enrollments have been as high as 20 students per section. Foundations is a prerequisite for all advanced math courses and is usually taken by math majors and minors in the spring of their freshman year after completing multivariable or integral calculus. It is in Foundations where many of our students first encounter group exams, where they are given in addition to weekly homework and individual midterm and final exams.

2 Description and Implementation

A sample group exam with instructor notes is included in the Appendix. Below we describe the exam process, including preparation and grading, followed by a discussion of what makes a good group-exam problem.

On exam day, each group of three is seated at a separate table (usually with two students on one side and one facing them on the other). We ask students to move the tables and chairs so they are spaced throughout the classroom. This allows room for groups to discuss their work without the distraction of overhearing the conversations of other groups.

Working in groups of three, students are given sixty minutes to complete a three-page exam. Each group member is responsible for proving a set of propositions from one page of the exam. Students are required to spend the first third of the exam period working on the questions from their individual exam page. The period of quiet gives students the opportunity to focus without distraction. After approximately twenty minutes, we inform students that they may engage in quiet discussions with their group members. For the remainder of the period, students actively share their work and review the work of their peers. They read critically to ensure the argument is valid and to give feedback on the clarity of their peers' writing. Students must then assess the suggestions made by their group members and incorporate them into their work accordingly. Students are only allowed to write on their individual exam pages, but are encouraged to collaborate with each other through discussion of all exam problems. Once students are satisfied that a group member's proof is valid and clearly written, they sign their names at the bottom of that group member's exam signifying their agreement with the argument presented. At the end of the exam, all three group members should know how to prove the propositions on all three pages of the exam.

We incentivize and hold students accountable for their exam preparation by requiring that each student bring to the exam a written page of notes summarizing the current class material. A thorough page of notes benefits students directly as it may be referenced by all group members during the exam. The note page is turned in with the student's completed exam and counts for a small portion of his or her exam grade. Making this page of notes a requirement minimizes the likelihood of social loafing during the exam, or at the very least makes ill-prepared students easy to identify. Note pages can include statements of important theorems, definitions, and examples from class, but examples of completed proofs are not allowed. Students are informed that note pages and exam scores will be used to make future group assignments. In particular, we emphasize that students who are less prepared will be grouped with other less prepared students on subsequent exams. Knowing their future group assignments are based on their exam preparedness provides additional incentive for students to study for the group exam.

The grading process is two-fold, but not double the work. Each group exam is graded first on its individual merit, then it is assessed for peer-review points. We assign roughly 75% of the exam score based on the student's individual work and the remaining 25% on his or her peer-review work, which emphasizes to students that, although it is a group exam, they will be held accountable for their individual work. For example, on a 40-point exam, 30 points are earned for individual work and 5 points are earned for proof-reading the work of each of the other two group members. This distribution of points addresses a common

student discomfort with group grading where all students earn the same grade though the quality of student work may vary. A student's peer-review points are lost when a group member makes a mistake that careful proofreading should have caught. We separate errors on the exam into two types: gross errors, such as conceptual or structural flaws, and subtle detail errors, for example minor calculation or small grammatical errors. Students lose up to half the points lost by their group member for each mistake of the former type and few points, if any, are lost for the latter. Specifically, missing a logical step or skipping a key justification that reduces a student's individual score by 2 points results in the other group members' each losing one peer-review point. Using the example of a 40-point exam, no more than 5 points are lost by any student for mistakes made by another member of the group. Minor mistakes in algebra, calculation, or grammar are often at a level more detailed than that at which the peer-reviewer has time to catch, so points are rarely lost for missing them. If a member of the group does not know how to do the proof and leaves it blank, then the other group members will not lose peer-review points as there was nothing to review. Each team member is expected to pull his or her own weight. If a group member is stuck on his or her proof, the group is encouraged to work together to find a solution, but is not required to do so. If there is an unresolved disagreement about the validity of a proof, group members may sign partially down the exam page indicating the portion of the proof with which they agree. However, if the remainder of the proof is correct, students will lose peer-review points for not recognizing its validity. The collaborative nature of the exam and the instructor's guidance make disagreements and blank papers rare.

The instructor's role during the exam is active. We circulate through the class checking in with each group, especially during the first group exam when students are still gaining familiarity with the format. While we do not offer the kind of direct help we would provide on an in-class worksheet, we often facilitate group discussion and, when necessary, mediate disagreements by asking questions or suggesting questions that students could ask their group members. We frequently remind students of the time remaining and where they should be in the exam process. Proofreading and discussion take significantly longer than students expect. Consequently, time management is an important factor in successfully completing the exam.

When writing a group exam we carefully consider problem difficulty and length, in addition to content coverage. Since each group member has a different set of propositions to prove, it is important that the three pages of the exam be similar in length and difficulty while not overlapping significantly in content. We admit this is not an exact science. However, we try to assign three proofs that are challenging yet accessible with each having a degree of newness that makes it unlike any of the proofs students have seen previously. One way to approach the group exam writing process is to reserve certain exercises, such as those that combine two topics in a new way, by not including them in lecture or homework so that they may be used on the exam. In this way, though each group exam takes sixty minutes of in-class time, we have found that they do not lead to a large loss in content coverage since the problems included would have been discussed in class if not introduced on the exam. The sample group exam included in the Appendix covers function image, preimage, composition, and inverses. At the time of the exam, the topics had only recently been introduced to the students. In our course, we study functions after sets and power sets. The questions come from our course textbook, *Proofs and Fundamentals* by Ethan D. Bloch [2], which contains a wealth of good exercises. The problems had not been assigned as homework or covered in class.

(Editors' note: For another approach to peer evaluation of students' exam solutions, see Bennett and Nguyen [1], in this volume.)

3 Outcomes

The strengths of the group-exam model stem from the incorporation of reading and vetting into the exam setting. Our students' writing is improved by the authentic and immediate feedback they receive from peers. Questions we commonly overhear, such as "What do you mean here?" or "Why can you conclude that?", alert proof-writers to a lack of justification or a logical incoherence that they are often unable to recognize

in their own writing. In our experience, comments from peers increase the proof-writer's attention to detail and explanation. Students quickly and naturally realize the importance of clarity in their writing. The vetting process makes students better able to appreciate the communicative power of a formal proof and less likely to view the proof-writing process as a series of hoops.

Another key outcome of group exams is students' increased ability to critically review proofs. In analyzing the proofs of their peers, students are exposed to authentic examples of potentially invalid proofs and a variety of writing styles and approaches. Not having previously considered the propositions proved by their group members, students must learn from their peers' writing and follow the argument presented. By necessity students must approach the proofs with greater skepticism, employing both critical-reading and analysis skills. On our individual exams (midterms and final) we include a problem asking students to read and assess the validity of a mathematical argument. Students often struggle with these problems. However, in courses with group exams, we have observed stronger performance by the final exam. In processing feedback from their group members, proof-writers engage in a second level of critical review. Because the feedback is not from an authority figure, students must determine the appropriateness and validity of the suggestions made. Furthermore, the close reading of their peers' proofs enhances students' ability to critically reflect on their own writing.

Gains in the quality of student proof-writing, reading, and analysis are reflected in higher individual cumulative final exam scores. Table 1 shows the average final exam scores for eight sections of Foundations taught by the authors. At least 75% of the final exam score is based on proof-writing problems (or problems asking students to prove or disprove a statement). Thus our final exams are a good indicator of a student's proof-writing ability. While we have an admittedly small sample of classes for comparison, there is a noticeable improvement in the exam averages of the five sections taught since implementing group exams.

Table 1: Section averages on Foundations of Advanced Mathematics cumulative final exams

Sections without group exams			Sections with group exams				
68%	77%	63%	84%	79%	79%	73%	84%



Figure 1: Students showing solidarity while working on a group exam.

The timed aspect of the exam forces nearly all discussion among group members to be on-task. Anecdotally, in sections where group exams are used we have noticed an increase in the number of students participating in class discussions, and improved student ability to articulate questions, explain answers, and

pinpoint areas of confusion. We believe the improvements are a result of the group exam activity. There is less risk involved in contributing to the discussion than in speaking before the class as a whole, as the specifics of what students say during the group exam are predominately heard only by their group members. Having gained confidence explaining complex mathematical ideas in the small group setting, students are more likely to raise questions in class and take part in class discussions.

Since implementing group exams the class climate has become more cohesive, supportive, and active. From our perspective, group exams help us get to know our students better. Circulating through the class during the exam, we can give personalized encouragement and guidance while gaining a better understanding of our students' learning styles and misconceptions. Student responses have been overwhelmingly positive. On anonymous mid- and post-semester surveys, when asked about their impressions of group exams, students reported that they led to increased confidence in the material and enjoyment of the course. The group-exam period is a time for both learning and assessment. Students expect and value the contributions of their peers. We have not observed the same level of collaboration and cooperation in classes without group exams, even when compared to classes with group projects or student presentations. Students in classes with group exams view their peers and the instructor as advocates for their success.

Student presentations are another common component of proof-based courses, and are frequently used by both authors. Presentations, like group exams, provide the authentic peer-review experience of judging the validity of a proof. We have found that student response to in-class presentations is more positive in courses where there are also group exams. We believe this is largely due to the cohesive and supportive classroom climate which group exams promote. For instructors concerned about the amount of class time required to have every student present, group exams provide an effective alternative. In the more intimate, collaborative and less stage-like environment of the group exam, all students are given the opportunity to critically read, analyze and critique the proofs of their peers, and receive feedback on their own work.

4 Extending the Method

Group exams have been successfully used by the authors in classes ranging from calculus to upper-level math courses. Although proof-writing is not a component of most calculus courses, students in all math classes benefit from focused discussion and proofreading solutions. Even in introductory-level courses, group exams promote better written communication of mathematics. In courses beyond the introduction-to-proofs level, group exams give students the opportunity to grapple with complex mathematical content and engage in conversations resembling those of experienced mathematicians.

Our recent experience using group exams has been at Willamette University where small class sizes are the norm. However, prior to our arrival at Willamette, we have used them successfully at the University of Oregon, with class sizes of 35-40 students, and at the University of Rochester with large lectures where group exams were administered by teaching assistants in recitation sections with approximately 20 students. In larger classes, it may be difficult for instructors to check in with all their students during the group exam. This may increase the likelihood of frustration felt by some students in groups that struggle to communicate effectively. Thus, in this setting, group exams may result in fewer of the class-climate benefits, but students will still benefit from discussion and peer review.

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Appendix

A Sample Group Exam with Instructor Notes

Page 1.

Problem 1A. Let A be a non-empty set and $\mathcal{P}(A)$ be its power set. Consider the function $F : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ defined by

$$F(X) = A - X$$

for all $X \in \mathcal{P}(A)$. Prove that F is its own inverse function.

Problem 1B. Find two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that neither f nor g is constant, but $f \circ g$ is a constant function.

Instructor's note: While prior to the exam students have seen examples of inverse function proofs, they have not seen a problem combining inverse functions and power sets. Students must realize that the solution includes a set equality proof. Problem 1B was added to balance the length of the three pages.

Page 2.

Problem 2A. Let A and B be sets, let $C \subseteq A$, $S \subseteq B$ and let $f : A \rightarrow B$ be a function. Prove that $f(C) \subseteq S$ if and only if $C \subseteq f^{-1}(S)$.

Problem 2B. In this part of the problem we show that it is not possible to strengthen the result above.

(i) Find an example of a function $f : A \rightarrow B$ together with sets $C \subseteq A$ and $S \subseteq B$ such that $f(C) = S$ and $C \neq f^{-1}(S)$.

(ii) Find an example of a function $f : A \rightarrow B$ together with sets $C \subseteq A$ and $S \subseteq B$ such that $f^{-1}(S) = C$ and $S \neq f(C)$.

Instructor's note: While prior to the exam students have seen proofs involving images and subsets, and proofs involving preimages and subsets, they have not seen this relationship linking subsets and images to subsets and preimages. Part B highlights a common misconception.

Page 3.

Problem 3. Let $f : A \rightarrow B$ be a function. For each $b \in B$, define the set $Q_b \subseteq A$ as

$$Q_b = \{x \in A \mid x \in f^{-1}(\{b\})\}.$$

Prove the following results about the family of sets $\{Q_b\}_{b \in B}$

(i) Prove that $A = \bigcup_{b \in B} Q_b$.

(ii) Prove that if $Q_{b_1} \cap Q_{b_2} \neq \emptyset$ for some $b_1, b_2 \in B$, then $b_1 = b_2$.

Instructor's note: While prior to the exam students have studied families of sets and inverse images, they have never considered a family of sets defined in terms of an inverse image. This problem also foreshadows the concept of partitions.

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Improving Proof-Writing Skills Through Weekly Student Presentations

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Abstract

Students acquire and hone proof-writing skills using an approach that features presenting proofs at the whiteboard in real analysis and in calculus. Preparing, presenting, and critiquing proofs engages students and effectively promotes their learning. The weekly schedule of instructor lectures and student presentations promotes a disciplined and efficient progression through the course content.

Difficulty Level: Medium; Course Level: Non-traditional, Advanced

1 Background and Context

I assign weekly student presentations to improve proof-writing skills at Bridgewater State University (BSU). BSU is a former teacher-preparation college now serving 11,000 students in southeastern Massachusetts. Approximately 300 students have mathematics as their declared major [2]. Most of them pursue careers in education.

I use the presentation technique to teach proof-writing to students in real analysis and calculus. In particular, I have used the approach for two semesters of Introduction to Real Analysis I, two sections of which were offered each semester. I have also used it in two sections of Calculus II, having students present proofs on the convergence or divergence of series. In the calculus class, the students not only identify which convergence test(s) to use, but also meticulously show how each hypothesis of the test is met. They state the conclusion in a complete sentence, citing the name of the test, for example, “Therefore, the series diverges by the limit comparison test.” This work provides a basis for the formal structure students need in upper-level proof-based courses.

The first class in real analysis with student presentations had twelve students, who presented in pairs each week. The other section taught by another professor with other techniques had a similar number of students. The second class in real analysis that I taught with student presentations had eight students. Another professor taught a full-capacity section of 30 students using other techniques. All sections consisted of mathematics majors, primarily seniors, as well as a few juniors. Analysis students have learned proof techniques in a required freshman-level course, Transition to Advanced Mathematics. They have taken multivariable calculus and linear algebra as prerequisites and have had periodic exposure to proofs throughout their careers. However, at the beginning of the course, many of my students in real analysis reported that they did not feel confident in writing proofs.

In recent semesters, BSU has offered a few sections of Calculus II, generally capped at 25 students each. The two calculus classes in which I required student presentations had about 20 students each and consisted of approximately half mathematics majors and half science majors. Of the mathematics majors, most had taken Transition to Advanced Mathematics or were taking it concurrently with Calculus II. About half of the Calculus II students were sophomores, with the remaining students primarily first-year students for one section and primarily juniors for the other section.

All classes doing presentations had mathematics-education students well represented. About half of the students presenting proofs had double majors in elementary education or minors in secondary education.

Planning my course in real analysis, I believed that student presentations would engage my class deeply in learning to write proofs. I learned about implementing student presentations in a Project NExT session “Getting Started in Inquiry-Based Learning” at the Joint Mathematics Meetings in 2012. I based my system on that of Dr. Robert Vallin at Slippery Rock University, who requires student presentations in a course on advanced calculus [3]. Dr. Vallin had reported in correspondence with me that the vast majority of his students in real analysis were grateful for how much their abilities and confidence in writing proofs increased through giving presentations. I found the method so effective that I implemented it in Calculus II the next semester for teaching writing proofs on the convergence of series. I had taught such proofs over the previous ten years using traditional methods.

2 Description and Implementation

2.1 Overview

My presentation method requires devoting approximately 50 minutes per week to student presentations at the whiteboard. The time slot should occur at the same time every week to help the students and instructor get into a rhythm of preparing for class activities. For example, the presentations can take place during a 50-minute class meeting on a particular day of the week. For a class with a 75-minute meeting, the presentations can constitute the first 50 minutes of a class.

The rhythm of regular presentations promotes efficiency in lectures during the remaining class meetings. Following a careful weekly schedule promotes a prompt progression to new material and a comfortable pacing of the same course content that is covered without presentations. (See Section 2.5 on content for details.)

In my method of proof-writing instruction, the students work on homework prior to presentation of their proofs. I use different modes and timelines for initial homework in analysis and calculus. (See Section 2.3 on homework drafts for further details.)

In real analysis I collect a homework draft in the class period prior to presentations. Then I return graded homework drafts at the beginning of class on presentation day. Students receive grades for written homework drafts and presentations.

When students in Calculus II arrive in class on presentation day, they have been working on their online homework, which will come due at the start of the next class. Three tries per problem are available, and the online system provides feedback on whether each answer submitted is correct. So, students already know which of the online homework answers they have submitted so far are right and which are wrong when they present, and they may have additional attempts still available on each problem in their online homework. Not surprisingly, students who work on drafts before presentation day find presentations less stressful, and the vast majority of students do the advance work.

Before class on presentation day, I write approximately six problem numbers from the homework on the whiteboard, one problem per panel of the board, followed by the name(s) of the presenter(s), as well as, for a larger class, the name(s) of student(s) spearheading the class discussion of the problem. Explicitly flagging some students to ask questions promotes more discussion.

At the beginning of class on presentation day, the students look over their work on the homework and the feedback from me (Real Analysis) or online homework system (Calculus II) and compare notes with their partners if applicable. The presenters write their proofs on the board, finding a fair way to share the writing and speaking responsibilities. Sometimes one writes and the other speaks. Other times they split the writing and speaking. A student whose partner unexpectedly misses class must write and present solo. The absent student receives a score of zero. There are no make-up presentations.

After the presentation of a proof, the questioner(s) assigned to the problem must ask the presenter(s) a question or give a summary of the solution. They are graded on their participation. (See Section 2.4.) To promote honest and challenging discussions, student questions and critiques do not have a negative impact on the presenters' grades.

2.2 Partners

For large enough classes, I have students present with partners. The collaboration provides an additional layer of communication about proof-writing, which precedes the presentations and the subsequent discussion period. Working in pairs also helps each student feel less like a single individual put on the spot. Furthermore, teaming up with another student is a valuable networking opportunity, particularly on a campus like BSU, where most students are commuters.

In my experience, pairing responsible students together often results in respectful collaboration and complementary insights. I have seen less responsible students become somewhat more self-aware when working with similarly unreliable partners.

Switching the groupings week to week gives the students the opportunity to interact with a variety of collaborators. With changing partners, students can't rely on a particular skilled classmate every week. In larger classes with designated questioners, one should mix up the questioner and presenter roles from one presentation day to the next.

In my classes, the presentation method allowed for accommodation of and capitalization on a variety of cognitive styles. For example, two of my students who professed and manifested extreme anxiety over presentations took ownership of questioner roles; I never swapped them into the presenter role. They raised some of the best questions and consistently made helpful comments. Their forthright private disclosures of what was essentially a distracting disability led to enriched educational experiences, for the students directly affected and for their classmates.

I tell students that they can sell the skills they have used in class during job interviews. Employers value candidates' experience in preparation, collaboration, presentation, and critique.

2.3 Homework Drafts

Before presentations, students write an individual homework draft. In Real Analysis, it is submitted for a grade. In Calculus II, the students work on their online homework, writing a draft of the work in a homework notebook, which is not submitted for grading. After presentation day, students in both classes should polish their drafts in preparation for taking exams, but they do not submit a final written version of the homework for grading. In Calculus II, given that students can try each online problem three times, they may use any additional attempts on their online homework that are still available between presentation day and the due date.

Initially creating a draft rather than a finished product can set good priorities in the creative process of the writing of proofs: figure out the key ideas first; polish later. Students learn that the planning phase should precede the writing of a line-by-line proof. The time students invest in messy work on scratch paper improves their understanding of the problem, clarifying the task of designing a proof.

A draft in real analysis exposes starkly what the student knows and does not yet know. In calculus, a student can take an initial pass at developing an intuition for whether series converge or diverge before plunging into the proofs. The instructor can require students to write drafts in less time than the full week typically allotted for complete assignments, keeping the course moving along briskly.

2.3.1 Real Analysis

In Real Analysis, I grade a written draft, according to a simple rubric, and return the work in the next class. On a scale of a few points, maximum points correspond to having virtually all of the assignment

completed, with answers clearly written to show excellent understanding. One point indicates a poor start on the homework or poor understanding. I write general comments on the drafts. The more detailed discussion takes place between partners and with the class during student presentations.

While grading drafts, I take notes about which students should present which problems. Sometimes a student with a particularly interesting solution should present it. Other times class presentations should bring to light a common error, which can then be discussed in a supportive environment in which I emphasize that the mistake was widespread and easy to make. Students who should work harder on a final solution than they did on the draft can present the problem and answer follow-up questions. One can assign a harder problem to a student who did a more routine presentation the previous week and vice versa. Students should email the professor if possible if they know they will miss presentation day to help with planning and organization and make the presentation session run smoothly.

In Real Analysis, the homework drafts and presentations each count for 15% of the grade in the course. Three exams count for 45% of the grade and a final exam for 25%. To reduce the pressure of the tight timeline for weekly homework drafts, I drop the lowest grade(s) for each student at the end of the semester. Both homework drafts and presentations are given significant weight to motivate the students to take them seriously and to do good work.

2.3.2 Calculus II

Prior to presenting proofs in Calculus II, students work on preliminary homework assignments online. They write accompanying work in a homework notebook but do not submit it for grading. The online system gives automatic feedback on their determination of convergence or divergence of a series and various aspects of the application of tests for convergence. The online homework comes due at the start of the class following presentation day.

In Calculus II, quizzes can help motivate students to do preliminary work before presentations. I give quizzes during the class prior to presentation day. The quizzes often cover the first couple of problems from the online homework and count for 15% of the course grade. Presentations count for ten percent. The online homework counts five percent. As in Real Analysis, three exams count for 45% of the grade and a final exam for 25%.

2.4 Student Presentations

To launch the presentations, I write approximately six problem numbers on the whiteboards around the room before class. Next to each problem, I write the names of the student(s) assigned to the problem. Depending on the material, one can assign two problems to one pair of students. Each student picks up his or her graded homework draft and discusses the presentation assignment with a partner if applicable, potentially even before class officially starts.

For Real Analysis, the drafts show me which problems are most beneficial to discuss. In Calculus II, students can anonymously submit requests for presentation problems during the preceding class.

Presentations begin when the students have completely written problem solutions on the board, usually ten minutes into the class period, and when everyone sits down. Students who don't quite finish writing sometimes do a bit of writing during their presentations. Presentation and subsequent discussion times vary according to the nature of the problem but can average about five minutes per presentation, with the whole process taking about 50 minutes of class time.

In the initial discussion between partners, students often find that they received a different written comment on the graded draft. Talking about the instructor feedback helps the team correct and improve their drafts, leading to consensus on how to write the proof on the board.

Occasionally students cannot write a proof and leave the board essentially blank. In this case, the problem receives minimal attention during presentations and I make encouraging comments about next time. In

this situation, the professor has the option to provide or require some follow-up materials for distribution in the next class. The “failure” of a presentation can point the instructor to issues that befuddle the class as a whole, not just one pair of students.

Students receive a grade on a scale of five points with five corresponding to a fully correct presentation with good understanding. A student who requires tips or hints to give a correct presentation receives four points. A partially correct presentation earns three points. One showing weak understanding gets two points. Students who decline to present receive one point for attendance.

A similar five-point rubric applies to the work of questioners with five points for a question, comment, or summary that provides the class with additional insight, allows the class to clear up a misunderstanding, or reinforces or illuminates how to structure the proof, how to apply proof techniques, how to do the proof another way, or how to avoid errors. A student who is less able to raise a pertinent question or provide additional insight into the proof receives four points. Three points go to a questioner who is vague or shows weak understanding of the proof, two points for substantially vague input or a summary that shows substantially limited understanding of the proof. Students who decline to ask a question or to summarize receive one point for attendance.

One can post the presentation-day grades on the course-management website before the next class. An accompanying comment can come directly from the rubrics. Alternatively, one can write the presentation score on the next assignment the student submits. During presentations, I aim particular probing questions at each student, and partners receive independent grades, depending on their performance and understanding. I lower the stakes for presentations, putting the students more at ease, with a policy to drop each student’s lowest presentation grade(s) at the end of the semester.

2.5 Course Content

To compensate for time spent doing presentations, I distribute notes on the new material and present them at the document camera rather than writing them out in real time at the board. This strategy emphasizes the content over the act of writing. Examples in lectures are chosen judiciously, as many problems will receive detailed attention on presentation day.

A significant portion of presentation time comes from time that I would otherwise devote to miscellaneous questions during class throughout the week. Students must ask their homework questions during presentations, before class, after class, or in office hours.

One can also eliminate review days. Also, students continue to have homework in progress during exam weeks, but draft submission is optional. Presentations can take place on a volunteer basis or after more extensive discussion in pairs.

Using these strategies and keeping to a strict rhythm makes the course pacing more predictable and also more streamlined. In one semester I actually managed to cover two more sections in Calculus II than I had for all the previous sections I’d taught that didn’t do presentations, despite two days of campus closure due to snow storms. I covered approximately the same material as the other faculty, who taught the course without presentations.

3 Outcomes

3.1 Fewer Errors

This method seems to be effective in virtually eliminating my “favorite” error in proof-writing in each course. I have also seen a reduction in other common errors.

Before I implemented student presentations, my calculus students habitually abused the n th-term test for divergence. I had tried a variety of strategies to mitigate this error, such as having students meet with me

for one-on-one discussions of the theorem, do targeted assignments, write up exam corrections, and recite the theorem like the pledge of allegiance. The sweat, tears, and humor did not work as well as I wanted. Now my students rarely if ever misapply n th-term test for divergence.

One reason is that a collection of my students' presentations on convergence and divergence proofs can serve to introduce and reinforce proof ideas from the BSU course on transition to advanced mathematics. The presentations provide strong vivid repetitive illustrations of how to apply a theorem: show the hypotheses are satisfied, state the conclusion, and cite the theorem by name. Also, the associated class conversations let us state the n th-term test for divergence over and over again in context, assess its applicability, experience its power, and reinforce an intuitive understanding of the theorem on a variety of problems.

When preparing for Real Analysis, I dreaded having to repeatedly correct standard errors of proof. For me a particular point of sensitivity is a student's assumption that p implies q while attempting to prove p implies q . Some professors in my department have had endemic cases of this error in upper-level courses with the same prerequisites. Fear of this error spurred me to try the student-presentation method.

My students in Real Analysis rarely make this error. In discussions during student presentations we explicitly emphasize the use of valid techniques, and we note avoidance of invalid techniques, referring repeatedly to a handout on the "Dos and Don'ts of Proof."

In both Calculus and Real Analysis, students learn from their peers' additional errors during presentations. They subsequently avoid them.

By way of example, in calculus a pair of presenters wrote very few words in applying the limit comparison test. The students had failed to justify several of the steps in their proof. Class discussion included a comparison between the test itself and how the students had applied it. Conveniently, another pair of students on the same day showed a very thorough application of the limit comparison test, and a comparison/contrast of their work with the less satisfactory proof drove home how to supply all the required reasoning when applying the test.

3.2 Greater Clarity

Students refine their proofs in presenting them. I watch this process in action when I see a student in real analysis read my feedback on a homework draft, rethink the problem, discuss the problem with a classmate, and present an improved proof. Students notice their own improvement, too. About a quarter of one class mentioned on exit surveys that explaining their work clarified their own understanding when asked "What did you like about the course?"

In calculus, preliminary work on online homework can give students the wrong idea that determining convergence or divergence is the extent of the task. My quizzes before presentation day often reveal inadequate proofs. The next day, I hold presenters to a high standard in the writing of a proof to support the convergence or divergence "answer," and students rise to the occasion, finding and appropriately using the tools that justify their conclusions. They improve upon their quiz proofs.

My students often appear to obtain a deeper understanding from their fellow students' correct answers to their questions than from my answers. They learn effectively from peers, incorporating the acquired knowledge into their proof-writing.

3.3 More Diverse Problem-Solving Strategies

Often for a problem, different students write different proofs and use different organizational strategies. Classmates gain a broad exposure to problem-solving strategies through a student presentation and associated discussion, which brings in ideas and comments on varied approaches to proof.

Presentations on calculus proofs lead to rich discussions of which other convergence tests are and are not applicable to a particular series. Students also discuss how they got an intuition for the behavior of the series as a springboard to selecting an applicable convergence theorem and constructing a proof.

Students in real analysis similarly engage with the question of how to design a bridge from hypotheses to conclusion. For example, one student started her presentation by saying, “At first I thought my partner’s answer was wrong and mine was right because I got full credit on my draft. But she thought I was wrong because she got full credit. Then we realized we were both right.” Then the students presented the two ways and explained their compatibility.

When classmates write proofs after such presentation sessions, they bring more experience, tools, and perspectives to bear. Sometimes they discuss and compare them with me in office hours. Then they write a strong proof. In addition, discussion of a variety of student approaches lets students overcome a fear of not knowing “the” right way. Such anxiety can be a barrier to seeking an effective proof strategy.

3.4 Improved Attitudes

As it happens, consolidating homework discussion to presentation day serves my students well. Most arrive in class ready to meaningfully discuss the problems. Prior to my implementation of the method, some students noted on course evaluations that they found in-class Q&A too individually targeted and too repetitive. At the time, most students had not benefited from those teachable moments on proof-writing because they had not worked on the problem and did not yet feel invested in it.

As an added benefit, the presentation method appears to increase my students’ motivation. For example, unlike students in the other sections of real analysis, almost all my students submit their drafts religiously even though they know I will drop the lowest three grades at the end of the semester.

In my classes, the presentation method has dispelled the culture of “I can’t do proofs” or “I don’t get series.” The class environment makes these assertions socially unacceptable. Doing proofs becomes part of the expected routine. Students must and do spend significant time on the task. Ultimately, the presentation/discussion period provides an outlet for the enthusiasm of the instructor and the students. In both real analysis and calculus, the presentations produce an enjoyable atmosphere in the classroom.

4 Extending the Method

The description of the method in Section 2 indicates that the number of presenters on each of about six problems depends on the size of the class. In my calculus classes of about 20, I assign two presenters to each problem and also designate a student to lead class discussion of the problem with a question or a summary. In larger classes, one can assign additional students to ask questions about every problem presented.

This method works best with at least six panels of board space available for putting up a problem on each panel. A room with only four panels of board space can also accommodate the technique if a couple of presentations can be done via a document camera. Alternatively, the number of proofs presented can shrink from about six to four if the room has four panels and no document camera.

If the size of the class permits, all students should have roles presenting or questioning on each presentation day. Otherwise, larger classes can have each student participating in presentations every other week instead of every week. (Editors’ note: For another approach to student presentations, see Mathews’s paper [1] in this volume.)

Also, the presentations can be embedded in an inquiry-based course. For example, Dr. Vallin teaches *Advanced Calculus With Generalizations* in this form, covering similar topics in the first semester to the course *Introduction to Real Analysis I*, addressed here. He has published a full set of notes [3].

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Individual Presentations of Group-Written Proofs

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Abstract

This method devotes one class hour each week to student presentations of proofs. For each section of homework problems, a designated group works together outside of class to discover and typeset proofs. For each problem, I choose one student from the group to project her typeset proof in front of the class. She explains the structure and key ideas of the proof, drawing any diagrams on the board that help to illustrate it. After a class discussion, I highlight important lessons to be learned. Students report that the presentation days help them learn to write clearer proofs and to understand proofs at a deeper level.

Difficulty Level: Medium; Course Level: Advanced

1 Background and Context

I teach at Azusa Pacific University, a comprehensive university near Los Angeles with over 5,500 full-time undergraduates. The Department of Mathematics and Physics is located within the College of Liberal Arts and Sciences and serves about 70 math majors. Our class sections usually contain between 15 and 30 students. About half of our majors go on to pursue credentials in secondary education. The others enter a variety of graduate programs and careers in quantitative fields.

As our mathematics faculty has grown, we have worked hard to increase the rigor of our curriculum. Students now take our Discrete Mathematics and Proof bridge course, based on Daniel Velleman's *How to Prove It: A Structured Approach* [10], during their second or third semester. After this intensive introduction to the logic, language, and technique of proof-writing, students advance through proof-oriented semester courses in linear algebra, number theory, abstract algebra, real analysis, and topology. Our goal is for students to master the nuts and bolts of proof-writing early on, so that proof can become a familiar vehicle for understanding and communicating new concepts and arguments in the later courses. As students gain fluency in proof-writing, their primary focus should shift from the grammar and technical details of proofs to the stories that the proofs tell. We want them to see that proofs don't just bestow certainty: they lay bare the hidden connections between mathematical ideas.

In particular, we want students in our real analysis course to encounter proof as something more than a means of verifying calculus theorems that they already believe to be true. Following Stephen Abbott's *Understanding Real Analysis* [1], we want our students to wrestle with the fascinating questions and paradoxes that led to the development of real analysis in the first place. Our course should help them to refine their intuition regarding limiting processes and continuity and the infinite. It should enable them to solve problems in new contexts, to illustrate their ideas using diagrams, and to communicate their insights to others. Our students do need to be able to formalize their arguments in precisely written proofs, full of quantifiers, inequalities, and indices. But then we want to make sure that they can clearly explain and evaluate the strategy, structure, and significance of what they have written.

How can we best help our students to achieve this deeper understanding of the proofs they write? Explanation and demonstration are not enough. Ambrose, et al., summarize a sizable body of literature with the principle that "*Goal-directed* practice coupled with *targeted* feedback are critical to learning" [2, p. 125]. We need to design practice opportunities that push our students to strive for deep understanding of

proofs, and then to help our students to see whether they have attained that level of understanding. I implement this practice-feedback cycle in our real analysis course via carefully structured student presentations of group-written proofs, followed by peer discussion and instructor comment.

2 Description and Implementation

Our real analysis course meets for three hours each week, two of which I use for interactive lecture. My aim during lecture is to frame the questions that will drive our investigation, explain key results, present visualizations, and model the processes of proof discovery, writing, and explanation. Each lecture day covers one section from the text and prepares students to work on four or five (often multi-part) homework problems that are assigned from that section.

For the third hour each week, the students are in the driver's seat for presentation day. To organize them, I assign a group of three or four students to each section of assigned homework problems. This group of students works together to typeset collective solutions (in \LaTeX or Word) to the assigned problems for that section. I provide a \LaTeX "cheat sheet" [9] and a list of keyboard shortcuts for Microsoft Word Equation Tools [8] to remind students how to use these typesetting programs. In order for the groups to have sufficient time to meet and to seek help if needed, I schedule at least a full week between the lecture on a section and the corresponding student presentations. The students not in the presentation group for a section submit their personal solutions (which may be handwritten) to the problems for that section at the beginning of class on the day they are to be presented. Each student submits eight to ten problems each week.

Each presentation day covers problems from two sections of the textbook, with two groups presenting solutions to the class, so that there is time for each presenter to speak for approximately seven to nine minutes. For each problem, I choose a student presenter at random from the assigned group. This means that each member of the group must be prepared to present the group's solutions. If there is not enough time for all problems to be presented, I select those that I think will lead to the best discussion. After a student presents a proof, I invite the class to offer comments, questions, or alternative strategies. In the end, I seek to clear up any persistent misconceptions and highlight important lessons to be learned. My remarks address the logic and intuition behind the proof and also the notation and style of the writing.

When student presenters project their completed proofs in front of the class, their tendency is to simply read aloud what they have written. I challenge them to read word-for-word only when necessary and to focus on communicating the intuition and strategy behind their proofs. They should work to become experts on their proofs so that they can share their deep understanding of them with their classmates. To encourage students in this direction, I give detailed instructions for their presentations and assess their work using a carefully designed rubric (see the Appendix). The instructions, which mirror the process that I myself follow while lecturing, are

1. Clearly summarize the **statement** you are proving.
2. Introduce the overarching **strategy** and/or **structure** employed by your proof and explain why you have chosen it.
3. If you think it will aid your audience's understanding, sketch a **diagram** on the board illustrating the statement and/or your proof.
4. Walk your audience through the **main steps** of the proof, summarizing when possible and providing careful **justification**.
5. Close your presentation by identifying what you consider to be the **key insight** behind your proof.

The rubric assigns a score for each of the five steps in the instructions, and scores for proper notation and writing style. The last two scores are averaged among group members, since the written proofs are their collective work. I mark the first five scores during the presentation and the final two later, occasionally making adjustments to the scores of students who have been chosen to present particularly difficult proofs. When appropriate, I use half-point scores. The style rules referenced in the last row of the rubric are adapted from Anders Hendrickson's more extensive list, "*Elements of Style for Proofs*" [5], included in this volume, and are explained to the students in a handout at the beginning of the term. The rules are

1. Proceed from hypothesis to conclusion.
2. Use paragraphs and signpost sentences to clarify the structure of your proof.
3. Use logical transition words to indicate relationships between sentences.
4. Write in complete sentences, with proper grammar and punctuation.
5. Type all mathematical notation in math mode.
6. Use displayed equations to help the page breathe.
7. Be concise.
8. Introduce every symbol you use.
9. Avoid symbols when words would be easier to read.
10. Avoid weasel words ("obviously," "clearly," etc.).

With a class of about 20 students, there is space on the calendar for each student to present four times. A student's presentation scores (from the rubric) account for 15 percent of her course grade. A group's typeset proofs are also graded for correctness along with those (usually handwritten) of the rest of the class, with all group members receiving identical homework scores for their problems. A student's written homework scores account for 45 percent of her course grade.

This method is simple to manage and requires relatively little time from the instructor outside of class beyond the time that is normally devoted to grading students' written work. Presentation groups must be assigned to sections of the text, and sufficient assistance must be given during office hours when a group cannot solve all of its assigned problems. Most importantly, the instructor must fully digest the homework problems before each presentation day, to be prepared to effectively moderate the discussion.

Some extra investment is required by students during their presentation weeks as they meet together to craft and typeset their proofs, and as they work to understand their proofs well enough to explain them with some fluency. The additional work is balanced by the slower pace of the course that results from covering two sections each week instead of three.

The greatest cost incurred by this method is reduced coverage. If one's primary goal is to maximize the number of topics covered, this method is likely not a good fit. In my one-semester real analysis course, I do not make it through Chapter 7 on integration (and we do not offer a second real analysis course at this time). I have judged it worthwhile to sacrifice some coverage in order to boost the proof comprehension and content mastery of my particular student population. I trust that those of my students who go on to study integration in more detail will be able to learn that material more deeply as a result of what they have experienced in my real analysis course.

I have been able to recover some time by testing less often than I otherwise might. I find regular group work and presentation responsibilities to be significant motivators for students to keep up with their course work, making frequent written testing less of a necessity.

3 Outcomes

I have taught real analysis two times using this method. Student feedback has been very encouraging. The first time, using the IDEA Student Ratings of Instruction instrument [6], students gave the course a perfect 5.0 raw score on “Progress on Relevant Objectives.” Many students personally encouraged me to continue employing structured student presentations in the future. The following comment, written on a course evaluation form, is representative:

I really appreciate how this course was laid out this semester. It seemed daunting at the beginning, but I feel like I learned a LOT. The presentations helped me learn more and understand more than a test would have, which is so important in this challenging of a course. . . . So the presentations were great for me. Also, we learned how to type math and be clear on what we say in our proofs.

The second time I taught real analysis with this method, I was not authorized to use the IDEA instrument (due to the cost), but I did collect anonymous feedback from students at the end of the semester, including

The requirement to present was very valuable. As we presented, I made a lot of connections that I hadn’t made previously even working through the homework.

I think presenting the homework each week was the most valuable for my learning in this course. I had a much better understanding of the homework when I met with my group, wrote the proofs together and discussed them in front of class. Listening to other groups present was equally as valuable because I was able to get a more thorough understanding of my homework by having it discussed aloud as a class.

As these comments demonstrate, the greatest strength of this method is that it sets high expectations for students’ level of comprehension. In order to follow the presentation instructions, students must be able to explain the big picture of a proof, rather than just the individual logical connections. Over time, they come to see the wisdom of William Basener’s advice in *Topology and Its Applications* that “A proof should be read not only step by step to see its logical progression, but as a whole. It is often helpful to try to summarize the proof in a single sentence” [3, p. xxvii]. Structured presentations lead presenters to identify the higher-level conceptual links that a proof embodies and brings to light. Students often comment early in the course that they thought they had fully grasped a proof before presenting it, only to discover during their presentation that their understanding remained incomplete. It is gratifying to see them aim for, and often display, a deeper level of comprehension on subsequent attempts.

As much as this method draws students’ attention to the stories proofs tell, it leaves plenty of room for continued instruction in the grammar of proofs. The class discussion and instructor comments can address any aspect of a presented proof, including its key insights, structure, notation, typesetting, and style. Without intervention, students tend to repeat the same mistakes, and so it is valuable to be able to point to, say, a misplaced quantifier, and to explain to the entire class at once why it results in a compromised proof. After three or four presentation days, many of the common errors occur much less frequently, and the proofs begin to appear more polished.

A side benefit of this method is that it causes students to repeatedly experience proof as an attempt at communication (and to organically discover ways in which it can succeed or fail). Students know, as they carefully craft their proofs, that their work is preparing them for a conversation, rather than simply being destined for a pile on their professor’s desk. This can be powerfully motivating for some students, and potentially unsettling for others, who may have grown accustomed to working on math problems in isolation. The latter students may especially benefit from this sort of collaborative work which, in the words of a recent Association of American Colleges and Universities document on high-impact educational practices, can help

students to make progress on two important goals: “learning to work and solve problems in the company of others, and sharpening one’s own understanding by listening seriously to the insights of others” [7, p. 20].

In summary, this method guides students as they:

- enter into the story of real analysis and the process of proof-writing through interactive lectures
- synthesize their ideas in a small-group setting
- typeset precise proofs, following the language and style conventions of the discipline
- communicate their logic and intuition to a classroom of their peers
- receive clear feedback on their own work and that of others.

Students learn to aim for a deeper comprehension of the proofs they write, to avoid common mistakes in logic, notation, and style, and to explain the ideas in their proofs with greater fluency.

One potential drawback I have considered is the high level of stress that some students experience when asked to speak about mathematics in front of their peers. While most students have expressed their enjoyment of this class structure, a small number of students have given evidence of being somewhat overwhelmed by it. For them, I suspect that the combination of public speaking with challenging mathematics may have taken away from their enjoyment of the class, rather than having added to it. I find that their anxieties can be partially allayed if I acknowledge the difficult nature of the task I am setting before them and encourage them to do the best they can to grow through the experience.

4 Extending the Method

With a smaller class, it may be preferable to have just a subset of the homework problems presented, so that students are not required to present every week. Alternatively, the group size could be reduced, to make it easier for students to arrange meeting times and to reduce the number of problems for which they are held responsible to a reasonable level. (Editors’ note: For another model of student presentations of proofs, see Gross’s paper [4] in this volume.)

A larger class could be split into smaller sections for the presentation days. Alternatively, after working with their groups, students could record their proof explanations in screencasts, made available to the class for viewing, and comment using technology available online. It would be particularly interesting for students to be able to watch multiple screencast proofs of a single statement in order to see the variations in their classmates’ approaches. Some class time could be used to comment on interesting lessons that have arisen via the screencasts and comments.

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Appendix

A Rubric for Presentations

	0	1	2
Problem Statement	unclear or incomplete statement	clear and complete statement	
Strategy/Structure	no mention of overall strategy or structure	unclear introduction of strategy/structure and/or reasons for choice	clear introduction of strategy/structure and/or reasons for choice
Diagram (if applicable)	does not illustrate statement or proof with a helpful diagram	illustrates statement or proof with a helpful diagram	
Main Steps	inadequate explanation of main logical steps and their justification	adequate explanation of main logical steps and their justification	excellent explanation of main logical steps and their justification
Key Insight	does not identify a key insight	properly identifies a key insight	
Notation	significant notational mistakes	minimal or no notational mistakes	
Writing Style	two or more style rules violated	one style rule violated	no style rules violated

Bryant G. Mathews: Azusa Pacific University, Azusa, CA

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3

Whole Class Activities

Use of Video Case-Studies in Learning to Prove

Connie Campbell, James Sandefur, Kay Somers

Abstract

The authors have developed an online library of over 40 videos of students attempting to prove statements that are typical for an introduction-to-proof course. This article describes two of the videos and how they might be used to promote in-class discussion of the proof process, supporting students' acquisition of the skills necessary to successfully learn to construct well-written proofs.

Difficulty Level: Medium; Course Level: Transitional
Technology Based

1 Background and Context

We have been teaching various versions of an introduction-to-proof course at our respective institutions for many years. Millsaps College is a small liberal arts college with fewer than 1000 undergraduate students; Moravian College is a liberal arts college with approximately 1500 students; Georgetown University is a Catholic research university with about 7000 undergraduate students. Each school offers an introduction-to-proof course that serves as the gateway to the mathematics major and is normally taken in the sophomore year. The course at Millsaps has one semester of calculus as a prerequisite, while the courses at Moravian and Georgetown require two semesters of calculus. The course sizes range from 10–12 students per class at Millsaps, 12–16 at Moravian, and up to 25–30 at Georgetown. The courses introduce basic approaches to writing proofs, with the course at Moravian focusing on discrete mathematics and also serving as a writing-intensive course for the major.

From our years of experience teaching introduction-to-proof courses, we have observed that students seem to lack strategies to help them complete a proof when their first idea does not work; that is, they do not have the cognitive maturity to identify what is causing them to be stuck, they have difficulty bringing to the task other methods or approaches, and they lack the metacognitive skills necessary to reflect on their own thinking processes.

In an effort to understand how to effectively help students develop their proof-writing skills, we, together with Manya Raman (University of Umea, Sweden), began making videos of our students as they worked, typically in pairs, to develop proofs. In watching and discussing them among ourselves, we noticed that many times students were stuck at the first or second step, often having only a vague idea why the statement was true and no mathematical techniques for approaching the problem. Based on our discussions and ideas originally developed by Professor Raman [1], we identified three steps to proof-writing:

1. Gaining an insight into why the statement is true, which we call the **key idea**.
2. Finding a mathematical tool or technique that allows one to translate this idea into a mathematically sound argument, called a **technical handle**.
3. Converting the key idea and/or technical handle into a well-written proof.

To illustrate, consider the problem of showing that if n is odd and not a multiple of three, then $n^2 - 1$ is divisible by 24. An example of a key idea for this problem would be noting that $n^2 - 1 = (n - 1)(n + 1)$ and that one of the factors is divisible by 3, one is divisible by 2 and one is divisible by 4, giving a product of 24. This idea offers clear insight as to why the statement is true, but does not provide an obvious way to write a proof. A technical handle for this problem would be to use cases, noting that there exists k such that $n = 6k + 1$ or $n = 6k - 1$. The technical handle gives the student a mechanism for constructing a systematic proof, but does not necessarily offer insight into why the statement is true.

Students have surprising difficulty with the third step, even when they have the key idea and a technical handle. In the discussion below, we describe how our videos can be used to help students focus not only on the first two steps, but also on this third step of the process.

An interesting thing happened when we showed some of our videos to our students: they would often ask why the students in the video did not try a different approach. Additionally, they seemed to be able to catch, and then correct, errors in the proofs presented even as they were not recognizing similar mistakes in their own work. What we realized is that our students could reflect on the thinking of others better than they could on their own thinking [2].

As a result of this early observation, and with funding provided by the National Science Foundation (DUE 1020161), we began fully developing an online library of over 40 video case-studies of students working on proofs of theorems that were new to them.¹ While it generally took the students in the videos, most of whom had recently completed an introduction-to-proof course, 20-40 minutes to work the problem, our videos have been edited down to between 3 and 10 minutes in length, eliminating periods of time when little progress was made. To let students who watch the videos know that this was done, we have added comments like “after 30 minutes of no progress” to our videos. Our students can then see that it is natural to struggle, and that eventually the students in the video can get back on track and make progress towards the solution.

When used with a directed discussion, the videos help students develop their critical-thinking skills related to proof construction. Recently, working with mathematics faculty at a variety of institutions (University of Texas at Arlington, Arizona State University, Athens State University, and Northwestern State University of Louisiana), we have found that use of the videos can be incorporated to a greater or lesser extent into many types of courses. The frequency of use ranges from three to eight videos per semester, depending on the content of the course, the instructor’s approach to the class, and the level of the students.

In the next section, we describe how we have incorporated the videos into our classrooms, using a brief description of two of them. Each description offers a classroom use of the video.

2 Description and Implementation

In the first video (under three minutes long), which we show early in the course, we focus on students’ struggles with understanding how to interpret and prove statements that involve multiple quantifiers. The students in the video are asked to prove or disprove that:

There exists a real number x , such that for every real number y , xy is an integer.

We typically give this statement to our students and allow them to think about it for a minute or two; then we start the video. Within one minute, the students in the video engage in a discussion as to whether they need to define y in terms of x or x in terms of y . Pausing the video just before the students realize they actually need to find one x that works for all y provides an excellent opportunity for a class discussion on

¹We obtained IRB approval for the development of the videos. Our students were surprisingly willing to be videotaped, knowing that the videos would be posted on a password-protected website for use by other instructors. Students were generally pleased to be invited to participate and felt they were contributing to the learning of future students.

the order of quantifiers. We find it amazing how quickly students identify errors made by the students in the videos, even though we know many of our students would likely make these same errors themselves if they were working on the problem. We conjecture this is because the students viewing the video are freed from the pressure of solving the problem themselves, but are instead focusing on the thinking of the students in the video.

In guiding the discussion, we usually allow our students to talk about the video in small groups before having a full-class discussion. This results in deeper thinking and more input from all students. They engage in reflective thinking if we ask open-ended questions such as:

- What are the students thinking?
- Why are they confused about which variable to write in terms of the other?
- What advice would you give them?

While some students may realize they just need to find one x , the emphasis of the class conversation should be on what created the confusion, thereby giving them the tools to process such statements on their own.

Our second scenario is a six-minute video that helps students learn how to process and use a definition, think about how examples can be used to gain insights, and gain experience in how to transition from a key idea to a well-written proof. The students in the video are given the following definition, asked to describe what it means to them, and asked to sketch some examples:

A function f is *bounded above* if and only if there exists an $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in D$, the domain of f . In this case, M is an upper bound for the function.

Before we view this video with our classes, we give our students the definition and ask them “What does this definition mean to you? Find some examples that satisfy the definition and some that do not.” We allow our students to work in small groups until most groups have processed the definition, which usually takes about five minutes. When we discuss their examples as a class, we generally find that some groups have created graphical examples while others come up with algebraic examples. In addition, we may ask students to try to write a formal definition of a function f that is not bounded above and discuss their attempts.

We then watch the students in the video as they process the definition by sketching a graph of a function, marking M on the vertical axis, and writing an algebraic representation of their function as $-x^2 + 2$. They observe that M can equal 2. One of the students says “ M can equal 2 or” (voice fades). He writes something and crosses it out before we can read it, but it clearly begins “ $M \dots$ ” It appears to us that the student in the video is confused between upper bound and least upper bound in that he was not sure if he could write $M > 2$ or not. This is a point (about one minute into the video) where we can pause and ask “What is this student thinking?” or “Why does he seem confused?” If this goes nowhere, we can ask “Is M an upper bound or the upper bound?” We would also discuss any ideas that may have been raised in the video and not addressed in the examples generated by students in the class.

This pause is a good time to give students the statement of the theorem addressed in the rest of the video:

Let f and g be real-valued functions with domain $D \subseteq \mathbb{R}$. Show that if f and g are bounded above then $h = f + g$ is also bounded above.

We then give them a few minutes to think about the theorem, either individually or in small groups. We like to allow students enough time to understand the statement of the theorem, but not so much time that they will complete a proof.

At this point, we watch the students in the video begin to consider the theorem, and we hear one of them say “This is going to be fun!” which usually makes our students smile. The students in the video write their assumptions,

$$f(x) \leq M, g(x) \leq N, \text{ for } M, N \in \mathbb{R}$$

as they begin to focus on the statement. They draw rough sketches of two functions f and g , one increasing to a horizontal asymptote and the other resembling a concave down parabola. It is interesting that one of their examples does not attain its least upper bound and the other does. One student laughs and says “In general, we should have functions where we can picture an $h(= f + g)$. I can’t picture an h there.” This is a good place to pause (at approximately three minutes into the video) and help our students focus on the thinking of the students in the video. Questions might be “How would you add these two functions (from the graphs)? What advice would you give to these students?” We usually give our students time to attempt to graphically construct h themselves from the two graphs on the monitor. An interesting question for them is to determine if the peak of h is at the same x -value as the peak of g , is to the right of it, or if there is insufficient information to determine this.

As we return to the students in the video, we see them abandon trying to add their graphical functions and turn to algebraic examples; they define two functions, $f(x) = -x^2 + 3$ and $g(x) = -x^2 + 5$ with $h(x) = -2x^2 + 8$. They observe that M is three or greater and N is five or greater. This gives them insight into an upper bound for $h = f + g$ of eight or greater. At approximately four and a half minutes into the video, one of the students says “I don’t know if you can just add them, you know, $M + N$. I mean, that makes sense but I don’t know if it’s mathematically rigorous enough.” This is a good point at which to pause and ask the class some general questions, such as, “Are the students on the right track? Will this key idea work for other examples? Will adding $M + N$ always work or does it only work because the ‘peaks’ in f and g occur at the same x -value? Would it work for f and g in their first example?” The students in the video finish by writing

$f(x) < M, g(x) < N$. Then $h = f + g < M + N = P$. $h \leq P \forall x \in D \subseteq \mathbb{R}$ then h is bounded above by P ,

and one of the students says “It works for me.” At this point we might ask our students to think about the difference between the notation f and the notation $f(x)$. We would also ask our students to give us a well-written proof, produced, if time allows, in their small groups during class or individually for the next meeting.

As indicated in the previous example, after class discussion on the ideas for a proof developed in a video, we have had success with either writing a collective class proof, asking students to produce a well-written proof with their groups in or out of class, or having students individually write a final proof of the statement for homework. Producing a well-written proof is multi-faceted, and students often have a hard time decoupling *what to do* and *how to write it*. By having students write a proof after the ideas have been developed and discussed together, the instructor can separate out the *how to write it* component and give the students feedback on this aspect.

In a companion video to our second example video, students are asked to:

Prove or disprove: if f and g are bounded above functions with domain D , then $f - g$ is bounded above.

Because the statement is not true, the video provides an opportunity for our students to focus on the thinking of the students in the video as they decide whether the theorem is true or not and attempt to disprove it.

As described in the examples, the videos can be shown in class with pauses for small-group discussion, followed by full-class discussion. Using a video in this way will typically take 20 to 30 minutes of class time, depending on the direction of the discussion. Showing from three to eight videos during a semester brings variety to the classroom without taking too much time from other class activities.

3 Outcomes

One key benefit of using the videos we have created is that, when used in the classroom together with a well-guided discussion, they challenge students to articulate why a particular approach may or may not be

working. In one student's words "It helps with your thinking, and identifying which proof to use and why they are thinking that is the right kind of proof to use." We have observed that the classroom discussions are especially helpful for the majority of our students who are mid-level performers. Our top-performing students can typically develop the ideas for a proof independently; students who are really struggling to understand the proof process sometimes have difficulty following the thinking of the students in the videos.

Another benefit is that students viewing the videos are able to critique the work of peers, without feeling they are criticizing their classmates. One student noted "It's not fair to pick a kid out of class and like, 'Do this problem on the board right now, let's see you go for it.' But to watch a video of someone doing it, it's nice. It brings something to the class that you can't get without embarrassing someone."

The language in the videos is the language of their peers, so students may find the videos more understandable than other approaches. A student responded "I've always found textbooks are really difficult... they take a long time, for me, to parse through the language, especially. The videos are nice because it's students. And they always speak in the same language. And that makes it really helpful."

Dr. Gulden Karakok, at the University of Northern Colorado, conducted an independent assessment of our project. She interviewed students and gave pre-and post-class surveys to students enrolled in introduction-to-proof classes that used our videos and to a limited number of control students. Quantitatively, over two academic years, 86% of the students in the video case-studies sections found the videos to be somewhat valuable to very valuable. Professor Karakok reported to us some general qualitative observations about students who have participated in classes where these videos were incorporated:

- The videos have helped students understand "it is OK to make mistakes and others struggle too" with proof-writing.
- The experience of watching the videos has helped them learn to "take a step back and check your work" before proceeding or when you become stuck in the proof-writing process.
- About 30% of students mentioned that observing the students in the video helped them learn to break the statement to be proven into smaller parts and to explore the meaning of each part.

In addition, she noted that while students in both the video case-studies and control group sections learned that using examples is important in proof construction, some students mentioned that they observed the use of examples in many videos and this stuck with them as a helpful tool. Students also appreciated that using videos in the classroom added pedagogical variety to their course.

These findings have been commensurate with our observations. It has been our general sense that students who engage in classroom discussions with the videos develop a greater sense of confidence in their own abilities, learn strategies to interpret statements, and try multiple approaches to a proof. In addition, students are better able to understand the importance of, as well as the level of, details that should be included in a proof.

Even if not used in the classroom, our videos are helpful for faculty teaching introduction-to-proof courses. In our work with colleagues in our project and with giving workshops and presentations, faculty who have watched the videos have reported gaining greater insight into their students' thinking processes and greater understanding of where their difficulties lie.

Our complete library consists of over 40 videos. In this paper we have detailed the contents of two of them and how one might use them in the classroom. The remaining videos contain a variety of statements involving functions, sets, multiple quantifiers, inequalities, and divisibility. The statements vary widely in level of difficulty, which allows instructors to use them in a variety of classes and at different points in their courses. In general, we tried to utilize the types of statements with which our own students often struggle.

The videos are currently available for online streaming at

blogs.commonsgeorgetown.edu/proofs

As they include student work, IRB requires they only be used for educational purposes. Consequently, this is a password-protected site and we ask that people share access information judiciously. Interested users can access the site with the guest ID “proofguest” and password “proofsCm5e”. Faculty are free to use the guest ID to review the videos and teaching materials, but we encourage those who plan to use them on an ongoing basis to register with us and obtain their own ID and password by contacting James Sandefur at

sandefur@georgetown.edu

4 Extending the Method

In addition to the in-class approach we described, we have also assigned videos for out-of-class group or individual viewing, with reflection questions to which students respond in writing. We might ask our students to use the work done by the students in the video to answer the questions:

- What was the key idea of this problem and what key insights were necessary to develop the proof?
- Have students developed all the ideas of the proof?
- What insights can you take away from the work of the students on this theorem?

We can also ask our students to submit a well-written proof of the theorem, using either the ideas discussed in the video or their own method.

A third way to use a video is to view it in class after students have handed in their own proofs. This approach allows students to compare their thinking with that of students in the video and to potentially see another way to complete a proof.

In summary, our videos can be used in or out of class with classes of just about any size. They provide a unique opportunity to engage students in metacognitive thinking about the proof process.

References

- [1] Raman, M., “Key ideas: What are they and how can they help us understand how people view proof?” in *Educational Studies in Mathematics*, 52, (2003) 319–325.
- [2] M. Raman, J. Sandefur, G. Birky, C. Campbell, and K. Somers, “Is that a proof? Using video to teach and learn how to prove at the university level,” in *Proof and Proving in Mathematics Education: ICMI Study 19 Conference Proceedings*, 2 (2009) 154–159.

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Pass-the-Proof

Tom Sibley

Abstract

Pass-the-proof is an effective way to engage students in developing a proof and discussing proof-writing as a class. In brief, different students build an in-class proof by contributing steps or ideas. After students give their contributions (or correct an earlier step), they “pass the proof” to a student of their choice.

Difficulty Level: Low; Course Level: Transitional, Advanced

1 Background and Context

Teaching proof-writing can require a range of activities, from those at the level of the individual student to those involving the whole class. While I present some proofs to the entire class, I find students in general learn better when they are actively engaged. This article focuses on a method, called “pass-the-proof”, that seeks to engage the entire class in developing a proof. I have successfully used this technique in all courses where proofs are a major component, especially our introduction-to-proofs course.

I teach at St. John’s University, a liberal arts college of about 3800 students, where most proof-oriented mathematics classes have from 15 to 20 students, although they have had as few as six and as many as 25. The plurality of our 20 to 25 senior majors per year find jobs in business and industry after they graduate. Another big segment become secondary school mathematics teachers. Others volunteer, go on to graduate school in mathematics or statistics, or other pursuits.

2 Description and Implementation

Previously, to engage students in class proofs, I had relied on asking for volunteers or calling on particular students. However, too few students volunteered, and when called on, too many students seemed intimidated. A student’s choosing the next student to contribute has seemed to alleviate both problems.

I plan ahead of time what statement(s) I want the class as a whole to prove, choosing the statement by the proof technique and the appropriate level of difficulty. (On occasion I will use pass-the-proof in response to a student’s question.) Students don’t know what the statement will be beforehand, but the proof will build on their previous homework and general preparation. Developing a proof is hard, so it is important for students to experience the process in class, not just preparing it beforehand (or looking it up online). Since all of them are potentially “on call” for any step, they have an incentive to think about how to prove the statement, not just copy down the steps.

Once I write the statement on the board, I call on a student to offer any part of the proof. They quickly learn that this step is the easiest one: the first student can just state the hypotheses. (They also realize the last sentence, the conclusion, is just as easy and so is usually offered next.) Then that student chooses the next student, “passing the proof.” Subsequent students offer another step in the proof, or comment on, reorder or correct previous step(s), and then pass the proof to a student of their choice. A student may, without penalty, pass the proof on without contributing. In my experience unspoken peer pressure minimizes such passes, although I will speak to individuals privately as warranted. At or after his or her turn, if a student thinks

the proof is finished, I generally ask the class whether they agree. I don't write the end-of-a-proof symbol until there is consensus, although with a beginning class I am willing to give the verdict. And sometimes with a more interesting proof, I'm excited enough that they already know they have succeeded before I can compliment them.

If at some point more than one student passes without contributing, I ask whether they need a more general discussion, a volunteer or a hint. This situation often alerts me to a point on which the class needs extra clarification. In that case, I can give a short explanation right where it is needed.

Without my mentioning it, students tend to pick those who haven't yet offered a step. Ensuring everyone is able to contribute is, indeed, one of my pedagogical goals, although I believe they consider it more a matter of fairness. With a large class not every student will have an opportunity to contribute on any given day, but that would happen with other approaches. I have noticed that some students will remember from one class to the next who hadn't contributed and so are more likely to pass the proof to someone new.

During pass-the-proof I act as scribe, writing students' suggestions on the board. My role keeps the process moving along and ensures correct grammar, spelling, and notation. While there is pedagogical value in making students aware of low-level errors they often make, I think that class time is better used focusing on the higher-order thinking skills needed to generate proofs. They include proof formats, definitions and the idea guiding a proof. Errors in these areas need explicit correction fairly quickly. When one occurs, I hope that the next student points it out or looks puzzled by it so that I can ask a question. However, I will intervene more directly if a student doesn't.

For students with very little experience, I write their contributions on the board roughly where I think they will go in the final proof. The placement can give a nonverbal cue about a gap they should consider. With more experienced students, I ask them where their steps belong. I also adjust the amount of my comments during the proof to the level of experience the class has had with proofs. For example, early in a sophomore-level class, I will ask leading questions to solicit an appropriate definition or to refocus attention on the technique or the conclusion. As students develop experience in proving, I encourage them to take over that role. Incorrect approaches or steps provide potential learning opportunities. Of course, they require some judgment in deciding how long it is pedagogically beneficial to allow students to pursue what appears to be a dead end to the instructor.

For example, suppose that students have recently learned proof formats including direct, contrapositive, and contradiction in the introduction-to-proof course. I might ask them to prove "For all integers n , n is odd if and only if n^2 is odd." Students readily find a direct proof for the forward direction. Generally the student starting the other direction will try a direct proof as well by supposing n^2 is odd. The next student will (I hope) add "So there is some k so that $n^2 = 2k + 1$." After a couple of students struggle to build on that step, I would intervene to say, "That first step doesn't seem to be getting us anywhere. Can anyone think of an alternative?" After a pause, often some student has the idea of using one of the other proof formats and gives some indication of it. The last student will (gratefully) pass the proof on.

In the introduction-to-proofs course, I use pass-the-proof regularly to reinforce each proof technique the class after I introduce it. If the class seems ready, I may use pass-the-proof the same day we work on a new proof method, but definitely the following class. I use it on some other days as well. Several of my colleagues also use pass-the-proof in this course. Because this course is as much about process as it is content, we regularly include proving activities in class. As a result, we expect to cover less content than in our other sophomore-level course, Linear Algebra. Thus the regular use of pass-the-proof doesn't interfere with covering the content here.

Our other sophomore-level class, Linear Algebra, has a smaller emphasis on proofs. I use pass-the-proof only occasionally there, and I wait until I have given students individual feedback about their proofs on homework. Certain proof formats, such as those about subspaces or independence, occur sufficiently often in this course that I want the students to be able to produce such proofs on demand. Pass-the-proof provides a good way to ensure that students can make such proofs and reinforces the formats more effectively than

lecturing.

In upper-division proof-based courses (Abstract Algebra, Real Analysis, and Geometry) I use pass-the-proof for much the same purpose as in our introduction-to-proofs course. I expect these students to be able to use definitions and proof techniques more fluently, so I can also emphasize having them look for the idea underlying the proof. In these courses I choose among calling on students, asking for volunteers, using pass-the-proof and presenting a proof myself, depending on the time available, the classroom rapport, and my pedagogical goals that day. With the smallest class of six, which was the second semester of abstract algebra, I asked the students to offer more substantive inputs.

3 Outcomes

As students get used to contributing to the whole class, I observe them using notation more accurately and paying attention to organization and logical format, especially as students start making more suggestions to improve the proof when it is their turn. Students, I suspect, pay more attention to another student's correction than to mine. Perhaps they unconsciously think that if another student can recognize inadequacies, then they should be doing so as well.

More students seem more engaged in the proofs with pass-the-proof than when the instructor calls for volunteers, who tend to be the stronger, more vocal students. Pass-the-proof also seems to reduce the anxiety of weaker students compared to when the instructor calls on specific students. Students are more comfortable having a classmate choose them, rather than the instructor's choosing them. Also, students clearly enjoy getting to choose the next student and often times think of pass-the-proof as game-like. They tend to choose students who haven't contributed. In the process of looking for others who haven't yet contributed, they work harder to learn everyone else's name. As a result I think students interact with a bigger percentage of the other students, including outside of class.

Pass-the-proof worked best with 10 to 18 students, although it worked satisfactorily in classes of all sizes. In a relatively small class, students get to know each other fairly quickly and can remember who has been called on already. Also, it is harder for a student to hide in a class of that size.

My acting as a scribe may increase the awkwardness for less confident students if they later present a proof with deficiencies. However, I believe my use of good notation and other mechanics enables the class to concentrate on learning harder aspects of proofs. To a certain extent pass-the-proof permits presenting different styles of writing proofs or approaches to a proof without devoting lots of time to students presenting whole proofs.

4 Extending the Method

The following are some possible alterations to pass-the-proof that may help instructors emphasize other aspects of learning to write a proof:

Recently my geometry students spontaneously modified pass-the-proof by having a short class discussion whenever one of them couldn't think of a useful next step. After each short discussion, they reverted to pass-the-proof without my saying anything.

The instructor could have the students write their contributions on the board. Since there might well be more notational and other mechanical errors, this variation might make students more aware of these aspects of a proof. To save time, an alternative might be to use technology, such as iPads, to enable students to write their contributions from their seats. However, in this case, there might be an issue of not having the entire proof visible on the screen. See the companion article by Hindeleh [1] for an adaptation of this method using wireless technology.

The instructor could, at the end of one class, give a statement to be proved during the next class with the pass-the-proof method. In this setting, subsequent students could well have prepared different approaches to the same proof. Such a situation would provide a teaching opportunity for students or the instructor to discuss valid, but different approaches to a proof.

While I teach at a liberal arts college, I think this technique would work in a proof-oriented course in any institution.

References

- [1] F. Hindeleh, “Engaging Students in Interactive Peer-Review Proof-Writing Activities Using Wireless Technology,” in this volume.

Appendix

A Examples for Introduction-to-Proofs Courses

Early in an introduction-to-proofs course:

For integers a , b , and c , prove that if a divides b and a divides c , a divides $b + c$.

For induction in an introduction-to-proofs course:

Prove for all non-negative integers n that $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$.

Late in an introduction-to-proofs course:

For any sets A and B , prove that if the power set of A is a subset of the power set of B , A is a subset of B .

Students often mistakenly try to convert the hypothesis into the conclusion. Since each is an implication, such a conversion is quite difficult. I explicitly teach the format of such implications of implications. The focus should be on how to prove $A \subset B$ by choosing an arbitrary element $x \in A$. Then with a little creativity and discussion they can see that $\{x\} \subset A$ and so $\{x\}$ is an element of the power set of A .

B Examples for Abstract Algebra Courses

Early in an abstract algebra course:

Prove that the intersection of any collection of subgroups of a group is again a subgroup.

Later in an abstract algebra course:

Prove that a factor group of an Abelian group is Abelian.

C Examples for Geometry Courses

Early in a geometry course:

Prove that if $ABCD$ is a parallelogram, $\triangle ABC$ is congruent to $\triangle CDA$.

Late in a geometry course:

Let P be the set of all (affine transformations or similarities, as appropriate) that take every line k to a line parallel to k . Prove that P is a group of transformations.

D Examples for Analysis Courses

Early in an analysis course:

Suppose $\lim_{n \rightarrow \infty} b_n = L$. Prove that $\lim_{n \rightarrow \infty} |b_n| = |L|$.

Later in an analysis course:

Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on \mathbb{R} . Let $S = \{x : f(x) = 0\}$. Prove that S is a closed set.

Tom Sibley: St. John's University, Collegeville, Minnesota

Engaging Students in Interactive Peer-Review Proof-Writing Activities Using Wireless Technology

Firas Hindeleh

Abstract

In this article, I share my experience with an in-class peer-review proof-writing activity. Students collaborate in typesetting their mathematical proofs using a wireless keyboard and mouse, and give each other instantaneous feedback. The final product is a polished \LaTeX document which is produced and reviewed by the class as one big group.

Difficulty Level: Medium; Course Level: Transitional
Technology Based

1 Background and Context

Grand Valley State University (GVSU) is a liberal arts school with 24,600 students, of whom 21,000 are pursuing an undergraduate degree. Our mathematics department serves an average of 500 mathematics majors, of whom roughly 72% pursue an emphasis in elementary or secondary education. Class sizes are small with all lower-division courses having a cap of 30 and upper-division courses having a cap of 20 or 24.

Communicating in Mathematics is a typical introduction-to-proofs class. Students in this class are primarily sophomores who have successfully completed Calculus I and Strategies in Writing¹ courses.

At the beginning of a semester, students in the class are introduced to the basic structure and proof-writing guidelines through examples with simple mathematical content. They are also provided with a \LaTeX template file and a set of screencasts from our department's YouTube channel describing how to install and use \LaTeX [4]. It is not surprising that students are still confused about their first proof by the end of the first week. The quality of the first Portfolio² proof draft is usually poor. The goal is that students will rise up to the expectations after receiving constructive feedback on two drafts.

Students learn better when they actively participate in the classroom learning process [6, 7]. This motivated me to use a method that actively engages my Communicating in Mathematics students in interactive proof-writing and peer-review in an in-class activity using scientific-capable word processor software such as \LaTeX . Another motivation for this method was to help students overcome the technical and conceptual challenges they face preparing their first \LaTeX -typeset proof by the end of the first week of class.

2 Description and Implementation

To ensure a lively classroom discussion, students are expected to read the section before class and turn in a short-answer preview activity. Some preview activities review prior mathematical work that is necessary

¹A university requirement writing class where students practice a variety of rhetorical forms and develop structure, style, and voice.

²The Proof Portfolio is a semester-long project consisting of 10 proofs that students must typeset. Proofs must follow certain mathematical writing guidelines from the textbook [9], and students are encouraged to submit drafts of their work for preliminary feedback before the problem is formally graded. At most two drafts can be turned in for feedback before a final grade is awarded.

for a new section. Other preview activities introduce concepts and definitions that will be used when that section is discussed in class. I do not expect students to completely understand the new topic, but having some idea gives them an opportunity to become involved in the teaching and learning process. For example, a preview activity question might ask a student to explain why 8 is an even integer by relating it to the recently introduced definition. To make sure that students come well prepared, I collect the activities at the beginning of each meeting and grade them based on attempt and completeness for 10% of the course grade.

In order to prepare students for the proof activity, I ask them to look over three problems from the textbook for the next meeting in place of their daily reading assignment and preview activities. I explain that they need to seriously try to solve them and that we will be discussing one of the problems in the next class. On the day of the activity, I bring my laptop, wireless mouse, and wireless keyboard. I connect my laptop to the classroom's projector, and have an empty \LaTeX template file opened. Then I ask the students to choose a challenging problem from the assigned exercises that they would like to discuss. The majority agrees on a problem, and we discuss it as a group with a few guided questions from me. Using input from students, I sketch on the board a plan for the proof.

One example of a problem we have used is

Prove that for each integer a , if $a^2 - 1$ is even, then 4 divides $a^2 - 1$.

After the 10-minute discussion phase, I turn the projector on, pick up my wireless keyboard and mouse and ask the class: "How about we write a portfolio-quality proof? Who wants to take control and get us started?". The following is a script³ of what happened next for this particular question.

STUDENT 1: Sure!

[*Student 1 grabbed the keyboard and mouse, and started typing.*]

Proof.

STUDENT 2: Don't you want to restate the question into a proposition?

STUDENT 1: Fine.

[*Student 1 wrote the proposition with confidence.*]

Proposition: For every integer a , if $a^2 - 1$ is even, then 4 divides $a^2 - 1$.

Proof.

INSTRUCTOR How about we give the keyboard and mouse to someone else?

[*With a smile Student 3 picked up the keyboard and mouse and started typing.*]

Proof. Since $a^2 - 1$ is even, then

STUDENT 4: You need to keep the reader informed of what you are assuming and what you need to show.

[*The instructor gave the keyboard and mouse to Student 4 who wrote.*]

Proof. We assume that $a^2 - 1$ is even. We will show that 4 divides $a^2 - 1$. Since $a^2 - 1$ is even, then

INSTRUCTOR Anything missing so far?

[*Students looked again but seem to find nothing.*]

INSTRUCTOR Does your reader know what your a stands for?

STUDENT 5: But it says in the proposition that a is an integer.

STUDENT 6: We have to state that at the very beginning.

[*The instructor gave the keyboard and mouse to Student 6 who fixed that.*]

³Thanks to my colleague Salim Haidar for transcribing this dialogue as part of a personnel review.

Proof. We assume that a is an integer, and $a^2 - 1$ is even. We will show that 4 divides $a^2 - 1$. Since $a^2 - 1$ is even, then . . .

The majority of the 20-student class was working as a team. Students were giving their peers constructive feedback that put writing guidelines into a context. The feedback language they were using was similar to the language I used on Proof Portfolio problem drafts, so students were connecting this activity to individual work they were doing outside of class. The final product can be found in the Appendix.

This method requires minimal preparation from the students. As previously mentioned, I usually ask them to look over three problems from their text to discuss at the next meeting, and turn in a serious attempt at the beginning of class. I walk into class with no idea of which problem the students want to discuss. I give the students the choice of the topic for that hour to stimulate their interest in the discussion.

Even with an active class, the method can take the whole hour to finish. As much as students valued the experience, I knew that I could not afford to use another hour to do a similar activity and finish the seven required chapters in the textbook [9]. In later semesters I experimented with other variations that would require less class time. For example, in a second similar activity a month later, I set up in advance a generic L^AT_EX template for a mathematical induction proof, and shifted the focus to the mathematical content. This allowed students to deepen their content knowledge, without spending as much time on the writing.

In order to cover all the course topics, I was able to do only three hours' worth of these activities. After recently reviewing our program, we observed that our students needed more class time for proof-writing and discussion, so we increased the course credits to four starting the academic year during which this article was written. The credit increase was compensated with a credit decrease in Calculus I, which used to be five credits. An extra hour will allow weekly opportunities for similar in-class proof-writing activities and still cover all the topics at a relaxed pace.

I also tried this method in my Euclidean geometry classes. Students were provided with guided questions on a handout at the beginning of class. They used Geometer's sketchpad [3] or GeoGebra [5] on my computer to discuss and interactively construct pictures of a certain proof as one big group. They were then expected to finish typesetting their proofs in smaller groups outside the classroom.

3 Outcomes

I have been using this method in this class for four years in a total of eight different classes. There are many advantages and strengths to using this method. Here are some of the highlights.

3.1 Benefits Directly Related to Proof-Writing

I noticed a significant improvement in students' writing after this activity. My feedback on Proof Portfolio problem drafts became more centered on the mathematical content rather than the mathematical writing and technical formatting. For example, I would notice that some proofs had a well-written opening and closure, appropriate use of definitions and theorems, but might need more work on the content. After using the activity, it was clear that students understood that proofs have certain writing guidelines that they need to follow even if students were not sure which proof method to use yet.

Additionally, when students are given the power to choose a question that interests them, the class becomes theirs and not mine. Proposition 1 in the Appendix was produced, written, and typeset by the students. They were engaged in discussion and review rather than taking notes. The instructor's role was more as a moderator than a lecturer while students were the peer-instructors. They gained more skill and confidence while I gained a classroom of teaching assistants who can help each other learn. Also, there were numerous positive student comments about this method in the end-of-term evaluations. For example, one person wrote: "I really like the day we passed around his bluetooth keyboard and wrote a full proof as a class rather than him doing it for us".

3.2 Classroom Dynamics

The method creates a lively classroom. The first time I tried it, I had a late evening section and slightly feared that people would fall asleep if the lights were dimmed for the projector. I had never seen my proof-writing class as active and enthusiastic about writing proofs as they were that evening. Students were excited to be handed the keyboard and mouse and to add their input to the document. They knew that they would receive a turn if they provided feedback to their peers. In later semesters, I experimented in passing the keyboard and mouse to all students, and allowed each student to contribute even if it was typing one sentence. That being said, students understood that they were expected to provide feedback on what was being typed. This would ensure that even shy students participated at least in typing, and had the class support them.

In the past, I used other traditional methods such as having a group of students go to the board and write a proof while receiving feedback from the class. I found two issues with this: students needed a lot of encouragement to go to the board, knowing that their contribution would soon be critiqued. It was also frustrating having to copy and erase multiple times. Using the wireless keyboard and mouse brings the board to students at their seats!

With regards to technical aspects, students had never used \LaTeX before this course, so the activity served as a good interactive demonstration and anxiety breaker. Also, wireless technology gives a wide range of connectivity. A Bluetooth[®] keyboard and mouse give a 33-foot range of wireless connection [1, 2], so even students sitting at opposite ends of the room can contribute to writing the proof.

The only challenge that I face using the method is that a fair amount of time is needed to complete the first proof in class. As students become more comfortable with the activity, it takes less time and students can easily finish two proofs in an hour. However I believe that these activities are worth every minute as they drastically improve the students' proof-writing, and thus reduce the amount of time I spend reinforcing proof-writing throughout the course.

4 Extending the Method

This would be an excellent activity to implement in our latest LearnLab classrooms, which are designed for a different experience in learning. Tables form an X shape, so there is no front or back to the room. With four mirror-display projectors around the room, students connect their devices to the projector's port on their table to share and discuss with other groups. The instructor can switch the display between different tables from the main station.

The technique is ideal for small classes. One can easily get input from almost every student, whether verbally or in writing; however, for larger classes of 40-50 students, I would split the class into smaller groups of 10, and have each group work on a problem. In general, the method can be used in any class where the instructor would like students to work interactively using software. (Editors' note: For a similar approach in this volume that does not use technology, see Sibley [8].)

References

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- [8] Tom Sibley, “Pass-the-Proof,” in this volume.
- [9] T. Sundstrom, *Mathematical reasoning: writing and proofs*, Pearson Prentice Hall, New Jersey, 2007.

Appendix

Proposition 1 For every integer a , if $a^2 - 1$ is even, then 4 divides $a^2 - 1$.

Proof. We assume that a is an integer, and $a^2 - 1$ is even. We will show that 4 divides $a^2 - 1$. Since $a^2 - 1$ is even, there exists an integer k such that $a^2 - 1 = 2k$. Adding 1 to both sides of the previous equation we obtain

$$a^2 = 2k + 1.$$

Therefore, a^2 is odd. Since a^2 is odd, then a is odd by Theorem 1.6 which states that “For all integers n , if n^2 is odd, then n is odd”. Since a is odd, then there exists an integer x such that $a = 2x + 1$. Then

$$\begin{aligned} a^2 - 1 &= (2x + 1)^2 - 1 \\ &= 4x^2 + 4x + 1 - 1 \\ &= 4(x^2 + x). \end{aligned}$$

Since x is an integer and integers are closed under addition and multiplication, $x^2 + x$ is an integer. This means that $a^2 - 1$ has been written in the form $4z$ for some integer z . Therefore 4 divides $a^2 - 1$. Consequently, it has been proven that for every integer a , if $a^2 - 1$ is even, then 4 divides $a^2 - 1$. \square

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Proof-Writing through Electronic Document Editing

Tom McNamara

Abstract

We discuss technologies that allow multiple users to access and edit documents. Further, we elaborate on methods that employ these tools to enhance students' proof-writing skills. The methods have received classroom testing in courses for junior and senior mathematics majors at Southwestern Oklahoma State University, a regional comprehensive university serving nearly 5,000 students. We also examine how these methods can enhance more traditional ones in the teaching of deductive proof construction.

Difficulty Level: Medium; Course Level: Advanced
Technology Based

1 Background and Context

Readers may be familiar with Wikipedia, a freely accessible online encyclopedia. As the name implies, it is a wiki—software that allows users to create, edit, and manage content online. What makes a wiki different from other platforms is that there is no owner of the content. Any user can edit anything. This might sound like an invitation to anarchy, but the system keeps track of changes. If desired, users can revert to previous versions of documents. Content can also be reorganized in many ways. This contrasts with other forums like blogs and discussion boards where material is presented based on creation date.

My first experience using a wiki was while teaching the Advanced Calculus course at Southwestern Oklahoma State University (SWOSU). This is a regional, comprehensive university serving nearly 5,000 undergraduate students. Around 1% of them are mathematics majors. SWOSU math students are evenly split between those intending to be mathematics teachers in secondary schools, and those pursuing other goals, which include pursuing jobs in the energy industry and attending graduate school.

2 Description and Implementation

2.1 Introduction

The impetus for experimenting with wikis came from discussions with colleagues. Initially, the concerns dealt with lower-division students. There was a consensus that to really learn the course material, students needed to be more active in creating examples. It was suggested that we give students a space online where they would create, say, a mixture problem. Then other students would be tasked with solving this problem posed by their peers.

While this goal might be laudable, there was concern about the students' motivation. Our lower-division courses are populated by students from all fields of study. We feared these students would balk at something novel in their mathematics courses. Further, the learning curve involved with introducing a new medium could be problematic. It was decided that students would be more likely to invest time and effort for courses that were part of their majors. Thus, an upper-division mathematics course was chosen, and as a result, I created a wiki for use by the students in an advanced calculus course.

2.2 Setup and Initial Assignments

There is one section of advanced calculus offered at SWOSU each year. The last time I taught it, it was a small class with only six students. Some of them were highly motivated, so it seemed an ideal situation for experimenting with a new method. Shortly after the course began, I decided to set up a wiki for use by the class. The first place I checked for resources to help with this project was the online course management system provided by the university. At that time the system had no resources for setting up a wiki, so a third-party provider was needed to create it. There are several places online that host wikis, and some of them are free to use. We were looking for a host who provided software that supported the use of \LaTeX in wiki posts, and we created an account with PBWorks. This process took only a few hours, which included contacting our Information Technology department to see if our university's system supported wiki software, finding an outside provider, and creating an account.

Once the wiki was created, I invited the students to register. The arrangement that I chose was that anyone with access to the web would be able to view the content, but only invited users could create and edit it. This option is available with other platforms as well. There are also provisions for making the content viewable only to users who have been invited to participate.

The students had no trouble creating their accounts. The tools provided online also made it fairly simple to create new documents. The interface for typing regular text was similar to a word processor. Typesetting the mathematics was more challenging. Some students had learned typesetting with \LaTeX during an earlier course that is required for mathematics education majors, though some mathematics majors do take it. Those students who were already familiar with mathematical typesetting had no difficulty incorporating symbols in their documents. For those unfamiliar with \LaTeX , links to online resources were provided on the front page of the wiki. They contained enough examples to get students started.

My initial plan was for the wiki to serve as an online notebook, creating a place that students could go to see the important ideas covered in the course. Thus, students were offered bonus points for posting to the wiki in addition to their standard pencil-and-paper work. I suggested that students create wiki pages for instructive proofs from the class, examples, and homework that had already been collected through traditional channels. I hoped that, for example, one of the students might post a page concerning bounded sequences that do not converge, while another could cover the proof of the Bolzano-Weierstrass theorem discussed in class.

Unfortunately, there were problems with the structure. The documents on the wiki were created almost exclusively by students who were doing quite well in the class. While we wanted our strong students to excel, we had hoped that weaker students would have used the wiki posts to strengthen their understanding while boosting their scores. In hindsight, I should have seen this idea was flawed. There was a lack of structure. Students who were struggling with the course were not doing so because of the medium being used to record their work but rather were having difficulty with the creation of a deductive proof. Changing *how* students communicated their work was not going to address this issue.

2.3 Adapting Expectations

Thus, after a few weeks during which the top students solidified their high grades and weaker students continued to flounder, I decided to take a different approach. I required posts to the wiki as part of the assigned work. Thus, students were given work online in addition to traditional pencil-and-paper problems. Assignments were split about 80%-20% in favor of standard written exercises. Typically students would have one week to complete a set, which would include roughly twelve pencil-and-paper proofs and three assignments involving online material.

I also changed what students were asked to do on the wiki. Instead of asking them to create new documents with proofs of theorems, we gave each student her own folder on the wiki. They contained individualized assignments.

2.4 Types of Online Assignments

1. We borrowed an idea used by many authors which we will refer to as a “skeleton” proof. There are many examples in texts used at the advanced undergraduate level. It features prominently in the work by Gaskill and Narayanaswami [2, p. 28]. To create an assignment like this the instructor supplies a new document on the wiki that will be edited by a student. One example is:

Theorem 1. *Let H be a non-empty subset of \mathbb{R} bounded above by an element $b \in \mathbb{R}$. Then $b = \sup H$ if and only if for every positive ε there exists an $x \in H$ with $b - \varepsilon < x \leq b$.*

Proof. (**Direction?**) Let $H \subset \mathbb{R}$ be non-empty with $b = \sup H$. Given any positive ε , we have that $b - \varepsilon$ is not an upper bound for H (**Why?**). Thus we must have an $x \in H$ such that $b - \varepsilon < x$ and hence

$$b - \varepsilon < x \leq b.$$

Supply a proof of the converse. □

There are three boldface items in the proof. They are invitations to the student to edit the page. He can either supply the details being requested right in the document or provide a link to the information if it exists somewhere else on the wiki. The first of the three parts in the assignment is the (**Direction?**) question. The statement of the theorem involves *if and only if* so, the proof needs to deal with both the “if” and “only if” parts. Only one piece is supplied. The student needs to determine which part is supplied and replace (**Direction?**) with either \implies or \impliedby as appropriate.

The next question in the assignment is the (**Why?**) at the end of the second sentence. The student can either supply the reason on the page, or link to a page containing the definition of $\sup H$. We consider the last question on the assignment to be the hardest, as the student is asked to supply a deductive proof for the other implication in the claim.

As can be seen from the example, the wiki can provide the student with a new type of assignment. Many students have trouble understanding how to start their work when having to prove a theorem. A partially completed document like the one above works around this issue by giving the students a springboard.

2. A variation on the skeleton assignment involves starting with a previously created document that lacks some details. The instructor places the prompts throughout the existing post, with students assigned to supply the details. Students can either be tasked with revising the work of others or revisiting material they created themselves.
3. Once there are enough posts available, students can be assigned to organize them. This can involve linking between theorems and their supporting lemmas. Documents that contain examples can be linked to the theorems they illustrate.

2.5 Grading for Wiki Assignments

Students received feedback for their online work in the form of reports. Here is an example.

Wiki Report

For: _____ Grade: _____

The grade for your project has been determined based on the following categories. Subsequent projects will be graded on similar criteria.

1. Accuracy: _____/15
2. Clarity: _____/10
3. Presentation: _____/10
4. Linking to other pages: _____/7
5. Editing: _____/8

The assignment's accuracy component is based of course on whether or not the student used correct mathematics. Clarity is based on the use of appropriate connectives, complete sentences, and general readability. I feel this component is very important, and deserves as much emphasis as it can get. The interested reader is encouraged to consult chapter three of Houston's book [1] for more about this idea. The presentation grade is based on whether or not the student made appropriate use of \LaTeX commands in the post. Lastly, some assignments require the students to edit existing posts, and to make sure that a new post is linked to at least one other post. The Editing component on the report gives the student feedback on this.

3 Outcomes

3.1 Introduction

I have used wikis in three courses so far. The Advanced Calculus course mentioned above was the first. In addition, it has been used for two sections of a class which shows students the tools available for communicating, creating, and teaching mathematics. I have also used a wiki in Foundations of Mathematics, which is our bridge course. This eases the transition from the computational focus of lower-division mathematics to the construction of deductive proofs that is at the heart of modern mathematics.

3.2 Effectiveness

I feel that our method has several pedagogical advantages. When students make mistakes in their proofs, the instructor can access the wiki document and indicate where additional detail is needed. The student can then revisit the post, keeping all the pieces that were correct and adding the necessary revisions. This is tedious with pencil and paper. In the traditional setting, a rewrite involves fixing the mistakes and recopying. It can be all too easy to introduce additional errors or leave out something essential. In the electronic format, students can focus their efforts on improving the finished product. Thus, getting back graded material becomes more of a learning experience.

It is unfortunate, but students have a tendency to look at their score on an assignment as a whole, then move on without giving a second thought to the topics not yet mastered. My technique gives them the

opportunity to earn back some points, creating an incentive to review incorrect proofs and find out how to do them right.

Another feature is that students no longer get credit for our work. In other words, if a student hands in muddled, imprecise work, we are tempted to struggle and find out what he really meant. Giving partial points rewards the student for our efforts, not his. Since my method allows students to revise more conveniently, there is a safety net. Thus, anything the student earns is due to the fact that he produced a correct proof, even if it wasn't on the first try.

3.3 Strengths and Weaknesses

In each course where wikis were used, I have had mostly positive responses on student evaluations concerning the technology. Students are comfortable with word processing programs for their writing classes, and find the interface familiar. They also appreciate the convenience when they are asked to edit material. Since the upper-division mathematics courses at our university are rather small, it is difficult to say with certainty if this would hold true for students in general. I do, however, feel that students in any setting will appreciate breaking the task of proof-writing into smaller steps. Indeed, the assignments discussed are variations on reading a mathematics textbook with pencil and paper in hand, ready to supply any details needed. My method helps the student along this path.

Another advantage comes from the fact that material on the wiki has no owner. As any user is allowed to edit any document on the wiki, the work from one semester can be carried over to another. We simply invite these new students to register as users. They are then free to revise existing posts. This gives the instructor a growing pool of posts that can be used as the basis for new assignments.

The weaknesses of the methods I have used come from the scalability. The way I structured assignments involved giving each student a specific task on the wiki. This might become problematic with large groups. It is possible that with a large class, we could run out of theorems that are accessible to students. One suggestion for working around this is to use the wiki as a repository for examples rather than theorems. Indeed, a recurring theme in Houston's book, [1], is that advanced mathematics is learned best by the construction of examples. Thus, we might ask a student to create a sequence that is convergent but not monotone. Several students could be asked to supply different examples of this type.

4 Extending the Method

Compared to other technologies for managing online content, I feel that wikis have advantages. One common option, used for garnering student input online is a discussion board. Users of this tool can create threads dedicated to each topic, and then post messages to the appropriate thread. However, the discussion board software that I have seen sorts posts in a thread chronologically. There are some mechanisms for moving posts around, but they typically require administrator privileges. They are also limited in that these devices usually involve moving a post from one thread to another, not revising or reorganizing material within a thread. Further, each entry is editable only by the user who created it or by the administrators of the forum. This makes the skeleton type of assignments that we discussed above problematic, as the instructors' posts would be essentially etched in stone. Other users would not be able to make any changes.

This assignment structure might be possible with a blog, provided each member of the class was designated as author for all the posts. I have not looked into this possibility because blogs typically have a single author, rather than being a collaborative effort.

While some platforms are ill suited to these methods, there are other tools becoming available. All that is needed to make use of the skeleton assignments is the ability for multiple people to access and edit documents that have been created by others. They also need the ability to post these newly edited documents online so that others can access and edit them. This could be accomplished through the use of shared folders

or dropboxes in online course management systems. Indeed, SWOSU has recently implemented a course management system that allows instructors to post documents that can be edited by any user. We did not use it for my experiment because the university supplied different tools at the time. As online collaboration becomes more pervasive, we expect tools that allow multiple owners to access and edit the creations of others to multiply.

References

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A Humanistic Reading Component for an Introduction-to-Proofs Course

Gizem Karaali

Abstract

I use a reading component focused on the nature of mathematical proof in my Introduction to Analysis course, which is a transition-to-upper-division-math course that emphasizes writing proofs. Students are expected to read a variety of articles and essays about proofs and then create a joint document (currently using a class wiki) that allows them to consider a range of perspectives about what makes a mathematical proof work. The collaborative nature of the project seems to improve student engagement and the quality of student contributions. At the end students leave with a more nuanced understanding of proof. In this note I describe the project, my reasons for using it, and what my students appear to get out of it. Any instructor who has access to some technology might be able to incorporate this project with a class size ranging from 10 to 40.

Difficulty Level: Medium; Course Level: Transitional
Technology Based

1 Background and Context

Mathematicians like to think that proof is the center of the mathematical enterprise. And many of our aspiring majors agree, even if they have not had much experience with the notion. The collaborative project described here is my attempt to encourage students to discover the philosophical, the subjective, the humanistic, and the imprecise dimensions of proof.

1.1 Our School and Introduction to Analysis Course

I teach at Pomona College, a small and selective liberal arts college serving approximately 1500 students. Our student-faculty ratio is eight-to-one and the average class size is 15 [1]. Our students are predominantly traditional (coming straight out of high school). They often go on to professional schools after graduation, many go into graduate schools, and some eventually find themselves teaching.

This paper focuses on a course titled *Introduction to Analysis*, described in the Pomona College Catalog [6] as follows:

A workshop course on how to write proofs in the context of analysis. Focus on the construction and presentation of rigorous proofs. Learn how to use the language of analysis to prove results about sequences, limits and continuity. Students regularly present proofs in both written and oral form.

The course in its general framework is a writing workshop, mathematics-department style. In other words, students learn to write proofs by writing and rewriting them (as opposed to observing a professor present complete proofs on the board, or reading them in their textbooks). They are given proof assignments, and they revise and edit their proofs several times before the end of the semester. The mathematical content

of the course is basic real analysis (including sequences, limits, and continuity), but each instructor covers a different subset of the material; our common focus is on the proof-writing activities.

The course is Pomona's only proofs workshop course, though there are now a few other transitional courses intended to provide students some exposure to proofs before jumping into the full rigor of upper-division mathematics. Students focus on proofs to this extent in no other course.

The only prerequisite for the course is our introductory linear algebra course. Students will mostly have seen proofs, and in most sections of linear algebra but not all, they will have written some of their own. In other words, the proof content of the prerequisite course depends on the instructor and is highly variable.

One section of *Introduction to Analysis* is offered every semester though it is not required in any major or track. In the last few years, each section has been well populated; the typical section size lies in the range of 18-35. Students who take it fall mostly into two categories: those who are interested in a mathematics major and want to practice writing proofs before they move on to the advanced major courses, and those who are following the mathematical economics track of the economics major and do not want to take our standard real analysis course. Thus in the course we have students ranging from freshmen to seniors, and all have some experience with proofs but, usually, not much confidence.

1.2 Motivation and Goals

Mathematics students in college take a variety of courses introducing them to different fields and areas of study within the wide world of mathematics. But one of the main tasks of the mathematics major is to reveal the true nature of mathematics. This is not (only) to be found in the exhaustive searches for the correct answer via following a set of standard algorithms, painstakingly detailed computations, and clever tricks that occasionally save the day after a long and arduous struggle with known methods. The mathematics student is a young mathematician in training. Even if she does not intend to become a professional mathematician by declaring a major in mathematics, she commits to learning the fundamental principles and the framework of mathematical thought in general and of mathematical proof in particular.

Students in *Introduction to Analysis* are enthusiastic and eager to learn how to write proofs, and they are willing to work hard for it. They do not, however, understand the similarities and the analogies between writing a mathematical proof and an analytical academic essay. They believe, at least initially, that the main point is to get “*the right answer*,” and their focus is not on clear communication.

In this article I describe an activity I include in my version of *Introduction to Analysis*. There is a variety of reasons for readers to consider this activity for their classes:

- The activity **engages students in a collaborative learning and writing activity**. Our departmental stance is that mathematics is a collaborative and cooperative activity, and I developed this activity in order to encourage our students to experience this at a new level.
- The activity **introduces a liberal-arts component** to what could otherwise be a rather standard mathematics course. The humanistic dimensions of mathematics (including but not limited to philosophical, historical, and psychological approaches to the field) can enrich students' mathematical experiences.
- The activity **encourages students to tackle ideas about mathematical proof on their own turf**. Students are urged to construct their own understanding of the notion of proof by engaging with the resources they themselves locate and by reflecting upon their own experiences.

The third reason might need further explanation. Often students at this stage are still looking for binaries, dualities, and easy definitions; something is either right or wrong, either there is an answer or not. College gradually acquaints them with the nuances and the relativisms involved in most fields and disciplines; many come to accept that there are domains of knowledge where the answer depends on where the questioner stands. In mathematics, most think, this will not happen. In mathematics, they expect, there will be clear-cut

precise ways of doing things. The procedures will be easy to adopt once they have been learned. There is a correct way to write proofs. And perhaps there is only one way to do so.

One of my goals in this course is to shake that worldview a bit, possibly modify it, or even hopefully replace it with a more constructivist perspective, a perspective that allows for an understanding of knowledge as it is gained or constructed by the learner. In this context, mathematical proof is a human activity; it is a tool for communication, and as such, many of its features change and adapt relative to the context in which it is conceived or presented. There is not a single correct proof for a given statement, and subjective elements like elegance and surprise are valued as well as precision and logical structure.¹

2 Description and Implementation

The main idea is to create a joint collaborative document on the nature of mathematical proof. As they share their personal opinions and experiences related to the notion, students become aware of some of the subjective components of proof. As they engage with the resources they find on the topic, they develop a more sophisticated understanding of the philosophical and historical aspects of mathematics.

I use a free wiki to host the project, but other instructors could easily use a wiki tool available through their course management system or other providers. A wiki is a website that allows its users to edit its content and its structure in a collaborative way. Figure 1 is for a screenshot of the welcome page of a version I used in my classes.



Figure 1: Screenshot of the welcome page for the project in one installment of the course.

In the course we use the wiki in several ways. I use it to set up and announce the assignments and provide space for the students to share their thoughts. Students then visit the website and update it to include their contributions to the project. Often they will read what others wrote and respond with these previous remarks in mind. Through the course of the semester, the wiki is built as a team project for the whole class into a record of the students' thoughts and experiences related to the theme.

The collaborative project consists of three assignments given throughout the semester. In the first assignment students are required to write at least two paragraphs in the project wiki on their personal experiences in and current opinions on mathematical proof. In the second assignment students are expected to contribute to a joint bibliography on the nature of mathematical proof that is built in this collaborative manner on the project wiki. In the final assignment, which is a capstone component of the project, students are supposed to

¹Computers prove mathematical statements too [3]. However my focus is on my human students and in conveying to them that their species does mathematics with particular considerations in mind.

reflect upon and write two paragraphs about how their ideas of mathematical proof evolved as the semester progressed. Complete prompts I use for the assignments can be found in Appendix A.

For the first and third assignments, which both require two paragraphs from each student, I provide a link to a web source [7] that emphasizes that the length is not the defining characteristic of a paragraph. Rather “the unity and coherence of ideas among sentences is what constitutes a paragraph.” The link leads to a writing center website, as a clue to the students to remind them of the commonalities between writing for mathematics and writing for any other academic context.

In my classes, I create the wiki site for the course; in a typical adaptation of this project, the instructor would be required to prepare the project site and set up the underlying structure. The students need access to the internet in the case of a wiki platform, and access to the library (stacks or electronic databases) for the bibliographic research component (second assignment).

Instructors would need to be familiar with the wiki site or the assignment platform in which they choose to work. Wikis are simple platforms, and even those who do not know how to work with one can learn to do so easily and quickly. Furthermore, students would need to learn how to work on a wiki site, but once again, the learning process is smooth and straightforward. In my classes I initiate the conversation by referring students to the wiki site and informing them of the basic logistics, and I find that after a five-minute introduction, students are always able to move forward. Instructors who have already been working with a course management system may also consider using their wiki or discussion forum capabilities.

In terms of time requirements, besides the five-minute introduction to the wikis, I do not spend much time in class on the project. The assignments are completed outside of class meeting hours. In future versions of the project I would prefer to incorporate some discussions in the class about the joint project if at all possible, which could take up to half an hour at three points in the semester. In any case, the project adds value to the course and allows me to reach my goal of providing students with a more nuanced understanding of mathematical proof (see Section 3 for a detailed discussion), so I find it worth the time investment.

In terms of grading, the project adds one more component (or three, if one thinks of it in terms of the three tasks that are assigned). Instructors will need to find the time to do the extra grading, but perhaps they can consider weighing the benefits of the project against those of more traditional components of the course, and if necessary, cut down on the other components.

3 Outcomes

I taught *Introduction to Analysis* three times at Pomona College, and I introduced the project described here the second time around. In the second version, the assignments were much more polished, but the idea remained the same. The assignments as described in Appendix A were taken from this second version. The main student outcomes of the project follow.

3.1 Students’ view of proof changes drastically.

Students start my class with a simplistic approach to proof, assuming and explicitly asserting that a mathematical proof is a conclusive and concrete path to truth. Here are some student quotations from the first assignment that reflect this attitude:

[W]e can describe a proof as the art of combining statements to make new ones.

[W]ith proofs we can establish truths, and thus expand the study and field of mathematics such that all mathematics is based on few given truths and ideas, and all new facts and discoveries are firmly based on rigorous logic and proofs before them.

The final assignments are often much different in tone. Students often reflect upon how they used to think of proofs in their commentaries:

Upon entering the class, my basic views of what a proof is lined up most easily with the idea of a formal proof. A finite series of statements that lead logically to one another, based off established formal logic and / or existing theorems. The entire purpose of a proof was to know that something was true.

My initial understanding of a proof when I entered this class was a series of logical steps which sought to provide mathematical reasoning as to why something is a “truth” in mathematics.

They then point out that there is much more to proof than just that:

We write and study . . . proofs in order that we can gain some sort of insight not only to the consequences of the proof but the real inner workings of the mathematics involved.

[T]here is more to proof than soundness / validity; certain contexts require concision / brevity or thoroughness, explicitness / clarity or sneakiness, etc. Because proofs are a way to communicate ideas, they morph according to the purpose they hold in context, just as all other forms of communication do.

More explicitly, proof is a means of communication and should be context- and audience-dependent:

Proof is a means by which we communicate to other humans in our field. Proofs are meant for an audience. We write proofs because we want to establish truths not only for ourselves, but for others, to formally define ideas and concepts.

[Y]ou don’t need to lay every single step down for your reader, but you only have to give them enough for their own flash of insight and understanding. In a sense, to me, this is what a proof is: a tool to communicate to others and develop your own ideas.

In fact, proof is a large part of what makes mathematics a human endeavor.

I primarily quoted students who developed through the course of the semester a more humanized sense of mathematical proof, one that focused on its communicative dimension and dependence on audience. There were many other sophisticated ideas that developed. In particular, students have observed that personal styles can be reflected in proofs, that what is deemed to be true or false may be relative, and that writing a proof involves several stages including private and public subprocesses.

3.2 Students write better proofs when they have thought deeply about what a proof is supposed to be.

In my experience the project contributes substantively to the main announced goal of the course, that students learn to write better proofs. As their understanding of the meaning and purpose of mathematical proof develops, students become more interested in communicating their thoughts with more clarity. Some of my students reported on this in their comments, as in:

[M]y concept of the mathematical proof has broadened in scope (I now consider digestability [sic], readability of my proof) and yet also tightened in rigor (I’m very apprehensive of even the most minute gaps!).

Pascal believed that if you want to persuade someone else of something, you need to appeal to their mind and to their heart. Before I took this class, I never thought of a proof appealing to someone else’s mind. . . . I learned in my writing how to appeal to others’ minds and hearts. For example, being extremely blunt in a proof may make the readers seem unintelligent (using the phrase “clearly” too often in writing proofs).

As the instructor reading their proofs and revisions through the semester I could plainly see the improvement in my students. As a concrete outcome of this, my revision requests became rarer as we reached the end of the semester. A student pointed out that the project even affected his approach to other courses:

I have changed how I present my information in my other classes. That is not to say I write proofs for my problem sets in probability, but rather, I try to make my reasoning apparent, I point out my relevant information, and I explain my steps quite explicitly. Thus my goal is to convince the grader, as well as myself, that my work is correct. I realize now that a series of scrawled equations does not demonstrate critical reasoning, and I am glad that my homework habits have changed because I now am better at convincing people I am right.

Indeed years later I have had students from this course come back and tell me that this class was a turning point in their mathematical training, that it was the place where they learned that context and style matter, even in mathematics, and that their proof-writing became much more sophisticated and mature.

3.3 Students have the chance to think philosophically about mathematics and its nature.

The project encourages students to move above the daily concerns of a math class and think more broadly about what they are learning. They are forced to engage with the more philosophical aspects of writing proofs, and in the end they gain a deeper and more personal understanding of the concept. The work associated with the project allows students to engage with the various ideas about mathematical proof on their own turfs. Instead of the instructor's explicitly telling them that proofs do not originate in some void, they are not out there to be found, but more accurately they are a product of human effort, thought, and the need to communicate, through this project students discover these ideas themselves and teach them to one another. The instructor becomes a spectator while students become the actors on stage.

3.4 Students read about proofs from a variety of sources and create a class community around a philosophically substantive discussion topic.

As readers can see from the bibliography included in Appendix B, which was created by one of my classes for their second assignment, students searched for, found, and read carefully a wide range of sources, and many were able to incorporate ideas from them into their understanding and conception of proofs. However, my students did not merely find external sources from which to learn. The component also allowed them to create a class community! As I could see in the student submissions, each read their classmates' contributions and engaged with them in their own comments. There were some personal references (in the form of inside jokes and oft-repeated phrases), as well as agreements and, to a lesser extent, disagreements, which led to a more interesting discussion.

4 Extending the Method

This semester-long project could be easily adapted to smaller classes with no changes, as long as there are at least five students involved. Fewer than five students may find the web interface unnecessary; instead they might meet in person to discuss the assigned questions. If there are more than 30 students, the instructor can split them into teams of 5, 10, or 15, and assign teams separate wikis or separate pages of one class wiki.

Another idea would be to incorporate some conversations in class to bring the project home, so as to make clear the connections with what we do there. Holding a half-hour in-class discussion after the due date of the assignments might be a good way to do this.

The project depends only on an online platform that allows collaborative work. Several such platforms exist [2]. Wikis seem to be a good fit for the project and there are free and free-standing wiki providers, as

well as wiki components in most course management systems. Other authors describe in detail how wikis can be used to accompany and enhance a mathematics course [5]. (Editors' note: see also McNamara's paper [4] in this volume for a way to do this.)

Lastly, let me expand upon a comment I made in Subsection 1.2. I state there that my students are enthusiastic and eager to learn about proofs. This may have raised a question in the mind of the reader as to the transferability of the project. It is plausible that I might just be lucky to be teaching these students; perhaps with a different set of students in a drastically different context of another college or university, the project would not succeed. I am unable to respond to this with concrete data, as I have not used the project in any other institution besides my current one. Furthermore, it is indeed true that many of my students at Pomona are interested in learning for the sake of learning, many more than in any of my previous institutions. However we have our fair share of uninterested students who are looking for easy shortcuts, minimal workloads, and higher fun-to-work ratios. In general I think it helps to focus on what it is in students' attitudes that would help a class or a particular project work well.

Student enthusiasm is directly related to motivation, and I have found that it can, at least partially, be addressed by emphasizing the centrality of proof as the ultimate method of mathematical work. My students are often enthusiastic, but most of them are eager to learn to write proofs because of less idealistic reasons than readers might assume. Many students come to *Introduction to Analysis* with much anxiety about their place in the mathematics major. They are worried about whether or not they can succeed. They are unsure of their proving capabilities; they think that there has to be one right way to the answer, and that they are not smart enough if they cannot figure it out on their own; indeed that is why they think they need to take this course. Some feel vulnerable, even wounded, seeing themselves as failures who are not yet ready to cut it in "real" analysis. I work to alleviate this anxiety by underlining the role that good proving skills will play in their mathematical lives. This course is, in reality, a writing bootcamp, and we will all be stronger in the end as a result of our joint effort. In particular students will be much better equipped to handle the challenges of their chosen major. I believe, optimistically, that such an attitude toward the course from the instructor can set a tone that makes a difference; that attitude can be transferred to most, if not all, institutional contexts.

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Appendix

A Specific Assignment Descriptions

A.1 First assignment.

Here is a statement of the first assignment, directly from the class wiki:

OUR FIRST ASSIGNMENT WILL BE TO COLLECT TOGETHER OUR INTUITIVE IDEAS ABOUT MATHEMATICAL PROOFS. THUS YOUR FIRST ASSIGNMENT IS TO WRITE AT LEAST TWO PARAGRAPHS HERE IN YOUR OWN WORDS.

YOU MIGHT CHOOSE TO RESPOND TO ONE OR MORE OF THE FOLLOWING QUESTIONS:

1. WHAT IS THE PURPOSE OF MATHEMATICAL PROOF?
2. HOW IS A MATHEMATICAL PROOF DIFFERENT FROM PROOFS IN OTHER CONTEXTS?
3. WHAT MAKES A MATHEMATICAL PROOF A MATHEMATICAL PROOF?
4. WHEN WAS THE FIRST TIME YOU READ AND UNDERSTOOD A PROOF? WHEN WAS THE FIRST TIME YOU PROVED A STATEMENT ON YOUR OWN? HOW DID THAT FEEL? [IF YOU CHOOSE TO RESPOND TO THIS QUESTION, YOU SHOULD ALSO RESPOND TO ONE OF THE MORE OBJECTIVE QUESTIONS 1-3.]
5. WHAT IS THE ROLE OF MATHEMATICAL PROOF IN YOUR LIFE AS A STUDENT OF MATHEMATICS? [IF YOU CHOOSE TO RESPOND TO THIS QUESTION, YOU SHOULD ALSO RESPOND TO ONE OF THE MORE OBJECTIVE QUESTIONS 1-3.]

MAKE SURE THAT YOUR PARAGRAPHS ARE COHERENT, RELEVANT TO THE TOPIC AND WILL ADD TO OUR DISCUSSION (WHICH WE ARE JUST BEGINNING, SO IT IS IMPORTANT TO START WELL).

EACH STUDENT SHOULD CHOOSE TO USE A DIFFERENT COLOR THAN THE PERSON BEFORE HIM OR HER AND SIGN HIS OR HER NAME, SO THAT THE READER TO THE PAGE CAN EASILY AND VISUALLY DISTINGUISH BETWEEN DIFFERENT CONTRIBUTORS. LATER WE CAN TRY AND CREATE A SINGLE COHERENT DOCUMENT, BUT AT THIS STAGE, I WANT TO HEAR YOUR SEPARATE AND DISTINCT VOICES.

A.2 Second assignment.

Here is the second assignment, taken from the class wiki:

OUR SECOND ASSIGNMENT IS TO CREATE A BIBLIOGRAPHY OF SORTS FOR OUR PROJECT. PLEASE ADD IN AT LEAST TWO REFERENCES ON THE TOPIC OF MATHEMATICAL PROOF AND ITS NATURE. THESE DO NOT ALL HAVE TO BE TECHNICAL OR ACADEMIC, BUT SHOULD BE SOPHISTICATED ENOUGH TO CONTRIBUTE TO A DISCUSSION ON THE TOPIC. IN YOUR FUTURE COMMENTS ON THE TOPIC, MAKE SURE TO REFER TO AT LEAST THE TWO REFERENCES YOU HAVE CONTRIBUTED TO THE LIST AND TO AT LEAST TWO MORE ADDED BY SOMEONE OTHER THAN YOURSELF.

A.3 Third assignment.

Here is the description of the last assignment:

YOU HAVE AT THIS POINT IN THE SEMESTER READ SEVERAL PROOFS, FROM OUR BOOK AND MY LECTURE NOTES. YOU HAVE WITNESSED ME IN CLASS WRITING PROOFS, IN SOME CASES, RELATIVELY POLISHED ONES, AND IN OTHERS, KIND OF FEELING OUR WAY, KIND OF AWKWARDLY. YOU HAVE SEEN YOUR CLASSMATES PRESENT PROOFS OF VARIOUS STATEMENTS THAT YOU ALSO HAD SPENT SOME TIME THINKING ABOUT. AND YOU HAVE WRITTEN AND REWRITTEN AND REWRITTEN ALL OVER AGAIN MANY MANY PROOFS ON YOUR OWN. YOU HAVE ALSO FOUND AND READ AT LEAST TWO ARTICLES ABOUT THE NATURE OF MATHEMATICAL PROOF.

PRESUMABLY YOUR IDEAS OF WHAT MATHEMATICAL PROOF IS (OR WHAT IT SHOULD BE) HAVE BEEN REVISED, AT LEAST TO AN EXTENT. THE PURPOSE OF YOUR LAST COMMENT FOR THIS PROJECT IS TO DESCRIBE THIS REVISION. THUS YOUR TASK IS TO WRITE AT LEAST TWO FULL PARAGRAPHS ADDRESSING THE FOLLOWING QUESTION:

HOW HAS YOUR IDEA OF MATHEMATICAL PROOF EVOLVED IN THESE PAST FEW MONTHS? (REFINED? REVAMPED?)

MAKE SURE THAT YOUR PARAGRAPHS ARE COHERENT, RELEVANT TO THE TOPIC, AND WILL ADD TO OUR DISCUSSION.

EACH STUDENT SHOULD USE A DIFFERENT COLOR THAN THE PERSON BEFORE HIM OR HER AND SIGN HIS OR HER NAME, SO THAT THE READER TO THE PAGE CAN EASILY AND VISUALLY DISTINGUISH BETWEEN DIFFERENT CONTRIBUTORS. LATER IF ANYONE IS INTERESTED, WE COULD TRY AND CREATE A SINGLE COHERENT DOCUMENT, BUT AT THIS STAGE, I STILL WANT TO HEAR YOUR SEPARATE AND DISTINCT VOICES.

YOU NEED TO REFER TO AT LEAST TWO ITEMS FROM THE BIBLIOGRAPHY (ONE MAY BE YOUR OWN CONTRIBUTION, BUT AT LEAST ONE SHOULD BE CONTRIBUTED BY SOMEONE OTHER THAN YOURSELF).

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Teaching Proof-Writing by Public Grading

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Abstract

I grade copies of unmarked student homework assignments in class, using an overhead projector or document camera. This analysis of student proofs addresses logical exposition and writing style, and includes suggestions for improvement. Students get an immediate sense of expectations as well as guidance for writing proofs in future assignments.

Difficulty Level: Low; Course Level: Transitional

1 Background and Context

Colorado Mesa University is a moderately selective public four-year liberal arts university with about 9,500 students. Most of our students are traditional students living on or close to campus. Our curriculum requires mathematics majors to start writing proofs in their sophomore year in classes such as Introduction to Advanced Mathematics or Linear Algebra. The audience of our proof-based courses consists mainly of mathematics, statistics, and mathematics education majors, but some courses are taken by computer science or physics majors. The method described below can be used in all proof-based courses, beginning with Introduction to Advanced Mathematics, which is our transitions course. We teach one section of Introduction to Advanced Mathematics every semester, with a class size of 10–15 students.

The motivation for using this method came from some high school classes I took in Germany, where teachers in various subjects (especially German and English) would read students' homework and discuss or critique their essays in class. I have adopted a similar method for my college classes because I feel that it is an efficient way to teach proof-writing and uses less class time than other methods such as peer review.

2 Description and Implementation

The method of public grading works as follows: I pick the written homework of one student and grade it in class, using an overhead projector or document camera. To maintain privacy, the student's name remains confidential. I achieve this by making a photocopy of the ungraded assignment, grading the original and returning the graded paper to the student, together with the other students' papers. I then re-grade the photocopy in class, both verbally and in writing. It is a process of "thinking aloud" in which I convey to my students what is good or bad about their work, how much credit they will receive, and how their work can be improved.

The method requires me only to briefly read over the assignments before grading them in order to pick one or two problems to grade in class. It typically takes about five minutes to grade a one-page write-up of one proof in public, though this can obviously vary depending on the problem and student participation. Thus, picking two problems to grade in public will take about ten minutes of class time. The method can be used whenever graded homework is returned, which is typically once a week in my classes. Of course, I occasionally choose not to use the method, depending on whether the nature of the students' work makes it worthwhile to spend class time on it. For example, if an assignment contains mostly routine problems that are generally well done, I would skip the public grading.

If possible, I try to pick one solution that is elegant and well-written, and another one that exposes the pitfalls of reasoning and exposition. Ideally, I would find an A-level solution that is an example of good proof-writing, and a C- or D-level solution that contains an error that many other students made; of course this may not always be possible due to the varying quality of solutions.

I try to avoid making eye contact with the student whose proof is being graded. Some students choose to be quiet, while others identify themselves as the author and ask questions, trying to clarify what they meant or point out where and why they struggled. The other students are generally attentive and sometimes contribute to the grading by pointing out mistakes or making suggestions for improvement. The audience's tone has always been polite and supportive of the assignment being graded.

3 Outcomes

I have used the method in several courses, such as Introduction to Advanced Mathematics, Introduction to Topology, and Abstract Algebra. It is especially suitable for large classes where group work may be less feasible. In such settings, a student's anonymity is not jeopardized, and yet the whole class can learn from seeing one student's paper graded.

The method of public grading catches the students' attention whenever I employ it; not only do the students listen, but they also ask many questions. This provides a nice change of pace from lecturing. I believe there are several reasons for the students' elevated attention when I use public grading:

- The students typically worked on the problem recently, so it is fresh in their minds.
- Each student naturally wants to know how well his/her peers are doing, even though the authorship of a graded paper is typically unknown to them.
- The students get an immediate sense of what expectations are in terms of logical reasoning, citing theorems, writing style, level of detail, etc.; this helps them improve their own performances next time.

As a consequence of their paying close attention, I noticed that the students retain the points of discussion very well. For example, when I used the method in Introduction to Advanced Mathematics, I graded two induction proofs in public; in subsequent assignments the students knew exactly how to write such a proof. I had not used the method in the same class the year before, and in that class it took a long time for the students to fully grasp the concept and style of induction proofs.

Public grading allows me to address reasoning skills and writing style in an efficient way, by saying verbally how to fill in logical gaps, how to help the reader understand an argument, how to reduce redundancies, etc. Thus, the method uses little class time to communicate a great deal of information. One could argue that the time spent on the method pays for itself; for example, grading an induction proof in public means that I do not have to explain the method of induction over and over as the semester progresses.

I use the method to share my insight with all students, providing much needed feedback. Being able to reach all students at the same time makes the method more efficient than others such as group-work activities where the instructor interacts with only one group of students at a time. Using the method also simplifies my grading, since it allows me to verbally point out common mistakes to everyone, instead of writing the same comments on the margins of several assignments.

I established strong anecdotal evidence of the success of the method by teaching the course Introduction to Advanced Mathematics twice within one year, the first time without using the method and the second time including it. Parameters such as class size, textbook, teaching style, and so on were the same in both sections. Though I did not formally collect data comparing the classes, I noticed improvements in the proof-writing skills of the second group, as illustrated with the example of induction proofs. The second group also scored higher on the final exam due to overall better performance in the proof problems.

Possible drawbacks of the method are:

- Student intimidation if strong words are used when grading a poor solution.
- Lack of anonymity in small classes since the students might know each other's handwriting.
- Possibility of a heated discussion erupting if a student disagrees with the grading.

To put these concerns in perspective, one should keep in mind that public grading is not substantially different from having a student present a proof on the board with the instructor asking questions and critiquing the student. I use both public grading and in-class presentations in my upper-division classes, and would not want to part with either method since their benefits seem to outweigh their drawbacks. Both methods provide instant feedback and yield a detailed analysis of student communication skills (written for public grading and verbal for in-class presentations).

4 Extending the Method

The method of public grading is an efficient teaching tool that works well in proof-based mathematics classes at all levels. It is especially useful for large classes, but somewhat less beneficial for small classes of up to five students, where it may be better to address each student individually.

The method is suitable for any kind of written assignment (not just proofs) and even for non-mathematics classes. For example, in a calculus class, there may be messy and elegant homework solutions for finding a complicated indefinite integral, both of which could be graded in public. A similar argument can be made for publicly grading a piece of code in a computer science class. In fact, independent of the content we teach or the technology we use, the method of public grading will always be applicable to a variety of writing assignments. I hope that further testing and study will reveal its full potential.

Appendix

A A Sample Assignment

The scan below shows a graded proof, which is included with the student's permission. My verbal in-class comments are paraphrased as follows:

- Using the equals sign after $P(n)$ is not proper language; you could say "let $P(n)$ be the statement ' 11^n is divisible by 5' " or leave out the $P(n)$ altogether like I showed you in class.
- Replace "works" by "is true" or say "the case $n = 1$ works".
- You are *proving* the induction step, not using it.
- Why switch from n to k ? I would stick with n ; this is not a strong induction proof after all.
- The grammar and punctuation of the induction step need improvement (see markup for explanation).
- Having " $11^k - 6$ " instead of " $11^{k+1} - 6$ " is an unnecessary mistake; proofread!
- The score is 4 out of 5 for this problem.

Markus Reitenbach: Colorado Mesa University, Grand Junction, Colorado

#6 Prove by induction that $11^n - 6$ is divisible by 5 for every positive integer n .

Proof

Let $P(n) \equiv 11^n - 6$ be divisible by 5.

Base case

Let $n=1$, then we have $11^1 - 6 = 5$
so $P(1)$ works!

∴ using induction step.

Assume $P(k)$ works for some $k \in \mathbb{Z}^+$

i.e. $11^k - 6 = 5m$ for some $m \in \mathbb{Z}$

so $11^k = 5m + 6$.

We want to show that it also works for $P(k+1)$, i.e. that

~~so~~ $11^{k+1} - 6$ can be expressed as a multiple of 5 and ^Aassuming that

$11^k = 5m + 6$ we have:

$$11^{k+1} - 6 = (11 \times 11^k) - 6$$

$$= 11(5m + 6) - 6$$

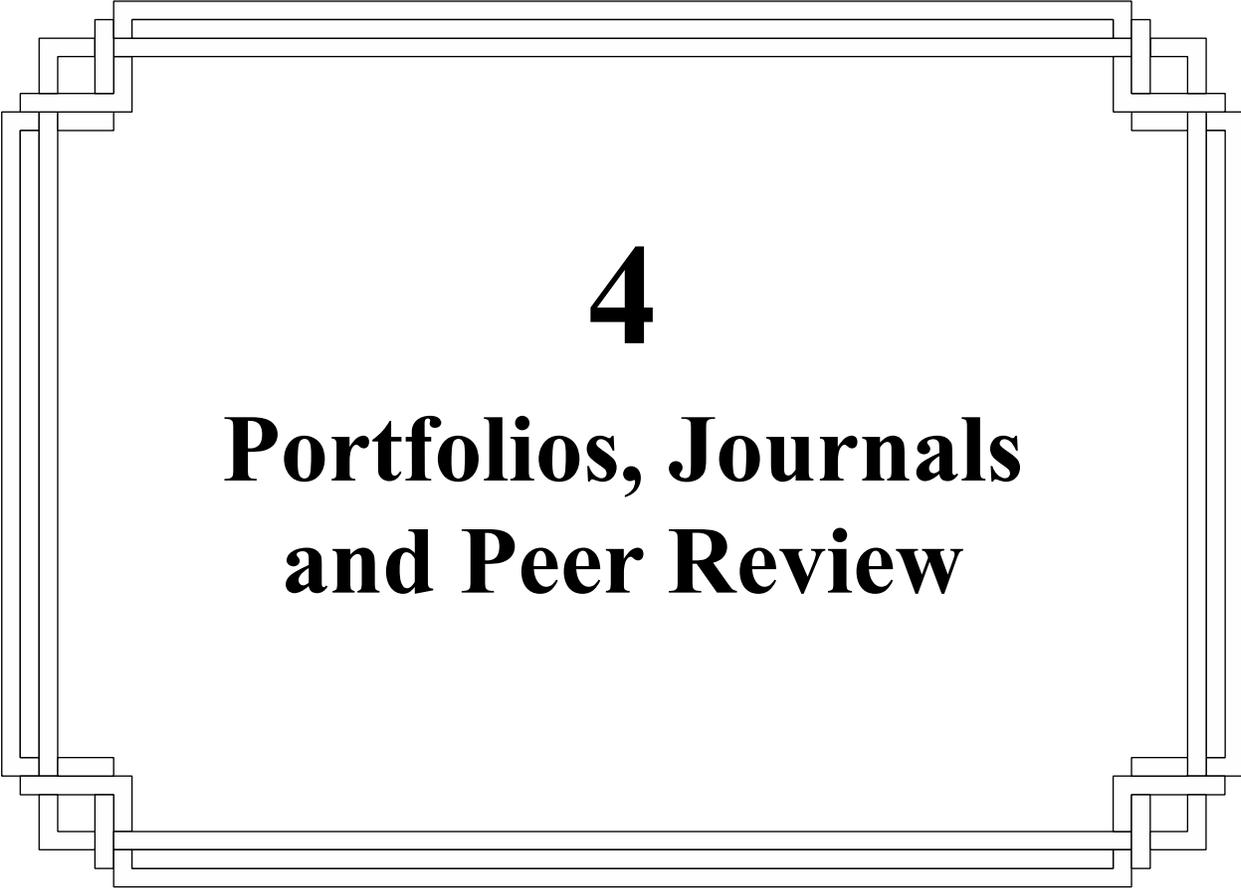
$$= 11(5m) + 66 - 6$$

$$= 5(11m) + 60$$

$$= 5(11m + 12)$$

Notice that $11m + 12$ is an integer so $11^k - 6$ is divisible by 5 and $P(k+1)$ works! \square

Figure 1: An actual proof



4

Portfolios, Journals and Peer Review

Using Proof Portfolios in a Bridge Course

Penelope Dunham

Abstract

Because writing well is an important factor in producing and communicating effective proofs, I have adopted methods from composition pedagogy to help students refine their proof-writing skills. In particular, portfolios have been especially beneficial for both students and instructor. Through portfolios, students can revise proofs, polish their writing, document their growth, and ultimately produce well-written arguments. Portfolios also provide an effective means of assessing student progress.

Difficulty Level: High; Course Level: Transitional

1 Background and Context

Muhlenberg College is a small liberal arts institution where most of the approximately 2200 undergraduates are traditional age and live on campus. All students at the college take three writing-intensive courses (or W-courses), one of which must be in the major. Our proof-writing course, Transition to Abstract Mathematics, satisfies the requirement for the mathematics major and serves as a gateway to upper-level courses such as abstract algebra, geometry, and analysis. It introduces students to abstraction and the mechanics of proof-writing early in the major and gives a view of mathematics beyond computational expertise. The course, with a prerequisite of Calculus 2, enrolls mostly sophomores and a few first-year students. It is taught in two 75-minute periods per week for 12 to 15 students every semester.¹ The text is most often Bond and Keane [2] or Smith, Eggen, and St. Andre [11].

The main goal of a W-course is to develop students' communication skills through critical and analytic writing with a process-oriented pedagogy that emphasizes frequent feedback and revision. In Transition to Abstract Mathematics, we use proof-writing as the vehicle to meet this goal. The course is designed to help students read and understand proofs, apply and see the role of definitions in proofs, craft valid statements and arguments, learn the rules of logic and the role of quantifiers, construct counterexamples, and master different proof types. At the same time, we focus on crafting elegant arguments as students develop the art of analytic writing through topics that are typical of many bridge courses: set theory, relations and functions, elementary number theory, counting arguments, mathematical induction, cardinality, and transfinite arithmetic. Throughout the course, we also emphasize the importance of revision through required rewrites of proofs and formative feedback (i.e., comments aimed at modifying students' thinking and learning). As Higham notes, "All writing benefits from revision. [The] first attempt can always be made clearer, more concise, and more forceful. Effective revision is a skill that is acquired through practice" [7, p. 94].

1.1 Rationale for Using Portfolios

The directors of our writing program offer annual training (usually a two-day workshop in May) for instructors of writing-intensive courses. In them, I learned that composition pedagogy emphasizes writing as a process. Though a well-written product is a desirable goal, the main focus for instruction and assessment in a W-course should be an ongoing process of writing, reflection, and frequent revision that leads to improved

¹W-courses are usually capped at 15 college-wide, though we have allowed as many as 20 in the course.

skills. The students' growth in the course is the main measure of success. A recommended way to assess growth for composition students is through course portfolios [8].

Portfolios have traditionally been used in the humanities, but they are becoming more common in mathematics instruction [9, 10]. After reading a few articles about portfolio use in mathematics classrooms [1, 5, 10], I realized that this assessment method would align well with the general emphases for Muhlenberg's W-courses as well as the particular goals of the proof course. Creating portfolios would let students show their progress in developing proof-writing skills while showcasing a polished final product at the end of the course.

1.2 Key Components of Portfolios

For readers unfamiliar with portfolio assessment, I shall describe some important aspects of the method. A portfolio is more than a random selection of student work gathered from a semester's output. Rather, it is "a purposeful collection of student work that tells the story of a student's efforts relative to specific instructional goals" [8, p. 252]. There are three essential aspects to consider: item selection should be based on instructional goals; students should control the content of the portfolio and provide rationales for selections; reflection is a necessary part of telling the story of an individual's progress.

Item selection. Here the phrase "purposeful collection" is key. The instructor determines the focus of the portfolio based on course goals. In a proof course, they might include understanding the role of proof in mathematics, mastery of proof types, development of communication skills, display of the writing process as well as final products, and assessment of individual student progress. Students then choose items intentionally to make the best case that they have met these goals and to provide evidence of their personal growth over time.

Student control. Instructors may provide guidelines for types of items, but allowing students to choose the items and provide reasons for their choices lets them exert control over their own evaluation [5, 8]. Moreover, creating the portfolio can become an opportunity for further growth. As Kuhs notes, "No system places so much responsibility on the student. . . . Perhaps the ability to self-assess and monitor one's own learning is the most important skill that students can acquire in school. If so, portfolios in mathematics classrooms may not only be a way to assess learning but an important outcome of instruction itself" [9, p. 335].

Reflection. A main goal of portfolio assessment is to encourage students to review their work and reflect on what they have learned and how they have matured [5]. To this end, Burks emphasizes the importance of a personal statement to accompany portfolio items, noting "A true portfolio requires reflection, writing, and self-critiquing to present a full assessment of learning and to fully realize all of [the student's] objectives" [4, p. 457]. While the brief rationale for each portfolio item allows for some reflection, requiring a final essay provides a further opportunity for self-assessment and reflection. Ending the portfolio with an essay lets students demonstrate more fully evidence of their achievement and progress.

2 Description and Implementation

The final assessment in our proof course is a portfolio worth 40% of the course grade. The portfolio consists of proofs, rationales², and a reflective essay. Students draw from weekly assignments throughout the term to select representative items that showcase their mastery of proof-writing. Early in the course, I provide a thorough description of what constitutes a good portfolio so that students can be aware of what lies ahead and start selecting candidates for the final product. (See Appendix A for my description.) Two or three weeks before the end of the term, students submit a draft portfolio. A few weeks before the draft is due, I

²The rationale is usually a sentence or two that tells the reader why the item is in the portfolio. For example, "This proof from the third assignment involves absolute value and shows how to do a proof by considering cases."

bring some former students to class to give advice about assembling a portfolio and show examples of their products. After I comment extensively on the draft, students revise and submit completed portfolios in lieu of a final exam.

Our weekly assignments usually have a few computational exercises (depending on the topic) or a “proof to grade” (see texts [2] and [11] for sample proofs that students can analyze), in addition to four to eight proofs. Revision is a required part of a W-course and, for the proof course, it begins early in the semester. For weekly assignments, students must revise all proofs with scores of 7 or less (out of 10); they may also submit revisions of any others as often as they like to improve the writing or the mathematics.³ At the beginning of the course, I grade proofs primarily on mathematical correctness and validity. (See Appendix B for my rubric.) By the fourth week, I start commenting on the quality of the writing, focusing on effectiveness, clarity, and style. The final score on a proof can be adjusted up or down by a few points, depending on the writing quality.

I expect revisions to address both the mathematics and the writing. To facilitate rewrites, students have my comments on graded assignments as well as feedback from fellow students during weekly classroom presentations.⁴ They also exchange papers for a more formal peer review on two assignments. In addition, I provide resources [6, 7] with advice on the specialized requirements for technical writing, and I encourage students to continue polishing their writing for the final portfolio submission. (Editors’ note: For details about another approach to student revision of proofs, see Strickland and Rand [12] in this volume.)

2.1 Assembling the Portfolio

The number of items in a portfolio should be limited because purposeful selection is a critical part of the process. Thus I require between 12 and 15 proofs. A rationale for the item’s inclusion, the original attempt, and all revisions also accompany each final polished version. Because mathematical understanding, writing quality, and variety are key objectives for the portfolio, selections should display the breadth and quality of each student’s work during the semester. I expect items that are representative of the entire course, including a wide selection of major proof styles, a variety of lengths from simple to complex arguments, proofs from a cross-section of the term’s assignments (but not necessarily from every assignment), and items from many of the mathematical topics we covered. The items should document growth in understanding of proof and in writing skills. Choices may also include proofs that were particularly interesting or difficult for the student. Accomplishing all that with just 12 to 15 items is a challenge and a learning experience for everyone.

Students usually assemble their selections in a binder with a table of contents listing the original assignment number, the proof style, and the mathematical topic for each entry. I recommend that proofs be well-written in condensed prose style using complete sentences, as one would find them in a textbook. The goal is concise but accurate exposition that avoids unnecessary steps. Students may submit photocopies of graded hand-written assignments for the original efforts, but they must type the final version of each proof using correct mathematical notation and conventions.⁵ The portfolio concludes with the reflective essay (two or three pages, double-spaced), a self-assessment in which students address what they have learned about proof-writing and point to specifics in their portfolios that document their progress; that is, they show how the portfolio provides evidence for their growth as mathematicians and writers.

2.2 Grading the Portfolio

Portfolio assessment encourages students to reflect on initial attempts and revisit proofs to improve their understanding. Therefore, compared to one-time grading of assignments or tests, portfolios put more em-

³I average the scores for the original proof and any rewrites to get an adjusted grade for each proof.

⁴Each student presents proofs on the board for a whole-class critique about five times during the semester.

⁵I offer a few training sessions on software for typing and formatting mathematics. Students tend to use the equation editor in Word, but a few take the time to learn \LaTeX , using references like Higham [7].

phasis on the value of revision and writing as a process. For instructors, however, grading portfolios can be more difficult, more time-consuming, and more subjective than other forms of assessment. There are ways to lessen the workload. For example, one can find rubrics to help with the grading task (see Burks [4], Jardine [8], and Schoenfeld [10]). Clearly communicating rubrics before students submit their work also helps; when students know what we expect, they are more likely to attend to those points. I find that recording forms can make grading a bit easier, as well. Thus I prepare checklists for the portfolio, with spaces to record assignment number, topic, proof type, plus comments on the mathematics, logic, and writing. The forms have places to tally items in each category so that I can quickly check variety. I also save time by distributing blank checklists before the portfolio draft is due. Then students fill in the columns for number, topic, and proof type and include the form with their portfolio.

When grading the portfolio, I look at four aspects. First, I consider whether each proof is mathematically correct and complete. Using my rubric, I check for accurate statements, appropriate proof method, logical development of the argument, organization, and correct use of notation and symbols.⁶ Next I judge the overall quality of writing in terms of clarity, tone, voice (relative to the audience), organization and unity, brevity, as well as mechanics (sentence and paragraph structure, transitions, grammar, diction).⁷ Then I take a more holistic approach, assessing the overall strength of selections for variety (proof style, length, complexity), topic coverage, item rationale, and mathematical understanding. Finally, I consider how well the student's self-evaluative essay shows reflection about proofs and the writing process and provides concrete evidence of growth in the course. A checklist helps me record comments for each of these four areas and arrive at a final letter grade for the portfolio.

3 Outcomes

Portfolio assessment is an effective tool in our proof course and has been successful on many levels, providing concrete documentation of improved proof-writing, satisfying a W-course goal of emphasizing writing as a process, and giving students a permanent record of accomplishments and a chance to reflect on what they have learned in the class. My personal observations, comments from departmental colleagues, and statements of the students in their final reflective essays⁸ provide evidence for the outcomes of this assessment method.

The most important outcome is improved skills in proof-writing. The nature of portfolio entries makes progress in this area immediately evident to the reader. In the revisions of each proof, one can easily see the development of an incomplete or marginal early attempt into a final polished proof. Similarly, improved skills in writing and communication are obvious when comparing the quality of initial efforts to the final products. Colleagues who teach later courses often comment on the difference the proof course made in their students' ability to communicate ideas and handle abstraction. I attribute much of that progress to the focus on constant revision that portfolios require. Student awareness of individual growth and achievement is also obvious, as this comment demonstrates: "By contrasting the first proof in my portfolio with the last few, the increased complexity is quite obvious . . . My portfolio demonstrates my progression from elementary proofs concerning even/odd numbers and inequalities to more complex proofs dealing with cardinality and equivalence relations."

Another outcome of portfolio assessment is the support it provides for the goals of a W-course. Using portfolios puts the focus on writing as a process as students hone their skills through revisiting and refining prior work. Evidence for this can be seen in this student observation: "In early assignments, my writing was poor along with the overall format of the proofs . . . When I was typing my proofs for the portfolio, I noticed small writing errors and changed them . . . When I can look back at old proofs and see writing issues right

⁶These are not usually issues because most items have been rewritten a few times.

⁷See Appendix C for characteristics of good writing.

⁸All student comments that follow are quotations from the self-evaluative essays.

away, I realize how much better a writer I am now.”

One outcome of asking students to reflect on their work is increased awareness of the mechanics of good writing. Their essays frequently comment on the importance of clear exposition, attention to detail, and attention to the audience. For example, “I have learned to first consider who will be reading a proof before I begin to write it. A particular statement may seem obvious to me, but to a reader with little mathematical background, a supporting argument may be required to make the statement clear.” This awareness often extends to other courses, as noted by this student: “[T]he most improved part of my writing is the use of my transitions within a paragraph . . . [I]t is more effective to use transitional introductions . . . I then noticed while editing a paper I had written for my education class that I was using many of these phrases, and it made my writing easier to read and more cohesive.”

Reflecting on portfolio entries also has led students to insights into particular aspects of proof-writing, such as “One of the most important attributes of a good proof is its clarity and succinctness. . . . A proof that is too confusing or verbose has failed to accomplish its task, even if it is based on completely sound mathematics.” Other observations included, “In early assignments, I would state the claim is true without fully proving it,” and “Observe that in the first few assignments I used improper notations. . . . Also notice in the second assignment I had assumed what I was to prove.”

Finally, an unexpected outcome was the pride that students got from constructing the portfolio. They often tell me that they have kept their portfolios for reference in subsequent courses, and it has always been easy to find former students pleased to share their portfolios and advice with a current class. Many students reported a sense of accomplishment that comes from making the portfolio, echoing the comment “Overall, I particularly enjoyed this class and working on the portfolio. I am especially proud of all the proof styles that I am now able to use in future classes.”

4 Extending the Method

What are the implementation concerns for those who want to use portfolios in their courses? Although I have found that portfolio assessment is an extremely effective tool for teaching proof in the bridge course, there are issues to consider for future adoptees. For example, assembly and grading are time-consuming tasks for student and instructor of the term. In order to lessen the crunch at the end, I have learned to give students more lead time for preparation and better instructions early in the process. Creating rubrics and grading forms speeds the evaluation process for the instructor as well.

Portfolios could be adapted for other courses that use proofs, e.g., analysis, geometry, or abstract algebra. Class size, however, is a concern. Our bridge course has limited enrollment because it is a writing-intensive class, but extending portfolio assessment to larger classes could be difficult because of the workload for instructors. Limiting revisions to a single time is one way to save grading time during the term. One could also significantly reduce the burden of final grading by asking for fewer items. A smaller portfolio of five or six items focused on a single topic or theme would not have the variety found in a comprehensive product spanning the course, but it could serve as a vehicle for students to show improvement. By reducing the size, however, I would expect a corresponding reduction in the proportion of the course grade determined by the portfolio.

I encourage other instructors to try portfolio assessment, even in a limited format. (Editors’ note: For another such approach in this volume, see Bryant and Schaefer [3].) Portfolios have been a valuable component of our proof course for over a decade, helping students improve their mathematical understanding and develop their ability to communicate. The benefits are summed up by a student reflecting on her achievements, who said “I have never loved math more than I did when I was able to prove a property using one of the many techniques I had learned. I found that the feeling of accomplishment not only came from mastering the techniques, but also from the improvement I saw in both my writing and mathematical skills.”

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- [12] Sharon Strickland and Betseygail Rand, "Learning Proofs via Composition Instruction Techniques," in this volume.

Appendix

A Portfolio Assignment Description

MATH 240, Transition to Abstract Mathematics

Fall 2012

Course Portfolio

Draft due: Tues., Nov. 27

Final due Monday, Dec. 10, 3:00

Part of your grade (40%) will be based on a portfolio of your best work in the course. The portfolio provides evidence for what you have learned in MATH 240.

Your portfolio should contain at least twelve—but no more than fifteen—proofs from the graded assignments. The proofs should be well-written in condensed prose style using complete sentences, as you would find them in a textbook. You should aim for a concise but accurate exposition. Avoid unnecessary steps. (See Gilman’s paper for suggestions on writing proofs well.) However, your portfolio must include longer proofs as well as a few short ones. The best portfolios exhibit variety in many ways: course chronology, mathematical topics, proof styles, writing styles.

Submit only proofs that you wrote as an individual. Avoid the “proofs-to-grade” critiques or problems that were part of a group assignment and exercises from our texts that were not proofs. The items you choose should be representative of the entire course, and they should reflect what you have learned about writing mathematical proofs. You should include:

- a wide selection of the major proof techniques
- a variety of lengths from simple to complex arguments
- proofs from a cross-section of the term’s assignments (but not necessarily from every assignment)
- proofs that you have rewritten (and improved!)
- proofs that you found particularly interesting or challenging.

With each item, include a beginning paragraph describing the item and explaining why you picked it as representative along with a copy of the original graded proof.

Format:

- Include a table of contents identifying the type of proof and the assignment number.
- Type a justification/description for each item, then provide the final polished version of each proof (typed using Equation Editor). Behind each final version, include photocopies of all prior attempts for that proof. (Originals usually will be hand written, of course.)
- Arrange the selected items in a portfolio (presentation folder, binder, etc.) in the following order for each entry: item description, rewritten proof, photocopy of the original graded proof, copies of all other drafts of the proofs that you submitted for re-grading.
- Conclude with the self-evaluative essay (described below).

Grading: The portfolio will be graded on accuracy of the proofs, quality of writing, and overall strength of the selected submissions.

More comments on the portfolio: Read Chapter 13 in Solow (on reserve). It reviews many of the proof types that we have studied and can give you a good basis for making selections for the portfolio.

The self-evaluative essay:

Write a brief (2–3 pages) reflective assessment of your growth in the course. You should address what you have learned about proof-writing and point to aspects of your portfolio that support your claims. You may, if you wish, discuss changes in your concept of what constitutes a proof, the role of proof in mathematics, or aspects of the course that had an effect on your writing.

The essay should provide evidence of your progress over the semester by telling me what you learned and pointing to specific examples in the portfolio that support your claim. The essay is your chance to convince me that you have matured as a writer of mathematics.

B Rubric for Grading Proofs

Score	Criteria
10	<p>The theorem and all assumptions are clearly stated.</p> <p>The proof method is appropriate.</p> <p>The logic and mathematics are completely correct.</p> <p>The proof is complete with all significant steps justified and all cases examined.</p> <p>The proof is well-organized with clear and efficient explanations.</p> <p>All quantifiers, notation, symbol use, and formatting are appropriate.</p>
9	<p>The theorem and all assumptions are stated.</p> <p>The proof method is appropriate.</p> <p>The logic and mathematics is mostly correct with only a minor error.</p> <p>The proof is mostly completely correct with most steps justified and all cases examined.</p> <p>The proof is well-organized but explanations could be clearer or more efficient.</p> <p>Most quantifiers, notation, symbol use, and formatting are appropriate.</p>
8	<p>The theorem and all assumptions are stated.</p> <p>The proof method is appropriate.</p> <p>There is a moderate error in the logic or mathematics.</p> <p>The proof is missing a case, an important step, or key justification.</p> <p>The proof is not well organized or explanations are unclear on a key point.</p> <p>Some quantifiers, notation, symbol use, or formatting is inappropriate.</p>
7	<p>Statement of the theorem and assumptions are incomplete.</p> <p>The proof method is not appropriate.</p> <p>There is a serious error in the logic or mathematics.</p> <p>The proof is missing one or more steps, a case, or several justifications.</p> <p>The proof is not well organized with missing or unclear explanations.</p> <p>There are several errors in quantifiers, notation, symbol use, or formatting.</p>
6	<p>Statement of the theorem or assumptions is incorrect or missing.</p> <p>The proof method is not appropriate.</p> <p>The logic or mathematics has several serious errors.</p> <p>The proof is missing several cases, key steps, or justifications.</p>

The proof is poorly organized with confusing or incorrect explanations.
There are many errors in quantifiers, notation, symbol use, or formatting.

Below 6

Statement of the theorem and assumptions is incorrect or missing.
The proof method is wrong.
The logic or mathematics has major errors.
The proof fails to reach the correct conclusion.
The proof is very poorly organized with missing or incorrect explanations.
There are frequent errors in terminology, notation, or symbols use.

C Characteristics of Good Writing

When I assess the quality of your writing, I look for:

- a narrative that flows clearly from assumptions to conclusion
- well-constructed sentences with varied form and effective word choice
- well-organized paragraphs that provide logical development and support for key ideas
- smooth transitions between and within paragraphs
- clear and effective descriptions and explanations
- appropriate tone and voice for the intended audience
- minimal errors in spelling, grammar, and diction

Penelope Dunham: retired, Muhlenberg College, Allentown, Pennsylvania

Becoming Successful Proof-Writers Through Peer Review, Journals, and Portfolios

Sarah N. Bryant and Jennifer B. Schaefer

Abstract

We discuss our experience at Dickinson College teaching an introduction-to-proofs course for our mathematics and computer science majors. We designed a course that satisfies the college's writing-intensive graduation requirement and teaches proof-writing in a way that utilizes writing pedagogies. We changed our usual course structure by adding group work, daily peer review, a weekly journal, and a final portfolio. The components were designed to place increased emphasis on revision and to improve student performance in this and future math courses.

Difficulty Level: High; Course Level: Transitional

1 Background and Context

We will share our experience teaching an introduction-to-proofs course at Dickinson College, a highly selective liberal arts college in central Pennsylvania with approximately 2,400 students. The mathematics and computer science disciplines comprise one joint department, and each year we graduate approximately 25 majors. Many of these students are double majors both in and outside of the department and follow a variety of career paths.

The introduction-to-proofs course, which our department calls Discrete Mathematics, provides an introduction to fundamental mathematical concepts used in mathematics and computer science, with an emphasis on the principles of logic and methods of proof. The prerequisite for Discrete Mathematics is Single-Variable Calculus or Introduction to Computer Science. Because the course is a prerequisite for all upper-level courses in the mathematics major, and is a requirement for the computer science major, the majority of students are freshmen or sophomores. The course is capped at 16. This class is often scheduled in blocks of 75 minutes, twice a week.

In addition to satisfying a requirement for our majors, Discrete Mathematics satisfies the college's writing-intensive graduation requirement. A designated writing-intensive course at our institution must fulfill three criteria: require a minimum of 15 pages of polished writing, teach the genres and conventions of the discipline, and build the writing process into the assignments — including planning, drafting, revising, and editing. Over the last few years, we have worked with Dickinson College's writing program director to develop new course components that address content and writing goals while complementing the usual assessment tools (homework, quizzes, and exams). We have been able to implement the new structure while still covering all the required mathematical content.

When teaching Real Analysis and Abstract Algebra (junior- and senior-level courses), we were surprised by the lack of student self-efficacy with regard to proof-writing. We had expected students to struggle with higher-level mathematical concepts but were surprised by how often we were pulled back into basic construction-of-proofs discussions. Though every student had taken Discrete Mathematics, the learning we expected from it had not automatically transferred to higher-level courses. Our goal in redesigning the Discrete Mathematics course was to build a stronger proof-writing foundation for our students to avoid the

setbacks we had seen in the higher-level courses. With the goal of supporting transfer of learning, we now integrate group work, peer review, journaling, and a final portfolio project with our other assessment tools, including homework, quizzes, and exams.

2 Description and Implementation

On the first day of the semester, we spend a considerable amount of time explaining the course structure and discussing students' notions of what proof-writing means. We emphasize the authentic nature of working in groups, writing for peers, reviewing peers' papers, reflecting on comments, and revising as they relate to writing in our discipline. We then read the following excerpt from the course syllabus that describes the course's writing components and explains why we assign them.

Throughout the semester you will be getting feedback on your work. On your homework you will not just receive a grade but comments and suggestions, too. In peer-review groups you will be giving feedback to each other on your proof-writing. In your journal you will be reflecting on this feedback. The purpose of the portfolio is to draw on all of these components to provide a capstone piece, which serves as evidence of your thoughtful reflections and showcases what you have learned in this course.

We give an overview of the course components and how they determine our course structure. For sample group-work problems, selected journal prompts, and the final portfolio assignment, see the Appendix.

2.1 Group Work

Group-work problems are ungraded problems related to concepts from the day's lecture. These are done in class and we encourage working in pairs or groups. In this course, we use group-work problems to prompt discussions about new mathematical ideas. We do not expect polished writing. Instead, students are meant to undertake the first few stages of mathematical writing: understanding definitions and related theorems, brainstorming about the mathematical analysis, and deciding which proof techniques to use.

2.2 Portfolio Problems and Peer Review

We assign peer-review groups at the beginning of the semester. When we post daily homework problems, we list an additional collection of problems from the same content; we call these "portfolio problems." Students select and complete one to three of them daily for peer review. The group members do not always choose the same problems. There is a variety of ways to structure the peer-review process. If they are not given guidance, we have found that our students' initial attempts entail exchanging papers and writing comments, with little interaction. We have had more successful peer review by leading class discussions on peer-review practices early in the semester, using sample work and providing handouts with tips gathered from multiple sources, including the rich guide on mathematical writing by Knuth, Larrabee, and Roberts [5]. After the activities, many of the groups use the effective and engaging practice of choosing a different group member's paper each day to read and discuss. Students then revise their problems and by the end of the semester select their best portfolio problems to submit in a final portfolio. Rather than merely telling our students that proof-writers should be writing for a peer audience, we now allow the peer-review process to organically elucidate this point. As one student stated, "It was good to be able to see the thought process of other classmates who were just learning as I was and contrast it with my own work and my own thought processes."

2.3 The Final Portfolio Project

We assess student proof-writing using portfolios because of the benefits described by Elbow and Sorcinelli [2, p. 207]:

If we grade on the basis of a portfolio (using either a grid or a single grade) we are drawing on a more trustworthy picture of the student's ability or learning (thus "validity" is enhanced). And portfolios have other advantages. The grade seems fairer because students can choose a *selection* of their best writing and they are not so penalized for having started out the semester unskilled. Most of all, portfolios greatly enhance student learning because they function as an occasion for retrospective metathinking.

Throughout the semester, students organize drafts of their portfolios. Each portfolio ultimately consists of ten problems written in \LaTeX . Six of the problems are proofs and must include one of each of the following types: direct proof, disproof by counterexample, proof by contradiction, proof by contraposition, proof by cases, and proof by mathematical induction. The remaining four are computational problems. Because this assignment is often the students' first exposure to \LaTeX , we provide supporting documents for \LaTeX , including templates and handouts of frequently-used syntax. We also hold optional \LaTeX workshops outside of class. In them, students learn how to open, edit, and compile sample \LaTeX documents. Because our course is meant to prepare students for future mathematical writing, it is important to us that they also learn \LaTeX , as it is the predominant typesetting tool used by mathematicians.

Mid-semester, each student submits his or her first draft of the portfolio, containing five problems. We then provide written feedback on the mathematical content, writing quality, and \LaTeX formatting of this draft. Before the second draft, each student turns in a list of the final problems to be included, with an understanding there may be some change. We review the list to ensure that the problems are in fact from the collection of portfolio problems and satisfy the types required. The second portfolio draft is due the second-to-last week of class. The students provide copies of their portfolio drafts for each member in their group, then read and comment on each member's portfolio. We allow one day in class for the review and exchange of comments. The portfolio also includes a typed one-to-two-page cover letter, which gives an overview of the included problems. Students submit their final portfolios the last day of the semester, and we grade them using a pre-distributed rubric that balances mathematical content and writing proficiency. See Gold, Keith, and Marion [4] for further information on assessment of mathematical assignments and rubric design. (Editors' note: See also Dunham [1], in this volume, for another approach to using proof portfolios.)

2.4 Journals

Students write at least one entry per week, summarizing a learning moment from that week. In addition to this regular journaling, they respond to approximately five to seven particular prompts throughout the semester. We collect the journals four times throughout the semester and check them for completion. The main goal of this type of journaling is to aid in the transfer of knowledge. The journals are a vehicle for students to analyze the patterns in instructor and peer feedback and then through their responsive writing, engage in the metacognition required to carry over their learning to future work. This is what Perkins and Salomon call "high road" transfer of learning [6]. To help encourage meaningful writing, we provide handouts on how to journal and supply leading questions.

2.5 Overall Organization and Structure

Now that we have described the course's writing components, let us elaborate on how they define the course organization and structure. On a typical day, students submit homework assignments that are collected and graded and bring one to three portfolio problems on similar content. Peer review of the problems begins

at the start of every class and lasts approximately 8-10 minutes. The day's lecture and lesson last another 45-60 minutes. In the remaining class time students work in self-organized groups on the group-work problems. Partial or complete solutions to the group-work problems are provided after class. Outside of lecture, students read the textbook before and after class, review comments on returned homework, complete homework and portfolio problems, examine group-work solutions, and study for quizzes and exams. At least once a week students reflect on feedback on returned homework and peer comments on portfolio problems and write a journal entry in response.

Our daily obligations include monitoring peer-review groups, presenting the new mathematical concepts for the day, modeling acceptable proof-writing techniques, and assisting group-work teams. After class, we assign daily homework sets and portfolio problems, post solutions for the day's group-work set, and develop new group-work handouts for the following period. Throughout the semester, we provide feedback on homework and journal entries and adapt and develop guides containing typical notational conventions and proof-writing tips. In our experience, by spacing assignments, quizzes, and exams wisely and selecting group work, homework, and portfolio problems carefully, students and faculty do not feel overwhelmed. This is due in part to the fact that we assign approximately the same number of proofs and computational problems that we did before implementing the new writing components. By breaking the problems into group work, homework, and portfolio problems we form categories that build on each other. This strengthens student learning by providing more opportunities for feedback and reflection, which support the transfer of learning from one type of problem to another.

Though the percentages have fluctuated a bit over the years, we have most recently used the following grade weight scheme for calculating final grades: each in-class exam 20%, final exam 25%, homework and class participation 10%, journal 5%, and portfolio 20%.

3 Outcomes

We have now used the model in Discrete Mathematics for four semesters, two by the first author and two by the second author. We highlight the positive outcomes for students and faculty and address issues to consider when implementing this method.

3.1 Positive Outcomes for Students

As had been our initial goal, we now feel students are developing a sense of ownership of the writing process. The following quotation from a portfolio cover letter gives evidence of this: "I feel as though the portfolio really tied the whole class together and helped achieve a genuine feeling of mathematical writing. As I typed up the problems, I sat back and realized how much my proof-writing skills had improved." As further confirmation, 16 of the 19 students in the second author's recent Abstract Algebra class had been freshman or sophomores in either the first author's or second author's Discrete Mathematics course. The second author found they were less likely to sidetrack the class with questions about how to construct proofs than the students in her previous Abstract Algebra class. The students who took the restructured Discrete Mathematics course were more confident about the stages of proof construction and revision, and the writing they turned in was of higher quality.

Another positive and useful outcome is that students can refer to their portfolios as proof-writing guides for future courses. Two students addressed this in their portfolio cover letters:

The proof that I've chosen to include as an example of contradiction is very memorable to me. The technique used in proving it is a sort of outside-the-box thinking I hope to be able to reproduce later.

I didn't choose the hardest or the easiest proofs that I had completed. Instead, within each category of proofs I tried to pick examples that best demonstrated each particular method and covered a variety of definitions This portfolio will be a great resource in the future when I need to review a proof method or definition.

An unexpected but favorable outcome is that our students have developed a sense of community in and outside of the classroom. Their conversations start long before we enter the room and it is clear from the following student comment that they see each other as true mathematical collaborators: "Being exposed to my peers' writing styles allowed me to regularly reevaluate my own, which has, I think, made me a better proof-writer. By letting the peers look over my portfolio problems, I received excellent feedback of what they think. So instead of just communicating with the professor, this activity increases the communication between my classmates and me."

This sense of community has carried over to subsequent courses. In the second author's Abstract Algebra course, she observed that the cohort of students had developed camaraderie and worked together on their mathematics homework. This is a great outcome for our department, as we strive to foster a positive and cooperative atmosphere for our majors.

In redesigning the writing-intensive course we have come to a deeper appreciation for the tools found in composition courses, such as journaling, peer review, and revising a portfolio. Though implementation of them requires careful planning, we are pleased with the benefits they provide. We want our students to have a successful introduction to writing in our discipline and to build a strong foundation for their future courses. Based on the positive outcomes described above and the following quotation by one of our students, we believe this has been the case:

I want to thank you for the way you structured the course. Not only did I learn the material but most importantly, I have learned the beginning of how to think like a mathematician.

3.2 Positive Outcomes for Faculty

Changing a course's structure is not always easy, and it is not always worthwhile. The Discrete Mathematics course, as we now implement it, works for us because rather than creating more work for our students and ourselves, its structure supports writing and content holistically. In undertaking the redesign of the course structure, we have found new avenues of thought and research about the interplay between writing and learning. We appreciate the insight, energy, and inspiration gained from working together on this course.

Our approach to responding to student work has changed because of this class. Because portfolio problems go through revisions, we have to give comments in a fashion different from our mathematical training. Rather than homing in on a proof's mistakes and rushing to correct them, we first address each proof's structure, tone, and organization. Throughout the revision process we expect students to truly re-evaluate their work rather than merely proofreading it. The efforts guide them toward clear and correct writing while emphasizing their role in the process.

One of the most rewarding aspects of planning the course has been working with the campus Writing Center. Through our work with the center we came to a greater understanding of the interplay between writing pedagogies and mathematical thought. In response to our collaboration, the Writing Center now has mathematics students on its staff to aid students with their quantitative writing assignments.

3.3 Issues to Consider

Based on student work, student feedback, and the outcomes described, it seems our course structure provides a successful invitation to mathematics and mathematical writing for our students. However, the course requires a lot of planning to accomplish this. While the course is running, organization is essential to keep

track of all of the components. Without group work, homework, peer review, portfolio problems, journals, quizzes, and exams all moving in rhythm, the workload for the students and the instructor could become unmanageable very quickly.

Student buy-in is another area of consideration. Many of our students are initially surprised to hear that journals, peer review, and a final portfolio are going to be a part of their math course. We spend a great deal of time and effort throughout the semester presenting the parallels between writing in mathematics and writing in other areas and explaining how the course components support the writing process in our discipline. We find that providing rationale behind each component encourages student investment.

Issues of how often to collect and view the journals and how to grade them continue to come up in our discussions. We recommend being careful with this assignment so that reflection on feedback can take place but the work does not seem like busy work. Because notebooks allow for in-class free writing and are easy to implement, we have had our students keep their journal entries in a notebook or a folder. However, if one has experience with online course management programs it would also work well to have students submit their journals electronically. This would improve the logistics of collecting them and quickly reviewing the entries.

4 Extending the Method

Many of the components of this method can be easily translated to larger sections of a similar course since they do not rely directly on instructor contact. The peer-review groups could continue to work together inside of class or meet outside of class. If one is at an institution with teaching assistants, advising peer-review groups could be part of their job. The students could reflect on comments they are receiving from the instructor and from peer-review meetings in their journal and later revise a selection of their best work for a final portfolio. The journals could be managed electronically, checked only for completion, or checked by student graders. If class size is a limiting factor we suggest decreasing the number of problems to choose from in order to reduce grading time of the portfolio.

One could also bring the same approach to other classes in the curriculum, such as Abstract Algebra and Real Analysis. The portfolios could then contain more substantial problems with lemmas and corollaries that bring them closer to article-level writing. For a senior capstone course with a final paper, one could certainly use regular peer review, journaling, drafting, and revision before collecting a final paper.

References

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- [2] Peter Elbow and Mary Deane Sorcinelli, "Using High-Stakes and Low-Stakes Writing to Enhance Learning," in *McKeachie's Teaching Tips: Strategies, Research, and Theory for College and University Teachers*, 12th ed., W. J. McKeachie and M. Svinicki, eds., Houghton Mifflin, Boston, 2006.
- [3] Susanna S. Epp, *Discrete Mathematics: Introduction to Mathematical Reasoning*, Brooks/Cole Cengage, Boston, 2011.
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- [5] Donald E. Knuth, Tracy L. Larrabee, and Paul M. Roberts, *Mathematical Writing*, Mathematical Association of America, Washington, D.C., 1989.
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Appendix

A Sample Group-work Problems

Discrete Mathematics

Group Work from Sections 4.6 and 4.7

Adapted from Susanna S. Epp's, *Discrete Mathematics: Introduction to Mathematical Reasoning* [3]

1. The following proof that every integer is rational is incorrect. Find the mistake.

Proof by Contradiction: Suppose not. Suppose that every integer is irrational. Then the integer 1 is irrational. But $1 = \frac{1}{1}$, which is rational. This is a contradiction. Hence the supposition is false and therefore every integer is rational.

2. Prove the following statement in two ways: by contraposition and by contradiction: For all integers a , b , and c , if $a \nmid bc$ then $a \nmid b$. (Recall that \nmid means “does not divide”.)
3. Prove that $6 - 7\sqrt{2}$ is irrational. (Hint: You may assume that $\sqrt{2}$ is irrational.)

(Group-work problems are prepared on handouts with space provided for a student's work. We have removed this space for the article.)

B Handout of Weekly Journal Ideas

Math Journals: Weekly

I will be giving you several prompts throughout the semester, but you are also required to write one journal entry a week on your own. The entries allow you to enhance your mathematical thinking and writing skills. They also allow you to reflect on your learning process. I recommend writing your entry after you have worked on a homework set or reviewed the week's materials. This will provide you with material to write about. The writing should take about 10-15 minutes. You are encouraged to focus on your strengths and weaknesses and your own learning process. Be as specific as possible, focusing on one or two concepts or problems and detailing how you worked through them.

Math Journal Prompts To Get You Started:

I knew I was right when

The thing you have to remember with this kind of problem is

Tips I would give a friend to solve this problem are

I wish I knew more about ...

How many times did you try to solve the problem? How did you finally solve it?

Could you have found the answer by doing something different? What?

What method did you use to solve this problem and why?

Where else could you use this type of problem-solving?

What other strategies could you use to solve the problem?

Write four steps for somebody else who will be solving the problem.

What would you like to do better next time?

Were you frustrated with the problem? Why or why not?

What decisions had to be made when solving the problem?

What do you like about this type of math? What don't you like about this type of math?

C Sample Journal Prompts

Journal Prompt 1 — Your Auto-math-ographical tale

Purpose: A light-hearted introduction to writing in your journal, this is meant to be an exercise in reflecting on where you've been and where you're going (mathematically, that is).

Guiding Questions:

1. What is your history with mathematics? What experiences, if any, in your education encouraged you to continue studying mathematics? Do any classes or teachers stand out (in a good or bad way, but please do not include real names!)?
2. Why are you in Discrete Mathematics? What are your goals in the class? How will you measure your success? What can I do to help you achieve your goals? What can you do to help you achieve your goals?

Journal Prompt 2 — Discrete Math: This isn't calculus anymore

Purpose: Discrete Mathematics is a gateway course to higher-level mathematics and will provide you with ideas and proof-writing techniques necessary to be successful in the world of mathematics. This entry is to help you think about the differences and similarities between Discrete Mathematics and your previous math courses and to reflect on the way you approach it.

Guiding Questions:

1. Before the semester started, what did you expect Discrete Mathematics would be like? Is it different than what you expected? How has it lived up to your expectations?
2. What skills and knowledge have you been able to use from your previous math courses? How have you had to approach the course differently?
3. Have you encountered any struggles in the course? If so, what are they? What are you doing to overcome them? What can I do to help you? What have you found to be your strengths so far in Discrete?
4. Describe a typical week in the course, especially how you negotiate the cycle of preparing for class, engaging in class, and completing homework questions.

Journal Prompt 3 — Peer-Review Groups So Far

Purpose: You will work in your peer-review group throughout the semester. For it to be successful, you must be giving and receiving constructive feedback. This entry is to help you think about the feedback you have received and how you and your group can work together so that you are benefitting from your interaction.

Guiding Questions:

1. How often have you been bringing portfolio questions to class to share with your peer-review group? How many do you usually bring? Have your peers been bringing portfolio questions to share with you?
2. What method has your group formed for distributing papers among yourselves and sharing feedback? Do you write on each other's papers or give feedback in the form of conversation?
3. What is one thing you have learned or changed because of feedback?

4. Now, as you think about the way you have been working with your group, what would you change so that it can be more helpful? After our discussion of types of concerns in mathematical writing, can you give an example of a higher-order concern and an example of a lower-order concern?

In your groups you will be giving your assessment of another student's work. While this may be uncomfortable at first, remember it is for everyone's benefit and will provide help only if comments and suggestions are offered.

Journal Prompt 4 — Patterns

Purpose: Now that we've been writing proofs in earnest for a few weeks, I'd like you to reflect on the feedback you've gotten from your peers and me. It is believed that transfer of learning is not an automatic process but is aided by self-reflection.

Guiding Questions: After gathering all of your returned homework, look for any patterns in the comments.

1. Are the comments aimed at higher-order concerns (e.g, fundamental flaws in the flow or structure, or mistakes in the mathematics) or lower-order concerns (e.g, format or mathematical notation convention)?
2. Reading a proof from Section 4.1 or 4.2, then comparing it with a later proof, what changes do you notice in your writing?
3. Elaborate on your writing process; then explain at what stage in the process you can make improvements based on the patterns.
4. In many ways, every homework set you submit is a paper, where the topic is set by its mathematical content. Of your work so far, which is your favorite paper and why?

Journal Prompt 5 — Final Reflections

Purpose: You are now finishing a course on mathematical writing. Though we have also learned a lot of new math along the way (the discrete mathematics topics of divisibility, parity, sets, functions, etc.), we have primarily focused on proof-writing. In this last journal entry, you are to provide a synopsis of your experience of learning to write mathematics. You will integrate it into your understanding of mathematics and your (first) journal entry about your own auto-math-ography.

Guiding Questions: Please answer **all** these questions in your journal response.

1. In a few sentences, give your best explanation (to a non-expert) about what it means to write a mathematical proof.
2. What exactly is a mathematician? How is a mathematician different from someone who just uses mathematics in his or her work like, say, an engineer, a theoretical physicist, an accountant, or a used-car salesperson?
3. What is mathematics? How can we define it? How do we decide if some activity should be considered mathematics?
4. In your first journal essay you wrote about your experiences in mathematics. After finishing this course, do you feel your trajectory has changed? In what ways have you changed?
5. What three pieces of advice would you give someone starting the class next semester?

D Handout of Portfolio Assignment

Discrete Mathematics: Portfolio Assignment

First draft due: Thursday, November 3

List of problems due: Thursday, November 17

Second draft due: Thursday, December 1

Final portfolio due: Thursday, December 8

The Assignment. Throughout the semester, you will be getting feedback on your work. On your homework you will not just receive a grade but comments and suggestions, too. In peer-review groups you will be giving feedback to each other on your proof-writing. In your journal you will be reflecting on this feedback. The purpose of the portfolio is to draw on all the components to provide a capstone piece which serves as evidence of your thoughtful reflections and showcases what you have learned in this course.

Your portfolio will include:

- a cover letter that stands as a reflection piece on why you chose the problems you included and how they demonstrate your strengths and your improvement in the course,
- the portfolio checklist which lists the problems that you have chosen, and
- Ten problems (six must be **proof** problems from Sections 4.2–7.3) chosen from among the recommended but uncollected and ungraded problems. The final portfolio will be collected at the end of the semester with drafts due twice during the semester.

Format

- The cover letter must be typed, single spaced, Times New Roman (or similar), 12-point font, and addressed to me, Professor Schaefer.
- Each of the ten problems must be written in LaTeX (using the Portfolio Template on Moodle) and submitted both in paper form and electronically.

First Draft Due: Thursday, November 3

- Five problems (two short-answer and three proofs) written in LaTeX. You will turn in one copy, and you will receive feedback from Professor Schaefer.

List of Problems Due: Thursday, November 17

- Portfolio checklist with final list of problems.

Second Draft Due: Thursday, December 1

- ten problems (four short-answer and six proofs) written in LaTeX. You should turn in three copies and treat this draft as your final draft.

Final Portfolio Due: Thursday, December 8 in your folder

- cover letter
- the portfolio checklist
- final versions of your best ten problems written in LaTeX (four short answer and six proofs) in paper form and electronically

Collaboration: Every day you have been bringing one to three problems to present for peer review. You have been given feedback during peer review and should be editing your work to include in the portfolio. Other than this collaboration, no further collaboration is allowed on this assignment. In this way, the problems are different from homework problems. You may speak with me about the problems, but you should not be working in pairs or in groups.

(Note: For the dates listed above, the course ran Tuesday, August 30 through Thursday, December 8.)

Sarah N. Bryant: Project Manager, STEM-UP PA

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Running a Class Journal

Theron J. Hitchman

Abstract

We describe a method for running a mock professional journal in an inquiry-based learning environment. The instructor plays the roles of managing editor and publisher, and the students play the roles of author and referee.

Difficulty Level: High; Course Level: Transitional

1 Background and Context

For seven years I have run an inquiry-based course called Euclidean Geometry. This course is an instance of what Timothy McNicholl calls the “extreme Moore Method” [2], where the guiding principle is to structure every part of the environment as a small mathematical community. In this direction, it is natural to teach mathematical writing through the use of a class journal.

The University of Northern Iowa is a comprehensive, public, regional university of about 12,000 students with small masters of arts programs in mathematics and mathematics education. The school started as a normal school, and it is still a significant part of its mission to educate future teachers. Nearly all students in the course are enrolled in a pre-service teaching program. The students tend to be sophomores or juniors, but the distinction is difficult to make because we have a large number of transfer students from community colleges. The course is required for all pre-service teachers, and its only prerequisite is a first semester of calculus.

2 Description and Implementation

The working process for students in the course mimics the phases of mathematical work as done by mathematicians. The students read parts of an English translation of Euclid’s *Elements*, and work to solve a carefully designed sequence of problems along the way. (I have designed my own sequence of problems.) The following is an example of how a course taught via the Moore Method is structured. All class meetings are devoted to individual student presentations of student work, with questions, critique, and commentary from the rest of the class. Though there is a perfunctory midterm exam and a rigorous take-home final exam, there are neither quizzes nor traditional homework. During the term, the students devote all of their effort to a cycle of making and sharing mathematical arguments to settle the conjectures set before the class. When a student has some confidence in an argument, he or she presents it to the rest of the class. If it survives and is accepted, the presenter writes a paper detailing it and submits it to the class journal by the next class meeting. This starts a process of peer review and revision, with the instructor or another student playing the role of referee. Eventually, the paper is judged as acceptable and published in an issue of the journal that is made available to all. For a single paper, the whole process, from initial presentation to acceptance, typically takes about a week and a half in a course with three meetings per week.

I do not grade the work in the journal in the traditional sense. Instead, the written work a student completes is used as part of a judgement about whether the student has met certain learning goals for the course,

and progress toward them is used to determine a final course grade. On the surface, the content of the course is classical planar geometry: triangles, polygons, circles, and area. Usually the class covers the first four books of Euclid's *Elements*, but it is not a big deal if that does not happen. Really, the goals for the course are about the mathematical process: learning to make sense of things, to use language precisely, to find solid axiomatic arguments, and to communicate them orally and in writing.

I find that enough material collects for an issue of the journal about every other week, so a 15-week semester will have seven issues, each with between 10 and 25 finished papers. The earlier issues are lighter, as students need extra revisions to get their first papers into a proper mathematical style. The last issue is usually quite large. I distribute issues electronically, except that those authors with papers appearing in a particular issue get a paper copy at the beginning of the next class. It makes for a pleasant not-quite-a-ceremony to hand them out with a message of congratulations in front of the whole class.

There are two significant differences from the circumstances of a professional mathematician. First, every paper will eventually get published. A paper may require a large number of revision cycles before it is ready, but it will never be outright rejected. Second, the referee process is not anonymous. Since students are all present when work is presented in class, there is no way to protect the anonymity of authors. Therefore, I do not allow the referees to be anonymous, either.

2.1 Roles for Instructor and Students

The instructor takes up all the roles that are usually handled by a journal staff: managing editor, administrator, clerical assistant, publisher, and distributor. From an educational standpoint, the most important role is the job of managing editor. The instructor retains control of all of the decision-making about assigning referees to papers, and through the process can help guide the learning of both the author and referee of a paper. At first, the instructor must serve as the sole referee.

Students serve as the pool of authors. After a few students have demonstrated an ability to write a reasonable paper in an acceptable mathematical style, they can be recruited as referees. A significant amount of support and mentoring is required to get students through the two roles, as everything about the process is new to them.

I find that students are ready to be asked to serve as referees after a second completed paper. At that point, they have some experience with the process, and have seen a few examples of a referee report on their own work. A student who is still struggling to produce a reasonable first draft might need more time to develop before being asked to contribute in a new way. Once a student is ready, I assign a paper to be refereed and give a speech about the dual mandate of a referee: a solemn duty to defend us all from confusion by giving honest but constructive feedback to the author, and caring about the author's feelings by focusing all criticism on the work (not the author), and by being specific. The referee submits a report to me, and as managing editor, I pass it along to the author. I read and evaluate the work of the referee, and police it for compliance with the dual mandate.

2.2 Materials Required

To smooth the process, I find it is helpful to create some guideline documents for students. I have included a few of my materials as appendices.

How a Professional Journal Works. An overview of how professional journals run the peer-review process, with a description of how our system deviates from the standard.

Instructions for Authors. A description of the submission process, with technical instructions for how to make the proper file format and turn it in.

Style Guide. A discussion of general nature about what professional mathematics writing looks like, aimed at an audience *who has never read a professional paper before*.

Template. A bare template of a paper that students can use as a starting point.

Example. An example paper, both the raw input file (L^AT_EX or otherwise), and the corresponding output acceptable for submission.

Instructions for Referees. A detailed list of instructions on how to referee a mathematical paper, aimed at an audience that has never seen a referee's report. I prefer for the students to write carefully sanitized and depersonalized referee reports. This requires some explanation and a few concrete examples of what to write and how to write criticism in an acceptable style.

Journal Cover. A cover page for the journal, with a professional-looking layout. I have titled my journal *Transactions in Euclidean Geometry* and made a colorful display figure out of a pretty theorem that we do not usually have time to cover.

As usual, the documents will be read and understood by some students, and largely ignored or misunderstood by others. The process of creating them prepares the instructor for part of the experience of mentoring students through the journal process. In the long run, it saves time to make them available and point students toward them at the appropriate times.

3 Outcomes

3.1 Educational Benefits

I have run a class journal in my Euclidean Geometry class in twelve semesters. I have observed three important benefits for the students, and a little something about how playing referee helps students understand the value of careful writing.

First, students must engage in a critical process of revising and rewriting. The process forces a student to react to criticisms from an external review, and think carefully about writing arguments to convince an audience.

The second and third benefits follow from the key idea of using student referees. The process of acting as a referee for another student's work is a wonderfully deep learning experience. Being responsible for the professional critique forces a student to think more deeply about how mathematics is written. This eventually helps the referee improve his or her own writing.

The less obvious benefit of using student referees comes from removing the instructor as sole arbiter of correct and acceptable work. There is a subtle change of focus for authors. Because the referee will likely be another student, the focus is no longer on satisfying the whims of the instructor, but instead on making sense and clearly explaining ideas to peers.

I learned a lesson about the way using students as referees helps their writing from the noticeable decline in the quality of student written work in some semesters where I chose not to have student referees. Generally, the daily experience of defending arguments at the chalkboard helps students find and discuss arguments more clearly, but not necessarily write them more clearly. Part of what happens is that each class builds a rapport and students gradually build an understanding of what needs to be said (and what can be left unsaid) to convince this set of peers. However, an author must be careful to write for a broader potential audience, which will include people outside the author's acquaintance. Without the experience of playing referee, some students stay in a mode where they write just as they would speak at the board. All the tacit understandings that make an oral presentation efficient and pleasing become mislaid justifications and confusing references without a touchstone for the reader. I found it challenging to convince students that writing

mathematics is a different task than talking about it. In my discussions with students, it became clear that serving as a referee helps students change perspective and consider a larger audience. Being asked to referee a paper on an argument that was presented a week in the past seems enough to convince a student to write arguments that are more clear and more complete.

3.2 Evidence of Effectiveness

I have gathered some evidence of student perceptions on the effectiveness of using a class journal.

Anecdotal Evidence

I have heard from colleagues in courses downstream from Euclidean Geometry that there has been a positive effect on student skill in writing complete proofs, and on student attitudes towards geometry. Most prominent is the change in attitude of pre-service teachers. A colleague who teaches a methods course keeps an informal poll of students about what parts of mathematics they are excited to teach when they find employment. She states that very few students in our program expressed a desire to teach geometry in high schools as of five years ago. She attributed this to the lack of comfort with formal mathematical reasoning and writing. Now, a significant portion (over a third) of students express a positive view of geometry and a desire to teach it themselves.

SALG Data

In a recent semester, I used the *Student Assessment of Learning Gains* website [3] to get more detailed feedback on my teaching in Euclidean Geometry. A few of the questions are relevant to the use of the class journal. There were twenty-five students in this section, but only thirteen responded to the anonymous online survey.

In the survey, students are asked to report their perceived gain in a skill set as a result of participating in the activities of the course. The responses available are no gain; a little gain; moderate gain; good gain; great gain; not applicable. All the respondents reported experiencing at least some gain on each of the questions pertaining to writing. Out of the thirteen responses, nine indicated a “good” or “great” gain in recognizing sound argument and appropriate use of evidence, ten indicated “good” or “great” gain in developing logical argument, eight indicated “good” or “great” gain in writing documents in a discipline-appropriate style, and nine reported a “good” or “great” gain in writing clear and concise arguments.

3.3 Challenges

The process of running a class journal is time-intensive. While the use of student referees leads to less time spent on marking papers, there is a significant amount of time that must be spent on basic administration and on mentoring students through their new roles as authors and referees.

It is important to keep a watchful eye on the referee process, and referees must be carefully vetted and prepared. They take on the critical job of providing educational feedback to their peers, but it is the instructor’s job to ensure this happens in a way that is beneficial to both the author and referee. This goes beyond the simple problem of getting the latter to complete the work in a timely fashion. Students must be taught the proper tone to use when writing feedback to protect fragile author egos.

4 Extending the Method

The class journal has worked well for an inquiry-based learning course with between fifteen and twenty-five students. It should work for smaller classes too, but with classes larger than twenty-five the administration

would be even more difficult. An advanced student who had completed the course could be turned into an administrative assistant to alleviate some of the clerical work.

I believe the idea of a class journal could also be extended to a course that is not inquiry-based. There would be extra challenges involved in assigning portions of writing to students, and helping them understand the process of explaining ideas that are not their own while avoiding plagiarism. This trouble could be avoided by using only homework as material for the journal. Still, the key components of writing and revising, and acting as a referee for the written work of others would remain. Another challenge would be a psychological intensification of the journal process. In my structure students get lots of feedback when they give an oral presentation. Students already have a feeling of success. So writing the paper is not about being correct any more, it is just about being clear and convincing. Without the inquiry environment, these two stages come back together and the likelihood of students having trouble dealing with criticism would increase. Some steps should be taken to mitigate this. (Editors' note: For a similar approach in a flipped classroom setting, with emphasis on reading and writing, see the article in this volume by Ellis-Monaghan [1].)

References

- [1] Joanna A. Ellis-Monaghan, "Reading, 'Riting, and Reals: Proofs in a Reading-and-Writing-Intensive Real Analysis Class," in this volume.
- [2] Timothy McNicholl, "The Extreme Moore Method," Educational Advancement Foundation, 2006 Available at docs.google.com/file/d/0B2t6ivhRzD_FQnN2MHBDCtRIaUE/edit?usp=sharing
- [3] Student Assessment of their Learning Gains, www.salgsite.org

Appendix

A How a Professional Journal Works

In general, here is how professional mathematics journals work.

1. Author writes a paper in a way to conform with the journal's declared style.
2. Author submits the paper to the editorial board.
3. Editorial board finds a referee (or perhaps two, or three). Usually, the author's name is removed.
4. Referee reads paper and checks it.
5. Referee writes a report to the editorial board, and takes a stand on whether the paper should be published.
6. Editorial board considers opinions of the referee, and forwards the report to the author.

If the referee suggests publication:

Usually, the authors have revisions to make to improve the paper. They make them and resubmit. Now the process starts again, but this time goes faster. When all are satisfied, the paper is published.

If the referee does not suggest publication:

The author revises the paper anyway, and looks for a new journal.

We have only one class journal, so getting rejected from the journal will not happen. Extensive rewriting happens pretty frequently, especially at the beginning of the semester.

At the beginning of the semester, I shall act as the managing editor and sole referee. After a few successful papers, you may be asked to referee also.

After submitting a paper, you should expect to get a referee's report and have some editing to do before the paper gets published. When you have more practice, the amount of reworking a paper requires will likely go down significantly.

B Instructions for Authors: Style Guide

To help potential authors produce the best work they can, *Transactions in Euclidean Geometry* has some style requirements for articles. Please adhere to these when writing.

- Mathematics is written in present tense. It is eternal and enduring.
- Mathematics is not a personal narrative. Do not choose the personal pronoun "I" when writing. If a personal pronoun must be used, please use "we."
- *Transactions in Euclidean Geometry* strongly prefers that authors use an active voice, not a passive voice. If you are unclear on this, consider if your sentence might still make sense if you add "by zombies" at the end of it. If so, it is likely a passive-voice construction.

For example: The brains were eaten *by zombies*. All men are created equal *by zombies*. I got kicked in the face *by zombies*.

- Each statement written should have a clear justification given from the literature. References to Euclid's *Elements* are given this way: "...by Euclid II.13..." That is a reference to Proposition 13 of Book II of the *Elements*. References to work produced for this journal should be given by the name of the author and the official sequence number/letter. For example: "...by Mr. Jones' Theorem 2.3..." and "...by Miss Marple's Theorem F..." are correct.
- Authors are encouraged to include figures that enhance the exposition. A visual aid can go a long way toward making an argument understandable. But authors are cautioned that a figure is not an argument, just a supplement. Write as if there is no figure, but include one anyway.
- *Transactions in Euclidean Geometry* strives for a classical feel, so authors are discouraged from using mathematical symbols when words will do. English prose is easier to read than long strings of strange glyphs.

For example: "Triangle ABC is congruent to triangle XYZ." is preferable to " $\triangle ABC \cong \triangle XYZ$."

C Instructions for Referees

A note about the proper tone

The peer-review process is important for improving the quality of written mathematical work, but receiving a referee's report is uncomfortable. The process is inherently critical and likely negative.

To reduce the strain on authors, referees should depersonalize the comments made. Comments should focus on the paper, and not on the author. The best mode is to focus each comment on some aspect of the paper, and to never mention the fact that the paper has an author.

For example: "...paragraph three is confusing. How does one conclude that..." is fine. But "...Mr. Jones is stupidly confusing the statement of theorem 4 in paragraph three..." is very bad.

In short, be specific, and talk about features of the paper. If you must refer to the author in some way, say "the author" with no name.

What goes in a report

Appropriate things to put in a referee report include

- comments about the logic of an argument
- comments about passages that you feel are confusing
- comments about poor grammar and spelling
- comments about the appropriateness of figures
- comments about the lack of adherence to the style guide of the journal.

A challenging thing is that you should limit yourself to pointing out deficiencies, and refrain from making corrections for the author. A suggestion *might* be allowable. But telling the author how to write a paper is not usually well received.

How to file a report

Make a simple text document, and type each comment on its own line. It helps if you start each comment with location information. For example,

"In the third paragraph, the second sentence: the grammar here is unclear."

"On page 2, third line from the bottom: Is that really 'triangle DEF'? I expected 'triangle EDF'."

When your comments are complete, send the file to the managing editor.

Theron J. Hitchman: University of Northern Iowa, Cedar Falls, Iowa

Peer Grading of Exams

Andrew G. Bennett and Xuan Hien Nguyen

Abstract

The goal of this technique is to teach students to grade proofs, which helps them critique their own work and become better writers. The central activity consists of getting students to grade their own midterm exams and the exams of two classmates. In this article, we describe how to prepare and motivate students for the task. We also discuss the logistics involved in organizing and grading students' work.

Difficulty Level: Medium; Course Level: Transitional

1 Background and Context

This peer-grading activity was designed by the first author for an introduction-to-proofs course called Foundations of Geometry at Kansas State University, which is a public university with an enrollment of over 20,000 students, the majority of whom are traditional students living on campus. The class size ranges from 30 to 40 students, most of whom are majoring in mathematics, secondary education, or both. It meets three times a week and one section is offered every fall semester. Although Calculus I is the only prerequisite, students typically take it in their third year. The course is one of three (along with Discrete Math and Foundations of Analysis) aimed at familiarizing students with rigorous proofs. Although the technique described here would be suitable for other introduction-to-proofs courses, Foundations of Geometry is the only one where we implement it because it is the one we teach on a regular basis.

We structure Foundations of Geometry to encourage students to work on problems. In the first couple of classes, we discuss with them the meaning of the words “proof”, “definition”, and “axiom”. We then introduce Euclid’s first four axioms of (neutral) geometry; the fifth axiom, which produces Euclidean geometry, is given later on, after a few weeks. For the first half of the semester, students solve problems from a prepared list in groups. The groups meet regularly outside of class to work on them; finding common weekly meeting times is a requirement for forming a group. In class, we randomly select groups to present their proofs at the board. Sometimes, we pretend the selection process is random by rolling a die so small that students can’t see the top face. After the midterm exam, the class moves to a computer lab once a week where, with the help of software such as Geometer’s Sketchpad and Geogebra, students explore properties of geometrical objects and form conjectures. We then ask them to prove their conjectures and present them in class. The last two weeks of classes are devoted to examining an example of non-Euclidean geometry for contrast. A student’s grade is determined by a combination of his/her group work on the presentations and his/her individual work on a midterm exam and a final exam.

The course combines many activities to help students practice discussing arguments, presenting at the board, and formulating their own conjectures. Despite these aids, it is difficult for students to recognize mistakes in their own reasoning. We designed the peer-grading task with two goals in mind. First, we wanted to give students examples of genuine proofs written by their classmates, with the idea being that, by reading others’ proofs and acting as editors, students would be more critical of their own work and be better writers. Second, because the majority of the students in the course are future teachers, they need to learn to grade proofs fairly and consistently.

2 Description and Implementation

The main grading exam activity happens right after the midterm. The day after the in-class exam, students are given back their own as well as copies of two of their classmates' to grade. While they complete this assignment outside of class, we also grade all the exams. A student's score on the midterm is determined as a weighted average of the scores given by the instructor and the three student graders. In addition, students get a midterm grading score that reflects the quality of their grading.

To prepare students beforehand, we give them a grading rubric (see the Appendix) and a couple of homework assignments where they grade several problems. At this point, students receive the same proofs so that we can discuss the grading in class to ensure that they understand the rubric. For this reason, the proofs are provided by the instructor and are not work from their peers, although some of the proofs may be inspired by work from students in previous semesters. Students not only have to determine whether an argument is correct, they must also judge if the mistakes are minor, or if the approach is intrinsically flawed. We grade the student grading according to a second rubric (see the Appendix) that is also available to students. From inspecting the work of students on the grading assignments, we have found that two preparatory assignments such as these are sufficient in general.

The midterm exam is usually composed of six questions and is worth 50 points. In each of the first four questions, students are asked to prove one of the problems from the list given at the beginning of the semester; the fifth question is a five-part true/false; and the last question is more computational in nature. The first four questions are worth 30 points combined (usually one of the questions is an equivalence or has two parts and the other three are direct proofs), and the last two questions are worth 10 points each.

Once students have taken the in-class exam, we make three copies, on which the name is replaced by an identifying number. Students are then given their own exams back and two additional ones to grade. The choice of the extra two exams is (mostly) random. It can be tempting to give weaker students the exam of a better proof-writer or challenge good students with subtle proofs; however, we have found that it is too time consuming to plan. Moreover, it is also unfair to knowingly give some students a more difficult assignment. For these reasons, the only thing we try to control is that no student gets two perfect exams to grade (besides his/her own).

We do not provide an answer key. Students are allowed to discuss solutions with their classmates, but they are asked to grade on their own. Before they take the exams home, we remind them to provide as much feedback as they would like to receive themselves and to write comments to support their decisions. Instructor and students have one week to complete the grading.

A student's exam is therefore graded four times and his/her grade is computed as follows: our score counts for 50% and the average of the three students' scores makes the other 50%. To keep students from giving themselves perfect scores and inflating grades, the quality of their grading is also taken into account (see the Appendix for a rubric). We go through all the exams graded by students, note the scores, and read comments. Checking the grading takes time, but it is not as tedious as the number of exams and problems suggest. When a student's score falls within one point of our own on a problem, which happens most of the time, a quick look can confirm that the grading is sound. Questions where scores differ by two or more points take more time on average. We found that it is not possible to judge the quality of a student's grading based solely on the distance to our scores, which is why we review every problem. Sometimes, students can give the same score as we do for misguided reasons, and at other times, a wide range of scores is perfectly valid. For example, some condensed proofs can receive very different grades depending on the grader's interpretation and expectations. If the grading is fair and justified by comments, a student gets full credit for it.

The total grade for the midterm exam and grading is out of 75 points and computed:

$$\begin{aligned} \text{exam score (out of 50)} &= \frac{1}{2}(\text{score given by instructor} + \text{average of three scores given by students}) \\ \text{grading score (out of 24)} &= \sum_{i=1}^6 (\text{grading score on the } i\text{th question}) \end{aligned}$$

total midterm score (out of 75) = 1 + exam score + grading score.

Once all the grading is completed (including the instructor's review of the student grading), students get the four copies of their own exams back with all the comments.

3 Outcomes

Students take the grading assignment very seriously because they know the scores they give count. This responsibility is crucial to the success of the task. In the past, we have given students several proofs to grade in the form of quizzes or homework. They were written beforehand by us and students usually saw them as trick questions, i.e., questions where there is a trick they must find. With peer-grading, students know that the proofs they receive are written by other students, who are certainly not trying to challenge them. They also know that anything can happen; for example, all the proofs could be correct, or completely wrong. Because they want to be fair to their classmates, students read the proofs more carefully and do not give up on tricky arguments.

The scores given by students are close to the scores given by the instructor (see Table 1). In addition, in fall 2012, out of 21 exams, the scores given by the instructor averaged 31.3 (out of 50) with a standard deviation of 7.8 and the ones given by students averaged 31.6 with a standard deviation of 7.7. In that semester, one student did not turn in his grading, so we had only 60 scores for students instead of 63. In fall 2009, the averages were 30.32 and 29.10 and the standard deviations were 6.23 and 6.04, respectively, for 28 exams. The data from fall 2011 is missing because we kept only the two most recent years (2012 and 2013). We managed to recover some numbers for 2009 and 2010 because we gave a presentation in January 2011. In spite of the missing data, we don't suspect that the fall 2011 scores would have shown wide differences between the grading of the instructor and the students.

Our technique focuses on teaching students to evaluate proofs. This helps them write better proofs. It is difficult for students to check their own reasoning. When students first learn to create proofs, they often make leaps in their arguments because they think something is obvious. Sometimes the missing step is indeed obvious, but it generally is not. Seeing examples of their classmates' work on the same problems can help students understand what makes a good proof and identify gaps. As students learn to distinguish a good proof from a bad one, they become better judges of their own writing. This does not automatically

# scores	Fall 2013				Fall 2010			
	Instructor		Student		Instructor		Student	
	27		81		30		90	
	mean	std. dev.	mean	std. dev.	mean	std. dev.	mean	std. dev.
Q1 /6	4.9	1.3	5.0	1.3	3.5	1.5	3.9	1.7
Q2 /6	5.1	1.3	5.1	1.4	7.2	3.3	7.9	3.5
Q3 /6	3.4	2.1	2.9	2.3	3.6	2.3	3.7	2.2
Q4 /6	4.0	2.4	4.1	2.3	3.3	1.8	3.4	1.9
Q5 /6	5.2	1.7	5.0	2.0	6.9	2.4	6.9	2.5
Q6 /10	7.5	2.0	7.5	1.9	5.5	2.4	5.3	2.4
Q7 /10	7.3	2.3	7.0	2.9				
Total /50	37.3	7.3	36.6	8.1	29.95	9.21	31.16	9.19

Table 1: Comparison of scores given by the instructor and by students. In fall 2013, the two-part question was split into two different questions. In fall 2010, Q2 and Q3 were combined into one question out of 12 points.

mean that they will be able to improve their writing, but they can better discern which parts work and how much editing is needed.

Students often ask how much detail they should include in a proof, and are generally told to write it so that other students in the class can understand it. However, in most courses, the instructor is the only other person reading the proof and the sole arbiter. For this reason, it is hard to convince students to rewrite a confusing proof or fill in missing steps. Students comply to improve their grades but they may not believe that the editing was necessary. After all, the reasoning was clear to them and the instructor knows the material, so the teacher is only pretending not to understand. Getting feedback from their classmates changes the dynamics. If other students also indicate that a proof is confusing or incomplete, the proof-writer is more likely to believe that the comments are legitimate, which gives motivation for improvement. The feedback from their peers also convinces students that the instructor is not the only authority in the classroom. Students can judge proofs themselves, including their own.

A good portion of our students are secondary education majors, who will teach geometry proofs. The peer-grading task is the only occasion where they are provided guidance for and hands-on experience with grading. On an exit survey conducted in 2010, students reported that the assignment was useful in terms of their future careers. On a scale from 1 (strongly agree) to 5 (strongly disagree), the average of the responses of 21 students was 2.14, with a standard deviation of 1.04.

The two drawbacks are time and paper usage. Each exam is graded four times, so for our class size, we have about 120–160 exams. Organizing them requires a substantial amount of time, which is the main commitment of this technique. In a regular course, the instructor would just grade the midterm exams and record the scores. For peer-grading, he/she goes over each exam three more times to check how students graded. For about 35 students, the second stage of grading takes us in general 3.5 hours, or about 2 minutes per exam.

4 Extending the Method

Technology can help reduce paper usage. Most students own a mobile device (smartphone or tablet) on which they are able to view files and write comments. It is therefore possible to let students grade and submit their work online. Scanning each exam should not take more time than making paper copies of them. After the initial set-up, the logistics of distributing and organizing the exams should be simpler. In addition, the instructor will be able to easily save electronic versions of students' grading and comments and access them in the future. However, giving students a ready-to-post copy of exams raises new challenges such as privacy and anonymity. (Editors' note: for an activity involving peer review in a group examination setting, see Johnson and McNicholas [1], in this volume.)

The peer-grading task works best for a class of about 15 to 35 students. The number is large enough to allow for a good variety of writing styles and possible approaches, yet it is not so overwhelming when the instructor has to check the grading. With fewer students, an instructor has time for peer grading on multiple exams or quizzes. Although it will likely be impossible to preserve anonymity in that setting, we do not think that peer grading will be negatively affected. We have never had a student be uncomfortable with sharing his/her work.

In classes larger than 35, it is less practical to implement the peer-grading task as is, because of the sheer number of exams to be handled. A possible alternative is to build a database of proofs written by students in previous smaller classes, then assign some of them to students in the larger class. The comments by student graders can highlight which proofs have common mistakes or difficult arguments. We have not taught a proof-writing class with more than 35 students, but doing so would give us a strong incentive to use technology.

References

- [1] Inga Johnson and Erin McNicholas, “Group Examinations in Introduction-to-Proof Courses,” in this volume.

Appendix

A Rubric for grading proofs

Statement of problem (2 points)

0	1	2
Incorrect statement of problem. May misinterpret what is given or what is to be shown. Might just recopy problem rather than give a precise restatement.	Correct but incomplete statement of the problem. Doesn't include a statement of either the “given” or the “to show” or fails to connect them to the diagram ¹ . Might be stated for indirect proof but a direct proof is given or vice-versa.	Correct statement with a labeled diagram and the “given” and “to show” stated in terms of the diagram.

An improperly drawn diagram may fall into either the first or second category, depending upon the extent of the error.

Correctness of proof (4 points)

0	1	2	3	4
Mainly incorrect consequences improperly deduced from the given. Little or no sense of how to prove the result.	Unconnected, mostly true statements properly deduced from the given. Listing facts without a sense of how to link them to get a correct proof. May just jump to the conclusion without justification.	Statements linked into a reasonable (though perhaps misguided) attempt to prove the theorem. The proof may be left incomplete or may depend upon a major unjustified leap.	A correct approach to proving the theorem is attempted. Some statements may be unjustified or improperly justified, but errors are minor and could be fixed without substantially changing the proof.	A correct and complete proof is given. Some irrelevant information may be included since the time limit precludes polishing up the presentation.

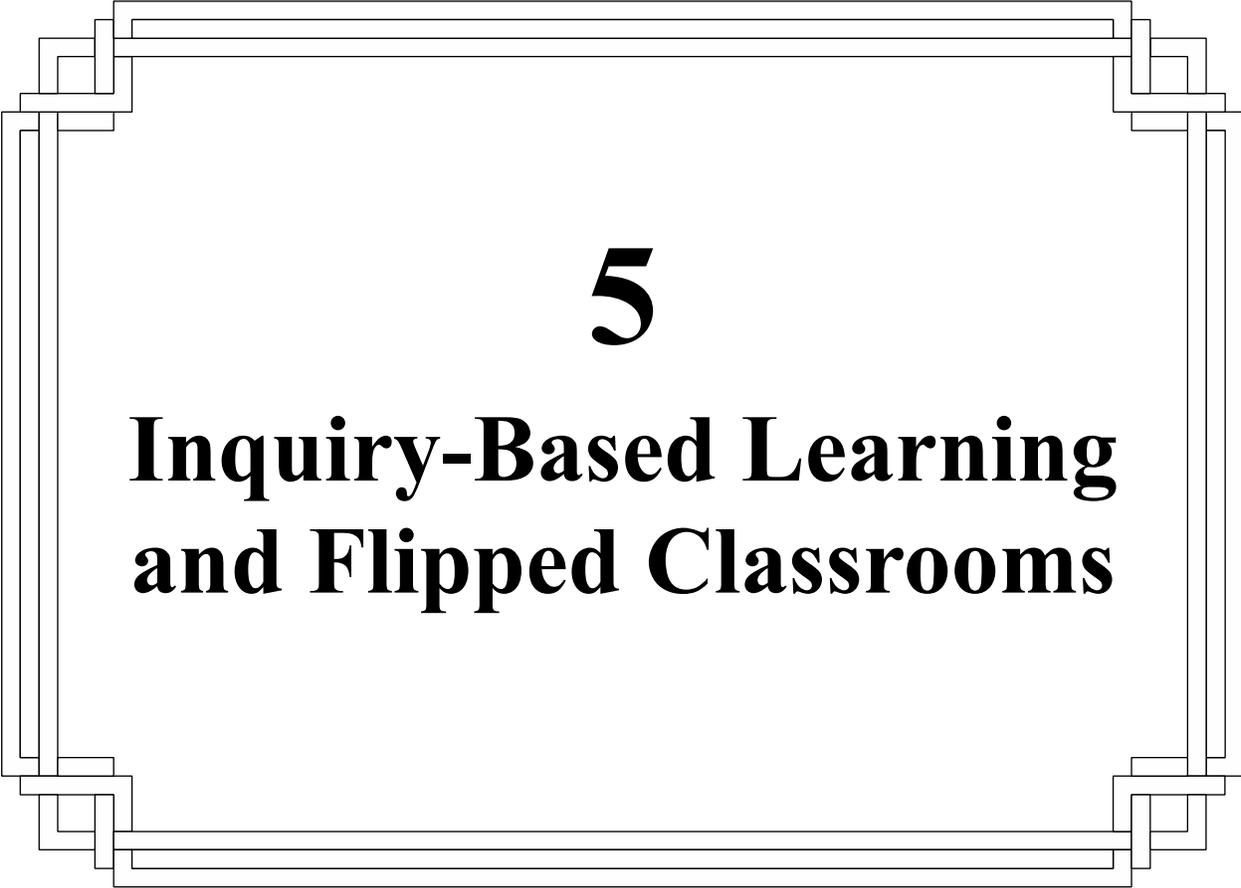
¹The rubric was written for a geometry course, where all the proofs are accompanied by pictures.

B Rubric for grading your grading

0	1	2	3	4
No response	Incorrect response	Unclear response	Incomplete response	Correct response
	An incorrect proof is marked correct or a correct response is marked incorrect due to a mistake in understanding (as opposed to an inaccurate grade due to an overemphasis on errors of detail as in an incomplete response).	It may be unclear whether the answer is considered correct or incorrect, or the answer may be accurately marked incorrect, but the reasons it is incorrect may be wrong or may not be given.	The answer is accurately marked correct or incorrect, but some inaccurate details may not be noted. Alternatively, mistakes in details may be weighted too heavily, leading to a correct answer being marked incorrect.	The answer is accurately marked correct or incorrect. Any errors in logic or detail are properly noted.

Andrew G. Bennett: Kansas State University, Manhattan, KS

Xuan Hien Nguyen: Iowa State University, Ames, IA

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5

Inquiry-Based Learning and Flipped Classrooms

An Inquiry-Based Introduction to Mathematical Proof

Patrick X. Rault

Abstract

In this article I convey my methods as an instructor for proof-based mathematics courses, with emphasis on an Introduction to Mathematical Proof course. The course is designed for students who have completed introductory calculus, as a bridge to abstract upper-division courses. The approach involves actively engaging students through a process called inquiry-based learning.

Difficulty Level: High; Course Level: Transitional

1 Background and Context

Many instructors wonder whether graduating seniors will make use of what they have learned in their mathematics courses. Michael Starbird, following in R. L. Moore's legacy of inquiry at the University of Texas, frequently asks new teachers to think about and share what we wish our students to retain from the experience twenty years from now. The reader may wish to do so before continuing, to be in a better position to reflect on whether or not my approach described below is appropriate. The reader is also invited to see the companion article for a description of the history of inquiry-based learning (IBL) and the Moore Method [2].

The State University of New York (SUNY) College at Geneseo is a liberal arts college, with over 5,000 undergraduates. Known as New York's public honors college, entering freshmen have the highest ACT scores in the SUNY system. The SUNY College at Geneseo is a traditional residential college, with most students between the ages of 18 and 22.

The mathematics programs are quite popular at the college, with the sixth largest major at 355 students and the largest minor with 102 students. One-third of majors plan to teach in secondary school, up to a dozen enroll in Ph.D. programs in mathematics each year, another handful become actuaries, and many pursue careers in industry or graduate programs in second majors.

The course Introduction to Mathematical Proof, nicknamed Proofs, is taken by all math majors and is recommended for minors. Three to five sections are offered each semester, each with about 25 students. The only prerequisite is Calculus II, while most core junior- and senior-level courses have the Proofs course as a prerequisite. Most majors therefore complete Proofs in their sophomore year. In addition, most minors take the course, and also all elementary education majors who have a concentration in mathematics.

A primary goal of mine in all proof-based courses, especially Introduction to Mathematical Proof, is for students to attain a higher level of independent thinking. The first time I taught Proofs via traditional lecture, I found that the students simply memorized proof templates and failed to apply them with the thought that was necessary. When I was a student, I too was lacking in drive and I attribute my own transformation to an inquiry-type experience in the Program in Mathematics for Young Scientists (PROMYS) at Boston University. Inspired by the IBL methodology of teaching, which is better suited than PROMYS for the classroom, I attended an MAA Prep workshop on the subject. After being mentored by experienced IBL practitioners Walker White of Cornell University and W. Ted Mahavier of Lamar University, I have collaborated with Ron Taylor of Berry College to publish a textbook for Introduction to Mathematical Proof [9].

It is important to note that no single teaching style will function perfectly for every instructor at every college. I have written this paper to share what might be called the Rault Method, which differs from the Moore Method and any other instructor's method. In choosing a personal teaching style, the reader should consider his or her department's learning outcomes, students' backgrounds, and personal goals for the class.

2 Description and Implementation

I use my own version of an IBL method, sometimes called a modified Moore Method, of teaching many of my proof-based courses. Students are given a series of small challenges to complete independently and then present, while receiving a healthy amount of formative feedback in the process. This is summarized well in the following:

The primary goal of such a course is to address the content in such a way as to develop in the students the ability to investigate problems independently. The instructor serves as a coach, mentor, collaborator, and guide, as well as a source of positive reinforcement.

— W. Ted Mahavier [4, p. 1]

Before each class period students are expected to read the relevant pages of the textbook and attempt all problems while preparing some solutions for presentation. An IBL textbook is sparser than a traditional textbook, in that it contains very few proofs. The problems are designed to challenge, hence deepening student understanding of the fundamentals.

Before going into detail about my implementation, the reader who is unfamiliar with IBL should consider looking at a sample course script through the *Journal of IBL in Mathematics* [6]. In the “try it for yourself” spirit of IBL, the reader is encouraged to choose a sample script related to his or her discipline, work through the first page of problems, and reflect on what a student would gain by doing so. For the reader without internet access, a few sequential problems from the chapters 1 and 4 of my textbook [9] are provided here.

Statement 1.24a. Let P and Q be statements. $\neg(P \wedge Q)$ is logically equivalent to $(\neg P) \vee (\neg Q)$.

Exercise 4.27. Let $A = \{2, 3, 4, 5\}$ and $B = \{1, 5\}$ be subsets of the universal set $U = \{1, 2, 3, 4, 5\}$. Write out the following sets. (a) A^c , (b) B^c , (c) $A \cup B$, (d) $A \cap B$, (e) $A \setminus B$, and (f) $B \setminus A$.

Statement 4.63a. Suppose A and B are subsets of a universal set U . Then $(A \cap B)^c = A^c \cup B^c$.

Statement 4.68. Let A , B , and C be sets. Then $(A \setminus B) \cap (A \setminus C) = A \setminus (B \cup C)$.

The core of the problems, and of most IBL scripts, are the Statements for students to prove or disprove. They should be important enough that students and instructor are willing to give enough class time for successful presentation by a student. That occasionally requires a presentation on a second day, sometimes after the instructor has approved a correct version (modulo minor issues that stimulate discussion) between classes. In addition to Statements, a second type of problem is usually necessary, which I call Exercises. This term is used primarily to refer to explorations of new definitions or calculations illustrating core algorithms, such as Exercise 4.27 above. While their outcomes are pedagogically critical, the topics are so natural that they often appear in earlier discussions. For this reason, I make it clear to students that these Exercises may be skipped if their insights were already brought up in other discussions. Since students put a lot of time into discovering and perfecting proofs of Statements, I find that it is important to distinguish between Statements and Exercises by guaranteeing at least one presentation opportunity per Statement; not doing so is demoralizing when students have spent hours in preparation.

Each IBL practitioner has a method for deciding which student presents the solution to which problem. At the start of class I use a computer projector to display a sorted class list, using a program created by teaching assistant Brian Knapp and available online via the following link.

www.math.geneseo.edu/~rault/Knapp/IBLProg.html

Students at the top of the list are asked whether they are prepared to present an informal solution to the next Exercise or a formal proof (or disproof) of the next Statement in the book; passing does not diminish their grades. After presentation, points are added to a spreadsheet that results in the student's moving down the list. A common alternative approach to matching students to problems is to first list the problems and the students who are prepared to present them, and then address priority [2]. The reader who is considering using IBL should pick a simple strategy that fits his or her personality.

Regardless of the choice of a student-problem pairing method, maintaining a very positive and friendly class atmosphere can help the many of us who suffer from math anxiety. In the first week of the semester one might ease students into the new system through some lecture and requests for volunteers. It may be a good idea to make it clear that “what happens in this class, stays in this class” just as in a sensitive discussion about race or gender in a philosophy class. One aim of this method of learning is that students see errors as being part of the process. Award-winning IBL professor Ed Burger was known to reserve a portion of his students' final grades for productive failure. It is well known that many of the best scientific discoveries occurred due to accidents, and it is important that our students be comfortable learning from their mistakes.

During student presentations, it is important to follow every detail and make notes of possible issues. As soon as the presenter is finished, students new to IBL will all turn their heads in a synchronized motion toward the instructor; for this reason, the instructor should avoid eye contact with the speaker by turning to face the class and asking something like “Who has questions or comments?” This helps to establish an environment where students seek affirmation from each other, and eventually from themselves. My ultimate goal is for students to rise to a level in which the instructor is unneeded, since after graduation no professor will be there to help. The use of words like “silly mistake” in place of “stupid mistake” and the avoidance of demeaning terms like “obvious” and “trivial” also help to create a safe classroom.

While informality helps to create a friendly environment for students to speak in front of their peers, a formal system of allowed questions can help reduce anxiety for the student at the board. Students may ask questions like

- Why is line three true?
- How did you come up with this idea?
- Could you give an explicit example (with numbers) of how the definitions in line five work?
- Which problem number are you referring to in line four?
- Can you write down the verbal clarification you just gave?

but not questions that

- are leading or are directed at the speaker instead of the mathematics, such as I think you're overcomplicating it — wouldn't it be easier to ...
- are tangential, such as asking about an entirely different approach to the proof, and whether it is also correct, or
- force the speaker to think on the spot several steps ahead.

The end of the questioning period can be awkward for students, so it is helpful to find something positive to say about their solutions. Here are some common comments that I give.¹

- It looks like Ryan has proven a significant case of the Statement. Let's call it Ryan's Lemma, and perhaps someone can prove the rest of it next time using this lemma as a black box.
- It sounds like we feel that if ____ is true, then Yanyan's proof works. Can we formulate that into a conjecture? Let's call it Yanyan's Conjecture, and perhaps someone can prove it next time.
- It sounds like the class feels that Margaret has a basically correct solution, but that there are a few issues that really need to be ironed out. Do we all agree — I don't want to put words into your mouths? Yes, I see a lot of nodding heads. Okay, so Margaret, can you have that on the board before the start of our next class, so that we can just recap it quickly?
- It sounds like we feel that this counterexample attempt was unsuccessful, but do we agree that the example was really helpful in understanding the new definition? Yeah, I see nods, so even if Yousuf hadn't tried the counterexample, it would have been necessary for us to come up with an example as a class.
- I see a lot of people's hands in the air (wanting to ask questions). Perhaps I can summarize the issue that several people have brought up . . .
- This is something that we often have issues with in learning this material, so I actually set up the problem to bring out the issue so we could discuss it. So don't blame Walker for making this mistake — I would have been upset with him for doing it too perfectly and depriving us of this fine discussion!

The examples include discussions of lemmas, conjectures, and counterexamples. This process is exactly how we mathematicians think, and is a strength of IBL since undergraduate students rarely find other opportunities to understand the true use of these terms. However, since some students will immediately see how to complete the proof, there is a danger in moving too fast. I always delay discussion of any new conjectures until the next class period, and indeed the "thinking time" has been a highlight of positive student feedback.

After students have finished asking questions, it is usually a good idea for the instructor to go over the key ingredients of the proof. It is easy to lose the forest for the trees: students sometimes spend 90% of their space in the proof discussing 10% of the difficulty, while leaving the magical steps to just one summary line. I make it a recurring theme in writing good proofs to ensure that this magic is both clear in the proof and clear to the entire class via a summary. However, one should avoid giving a recap of the entire proof, line by line, as this encroaches on the presenter's territory and sense of responsibility.

Upon completion of each presentation, I input a grade into the class list program on the following scale

- 4 points: correct proof of a Statement.
- 3 points: correct calculation for an Exercise; correct proof modulo a lemma; correct proof of a significant case.
- 2 points: solution has some significant issues, but moves the class forward in overall understanding.
- 1 point: solution has very little that adds to the class's overall understanding.
- 0 points: solution was confusing or incorrect, and possibly derailed class understanding.

¹For the sake of privacy, these names are not those of students.

Less credit is awarded for informally discussed computational Exercises than for formally written professional proofs of Statements; this encourages students to attempt the more difficult problems and helps demonstrate the difference in expectations. The sorted class list program also prevents students from presenting on consecutive days, hence removing some of the competitiveness from the original Moore Method. The absence of randomness in the system is seen by students as open and unbiased, and effectively moves responsibility for advancing class material from a few A students to the entire class. In fact, if only the students at the end of the list are able to present a solution, then to prevent demoralization and to share knowledge we change gears and work cooperatively: a class discussion, mini-lecture, or group exercise may be more appropriate than a student presentation that few are prepared to comment on. For those desiring more detail about grading presentations, a different approach is provided in the companion article [2].

This teaching style requires a significant amount of a professor's energy both in class and between classes. W. Ted Mahavier describes IBL using a metaphor of teaching someone to cross a stream for the first time: if one puts a stepping stone in the middle of the stream, the student will fall in with a splash and get soaked without learning anything; if one places stepping stones across the stream with only an inch of space between them, then the student will not even notice the stream and hence learn nothing; if one instead carefully places stones a few feet apart, then the student will cross with damp feet while learning some important aspects of balance and foot-eye coordination.

I aim to make every assigned problem at a level that the students can resolve, and only provide extra stepping stones when needed. Every class and every student is different, so the instructor provides positive reinforcement by suggesting lemmas to fill gaps in student reasoning. Following the advice of W. Ted Mahavier, I aim to restrict each office hour visit to 10 minutes to ensure that each student gains an important tip or a better understanding of a definition, without having the problem solved for him or her. Students are also encouraged to check in for five-minute chats any afternoon.

To ensure contact with every student, I have implemented an electronic feedback system. Students upload an informal proof sketch to a free online file-sharing service (e.g., www.dropbox.com) each evening after class (using L^AT_EX, a scanner, a camera, MS Word, or an electronic pen such as a Smartpen). Students are given full credit (1 point) for using relevant definitions and making some progress. No credit is given when a student relies on intuition instead of definitions or ignores a recent relevant class discussion (usually only in the first week of the semester). Feedback is given by annotating the PDF file (often copying and pasting repetitive comments) and uploading it by 5:00 p.m. the night before class. The result is that students use their new tips as if they had visited office hours, and are more likely to come to class prepared to present a difficult proof. In the event that all students have mastered an online submission, discussion of the problem may be skipped. Implementing the system has had the added effect of significantly increasing the pace, and hence the amount, of material covered by students. In one semester students covered an additional chapter of material which formerly was covered by an interactive lecture at the end of the semester.

A significant amount of preparation is required by IBL instructors, who often prepare multiple short lectures to be ready for multiple outcomes of a class discussion. Instructors spend substantial time writing course materials appropriate to their institution, even when basing the course largely on a pre-made script or textbook. Such course materials are significant enough to encompass a publication in the peer-reviewed *Journal of IBL in Mathematics* [6]. Immediately following class, it is important for an instructor to follow up on occurrences of the day by posting any newly posed lemmas or conjectures, scheduling check-ins with students who had almost correct solutions, and determining the upcoming class assignments. Speedy grading and feedback are also important, as is being available for brief check-in visits outside of official office hours; however, quick turn-around and limiting unofficial visits to 10 minutes remarkably decreases the need for office hours over the long run.

Students new to IBL can become demoralized when they are constantly resolving questions about new definitions common in IBL assignments: when nothing becomes as easy as following a calculus algorithm, students taking their first proof-based class sometimes feel that they have learned nothing. Students submit

a journal with their top accomplishments (usually which they presented in class) and top learning outcomes (usually learned from a presentation of a peer). Included in their journals are reflections, which on any given problem can go over a page in length while revealing much about a newfound appreciation of the nature of mathematics. However, early in the semester it can be difficult to convince students that a different way of instruction can be positive, as they are probably where they are now by succeeding in standard lecture-based courses. The simplest intervention is to fill downtime (when students are writing on the board) with short speeches about the benefits of IBL. Past students can help by sharing what they gained from the class and why current students should stick with it. The most time-consuming technique is highly effective: I spend 20 minutes before the semester looking over student schedules to find a new meeting time for half the students, and on the first day of class I surprise the students with a request for those chosen students to meet at the alternative time for the first month. Diminishing the size of the class lessens presentation anxiety, doubles the amount of time that students are expected to be at the board, and allows for more camaraderie. Students become highly motivated and appreciate this extra time so much that by the end one inevitably hears “can we just stay this way for the rest of the semester?”

3 Outcomes

This method has been implemented in various forms by me in eight sections of Introduction to Mathematical Proof, four sections of Theory of Numbers, and three sections of Abstract Algebra. The strength of the method is that it gets students thinking like research mathematicians. While student evaluations were more positive for my first lecture-based Proofs course than my first IBL course, the former was not fulfilling as students simply did not meet expectations of understanding. Overall, IBL was more effective, as is evidenced by grading data (including similar questions on exams across semesters), student evaluations in later iterations of the course, and my opinion of the mathematical aptitude of students finishing the course.

One shortcoming of IBL is that sometimes the students will see less material. A difficult topic might take one day to lecture on but one week for an IBL class to discuss. However, much of such a lecture is lost on the listener, while in an IBL class the students delve deeper into the material to internalize it. In any class, one should consider whether the topic is more important for the lecturer to cover or the students to cover. Every teaching style can go awry if followed too inflexibly, so I have nothing against lecturing on selected topics in IBL courses. It is common knowledge among IBL practitioners that more will be accomplished if students have already had an IBL experience, as it takes time to ease students in to a new way of learning. In fact, a two-semester IBL sequence (with all the same students) can cover more content than a traditional lecture-based course [4].

Educational research on IBL has been positive. We provide an excerpt of a description of a comparison study of students in traditional lecture-based courses and Moore Method (a type of IBL) courses:

The most striking differences were seen in the strategies the students employed when presented with a statement to prove and in their approaches to validating proofs of others. In general, the Moore Method students demonstrated a tendency to make sense of the mathematical ideas, while the traditional students were more concerned with finding quick and easy ways to complete tasks. This suggests that the structure of the Moore Method course can provide students with an opportunity to learn mathematics in a meaningful way. [4, p. 144]

Students in my IBL courses grow to appreciate that mathematics is a growing subject: if there were a list of proof techniques which could solve any problem, then there would be no unsolved problems in mathematics. The book goes into more detail about how students internalize the ideas of their proofs by writing them in clear and meaningful ways as opposed to merely identifying a proof technique from a long list of templates. When the quotation is read to students near the end of the semester, many nod their heads and vocally agree

that it resonates with their transition from previous course experiences in algorithmic-based calculus to a newfound relationship with mathematics in proof-based courses.

A more recent national study by Laursen et al. at the University of Colorado at Boulder describes the impacts of IBL for various groups of students, such as a positive impact for lower-achieving students and a lack of negative impact for high-achieving students [7]. The findings are described in more detail in the companion article by Ernst and Hodge [2].

The National Science Foundation (NSF) has recently made a statement about the impacts of active learning over lecturing in college science, technology, engineering, and math (STEM) courses. They made this press release as a result of the completion of the most comprehensive national comparison study of active learning and lecturing, by Freeman et al. The author Freeman stated that one of the findings was “If you have a course with 100 students signed up, about 34 fail if they get lectured to but only 22 fail if they do active learning, according to our analysis.” As the largest supporter of research on education in the United States, the fact that the NSF has taken this official position demonstrates that active learning will be critical to the future of education in this country [3, 5]. IBL is one way to bring such active learning to our classrooms.

4 Extending the Method

The idea of student centered learning is not unique to IBL. Both flipped classrooms and interactive lectures can actively engage students as well. One feature which seems to currently be unique to IBL and may be extendible to other environments is the focus on student progress between one class and the next. A stereotypical lecture plan concerns the topics, skills, and definitions to be learned in one day. In an IBL class, topics may begin halfway through one class period, while understanding is matured by the discussion and the ensuing work by students between classes. The topic is then mastered during the discussion in the first half of the next class period. Course observations in this setting are more difficult, as the observer should be encouraged to attend two class periods in a row. This alternative lesson plan might be implemented in lecture or flipped classrooms as well.

My method is easily adapted to smaller classes, as it will be easier to get each student involved. For slightly larger classes, a teaching assistant can be in charge of the daily electronic feedback submissions. An automated homework system (e.g., Webwork) can effectively grade some of the more routine Exercises; however, this feedback is limited to “correct” or “incorrect,” so it is important to give ample opportunities for students to make enough educated guesses to learn the correct answer — and an early deadline to allow reflection while preparing for the next class.

For significantly larger classes, frequent student presentations are impossible (except in recitation sections), but the electronic feedback system would increase the caliber of other homework submissions. Without recitation sessions, it may still be possible in large classes to partition students into groups of 20 and take turns judging each others’ submissions; such a system could be used as a way to create an IBL MOOC (massive open online course).

Given the constantly evolving nature of technology, the future may bring systems which make it easier for professors to see students’ daily work. In one semester my students purchased Smartpens for \$90 each, which, when plugged into a computer, uploaded their work automatically to an online file-sharing service. If students completed all their work with such a machine, and if it were inexpensive, had fewer bugs, and uploaded the work wirelessly, then professors could realize a dream of W. Ted Mahavier’s by looking into any struggling student’s work and providing a tip at the most opportune teaching moments.

Instructors interested in IBL are encouraged to contact the Academy of IBL for more information [1]. For those in the northeast, the Greater Upstate New York IBL Consortium is also an excellent resource [8].

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Within ϵ of Independence: An Attempt to Produce Independent Proof-Writers via IBL

Dana C. Ernst and Angie Hodge

Abstract

In many mathematics classrooms, “doing mathematics” means following the rules dictated by the teacher, and “knowing mathematics” means remembering and applying them. However, an inquiry-based-learning (IBL) approach challenges students to create or discover mathematics. According to the *Academy of Inquiry-Based Learning* [1], IBL is a method of teaching that engages students in sense-making activities. Students are given tasks requiring them to conjecture, experiment, explore, and solve problems. Rather than showing facts or a clear, smooth path to a solution, the instructor guides students via well-crafted problems through an adventure in mathematical discovery. In this paper, we discuss one possible implementation of an IBL approach with an emphasis on developing independent thinkers and proof-writers. Nuts-and-bolts details are provided to help instructors implement the method in their proof-based mathematics courses.

Difficulty Level: High; Course Level: Advanced

1 Background and Context

We have been employing an inquiry-based learning (IBL) approach in a variety of courses for several years. Prior to making the transition to IBL, by standard metrics we were excellent teachers. Glowing student and peer evaluations, teaching awards, indicated that we were effectively doing our job. However, two observations made each of us reconsider how well we were really doing. Namely, many of our students seemed to heavily depend on us to be successful and retain only some of what we had taught them. To address these concerns, we began transitioning away from direct-instruction (i.e., lecturing) and towards a more student-centered approach. The goals and philosophy behind IBL resonate deeply with our ideals, which is why we have embraced this paradigm (with no negative impact on our evaluations). In a 2013 article, D. Retsek conveys his shift to IBL as an attempt to deal with retention [14].

This article describes the nuts and bolts of one possible implementation of an IBL approach in proof-based mathematics courses. Its goal is that the reader be able to replicate our approach in his or her courses. While we firmly believe our approach is effective, we readily admit there is no ideal method for teaching that works for all instructors and for all students. Our expectation is that readers will see ways to modify and even improve on what we describe here. While we attempt to provide specific details about course structure, it is beyond the scope of this article to address all aspects of the course. We refer the reader to articles by D. Chalice and W.T. Mahavier for additional information [3, 11].

The first author has experience implementing the approach at two institutions—Plymouth State University and Northern Arizona University—and in several courses, including number theory, abstract algebra, real analysis, and an introduction-to-proof course. Plymouth State University is a regional comprehensive university located in central New Hampshire and is part of their university system. PSU is a residential and predominately undergraduate institution with an enrollment of approximately 4300 undergraduate students. The mathematics department at PSU offers a Bachelor of Science degree in mathematics, and each year there are roughly 50 mathematics majors (about 1% of the undergraduate population). Majors either pursue

a general degree in mathematics or one of two teacher certification options (secondary or middle school). The ratios vary from year to year, but typically the majority of the majors pursue the Secondary Teacher Certification option. Northern Arizona University's main campus, located in Flagstaff, AZ, has an enrollment of 18,000 students, which includes graduate students. The Carnegie Classification of Institutions of Higher Education classifies NAU as a research university with high research activity. Each year there are approximately 200 majors in the Department of Mathematics and Statistics split evenly between a Bachelor of Science in Mathematics and a Bachelor of Science in Secondary Education in Mathematics.

The classes in which the method has been employed have varied in size from 10 to 35 (mostly juniors and seniors) and typically include a mixture of mathematics and mathematics education majors, and the occasional non-mathematics major. In our experience, fewer than 30 students is ideal. In larger classes, modifications are likely necessary. Apart from class size, we believe that what we outline is appropriate for any proof-based course at any institution.

As our title suggests, one of our goals is for students to develop independence from the instructor. By “independence,” we mean that students are not reliant on the instructor for guidance and validation. In our view, nothing else that we teach them will be half as valuable or powerful as the ability to reach conclusions by reasoning logically and being able to justify them in clear, persuasive language. We want our students to experience the unmistakable feeling that comes when one truly understands something. For many students, the only way they know whether they are “getting it” comes from the grade they make on an exam. The goal is for our students to become less reliant on such externals.

2 Description and Implementation

2.1 What is Inquiry-Based Learning?

First, it is important to point out that there is no universal definition of IBL. One of the difficulties is that it manifests itself in different ways in different contexts. Boiled down to its essence, IBL is a method of teaching that engages students in sense-making activities. According to the *Academy of Inquiry-Based Learning* [1], students are given tasks requiring them to solve problems, conjecture, experiment, explore, create, and communicate—all those wonderful skills and habits of mind that mathematicians have. Rather than showing facts or a clear, smooth path to a solution, the instructor guides students via well-crafted problems through an adventure in mathematical discovery. In other words, get out of the way and see what your students can do.

IBL has its roots in an instructional delivery method known as the Moore Method, named after R.L. Moore. Loosely speaking, the majority of a Moore Method course consists of students' presenting proofs or solutions that they have produced independently from material provided by the instructor; other students in the class are then responsible for determining the validity of a presented solution and—in the case that a presented proof is flawed—in making suggestions for how a given proof could be adjusted so that it is valid. In a traditional Moore Method course, students are discouraged (in fact, forbidden) to collaborate in the solution process [3, 10]. This is one significant difference from the approach we describe here, where student collaboration on most assignments was allowed and encouraged.

Variations of the Moore Method take many forms and are often referred to by the generic name modified Moore Method. In fact, the approach to IBL we describe is certainly a modified Moore Method. In our companion article [13], P. Rault discusses in detail his modified Moore Method for an introduction-to-proof course. For more detailed information on Moore, his method, and its history, we refer the reader to the books by C. Coppin, et al. and J. Parker [4, 12].

2.2 Course Materials

For proof-based courses, an instructor may not be able to effectively adapt a traditional textbook to an IBL approach. One of the key differences between a traditional textbook and course materials designed for IBL is that the former typically proves the key (and often interesting) theorems of the course. In contrast, IBL materials usually ask the students to produce the proofs of the important theorems—and maybe even their statements.

Many seasoned practitioners of IBL create their own course materials that often take the form of a sequence of tasks. One set-up for such a sequence is that students encounter a definition, play with some examples, conjecture a theorem, and then attempt to prove it. This is in contrast to the typical textbook approach where a set of exercises appears after several pages of prose. The intent with most IBL materials is that students will complete the exercises and problems and write a proof for each of the theorems as they progress through the reading.

Creating IBL materials from scratch is a serious undertaking. For instructors that are new to IBL, we recommend utilizing pre-existing resources. In addition to numerous free course materials scattered across the internet (including the first author's webpage), the *Journal of Inquiry-Based Learning in Mathematics* [2] publishes university-level course notes that are freely downloadable, professionally refereed, and classroom-tested. If one prefers, there are also several textbooks available specifically for an IBL approach (e.g., Marshall, Odell, and Starbird [5] and Schumacher [15]).

In Appendix A, we have provided an example of a portion of IBL materials for an undergraduate abstract algebra course. We have included a chapter from the first author's open-source IBL task sequence for a first course in abstract algebra that focuses on groups and emphasizes visualization as a pedagogical approach [6]. We hope that the example will provide the reader with a sense of the types of activities that a student would work on outside of class and then present in class. As students read the notes, they are required to interact with the material in a meaningful way. The items labelled as Definition, Remark, and Example are meant to be read and digested. However, the items labelled as Exercise, Theorem, Corollary, and Problem require action on the student's part. Items labelled as Exercise are typically computational and are aimed at improving understanding of a concept. Items with the Theorem and Corollary designation are mathematical facts and the intention is for the students to produce a valid proof of the given statement. The items labelled as Problem are a mixed bag. It is important to point out that there are very few examples in the notes, which is intentional. One of the goals of the exercises and problems is for the students to produce the examples.

2.3 Overview of Day-to-Day Operation

The next few subsections provide details about the day-to-day operation of our implementation of IBL. Generally, students are responsible for digesting new material outside of class by completing assigned tasks from the course notes (or textbook). They make up the Daily Homework, which we describe in Section 2.5. In each class, Daily Homework is assigned that is due the next class meeting. Nearly all class time is then devoted to students' presenting their proposed solutions and/or proofs. We elaborate on the student-led presentations in Section 2.4.

Within reason, we allow students to request mini-lectures on topics at any time. The typical length of a mini-lecture is 5–10 minutes. On average, students make requests for direct instruction once or twice a week and with experience one can anticipate when the requests will occur. Students need to be appropriately supported, but instructors must take care not to immediately default back to lecturing when students are struggling. Otherwise, students may become dependent on the instructor rather than on themselves and each other.

Since the format of class is different from the standard direct-instruction approach with which most students are familiar, we spend time during the beginning of the semester (and, in smaller ways, during the

course of the semester) making sure the students understand the benefits of the method, and emphasizing the roles that the students will be performing in class.

2.4 Student Presentations

The student-led presentations form the backbone of the class as this is what we spend the bulk of time on each day. The norm is to have one student at a time presenting to the class. However, if there are several easy exercises to present, we may have multiple students go to the board at once to simultaneously write their solutions and then take turns discussing them.

Having experimented with different methods, the approach we now take is to have students write out their proposed proofs or solutions as they present. That is, presenters should explain their reasoning as they go along, not simply write everything down and then turn to explain. While this may seem slow to the instructor, we believe it provides the audience with time to process and it allows speakers to practice their presentation skills as opposed to rushing through a pre-written solution. The number of problems presented in a 50-minute class meeting varies, but the norm is roughly five. For proofs, presenters are required to write in complete sentences and use proper grammar. This provides ample opportunities to discuss the finer points of formal composition, which we have found to drastically improve student writing.

In general, the instructor's goal should be to minimize involvement during student presentations. However, his or her role is to keep the class on task, facilitate discussion, and provide feedback. The feedback should come in both the "coaching" and "cheerleading" varieties. When the class or presenter is stuck, we will often interject with some suggestions to keep the ball rolling.

Typically, students volunteer to present. The expectation is that at the beginning of class volunteers write their names on the board next to the problems they would like to present. If multiple students volunteer for the same problem, the student with the lowest number of presentations has priority. If no student volunteers, an attempt will be made to call on a student, or possibly the problem will be saved for small-group work at the end of class. As an alternative, Rault utilizes a computer program for choosing presenters [13]. We reserve the right to modify the list of volunteers if we feel that it will lead to more fruitful classroom discussions.

It is the job of the audience to determine the validity of a presented proof or solution, and make comments on the execution of the proof itself. A student's presentation could be interrupted at any time by questions or comments from the class, and the presenter is expected to act as the discussion leader.

Presentations are graded using the rubric given in Table 1. In practice, students almost never receive a grade below a 2 and the most common grade is a 3. Also, we often break the rubric into half-point increments (e.g., a score of 2.5 may be given). It is important that the rubric not deter students from presenting. If a person has an idea about a proof that he or she would like to present, but is concerned that the proof is incomplete or incorrect, the student should be rewarded for being courageous and sharing creative ideas. For this reason, a student's overall presentation grade is predominately a function of how many times he or she came to the board as opposed to the average of their presentation scores. Yet, students should not come to the board to present unless they have spent time thinking about the problem and have something meaningful to contribute.

Students should strive to present a few challenging problems instead of tackling only easier problems. However, they must be coached in this regard; the instructor should not be afraid to provide feedback to individual students about the frequency with which they are presenting and the difficulty level of the problems they are choosing to present. We make it clear that earning a 3 on a difficult problem that generated lots of discussion is more valuable to the class as a whole than scoring a perfect 4 on an easy exercise.

Unless a student asks, we do not share scores of presentations. If students are interested in seeing their presentation scores, we ask that they come to office hours to discuss their progress. Our reasoning is that we want to encourage risk-taking and emphasize the discussion of mathematics rather than have students focused on their grades.

Grade	Criteria
4	Completely correct and clear proof or solution. Yay!
3	Proof has minor technical flaws, some unclear language, or is lacking some details. Essentially correct.
2	A partial explanation or proof is provided but a significant gap still exists to reach a full solution or proof.
1	Minimal progress has been made that includes relevant information and could lead to a proof or solution.
0	You were completely unprepared.

Table 1: Grading rubric for student presentations.

A student must present at least twice prior to each exam (two midterms and one final) for a total of at least six times during the semester to receive a passing grade on presentations. The purpose here is to provide a few benchmarks. The minimum number may need to be modified in smaller or larger classes. In general, a student's presentation grade is determined by taking into account the number of times the individual presented, the average of the presentation scores, the level of difficulty of the problems that the student chose to present, and the level of interaction during others' presentations. A student's overall presentation grade is worth 30% of his or her overall grade.

Inevitably, there will be problems that no one volunteers to present. In this case, we try to get a sense of why there were no volunteers. Did students attempt the problem but not make any progress? Or, did they not even make an attempt? In the first scenario, we may try calling on someone to show the class what that student tried. We are prepared to adapt if no one is willing to present even after being called on. Another technique is to break students into small groups to work on the problem right there in class. Often we will break the problem up into multiple parts and have each group work on a different part.

As a final comment about presentations, we always find something positive to say about a presentation. Our goal is for the feedback to be more specific than "Nice job." We also encourage—and in some sense coach—our students to provide positive feedback. One simple tactic we employ is to conclude each presentation with "Any comments, questions, or compliments?"

2.5 Homework

We give two types of homework assignments (Daily and Weekly), each with an intended purpose. In our view, this structure is a solution to an optimization problem that all instructors are faced with: maximize the quality of the experience for the student, and the usefulness of feedback, while minimizing the amount of time spent grading. In a typical semester, we assign 30–35 Daily Homework assignments and 10–12 Weekly Homework assignments. To account for the variety of reasons students may not complete an assignment on time, we allow each student to turn in up to five late homework assignments (Daily or Weekly) with no questions asked. In our experience, it is rare that a large portion of the class takes advantage of this opportunity at the same time. A student's overall homework grade is worth 25% of his or her final grade. Rault's companion article outlines a more elaborate grading system that involves electronic feedback [13].

Daily Homework

Daily Homework is assigned at each class meeting, and students are expected to complete (or try their best to complete) each assignment before the next class session. All assignments should be carefully, clearly, and cleanly written. Among other things, this includes proper grammar, punctuation, and spelling. However, the

work done on the assignments is not necessarily intended to be perfect. Each Daily Homework assignment consists of tasks (e.g., completing exercises and proving theorems) from the course notes or textbook. On the day that a homework assignment is due, the majority of the class session is devoted to students presenting some subset (maybe all) of their proposed proofs or solutions.

Students are allowed—in fact, strongly encouraged—to modify their written work in light of presentations made in class; however, they are required to use a colored marker pen that is provided at the beginning of each class. Students can annotate their work as much as they like and there is no penalty for using the colored pen. At the end of class, students submit their work and the assignment is then graded $\checkmark -$, \checkmark , or $\checkmark +$ based on how much work they complete before class.

There are several advantages to our approach to Daily Homework. First, students are encouraged to attempt every problem, but are not severely punished for making an error on their first attempt to learn new material. At the end of each class, students have at least a sketch of a proof/solution for most of the assigned problems. The Daily Homework more or less becomes the students' notes. When they look back at their notes, they see their comments about what they were thinking together with their corrected mistakes. Moreover, when grading the homework, we know which problems were attempted before class began versus what annotations were made during class. In addition, the grading of the Daily Homework requires minimal effort, and as a result, Daily Homework is easily returned the next class session. Lastly, the colored-pen strategy is very popular among students. They have repeatedly indicated in course evaluations that they value being able to annotate their work.

Weekly Homework

In addition to the Daily Homework, students are required to submit two formally-written proofs each week. During week n , students submit any two problems marked with a \star from the Daily Homework that was turned in during week $n - 1$. The \star -problems are typically a subset of the medium-to-difficult proofs. In general, the Weekly Homework assignments are due on a non-class session day so that they do not interfere with the Daily Homework. Students have the option of dropping the homework off in person or emailing a PDF file.

One advantage is that students are forced to reflect on the previous week's work and it allows them another opportunity to learn the material if they did not master it the first time. Another advantage is that since students are essentially completing a problem for a second time, the quality of the work is quite good. As a consequence, the grading of the Weekly Homework goes smoothly and at the same time we feel comfortable holding high standards. The Weekly Homework assignments are graded with a 0–4 point rubric that is a slight modification of the one given in Table 1.

Beginning with the second assignment, students are required to typeset their Weekly Homework submissions using \LaTeX . In the past there were many barriers to this requirement, but with the advent of free online \LaTeX editors (e.g., writeLaTeX) many of the obstacles have been removed. The first author has a button on the course webpage that will open a pre-filled \LaTeX template designed specifically for his course. It includes the course title and placeholders for the student's name, the statements of two theorems, and their proofs. We provide students with substantial assistance for about two weeks and then they are more or less self-sufficient with regard to using \LaTeX for typing up their homework.

2.6 Exams

In a typical semester, there are two midterm exams and a cumulative final exam. Each exam is worth 15% of a student's overall grade and consists of an in-class portion and a take-home portion. The weights of the in-class versus take-home portions vary, but roughly 75% and 25%, respectively, is common. For both portions of the exam, the proof-type questions are graded using the same rubric for Weekly Homework.

The in-class portion of each exam consists of routine exercises, problems where students are asked to provide an example or counterexample having some set of desired properties, and theorems requiring a basic proof. Typically, students are provided with several theorems and asked to prove some subset from the list. As one would expect, students must work independently and without notes. For the take-home portion of each exam, students are provided with a list of theorems that often includes, but is not limited to, the theorems from the in-class exam. Students are then asked to prove some proper subset from this list. If theorems from the in-class exam appear on the list, a student is not allowed to choose a theorem he or she attempted on the in-class portion.

3 Outcomes

Much of the research on the impact and effectiveness of IBL (across a wider variety of courses than just proof-based) have been in the form of small-scale studies. Currently, the most compelling evidence comes from the longitudinal study of S. Laursen et al. [9] that was referred to in Section 2.1. On a research-based test of students' ability to evaluate proofs, IBL students showed evidence of greater skill in recognizing valid and invalid arguments and the use of expert-like reasoning in making evaluations. After an IBL or comparative course, the study found that IBL students reported higher learning gains than their non-IBL peers, across cognitive, affective, and collaborative domains of learning. In addition, IBL students' attitudes and beliefs changed pre- to post-course in ways that are known to be more supportive of learning, compared to students who took a non-IBL course. The results certainly resonate with our experience, where students in IBL classes appear to develop stronger proof-writing skills and are more receptive to learning mathematics in general than students in a traditional setting.

Additionally, the Laursen paper also indicates that non-IBL courses showed a marked gender gap—women reported lower learning gains and less supportive attitudes than did men. Moreover, women's interest in continuing to study mathematics was lower. However, this gender gap was erased in IBL classes, but with no harm to men's attitudes and interests.

For three consecutive semesters, the first author taught an introduction-to-proof course to both mathematics and mathematics education majors at Plymouth State University. The first two iterations of the course were taught via a traditional lecture-based approach, where students engaged in the process of proof while working on homework or exam problems. The third instance of the course was taught using an IBL approach similar to what we have described here. When he taught an abstract algebra course containing students from both styles of the introduction-to-proof course, anecdotal evidence suggested that the students taught via IBL were stronger proof-writers and more independent as learners than those introduced to proof via direct instruction. Inspired by the apparent effectiveness of IBL, he has chosen to implement IBL in all of his proof-based courses and to a lesser extent in lower-level courses.

On two occasions, the second author, a math education specialist, was recruited to assist with conducting small-scale studies aimed at exploring the issue of student independence, as well as student perception of the method's effectiveness [7]. Our results are consistent with those mentioned. For example, students in the first author's real analysis course were asked to respond to the following statement on a post-test survey: "Please describe any advantages to the method with which this class was taught that you haven't mentioned already." Below are two responses that directly address independence.

The advantage of participating in a class like this is I am more confident of [sic] myself being an independent learner.

It forces students to take on concepts on their own, and be independent.

In Section 1, we described two underlying concerns we had when predominately teaching via direct instruction. Namely, many of our students seemed to heavily depend on us to be successful, and retain only

some of what we had taught them. However, since transitioning to IBL, we strongly believe that we have taken great strides in mitigating both issues.

4 Extending the Method

The approach we described could be used in the same manner if the class were small. It would mean that each student would have more time at the board to present problems. If the class contained only two or three students, however, the instructor would have to occasionally help generate questions to presenters and give hints on proofs. It is when the class is larger than, say, 30 students, that our approach to IBL becomes more challenging. One suggestion for larger groups is to split the class in half and use a classroom with whiteboards or chalkboards on at least two sides of the room. The instructor could then employ the same teaching technique as described, going from one side of the room to the other. In addition, one could easily use more small-group work to accommodate the larger class. In any case, we see the model as replicable in most classrooms and would be happy to answer any questions others have about the teaching technique.

Despite the apparent success, there are always improvements that could be made to make a course even better. One possible improvement would be to include more structured in-class activities, where students are given an opportunity to collaborate with a small set of peers. Another possibility is to free up more class time by off-loading many of the more computational exercises to an online homework platform such as WeBWork. In addition, instructors could incorporate aspects of the flipped classroom approach by creating videos for the students to watch outside of class that contain the content of the mini-lectures.

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A Example of IBL Course Materials for Abstract Algebra

Below is an example of a portion of IBL materials for an undergraduate abstract algebra course. It is most of Section 5.6 of the first author’s open-source IBL task sequence for a first course in abstract algebra that focuses on groups and emphasizes visualization as a pedagogical approach [6]. The work is licensed under a Creative Commons Attribution-ShareAlike 3.0 license.

Chapter 5. A Formal Approach to Groups

5.6 Revisiting Isomorphisms

Suppose $(G_1, *)$ and (G_2, \circ) are two groups. Recall that G_1 and G_2 are isomorphic, written $G_1 \cong G_2$, provided that we can choose generating sets for G_1 and G_2 , respectively, so that the Cayley diagrams for both groups are identical (ignoring the labels on the vertices). When two groups are isomorphic, it means that they have identical structure up to relabeling the names of the elements of the groups.

One consequence of two groups being isomorphic is that there is a one-to-one correspondence between the elements of the groups. This correspondence is referred to as an isomorphism. In other words, an isomorphism is a one-to-one and onto function that preserves the structure of the two groups.

Having an isomorphism between two groups immediately implies that they have the same order, i.e., $|G_1| = |G_2|$ (see Theorem 4.19). However, it is extremely important to remember that two groups having the same order does *not* imply that the two groups are isomorphic. Said another way, having a one-to-one correspondence between two groups does not imply that the two groups are isomorphic. They must also have the same structure!

Exercise 5.74. Provide an example of two groups that have the same order but are not isomorphic.

After we introduced group tables, we also discussed the fact that $G_1 \cong G_2$ exactly when we can arrange the rows and columns and color elements in such a way that the colorings for the two group tables agree (see Problem 5.39). The upshot of this is that if $G_1 \cong G_2$, then

the product of corresponding elements yields the corresponding result.

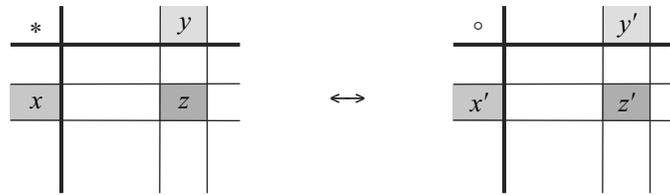


Figure 1:

This is the essence of what it means for two groups to have the same structure.

Let's try to make a little more sense of this. Suppose that $G_1 \cong G_2$ and imagine we have arranged the rows and columns of their respective group tables and colored the elements in such a way that the colorings for the two group tables agree. Now, let $x, y \in G_1$. Then these two elements have corresponding elements in the group table for G_2 , say x' and y' , respectively. In other words, x and x' have the same color while y and y' have the same color. Since G_1 is closed under its binary operation $*$, there exists $z \in G_1$ such that $z = x * y$. There must exist a $z' \in G_2$ such that z' has the same color as z . What must be true of $x' \circ y'$? Since the two tables exhibit the same color pattern, it must be the case that $z' = x' \circ y'$. This is what it means for the product of corresponding elements to yield the corresponding result. Figure 1 depicts this phenomenon for group tables.

We can describe the isomorphism between G_1 and G_2 using a function. Let $\phi : G_1 \rightarrow G_2$ be the one-to-one and onto function that maps elements of G_1 to their corresponding elements in G_2 . Then $\phi(x) = x'$, $\phi(y) = y'$, and $\phi(z) = z'$. Since $z' = x' \circ y'$, we can obtain

$$\phi(x * y) = \phi(z) = z' = x' \circ y' = \phi(x) \circ \phi(y).$$

In summary, it must be the case that

$$\phi(x * y) = \phi(x) \circ \phi(y).$$

We are now prepared to state a formal definition of what it means for two groups to be isomorphic.

Definition 5.75. Let $(G_1, *)$ and (G_2, \circ) be two groups. Then G_1 is **isomorphic** to G_2 , written $G_1 \cong G_2$, if and only if there exists a one-to-one and onto function $\phi : G_1 \rightarrow G_2$ such that

$$\phi(x * y) = \phi(x) \circ \phi(y). \quad (1)$$

The function ϕ is referred to as an **isomorphism**. Equation 1 is often referred to as the **homomorphic property**.

You should definitely take a few minutes to convince yourself that the above definition agrees with our previous informal approach to isomorphisms. For those of you that have had linear algebra, notice that our homomorphic property looks a lot like the requirement for a function on vector spaces to be a linear transformation. Linear transformations preserve the algebraic structure of vector spaces while the homomorphic property is preserving the algebraic structure of groups.

We've seen several instances of two groups being isomorphic, but now that we have a formal definition, we can open the door to more possibilities.

Problem 5.76. Consider the groups $(\mathbb{R}, +)$ and (\mathbb{R}^+, \cdot) , where \mathbb{R}^+ is the set of positive real numbers. It turns out that these two groups are isomorphic, but this would be difficult to discover using our previous techniques because the groups are infinite. Define $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$ via $\phi(r) = e^r$ (where e is the natural base, not the identity). Prove that ϕ is an isomorphism.

Exercise 5.77. For each of the following pairs of groups, determine whether the given function is an isomorphism from the first group to the second group.

- (a) $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$, $\phi(n) = n + 1$.
- (b) $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$, $\phi(n) = -n$.
- (c) $(\mathbb{Q}, +)$ and $(\mathbb{Q}, +)$, $\phi(x) = x/2$.

Problem 5.78. Show that the groups $(\mathbb{Z}, +)$ and $(2\mathbb{Z}, +)$ are isomorphic.

Perhaps one surprising consequence of the previous problem is that when dealing with infinite groups, a group can have a proper subgroup that it is isomorphic to. Of course, this never happens with finite groups.

Once we know that two groups are isomorphic, there are lots of interesting things we can say. The next theorem tells us that isomorphisms map the identity element of one group to the identity of the second group. It was already clear that this was the case using our informal definition of isomorphic. Prove the next theorem using Definition 5.75

Theorem 5.79. Suppose $\phi : G_1 \rightarrow G_2$ is an isomorphism from the group $(G_1, *)$ to the group (G_2, \circ) . If e and e' are the identity elements of G_1 and G_2 , respectively, then $\phi(e) = e'$.

Theorem 5.80. Suppose $\phi : G_1 \rightarrow G_2$ is an isomorphism from the group $(G_1, *)$ to the group (G_2, \circ) . Then $\phi(g^{-1}) = [\phi(g)]^{-1}$.

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Reading, ‘Riting, and Reals: Proofs in a Reading- and Writing-Intensive Real Analysis Class

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Abstract

I share an approach to proof-writing in a real analysis course that may be adapted to other upper-level courses. It emphasizes reading and writing mathematics to learn proof-writing in a flipped classroom setting, where students primarily work on problems in class while learning content independently outside of class through carefully scaffolded reading and writing activities. The reading and writing activities provide basic skills and models for writing mathematical proofs. Learning to write proofs then takes place in a framework wherein students draft, edit, rewrite, and eventually typeset, professional quality proofs. They become formal solution sets for the class, which are posted and used by the students to assess their individual work. Class resource materials and samples are included in the Appendix.

Difficulty Level: High; Course Level: Advanced

1 Background and Context

I would like to share here an approach to teaching proof-writing in the context of a real analysis course, with the core premise being that students must systematically read and write mathematics as the foundation for creating well-constructed proofs. Students first learn to read proofs, and to read mathematics in general, by actively engaging with the text to internalize the content. They then work to synthesize mathematical ideas, and to communicate them in writing effectively and with stylistic correctness. Finally, they learn to construct proofs by writing and rewriting them until their logic and exposition meet the level of the published mathematics they have been reading. Through the course, students should make a significant leap in the transition from novice to professional mathematician, in both their understanding and their communication of mathematics.

The course is a two-semester sequence, Real Analysis I and Real Analysis II, taught at Saint Michael’s College, a Catholic liberal arts college with about 2,000 students in northern Vermont. Real Analysis I is required for the major, and Real Analysis II is strongly recommended for students considering graduate school. They are among the most challenging and rigorous courses offered, and are typically cross-listed as graduate courses at institutions offering graduate degrees in mathematics. Class size ranges from about 4 to 18 students, generally with fewer students in the second course than in the first. Enrolled students are typically junior or senior mathematics majors or double majors, generally solid and hardworking, but not necessarily gifted. I chose *Mathematical Analysis, 2nd edition*, by Thomas Apostol [1], as the text for reasons discussed below.

When I developed the courses, Real Analysis I was the designated writing-intensive course for our major. Its aim is to build discipline-specific written communication skills, which in mathematics largely means being able to write good proofs. Because of the importance of this, I structured Real Analysis II as writing intensive as well. Thus, in addition to providing graduate-level content in real analysis, there are two major emphases for the courses. We move from a student-learning model to a professional model of doing mathematics, and we focus on reading and (especially) writing mathematics on a practitioner level.

The shift in emphasis led to a shift in pedagogy. In a lecture-style mathematics course, the professor may digest, augment, and present the text material to students so effectively that the students may only need to use the text as a reference and source of exercises. Thus, students who have had primarily lecture courses, particularly from gifted communicators, may become seniors without having really felt the need to read a text. However, as students mature mathematically, the source of new ideas should shift gradually away from the lecture to the text itself, since practicing mathematicians gain most of their new information from texts and journal articles. Furthermore, I wanted students to experience the repeated revision process central to publication-quality proof-writing and mathematical exposition.

As a result, the two courses have a very non-traditional structure. Although the terminology was not yet in vogue when I first designed this course, the pedagogical approach would probably now be described as a flipped or inverted classroom, with in-class problem-solving instead of lectures, and with highly structured out-of-class work for learning new content. The out-of-class work emphasizes reading mathematics, writing synthesis, and especially writing and rewriting proofs. I lecture very rarely, and then only upon specific request from the students. Instead students work with each other to read, digest, assimilate, present, and ultimately write, edit, and rewrite the material of the course. Despite the change of format, the breadth of coverage remained consistent with previous offerings of the course, while the depth increased somewhat, although at a cost of increased student work-hours.

Both Real Analysis I and II have three components, which, with a bit of wryness, I have come to think of as reading, 'riting, and 'rithmetic (or in this case, reals). Although the focus of both courses and of this article, is on writing proofs, because the reading and writing are so essential to the proof process they are weighted equally. For the 'reading' component of the course, students learn to read mathematics from a text independently through guided analysis of the text. In the ''riting' component, students write papers demonstrating synthesis and thorough understanding of the course materials. The ''rithmetic' component involves the mathematical problem sets at the heart of any mathematics course, and it is here that the explicit practice of proof-writing emerges. For the problems, students must ultimately produce proofs that are entirely correct in substance and in form.

2 Description and Implementation

The methodology for the course is similarly organized into the three components.

2.1 Reading: Understanding and Analyzing Good Proofs

In order to write good proofs, students must first study good proofs. This is the primary reason for the choice of Apostol's *Mathematical Analysis* as the text. It is very cleanly written, with clear mathematical formalism that provides an excellent model for students. Because there is no lecture component to the course, students learn the course content through reading the text, which is consistent with the course goal of moving students closer to working as professional mathematicians. The text is well-suited to this purpose, so that students demonstrate mastery of the material by the end of the course without having me interpret the text systematically for them.

Reading mathematics is strenuous; it can take an hour or more to read a single paragraph, often longer to actually understand it, and readers must generate examples for themselves and correlate material from several other sections of the text (not to mention previous courses). Since this is a new experience for most of the students in the class, I provide them with detailed reading guides for each chapter to help them begin to internalize this process (a sample reading guide for one chapter is included in the Appendix). Students eventually develop and articulate general reading guidelines for themselves and the class as a whole. They prepare reading notebooks of detailed glosses on the text from their own studies combined with the examples and examinations specified by the reading guides. Students analyze the proof structures to use them as a

model for their work. These notes serve the pedagogical function of lecture notes in a lecture-style class, and I collect and review them. They also form a basis for the writing assignments of the class.

Students seem to internalize the demands of reading a mathematical text quickly. After only the first two chapters, I ask them to include in their notes general reading guidelines based on their experience reading the book. Here is a representative individual response: 1) don't take definitions and theorems and proofs to be undoubtedly true; 2) investigate and consider other contexts of the material (other math classes, other focuses in general); 3) generate examples beyond what is given to grasp the material better; 4) try to prove theorems in different, sneakier ways; 5) don't leave a proof until you are fully satisfied, since most times you must make perfect sense of a proof on page 10 in order to understand a proof on page 22.

By the end of the course, the class consolidates a general set of reading guidelines: 1) When you read a theorem statement, stop and ask yourself what does this mean? Try to come up with examples that the theorem applies to (e.g., specific functions) and include these in your notes. Counter-examples are often even more helpful to understanding than examples. 2) Make sure you understand proofs at least line-by-line. If there are steps missing, and there often are, write these steps out in your notebook. If a proof refers to an earlier result or definition, write those down in your notebook to have them handy as you are studying the proof. Sometimes simply copying the proof out in your notebook will help you understand it. 3) Make a note of anything you would like to have discussed in class. Be as specific as possible, e.g. How does line 4 of the proof of Theorem Y follow from Definition X? 4) After you have read a proof, stop and just think about the theorem. What does it mean? How can it be used? What other questions does it make you think of? Can the proof be adapted to other applications?

2.2 'Riting: Synthesis of Concepts and Proof Tools

Because there are no lectures and no exams in this class, students must have some other mechanism to synthesize the material they are learning and to demonstrate their mastery of it. Also, good proof-writing relies on understanding many of the same basic principles as good expository writing, so students must develop facility with the latter to achieve the former.

I first provide a set of resources on writing formal mathematics (listed in the Appendix), both for expository conventions and for certain proof formalism, such as proof by induction, that students use for their own writing and for reference when refereeing. In Real Analysis I, students write three papers. The first is on foundational concepts of the formal definitions of limits and continuity, introduced in first-semester calculus but then developed fully in Real Analysis I. It serves two purposes: it provides a necessary review and lets students focus primarily on the exposition (and the nontrivial, and occasionally time-consuming, challenges of typesetting mathematics) without having to learn too much new content. The second and third papers cover the current class material. The framework is similar in Real Analysis II, except the papers are solely on the course content without a warm-up paper in the beginning. In their papers, students must explain why various concepts and theorems are important and how they interrelate. They give outlines of proofs and descriptions of proof techniques, and also illustrate central concepts with examples they have constructed themselves. I have included the paper guidelines for Real Analysis I in the Appendix.

Each of these papers is peer reviewed by two classmates. For the first round I evaluate, not so much the paper itself, but the quality of the peer reviewing, since an important part of working as a mathematician is developing the ability to read mathematics critically. The students then revise their papers based on the peer reviews and submit them to me. I provide further feedback, this time focusing on the paper, and, for the third round, the students do one or more revisions until the paper is satisfactory. The number of revisions for the final paper is restricted by the end of the term, so by the conclusion of the course, students must demonstrate independent writing skills without relying on my edits. I also provide mock referee reports modeled on actual journal reviews for the students to use as models for their reviewing.

The first time I taught the course, I found myself tediously repeating the same feedback for nearly all

students on the limits and continuity paper. Hence I subsequently compiled a list of comments so that I can give this feedback before the students start writing, and thus edit preemptively. This list is included in the Appendix. (Editors' note: alternatively, one might consider using the *Elements of Style* guide by Hendrickson, in this volume [2].)

2.3 'Rithmetic: Doing the 'Real' Proofs

From my experience as a student and from observing students in my classes, it seems that people recall most strongly what they last did completely for themselves. For example, if a problem was done incorrectly, that incorrect process seems to remain more strongly in memory than any red-ink corrections written over it by someone else, unless the students in some way process the corrections themselves. In order for students to most strongly retain exemplary proofs, they must revise their work until they achieve proficiency. Furthermore, the process of repeated revision models the typical efforts of working mathematicians, which is a valuable experience for students, particularly those considering graduate school.

The proof-writing and revision process begins during class time, which is largely devoted to problem-solving where the students put exercises on the board and discuss them. Each student writes up all the problems by hand. I do not formally grade them at this point, but I do check them over to be sure that each student is actively engaged in the process. The first-draft problems are circulated, and the students do a first markup of them for each other. The students are then responsible for typing up one or two problems in formal textbook language, following the posted resources on writing mathematics, but working from someone else's rough draft (not their own, and not the same person whose first draft they marked up). The typeset exercises are then given to two other students for peer review and editing. While this is going on, students each write by hand (neatly and carefully) their final versions of the entire problem set, incorporating the markup that was done by their peers. The formally typed solutions and proofs come to me, and I make a last round of detailed edits and return the changes to the students to finalize (note that I need only make detailed comments once for each exercise). When all the problems are typeset and meet professional standards, I collate them into a single document and post them to the course website. The goal is for this final class-generated solution set to be publishable perfect, that is, something that could appear in a textbook. As a final processing step, the students must then mark up their own final handwritten problem sets based on the class-generated solution set. This final version, with the markup, is what I finally grade for each student.

Course logistics with the repeated peer reviews of both the papers and the proofs are challenging. I devise a rotation scheme so that each student sees as many different problems and edits for as many different classmates as possible (this is a fun exercise in combinatorial design theory in and of itself!). I find it useful to publish a detailed calendar of due dates at the start of the semester, and use it to justify admittedly draconian penalties for missed deadlines. (One of these calendars is included in the Appendix.) However, students quickly realize the negative impact of any late work on their peers, and because of this generally work hard to keep everything on schedule.

This paradigm requires a lot of grading. However, I find the grading much more interesting and rewarding than in a more traditional class where I would be essentially repeatedly grading a large number of more or less similar first-draft problem sets in painstaking detail. Here, the work is not repetitive in that I grade each version of each problem only once, when I receive the (easy-to-read!) typeset version, and so can respond in depth without becoming frustrated by having to repeat comments on several papers. When I get the final, self-corrected, handwritten problem sets, I am looking for evidence that students have fully processed a *correct* and stylistically mature proof for each problem. However, since they have worked so intensively with each proof, this is usually a very quick check, and because they have made their own comments on any of their errors based on the class-generated solution set, there is often very little for me to add.

I also find providing feedback in this way well worth the effort because the students must thoroughly process the corrections to all the problems instead of just checking the grade at the top of their papers without

internalizing all the careful comments. The peer-review process is also essential for a deep understanding of the material, since students must not only read critically, but also provide correct alternatives, so again I find my time and effort well justified. There is a scanned image of peer-editing of a proof in the Appendix.

In addition to the logistical structure of the course, students at this level need some support in the thinking process for doing mathematical proofs. Most have had some exposure to proof techniques in other courses and have seen proofs demonstrated by professors in lecture classes, but are still often stymied at this level by just getting started on a proof. To help with this, I provide the resources on proof techniques included in the Appendix, and we spend some class time briefly reviewing generic techniques such as induction, set containment, and contradiction as they arise. I also include a set of “What to do when you don’t know what to do” tips on the course website. They are included in the Appendix.

3 Outcomes

I have taught Real Analysis I or II using the methodology described here five times. The greatest struggle of the course is the workload, for the students and for the instructor. The course got easier with time for me, especially as I learned to relax and trust in the students’ readiness, and in fact eagerness, to move to the next level as mathematicians. One of the greatest strengths of the course is intense student engagement and this seems to redeem the work it requires of the students and me. For the students, the concentration and independent learning that result from the course design seem to justify their efforts. For me, the fact that I get results, in the form of marked progress in reading, mathematical exposition, and competent proof-writing, makes it worth every hour of my time.

Both the students’ work and the students’ course evaluations give clear and compelling evidence that the course goals are being met through this pedagogical approach. I have found this course structure and approach to teaching proof-writing far more effective than a traditional course with lectures and unrevised homework. Generating the solution sets at the start of Real Analysis I is arduous for both the students and me, and initially the students need a lot of support and feedback. However, by the end of Real Analysis I, students are actually quite efficient at generating polished solution sets, and students who continue to Real Analysis II need very little guidance to craft well-written proofs. Also, the development of proficient writing skills from the first paper to the final paper is remarkable for many students. By the second semester, students are basically functioning as working mathematicians, routinely writing competent proofs, with several of them better referees than many working mathematicians. Student comments in exit surveys affirm great pride in what they achieved as independent learners and writers of mathematics, as for example in these excerpts:

- I have learned a lot about writing mathematics.
- I feel prepared to read (independently) and learn math. I know I am prepared to write at a professional level, and I understand every topic we covered.
- I am walking out of this class with the confidence to pick up a graduate level text, understand it, and demonstrate my comprehension through essays, problem sets, and working through proofs.

Students also responded positively to the non-traditional structure of the class, to having papers instead of exams, and to the emphasis on self-teaching. They also commented on the workload, but reflected that it was well justified by the commensurate amount they learned, the appreciation they gained of higher mathematics, and their preparation for graduate school.

4 Extending the Method

The pedagogical approach described here readily adapts to any upper-level course. The only critical element is a cleanly written text that provides both complete content and an excellent model for writing proofs.

Because of the flipped classroom, with lectures replaced by content-learning occurring outside of class, there was little reduction in coverage compared with more traditional sections, and this should be the case with any content, again assuming a robust text. Course logistics are challenging, and need to be developed carefully no matter where this approach is used.

Care must be taken in scaling the approach for different sized classes. Students should only be required to write and referee so many problems. If there are too few students, pre-written solution sets need to be provided for some of the problems, since there won't be enough students to generate solutions for all the problems they ought to do. On the other hand, if there are too many students, then students need to be grouped so that multiple students write up the same problem. Very large sections might use recitation sessions for subsets of the class.

While some forms of technology support this kind of course, it isn't essential. Probably some capacity for mathematical typesetting is most important. While students could certainly still write up 'perfect' solutions sets by hand, the result just does not have the same panache as typeset mathematics. Since my students go on to a variety of careers, it seems reasonable for them to use whatever typesetting software they are likely to use in the future. Thus, I accept and support both \LaTeX and Word with MathType. In both cases, students pick up the necessary technical skills quickly from templates, peers, and online resources. Both programs output PDFs, which I concatenate into the final problem set. Some years we used a SmartBoard, which has pros and cons. On the pro side, the board content is saved automatically. On the con side, it is a small board, and can hold only one problem at a time, so most of the other class problems go on the white boards anyway. Currently, it is not unusual for students simply to use cell phone cameras to take screenshots of the white boards anyway, so I have stopped requesting SmartBoard capability in my classrooms. I currently use a course management system to distribute logistical information, but this can be done in any way. Recently I have begun using DropBox to handle the logistics of distributing drafts and referee reports, which seems to work better than circulating hard copies.

It may be possible to carry out the the paradigm for just the proof-writing part of the course (the process modeled on professional proof-writing, with collaborative problem-solving, peer review, and a final published solution set) independently of the reading and writing components, but I have never tried it. However, conceivably a lecture course could achieve similar results by having out-of-class recitation sessions, perhaps guided by an advanced graduate student, and thus implement just the proof-writing portions of the pedagogical approach described here. (Editors' note: the reader may also see Hitchman [3], in this volume, for another model of a student-created course journal, via a modified Moore Method approach.)

References

- [1] Apostol, Thomas, *Mathematical Analysis*, 2nd edition, Pearson Addison-Wesley, 1974.
- [2] Anders O. F. Hendrickson, "*Elements of Style for Proofs*", in this volume.
- [3] Theron J. Hitchman, "Running a Class Journal", in this volume.

Appendix

A Sample Reading Guide

Apostol, Mathematical Analysis, 2nd ed. Chapter 3 Reading Guide.

Write your chapter notes in your notebook using some kind of navigation system to keep track of where in the book your notes refer (*e.g.*, page and section number, or page and paragraph number, or page and theorem number etc.)

These chapter notes *must* contain the items below as a subset, and you will be expected to discuss your reflections intelligently in class.

◆ indicates an item I will be specifically looking for when I review your notebook. Please write these items up especially carefully and make sure they are easy for me to find by marking the upper right-hand corner of the page with a ◆ and a label such as 'alternative proof of theorem 2.3.'

I have put a couple books on basic topology on reserve in the library. They may be helpful to look at in conjunction with this chapter.

Many of the activities below involve coming up with examples of sets with various properties. Be creative. We will put these up in class and maybe award prizes for good ones ☺.

◆ First make a catalog of the half-dozen or so most important results in this chapter (*e.g.*, those that are named after people!) to keep them straight. No more than 10-12 definitions or theorems. The point is to think about and choose what is really important — no mindless copying allowed.

Section 3.2. Look up vectors in your Calculus III book or Linear Algebra book to remind yourself that you are quite familiar with this. For Definition 3.2 and Theorem 3.3, use vectors in \mathbb{R}^3 with actual numbers to do an example for each item. Do at least two of them with vectors in \mathbb{R}^4 . Go back to Chapter 1 and review the Cauchy-Schwartz inequality, and convince yourself of 'immediate'. The triangle inequality is important, and we will be using it often. Do an example with actual vectors in \mathbb{R}^2 , with a diagram (by hand or on Maple), to see why it is called the triangle inequality. Do another example in \mathbb{R}^3 , with a diagram (by hand or on Maple) to see how it works in higher dimensions. Where have you seen Definition 3.4 before? What notation was used?

Section 3.3. Do specific examples (*i.e.*, pick a vector \mathbf{a} and some number r) of $\|\mathbf{x} - \mathbf{a}\| < r$ in \mathbb{R}^2 , \mathbb{R}^3 and \mathbb{R}^4 . Expand the equation you get, and rewrite it. Compare it to the equation for a circle, sphere, or hypersphere, to convince yourself that these really are balls, hence the name. Definitions 3.5 and 3.6 are really important — all the rest of the chapter will build on them.

◆ Use Definitions 3.5 and 3.6 to prove the interval $(0, 1)$ really is open. What does a 1-ball look like? Give an example.

Figure out why the empty set is open (hint: the defining condition is vacuously satisfied).

Section 3.4. Do the 'verification' in the proof of Theorem 3.10. Look at the second paragraph of the proof of Theorem 3.11. Is this exactly exercise 2.19b?

Section 3.5. Come up with at least three examples each of sets that are open, closed, or neither. ◆ Prove Theorem 3.13.

Section 3.6. Come up with three more examples like those on page 53, only more creative and maybe not just in \mathbb{R}^1 , identifying and distinguishing between accumulation points and adherent points. Write the difference between an accumulation point and an adherent point in your own words.

Section 3.7. Rewrite Theorem 3.18 in your own words to make sure you really understand it.

What is the closure of $S = \{(x, \sin(1/x)) : x \neq 0\}$? Hint: plot on Maple. Come up with 3 more creative examples of sets and their closures.

Section 3.8. This is an example of a very simple idea obfuscated by the miserable notation it requires. Kind of like Riemann sums. Remember those? The idea was easy: add up a bunch of rectangles. The notation for writing down and keeping track of all the little rectangles was a pain. But you survived. Go back and look at them, and remember surviving. The idea here is easy again: just keep subdividing the space, into halves for \mathbb{R}^1 , quarters for \mathbb{R}^2 , eighths for \mathbb{R}^3 , sixteenths for \mathbb{R}^4 , etc. Each time, at least one of the subdivisions contains an infinite subset. So just subdivide that piece again, and so on.

◆ Do the following:

1. For $S = \{1 + \frac{1}{n}, n \in \mathbb{Z} - \{0\}\} \subseteq \mathbb{R}$, find choices for $[a_i, b_i]$ for i from 1 to 5. Hint: Sketch this set on the number line first.

2. For $S = \{(\frac{1}{2} + \frac{1}{n}, \frac{-3}{2} - \frac{1}{n}), n \in \mathbb{Z} - \{0\}\} \subseteq \mathbb{R}^2$ find choices for J_i for i from 1 to 5. Hint: Use the pointplot and seq commands in Maple to graph this set.
3. For $S = \{(\frac{1}{4}, \frac{1}{4}, 1 - \frac{1}{n}), n \in \mathbb{Z} - \{0\}\} \subseteq \mathbb{R}^3$, find choices for J_i for i from 1 to 5. Hint: Use the pointplot3d and seq commands in Maple to graph this set.

Section 3.9. Give an example to illustrate Theorem 3.25. Line 3 of the proof says ‘otherwise the proof is trivial’. Prove the trivial case.

Section 3.10. Come up with 3 more examples of sets with coverings. Determine if the coverings are open coverings and/or countable. In Theorem 3.27, why is G countable? Hint: look at Exercise 2.19. Find a $B(\mathbf{y}, q)$ if $S = [0, 1] \times [0, 1]$ and $x = (\pi^{-1}, \pi^{-1})$ or $x = (1 - e^{-3}, 1 - e^{-3})$.

◆ In the proof of Theorem 3.27 it says $\|\mathbf{y} - \mathbf{x}\| \leq |y_1 - x_1| + \dots + |y_n - x_n|$. Prove it. This is a little different from the triangle inequality. Draw (by hand or on Maple) a picture in \mathbb{R}^3 illustrating this and another illustrating the triangle inequality. Another way of stating Theorem 3.27 is to say that if x is in an open set $S \subseteq \mathbb{R}^n$, then there is always an n -ball B with rational center and radius such that $\mathbf{x} \in B \subseteq S$.

Section 3.11. ◆ Work through the proof of 3.29 in the special case that $A = [0, 1] \times [0, 1]$, and $F = \{(x - 0.1, x + 0.1) \times (-1, 2) : x \in \mathbb{R}\}$. A countable subcollection of F is

$$\{(x - 0.1, x + 0.1) \times (-1, 2) : x = 0.1n \text{ for } n \text{ a nonnegative integer}\}.$$

In this case, what is S_m ? What is Q_m ? How large must m be for Q_m to be empty?

Section 3.12. Break the proof of Theorem 3.31 into three separate pieces and prove each piece separately.

Section 3.13. Come up with two more examples of metric spaces, and one example of an M and d that do *not* give a metric space (say why they fail).

Section 3.14. Think about exercise 3.27, but do not write up the gory details. Do the ‘verify’ in the last sentence of the proof of Theorem 3.33. See if you can prove Theorem 3.34 before looking at the book’s proof. Compare the definitions of adherent points, closure etc., in a metric space to the previous definitions.

Section 3.15 ◆ Prove part (i) of Theorem 3.38.

Section 3.16. Do the ‘verify’ below Definition 3.40. Give two more examples.

B Writing guide for the papers

Please conform to the following.

1. All writing assignments are in 12-point Times New Roman with 1-inch margins. You may use either \TeX or MathType to typeset the mathematics, but all mathematics must be professionally typeset. You may *not* exceed page limits.
2. Written work will be graded as 25% first draft, 25% quality of your refereeing work, 50% final draft.
3. Read carefully the course resources on how to write mathematics, and apply these principles in all your writing assignments. Model your exposition on our text (our text was chosen largely because it is an excellent model of very good basic mathematical exposition without any fancy frills).
4. You may use any resources you find useful for writing assignments. I highly recommend the other analysis books on reserve in the library, and advise you to use internet resources with extreme caution. It often takes far more effort to carefully check if something on the internet is actually correct than it is worth. Please share with the class as a whole any references you find particularly helpful. Remember that any resources you use must be properly cited in the paper and listed in the bibliography. The bibliography does not count in the page limits.

5. You may discuss the content of any of these papers as much as you want with anyone you choose, and in fact this is encouraged. However, you may *not* collaborate with anyone in the actual writing. Do not share files, and do not read anyone else's work (or allow your work to be read), outside of the formal refereeing step.
6. The limit and continuity paper:
 - Write a 3-page expository paper explaining the formal definitions of limit and continuity from calculus, with worked examples. Illustrate with screen captures from the limit applet.
 - Referee a version of the above paper.
 - Revise your original paper with respect to the referee report.
 - HINT: if you did not cover this material in your Calculus I class, you can find lecture notes and resources for it on my Calculus I website.
7. The Midterm paper:
 - Write a 5-page synopsis of the course content to date. You should know the content well enough to determine what is most important to emphasize, since you can not just cover it all in only 5 pages. You need to explain why various concepts and theorems are important and how they interrelate. Illustrate central concepts with examples (or counterexamples!) beyond what is in the book.
 - Referee two versions of the above paper.
 - Revise your original paper with respect to the referee report.
 - This counts as your midterm exam.
 - HINT: Start writing this now!
8. The Final paper:
 - Write a 6-page synopsis of the entire course contents, with emphasis on material covered since the Midterm paper. Only include that material from the Midterm paper which is necessary to carry your exposition forward. You should know the course content well enough to determine which is most important to emphasize, since you can't just cover it all in only 6 pages. You need to explain why various concepts and theorems are important and how they interrelate. Illustrate central concepts with examples (or counterexamples) you have constructed yourself. This first draft should be less than 6 pages, since you will need to add more material at the end (see below).
 - Referee two versions of the above paper.
 - Revise your original paper with respect to the referee report, and add the material covered in class since the draft was due. Due no later than the end of our scheduled exam time.
 - This counts as your final exam.
 - HINT: Start writing this now!

C Frequent comments for the papers, particularly the first paper on limits and continuity

Please check carefully that you have written carefully with regard to the recommendations below.

1. Read the resources provided for writing and typesetting mathematics, and adhere to the principles presented in them!
2. Typesetting and grammar:
 - Variable and function names should be italicized. A symbol should always appear in exactly the same font each time it is used.
 - Mathematical expressions should be separated by words, not just punctuation.
 - Do not begin a sentence with a mathematical symbol.
 - Use the symbol, not the word (*e.g.*, use δ and ϵ , not *delta* and *epsilon*).
 - Be very careful not to introduce or use undefined terms or symbols. Always define first, then discuss.
 - Read sentences with mathematical expressions in them out loud to be sure that they are grammatically correct.
 - Indicate definitions, theorems, lemmas, etc. by “Definition 1.”, etc., and offset them from the rest of the text by at least paragraph breaks. One- or two-word definitions may be embedded in a larger paragraph, but more detailed definitions should be separated from the rest of the text.
 - Proof read carefully to eliminate informal language. *E.g.*, “look at” should be “consider”, and “start off with” should be “beginning with”.
 - Do not use contractions in formal writing.
 - Avoid the second person (“you”). First person plural (“we”) may be used (it can help keep the exposition in the active voice), but keep it to a minimum. Similarly keep the use of “one” to a minimum. I personally do not care for ‘one’, although other people do use it
 - Be careful when using pronouns that the antecedents are clear.
 - Re-read each paragraph and eliminate repetitiveness, rambling, and unfocused sentences or phrases.
3. Content specific to the limits and continuity paper:
 - It is acceptable to do one example using a specific value of ϵ and finding a corresponding δ , but you must also prove that the limit in your example exists. That is, you must show that for any ϵ , there exists an appropriate δ .
 - Including left- and right-handed limits may support your discussion of continuity.
 - The choice of δ illustrated in your applet example should be sufficiently small to assure that all function values resulting from x within δ of a remain within the ϵ -band about L .
 - Make sure you address continuity separately from the formal definitions of limit. At least 1/3 of your paper should probably be devoted to continuity.
 - The definition for continuity can be broken down into three parts. Doing this will help your exposition.
 - Give both algebraic and graphical examples for continuity and for discontinuities.
 - For discontinuities, be sure to explain which parts of the definition are not satisfied.
4. General content for all papers:
 - Illustrate central ideas with examples of your own creation. This demonstrates that you understand and can apply them.

- Discuss the connections among the important theorems and definitions.
 - Giving extended proofs, especially verbatim from the text, generally takes more space than is available given the page limits. It is better to give the central ideas of how most of the theorems you include are proved, and give further details only for a small select group where the proof technique is especially important or tricky.
 - Because of the page limits, it is important to focus on the important ideas and not spend too much time on minor details.
 - Make sure to properly cite your sources (even if it is just our text over and over), and include the bibliography. Be sure to cite any internet resources you use.
5. When you are editing/refereeing:
- Check that the author has adhered to the recommendations provided above and those in the writing resources.
 - Be sure to let the author know if any critical ideas have been omitted.
 - Mark any mathematical errors, especially misunderstandings of definitions or applications of theorems.
 - The more detailed your markup, the better.

D Getting Started Tips

What to do when you don't know what to do...

Some things to try if you have been staring at a problem for too long and still don't have a freaking clue how to even get started:

1. Don't worry. This happens all the time, even to the most brilliant mathematicians in the world.
2. Copy the problem on a piece of paper. This may seem silly, but, for many people, the process of writing the words and thinking through the symbols seems to make them penetrate a little better.
3. Flip back through the chapter and write the statements of any results that seem to have any bearing on the problem at all. Also write any examples that seem related.
4. Go to the library and look in the books on reserve. Read the relevant chapters (sometimes it helps to have something explained in a slightly different way).
5. If you are called on in class to put a problem up on the board, you *must* have done at least this much and be able to deliver the material from your notes. It is unacceptable just to copy the problem out of the book and stop there. You then undoubtedly will be asked to share what else you tried and any thoughts you had even if they didn't immediately lead to a solution.

Some things to try after that:

1. See if there is anything similar (but presumably easier!) in your calculus book.
2. Talk to other people in the class about it. Try to talk about the ideas, not just "how to do it".
3. See if you can come up with any concrete examples of the problem. Sometimes if you can see how to do the problem for a particular situation (e.g. an explicit function instead for "any" function) you can see how to do it in general. Similarly for induction. If you can do the problem for, say, when n is 3 and then 4, you may be able to do it for n and $n + 1$.

4. Re-read the proofs in the chapter to see if you can adapt any of the proof techniques. Often textbook exercises are designed to have you do this.
5. You can try the internet, but be *very* cautious as there are a lot of highly unreliable solution sets out there written by other students who don't know any more than you do, even if the solutions are posted on some seemingly reputable websites.
6. If you happen to find a solution to a problem somewhere (another book, online, etc.) you may not just copy it. You will learn more if you pretend you never saw it and continue to work on the problem yourself. However, if you are at wit's end when you find it, and think you must look at it, please read it through, then put it aside while you write up the problem yourself in your own way.
7. Think about it very hard just before you go to sleep.
8. General good idea: immediately upon getting the assignment, try *all* of the problems at least briefly to familiarize yourself with each of them. You want to be thinking about them as we are talking about various theorems and concepts in class, because at any moment you may realize that one of these theorems or concepts could help with some one of the exercises. You want to be thinking about these problems as you are walking to the dining hall or in the shower, just kind of constantly turning them over in the back of your mind. If one of them is being recalcitrant, move to another. Thoughts you have on the later problem may help with the earlier one.

E Resources for proof techniques and writing mathematics

Students are encouraged to use ancillary resources for learning proof techniques – I provide some, and encourage students to find, evaluate, and share others. Some examples include:

- M. Beck and R. Geoghegan, *The Art of Proof: Basic Training for Deeper Mathematics* (*searchable pdf*), <http://math.sfsu.edu/beck/aop.html>, accessed July 14, 2013.
- R. Hammack, *Book of Proof*, <http://www.people.vcu.edu/~rhammack/BookOfProof/>, accessed July 14, 2013.
- Nipissing University Calculus Help Site, Mathematical Induction Tutorial, <http://www.nipissingu.ca/calculus/tutorials/induction.html>, accessed July 14, 2013.
- B. Turlach, *A list of invalid proof techniques*, <http://school.maths.uwa.edu.au/~berwin/humour/invalid.proofs.html>, accessed July 14, 2013.

I also provide the following resources on writing mathematics:

- E. Berry and J. Lawson, *Writing Mathematics Handout*, http://www.math.ualberta.ca/~orivasplata/writing_aids/purdue2_write.pdf, accessed July 14, 2013.
- J. Lee, *Notes on Writing Mathematics*, <http://www.math.washington.edu/~lee/Courses/583-2005/writing.pdf>, accessed July 14, 2013.

- S. Maurer, *A Short Guide to Writing Mathematics*, <http://www.swarthmore.edu/NatSci/smaurer1/WriteGuide/index.html>, accessed July 14, 2013.
- S. Maurer, Advice for undergraduates on special aspects of writing mathematics, *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, vol 1, no. 1 (March 91) 928.
- N. Steenrod, P. Halmos, M. Schiffer, and J. Dieudonne, *How to Write Mathematics*, American Mathematical Society, 1973.
- D. Knuth, T. Larrabee, and P. Roberts, *Mathematical Writing*, http://jmlr.csail.mit.edu/reviewing-papers/knuth_mathematical_writing.pdf, accessed July 14, 2013.

F Sample problem draft with edits, and final published solution

Figure 1 is a random example of a first typed draft of a problem, with peer edits, and Figure 2 is the corresponding segment of the final version of the problem. It is a random sample in that, since students

Proof. To prove that an absolutely continuous function f on the interval $[a, b]$ is also continuous on that interval, we aim to show is uniformly continuous, which we recall in the definition below:

The def of continuous

Definition: A real-valued function f on $[a, b]$ is uniformly continuous on $[a, b]$ if, for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all $x, y \in [a, b]$

$|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$

→ You can just use to prove

Thus, since our definition of absolute continuity states $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$ and $\sum_{k=1}^n (b_k - a_k) < \delta$, we know that for some $x, y \in [a, b]$ it holds true that $|x - y| < \delta$ will mean $|f(x) - f(y)| < \varepsilon$. With this, we have shown our function f is uniformly continuous, and this is also continuous.

on back

If want to show uniform cont need to use any

(b) *Let f be absolutely continuous.*
 To show that our function will also be of bounded variation on $[a, b]$, we begin by stating that for any subinterval $[c, d]$ of $[a, b]$, we have the total variation $V_f(c, d)$. If $c < e < d$, then by Theorem 6.11 we have $V_f(c, d) = V_f(c, e) + V_f(e, d)$.

Let P be a partition of $[a, b]$ and choose δ so
 Now choosing some $\delta > 0$ such that we have absolute continuity and $\varepsilon = 1$, we fix some number n so $\frac{b-a}{n} < \delta$ and let $P = \{a = x_0, x_1, \dots, x_n = b\}$.

that
 $\frac{b-a}{n} < \delta$ for some fixed n , where $\varepsilon = 1$.
 With this, we have $x_i - x_{i-1} = \frac{b-a}{n}$ and so $V_f(x_{i-1}, x_i) \leq 1$ since we took $\varepsilon = 1$.

Thus, we can say $V_f = V_f(a, b) = \sum_{i=1}^n V_f(x_{i-1}, x_i) \leq n$ and since we took n to be an integer, then this function must indeed be of bounded variation on the interval $[a, b]$ and we are done.

Therefore f is of bounded variation.

Figure 1: A typed draft with peer comments.

Proof. To prove that an absolutely continuous function f on the interval $[a, b]$ is also continuous on that interval, we aim to show f is uniformly continuous, which we recall in the definition below:

Definition: A real-valued function f on $[a, b]$ is *uniformly continuous* on $[a, b]$ if, for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all $x, y \in [a, b]$

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon$$

Our definition of absolute continuity states that given ε , then $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$ if

$\sum_{k=1}^n (b_k - a_k) < \delta$. Thus, we let $k=1$, and then for any $x, y \in [a, b]$, we let $x = a_1$ and $y = b_1$, and then it holds true that if $|x - y| < \delta$ then $|f(y) - f(x)| < \varepsilon$. With this, we have shown our function f is uniformly continuous, and hence continuous.

We now want to show that f is also of bounded variation. Let f be absolutely continuous. To show that f is of bounded variation on $[a, b]$, we begin by stating that for any subinterval $[c, d]$ of $[a, b]$, we have total variation $V_f(c, d)$. If $c < e < d$, then by Theorem

Figure 2: The corresponding section of the final published proof.

typically take all their work with them, it was the only example that I was able to find. Thus, it represents fairly typical work, rather than an example specifically chosen to showcase exemplary work.

G Calendar of due dates

Here is the schedule of due dates. It is essential that you meet every deadline listed here, since other people in the class depend on you finishing your part of the work (e.g. refereeing or editing problems or having solutions completed for posting) so that they can complete theirs.

The key to the schedule is as follows:

Letters—

- a. chapter notes due
- b. handwritten problems done (all of them!!)
- c. 1st pass mark up done (concentrate on the ‘to be typed’ problems)
- d. Typed problems due
- e. Second pass editing of typed problems
- f. Final solutions due (to be posted on eCollege)
- g. Final revised handwritten problems due

Note: for chapters 3 and 4, homework is split into two parts, so the subscripts on e.g. b_1 or b_2 refer to the first or second part of the homework set.

- A. 1st draft of the paper due
- B. Referee reports due
- C. Final draft of the paper due

Colors:

Chapter 1
Chapter 2
Chapter 3
Chapter 4

Delta-Epsilon paper
Chapters 1 & 2 paper
Whole thing paper

September 2010						
Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
	30	31	1	2	3	4
5	6	7	8	9	10 A	11
12	13 a, b	14	15 B c	16	17 d	18
19	20 C	21	22	23	24	25
26	27 e	28	29 f	30		

October 2010						
Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
					1	2
3	4 g	5	6 a, b	7	8	9
10	11	12	13 c	14	15 d	16
17	18 e, A	19	20 f	21	22 B	23
24	25 g	26	27 C	28	29 a, b₁	30
31						

November 2010						
Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
	1 c_1	2	3 d_1	4	5 e_1, b_2	6
7	8 f_1, c_2	9	10 d_2	11	12 e_2	13
14	15 g_1, f_2	16	17 a	18	19 g_2, b_1	20
21	22 c_1	23	24	25	26	27
28	29 d_1, b_2	30				

December 2010						
Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
			1 e_1, c_2	2	3 f_1, A	4
5	6 f_2	7	8 g_1, g_2	9	10 B	11
12	13 C (any time before Friday)	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

Introduction to Proofs Over Easy: A Low-Cost Alternative to the Flipped Classroom

Lewis D. Ludwig

Abstract

A modification of the flipped-classroom technique is presented, which is referred to as the “over-easy” method. In it, students use reading notes to engage in proof-writing by first learning to read mathematics, then transferring that skill to their proof-writing. A description of the technique including several examples is provided and several references from cognitive psychology supporting its advantages. The article concludes with the benefits and challenges of the over-easy method and other applications of it.

Difficulty Level: High; Course Level: Transitional

1 Background and Context

Eleven years ago when I arrived at Denison University, a small national liberal arts college of 2,100 students, I was asked to teach our introduction-to-proofs course. It is a bridge course, required for the math major and minor, that helps students transition from calculus to the higher-level proof-based courses. Traditionally we offer one section in the fall with 15–25 students who are math majors or minors in their sophomore year. The verbal/math SAT scores for these students average in the upper 1200s to lower 1300s.

I have taught the course five times in the last eleven years. When first assigned to it, like any young faculty member I was eager to challenge my students and help prepare them for this critical juncture in their mathematical careers. I carefully chose a text that was challenging, yet readable. I made detailed lecture notes by carefully following the text and adding interesting examples and exercises from other sources. When classes started, I gave interactive lectures, which included a presentation of my notes at the board and a series of well-crafted questions to make sure students were understanding and participating in the discussion. As we covered various proof techniques I would provide propositions that needed to be proved and would stand at the board coaching the class through the process arriving at a completed proof. All the while students took detailed notes.

At the end of the semester, my course evaluations were positive. Students liked how the class was structured, liked the interaction in the classroom, and felt they had learned a great deal. My classroom observations from senior members in the department were glowing, emphasizing how well I interacted with students and kept them engaged. The next time I taught the class, I took the same approach and got similar reviews from students and colleagues.

However, something didn't seem quite right. It felt like I was doing almost all the work. I did all the reading, found interesting examples, assimilated the new material, and then transferred this knowledge to students. They were doing little information-gathering or higher-level thinking aside from taking notes in class and chiming in when I asked questions. Moreover, when I had the same students in subsequent advanced courses such as real analysis, I found they had difficulty following the longer proofs from the text. They could create their own proofs of small propositions for homework, but struggled to read and comprehend a detailed proof from the text such as the Bolzano-Weierstrass or Heine-Borel theorem.

After a good deal of reflection and talking with others at Denison and beyond, it was clear that the course should not only teach students how to *write* mathematical proofs, but how to *read* mathematical proofs as

well. At the time, my daughter was in early elementary school. A large portion of her time and effort was given to *reading* instruction. While she knew her letters and had spelling tests, her writing instruction did not get serious until several years later, after she could read.

It seemed the same should be true for my proofs course. Before students can become effective writers of mathematics, they first need to become proficient readers of mathematics. This notion is supported by the work presented by Fitzgerald and Shanahan [4]. They provide research showing the strong link between developing reading and writing skills. For example, they look at the domain knowledge about substance and content. This includes the “meanings or ideas that are constructed through the context of the connected text.” They also discuss the importance of procedural knowledge which “refers to knowing how to access, use, and generate knowledge in any of the areas . . .” They argue that reading and writing are connected through domain knowledge and procedural knowledge, and that one informs the other.

Armed with this knowledge, I began looking for ways to adjust my students’ learning experience in my proofs class so that they developed their reading skills as well as their writing skills. In addition, I wanted them to have a more active role in their learning by taking ownership of the gathering and assimilation of information. Over the past nine years this has developed into what I call the “over-easy” classroom — a modification of the flipped classroom.

2 Description and Implementation

2.1 Motivation

Before I describe the method, let me first provide motivation. Considering my prior technique of creating class notes and delivering an interactive lecture, my colleagues from psychology would describe my students as passive learners. True, I did engage students in class by asking leading questions, but they did nothing before class to prepare for this. Moreover, their learning was dependent on me, the instructor. I was treating them as empty vessels, waiting to be filled when they came to class. The filling came in the form of their passively taking notes and answering questions.

In a sense, I was just transferring knowledge to my students. They had little or no responsibility in the information-gathering phase of their learning. This limited the amount of time and energy they spent assimilating the information. To truly engage in their learning, students need to fully understand the material and place it in the context of their existing knowledge [1] or domain knowledge, as described by Fitzgerald and Shanahan [4]. So my motivation for developing the over-easy method was two-fold: improve my students’ ability to read and assimilate information, and have them take greater ownership of their learning starting with the information-gathering stage.

2.2 Method: The Why and How

First, why the over-easy classroom? Traditionally, a flipped classroom involves a video lecture that students watch before coming to class. The information-gathering stage of the learning process is done before class instead of in a lecture, giving ownership of the process to the student. In class, the instructor focuses on clarifying the information gathered by having students work problems in class alone or in groups. This allows for immediate feedback and often closer instructor-student interaction [3]. We will see that the over-easy method is neither as time nor resource intensive as creating videos, but is just a modification of the development and use of traditional class notes. Compared to a traditional flipped class, it is over-easy.

So how does it work? As described in the introduction, I used to take detailed notes from the textbook and supplement them with additional exercises and questions. With the over-easy method I do the same thing, but without providing the answers or details. That is, I create a handout that is a skeletal outline of what would previously be considered my class lecture notes and have the students fill in the details as homework before

class. I call these handouts “reading notes.” There are two kinds of reading notes: information-gathering and assimilation of knowledge, or proof-reading practice.

Let’s consider a typical set of reading notes that focuses on information-gathering and assimilation of knowledge in my introduction-to-proofs class. Suppose I plan to cover the section on indexed sets on Wednesday. By the prior Monday, students will receive a handout that is partially depicted in Figure 1¹. Outside of class, the students use the text to complete at least 70% of the handout by the beginning of the following class, which in this case would be Wednesday.

Section 1.4 Indexed Collections of Sets

1. Give a working definition and example of:
 - (a) union of an indexed collection of sets
 - (b) intersection of an indexed collection of sets.
2. Find the union and intersection of each of the following families of sets.
 - (a) For $n \in \mathbb{N}$, let $A_n = \{1, 2, 3, \dots, n\}$.
 - (b) For $n \in \mathbb{Z}$, let $B_n = [n, n + 1)$.
 - (c) For $n \in \mathbb{N}$, let $C_n = [0, \frac{1}{n})$.
3. Give an example of an indexed collection of sets $\{A_\alpha : \alpha \in \Delta\}$ such that $A_\alpha \subseteq (0, 1)$, and for all $\alpha \in \Delta$ and $\beta \in \Delta$ we have $A_\alpha \cap A_\beta \neq \emptyset$ but $\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$.

Figure 1: A portion of a sample reading assignment.

I begin class on Wednesday by randomly putting students in pairs with their reading notes in hand. Students spend the first 10–15 minute of class comparing their notes, checking for agreement in their responses or discussing disagreements. I circulate through the room, checking for completion and listening to the discussions. If a student asks a question, I reply by first asking what his/her partner thinks about the question. (Initially, students are more comfortable asking me than their peers for help. The reminders help students better engage with one another.) Often times students are able to resolve their own questions, but I design the reading questions so that the students will be significantly challenged by roughly 30% of the questions, hence the 70%-completion threshold. As I circulate, I note which questions are giving difficulty and will center the day’s activities around them.

Let’s consider the questions from Figure 1. Question 1 asks for students to give a working definition. For a working definition, students should read the definition from the text and rewrite it as if they were explaining it to a friend. I discourage verbatim copying from the text as very little assimilation occurs if they do this. It is important for them to provide examples, different from the text, for each definition. This helps in assimilation and helps students put the new information into their existing framework of knowledge. That is,

¹Ample white space is usually provided so students can write on the handout. It has been removed to save space.

how does the new information relate to things they already know? For example, students already know about finite unions. How can they extend this understanding to infinite unions? What are the similarities? What are the differences? Anderson et al. [2] would refer to this as the second level of the cognitive domain of Bloom's taxonomy of learning objectives: understanding. I expect everyone to be able to sufficiently answer question 1 as it is straightforward information-gathering. A quick check with their peers will confirm this and I have saved five minutes of lecturing.

Question 2 is more involved. Here I am expecting students to put the new definitions into practice. This is the third level of cognitive domain in Bloom's Taxonomy: application. Most students will independently get 2(a). The majority will understand 2(b) and about 30-40% will get 2(c) since it is dealing with new definitions and a limiting process. By pairing the students at the start of class, 2(a) is confirmed, 2(b) is clarified, and 2(c) will be something we discuss as a whole class. The process also demonstrates instructional scaffolding where support is given during the learning process that is tailored to the needs of the learner in order to help the student achieve his/her learning goal [5]. In addition, I have saved at least another five minutes of lecturing.

Question 3 requires an even higher level of understanding. Unlike question 2 where I asked them to apply the new definitions to three concrete examples, in question 3 they must create their own examples. This encompasses the highest level of Bloom's taxonomy, the creation of new ideas. The observant student will notice that a slight modification of 2(c) will lead to a useful example, but generally only 20% or fewer students see this. This question will play a large part in my discussions with the whole class. By having students consider this question on their own and with a partner, I need less time to explain the notation, etc., and can focus on problem-solving techniques.

This is roughly how the over-easy method works for information-gathering. With the over-easy method, I spend the same amount of class time on the material as when I lectured. However, the time I used to spend on definitions and basic examples is now allocated to more challenging ideas and concepts.

Reading practice: There are Infinitely Many Primes

Theorem 1 *There are infinitely many primes.*

Proof. Suppose that $p_1 = 2 < p_2 = 3 < \dots < p_r$ are all the primes. Let $P = p_1 p_2 \dots p_r + 1$ and let p be a prime dividing P ; then p cannot be any of p_1, p_2, \dots, p_r , otherwise p would divide the difference $P - p_1 p_2 \dots p_r = 1$, which is impossible. So this prime p is still another prime, and p_1, p_2, \dots, p_r would not be all of the primes. QED

1. Define what it means for a number to be prime.

This proof technique is called *proof by contradiction*. The author assumes there are only a finite number of primes and based on the premise comes to a contradiction. Thus, the assumption must be incorrect and the alternative true.

2. For this proof by contradiction there are **only** two possible outcomes. What are they?
3. How many primes does the author assume exist?
4. Could P be prime? Sometimes? Always? Never? EXPLAIN.
5. Why can't p divide $P - p_1 p_2 \dots p_r = 1$?
6. What is the contradiction?

Figure 2: An example of proof-reading practice.

The reading notes that involve proof-reading practice are slightly different; an example is in Figure 2. This is the typical proof, attributed to Euclid, found in most introduction-to-proof texts. While it is concise and beautiful to those fluent in mathematics, it is a challenge for students new to mathematical proofs. The proof itself is only four lines long and takes about 30 seconds to read. However, students soon realize that the ability to read words does not guarantee understanding of the logical argument.

The questions for this type of exercise mimic how a mathematician would read a mathematical proof. First, do I know all the definitions? This is the intent of question 1. Students need to remind themselves that 1 is not considered prime. Next, what type of proof technique is being invoked? The notes explain the proof-by-contradiction technique to the students and then ask what the two outcomes are that establish the contradiction. What assumption is being made? If the students understand the two outcomes, infinitely many primes or finitely many primes, then question 2 should answer question 3. What role does P play? Herein lies the beauty of Euclid's argument. To understand the proof, the student must realize that P may or may not be prime, but in either case, the conclusion still follows. The last two questions secure the contradiction and conclude the proof.

I would start class with students' comparing their answers to the handout in Figure 2. The 70%-completion threshold still applies. Generally by the end of their discussion with their partners, most students will get 1–3 and 5 fairly easily. Questions 4 and 6 are the stumbling blocks where I will focus the class discussion.

With my prior interactive-lecture method, I would spend nearly the whole class on the proof. A good portion of time was spent just copying down the proof. With the reading practice, we can cover the proof in class in about 20 minutes with a fairly high level of confidence that everyone understands what is going on. To ensure this, we spend the rest of the 30 minutes in class tackling other proof-by-contradiction problems. This is usually done with a handout containing several additional statements that the students try to prove. They work in pairs on the questions for about 20 minutes while I circulate. The last 10 minutes is used to discuss their findings with the class.

A typical set of fact-gathering reading notes (Figure 1) is three to four pages. So in 10 minutes, my students can cover roughly 70% of the material I would have normally covered in a 50-minute interactive lecture. As described, a similar time savings occurs with the reading practice notes (Figure 2). By condensing the time spent on the more mundane information-gathering portion of the class, we have more time to tackle the issues and concepts that the students find most challenging. The over-easy method makes better use of class time and students are significantly more engaged in their learning. Some of the better class days are when the reading notes contain some challenging questions that no one got. My students arrive in class complaining, and I love it! I now have them emotionally hooked on their learning. They are no longer passively absorbing information, but now have a vested interest in seeing how a problem is solved. Making an emotional connection with learning is very powerful [1].

3 Outcomes

As anyone who has stood in front of a classroom knows, there is no one way to teach. However, I have incorporated the over-easy method into all my classes for majors (multivariable calculus on up). In them, at least 80% of the class meetings are conducted with this method. While I am a firm believer in the over-easy method, it does have its limitations. I will now discuss some its benefits and several of the challenges. I will also provide several approaches that address them.

3.1 Some Benefits

Students become better readers of mathematics. For example, in the subsequent real analysis course, the proof of the Heine-Borel theorem is still challenging, but the students are not as overwhelmed as they were in the past. When faced with a new statement to prove, instead of just reading the words, students now

transfer the skills they learned from the reading practice notes to better understand the new statement. The skills include considering concrete examples of the statement, the definitions needed, the proof technique used, the assumptions made, and the key element(s) of the proof. All the skills are developed and reinforced with the over-easy method.

Students became better writers of proofs. By developing their domain knowledge and procedural knowledge as described by Fitzgerald and Shanahan [4], my students no longer came to me saying “I have no idea how to get started on this problem.” Instead, they would say things like “I tried this” or “I have this idea.” That is, they were reading the questions and understanding the domain knowledge, and then trying to generate their own procedural knowledge. With one or two suggestions from me, they were usually on their way to a correct proof. Prior to the over-easy method, I often felt like I had to reteach the lecture during my office hour. This is no longer the case.

As I refined the over-easy process, I found I was able to ask my students more challenging and involved questions. Moreover, instructors, myself included, found that they were able to hit the ground running in courses like abstract algebra or real analysis. As another piece of interesting anecdotal proof, members of our Career Exploration and Development Department who help our students craft resumes and cover letters recently noted that they enjoy working with the students from my class as they are “such fine writers.”

Misconceptions are more easily identified and rectified before the midterm or final. Prior to this method, students would often have misconceptions that did not manifest themselves until a graded situation. With the over-easy method, many misconceptions are caught and corrected as the students compare their results with one another and the class. This also helps the students better organize their knowledge so they can more readily retrieve it for use and application [1].

Students are more willing to ask questions. In the past, one of my least answered questions to the class was “Are there any questions?” With the reading notes, students have a clear idea of what they need to understand and seek help when they do not.

I rarely see the comment “We need to do more exercises in class” on my course evaluations. This was a very common response when I used my interactive-lecture method. Since I had to present definitions and examples, we had less time as a class to go over exercises. Now the bulk of class time is spent working on examples and exercises.

3.2 Some Challenges

How do you hold students accountable? If not everyone gives a solid effort on the reading notes, the system crumbles. In a real analysis class of juniors and seniors who were familiar with the system, I asked them how I should hold them accountable for the reading notes. The class agreed that it was in their self-interest to do the work and I need not worry about holding them accountable. This turned into a New Year’s resolution. By the fifth week, five of my twelve students were not consistently doing the reading notes which totally derailed the whole system from the pairing-and-sharing down to the individual student’s ownership of his/her learning. I used this as a learning experience. Now my students get two “get-out-of-jail-free cards” for the semester. That is, they can miss two reading assignments over the semester. For n assignments missed beyond that, I deduct $\sum_{k=1}^n 2^k$ points from the total grade of about 400 points. Students quickly catch on that this is a series they don’t want to challenge. Moreover, as an incentive, if the students do not use their get-out-of-jail-free cards, they can cash them in on the final exam to skip one question.

Students must buy into this method. I often tell my junior colleagues that teaching is 57% salesmanship. You constantly have to remind students why you are using the method. You can’t just tell them it is good for them like broccoli. I share my pedagogical methods and reasonings with my students so they better understand why we are doing what we are doing.

The method requires more work from students. My classes meet four days a week for fifty minutes per class. I average two to three sets of reading notes per week. In addition, I have weekly homework and a test

every four weeks. Since adopting the over-easy method, I have shortened the weekly homework sets since many of the questions are now covered in the reading notes.

The method requires timely planning. Due to the larger work load and responsibility now placed on the students, I always give them at least two days to complete a reading exercise. This takes more planning and foresight than preparing a lecture the night before. If a typical set of lecture notes took me one hour to prepare, the over-easy method adds roughly an extra 30 minutes of preparation from me. But the time gained in instruction and the fact that I can easily edit the notes for successive classes makes the extra work well worth it.

4 Extending the Method

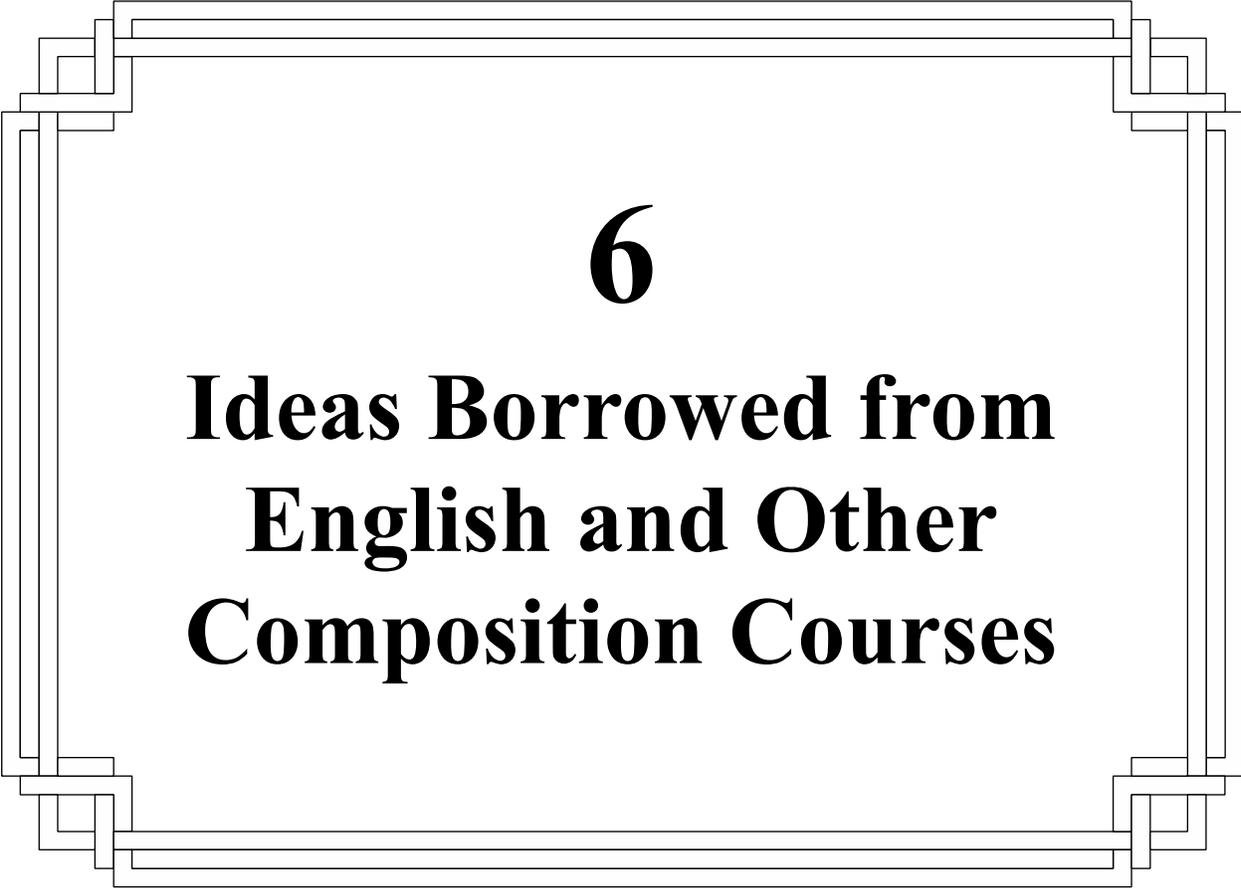
The over-easy method can easily be modified for other classes. The reader can see how this method can be used in other proof-based courses like real analysis or abstract algebra, but I have used it in other courses such as multi-variable calculus and linear algebra. The questions in multivariable calculus and linear algebra are often more computation-based, but the general idea is the same. In linear algebra, I often use true or false questions to test students' understanding of the definitions and theory, for example: "The columns of any 4×5 matrix are linearly dependent (EXPLAIN)." To receive full credit, students need the correct response and a supporting argument. This question tests the students' understanding of linear dependence and dimension.

While these are ideas that I have developed and refined over the past decade, they are not unique. I know David Pengelley of New Mexico State University uses a similar technique in a variety of classes and levels of students. While no system is perfect or will work for everyone, I have found the over-easy method to be a way to engage my students in their learning and help them become better readers and writers of mathematics.

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6

Ideas Borrowed from English and Other Composition Courses

Elements of Style for Proofs

Anders O. F. Hendrickson

Abstract

We describe a manual of mathematical style, in the spirit of William Strunk and E.B. White: a brief list of rules for writing clear, coherent, valid proofs in good mathematical style. We describe ways to use it in the classroom to help students improve their writing and to establish a vocabulary for discussing, and criteria for assessing, students' proofs.

Difficulty Level: Low; Course Level: Transitional

1 Background and Context

1.1 Introduction

Concordia College in Moorhead, Minnesota, is a residential Lutheran liberal arts college of about 2800 students. Of the twenty to thirty math majors who graduate from Concordia each year, about a third get positions teaching high school mathematics and about a quarter go on to graduate school in mathematics or the sciences, while the others typically find employment related to mathematics or to their other majors.

For all mathematics majors, the greatest shock of their undergraduate mathematical careers comes when they are first required to write proofs; at Concordia, this happens in a 200-level linear algebra class, which serves as the transition-to-proof course in the math major. Linear Algebra has Calculus II as a prerequisite and itself is a prerequisite for most upper-level courses. Students taking the course are typically sophomores, but occasionally freshmen, and are almost exclusively mathematics majors and minors. They have completed two or three semesters of calculus, they believe that mathematics is all about calculation and “getting the right answer,” and they generally take to proof-writing like a duck to icy rapids sweeping towards a waterfall.

When I first taught Linear Algebra at Concordia, I assigned proof problems from the textbook and dutifully bathed my students' proofs in red ink, to their and my mutual frustration. At last I realized that my students were writing proofs poorly because no one had ever taught them what a proof was, how to write one, or what criteria distinguish a good proof from a bad one. I further realized that, as the instructor of their first proofs course, *I* was responsible for teaching them the new language of proofs.

For sophomore math majors, proofs are indeed a new language: not quite the calculation-based mathematics at which they excelled, but not quite ordinary English either. For years they believed that the goal of mathematics was simply to find “the answer,” and although teachers pleaded with them to “show their work,” to earn full credit it often sufficed merely to leave visible scratchwork as evidence that they got the answer on their own. Now for the first time they are being required to articulate cogent, coherent arguments, much as in their English classes they passed from spelling tests to essay-writing.

If proofs are more like essays than calculations, and if our students' bad proofs stem as much from atrocious writing style as from bad logic, then it makes sense to borrow pedagogical ideas from those who teach writing style regularly, namely our colleagues in the English department. Several of the techniques for teaching proof-writing collected in this volume, such as revision of multiple drafts [2, 3, 4, 7, 8, 9], peer review [1, 2, 5, 7], journal-writing [5, 7], semester portfolios [2, 4], etc., are adaptations of pedagogical approaches that have long been used in humanities writing courses. My suggestion, which can be used in

conjunction with those other ideas, is to turn to one of the classics: the *Elements of Style* of Strunk and White.

1.2 Strunk and White

In 1918 William Strunk, Jr., of Cornell University self-published a 43-page list of rules as a required textbook for students in his classes [10]. Whereas most manuals of style, such as the one published by the University of Chicago, are lengthy and contain encyclopedic answers to questions of grammar, punctuation, and bibliographical formatting, the brief pages of Strunk's *Elements of Style* offer practical advice to help young writers produce lucid, vigorous, concise prose, while warning them against the more common grammatical errors. Strunk's keen emphasis on clarity, accuracy, and organization are perhaps unsurprising in a man who taught mathematics briefly before entering graduate school [6, p. 2].

Forty years later, Strunk's student E.B. White revised and enlarged his late teacher's work for publication by Macmillan. The resulting 1959 edition of *The Elements of Style* [11], commonly known as "Strunk and White," was immediately successful, as measured by sales and adoptions for college and high school classes [6, p. 106], and it quickly established itself as a fixture of English curricula across the United States, under the charming epithet of "the little book." Fifty years later, despite the grumblings of some critics, it remains a landmark text and continues to inform writers who wish to write clear, captivating prose.

One reason for the continued vitality of Strunk and White is its brevity. In twenty-two terse, aphoristic "rules of usage" and "principles of composition," the authors give aspiring writers strict, demanding guidelines: "Use the proper case of pronoun." "Do not join independent clauses by a comma." "Use the active voice." Standing above all with a critical eye is Rule 17, "Omit needless words." Only after the imperious phrase has made its impact do the authors go on to justify, amplify, qualify, and illustrate it, showing the reader why following the rule fosters clarity and vigor, while disobeying the rule often generates a morass of shifty, wordy, meaningless prose.

Of course the writing of good English, as of good mathematical proofs, cannot be reduced to a formulaic list of rules; indeed, all good writers break Strunk and White's rules at times. The book's format is nevertheless perfectly suited for its pedagogical context. For all their proverbial rebelliousness, undergraduates desire clear expectations against which they can judge themselves, and the "little book" offers nothing if not clear expectations. Only after students pass through a legalistic phase of knowing and obeying rules will they attain a sufficiently mature command of style to disregard rules judiciously for the sake, as Strunk puts it, of "some compensating merit, attained at the cost of the violation" [10, p. 6].

2 Description and Implementation

2.1 An *Elements of Style* for Proofs

Taking Strunk and White as my model, I set out to write an *Elements of Style* for proofs: a list of rules for writing clear, coherent proofs according to the conventions used by mathematicians, including warnings against a few common logical errors. I formulated my advice first as pithy, imperious rules that could be easily memorized and could serve as inspiration while writing or editing. I then developed each idea at length, providing examples of good and/or bad mathematical writing, and occasionally suggesting when exceptions to the rule might be warranted. My Rule 14, for example, is:

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14. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like

$$A = B \leq C = D,$$

he expects to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and he expects the *point* of the entire line to be the more complicated fact that $A \leq D$. For example, if you were computing the distance d of the point $(12, 5)$ from the origin, you could write

$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean Theorem, the second is just arithmetic, and the *point* is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write “ $\sqrt{12^2 + 5^2} = 13 = d$,” your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2 + 5^2} = d$.

I sought to list the most frequent logical and stylistic mistakes I encounter in student papers; the resulting rules fall into four general, somewhat overlapping categories. The first addresses general principles that could apply to almost all prose writing. Although rules such as “Use complete sentences” (Rule 4) and “Say exactly what you mean” (Rule 12) strike my students as self-evident, they nevertheless break them with astounding frequency. Most important in this category is my Rule 1, “The burden of communication lies on you, not on your reader,” because students’ previous mathematical communication has been based on precisely the opposite assumption. The greatest challenge for many students is realizing that it is not the professor’s job to guess what they mean; rather, it is their job to express it clearly in the first place.

The second category, less about style than content, warns against the logical mistakes that most often render students’ proofs invalid. These general principles, such as “Don’t ‘prove by example’ ” (Rule 22) and “Make your counterexamples concrete and specific” (Rule 24), help students to reason from hypotheses to conclusion with properly quantified variables. A student proof that follows the rules may be invalid, but its errors are more likely to do with the mathematical content than with mere logical floundering.

Once a student has obtained a logically valid proof, she still must communicate it according to the grammatical conventions both of English and of the mathematical community. A third category of rules teaches students some of these. When students define a new variable z equal to $x + y$, for example, Rule 23 teaches them to write “Let $z = x + y$ ” rather than “Let $x + y = z$.” Some of the grammatical rules overlap with logical principles, such as my admonition to “Introduce every symbol you use” (Rule 19).

A fourth class of rules refers to manuscript preparation. Left to their own devices, many of my students would turn in their proofs and scratchwork side by side or even commingled. Others, motivated perhaps by economy or environmentalism, prefer to fit each assignment on a single sheet of paper by writing in a very, very small hand from margin to margin. To avoid such unpleasant and unprofessional papers, I tell my students to “use scratch paper” (Rule 10) before writing their final copy, and to “use whitespace” (Rule 8) such as paragraph breaks or indentation to organize the logic of their sentences.

The full *Elements of Style* document I currently use is given in the Appendix.

2.2 Use in the Classroom

As an introductory codification of what good proof-writing looks like, the *Elements of Style* handout is a flexible tool and could be used in many ways. It could have some good effects even if an instructor merely gave it to students as a useful resource, but when used consistently and systematically by professor and students together, its value becomes far greater. Readers of this chapter will undoubtedly devise ways to integrate the *Elements of Style* into their own course structures, but in this section I will outline how I have used it in my courses to help students improve their proof-writing.

I print the *Elements of Style* on neon red paper, highly distinct from all other course materials, so that it will be hard to lose and easy to find. It must not get buried among the other papers in the students' folders, because I want them to use it almost every day of the week. I hand it out to them early in the course—usually on day one of an advanced class, or within the first week of students' introduction-to-proofs course—and I explain that it will be of vital importance to them throughout the semester. (If giving away the handout would exceed a department print budget, it could be sold instead as part of a course pack in the campus bookstore.)

In the early weeks of the course, I go over one or two rules at the beginning of each class period, explaining each rule in detail. I encourage students to use the *Elements of Style* to improve their homework in three ways. First, in reading its rules and hearing me speak about them in class, they gain an initial idea of how to write a proof well. Then when they begin working on a homework assignment, they can consult the rules to help them write their proofs. Finally, as they edit their proofs before turning them in, they can use the rules to evaluate and improve their work, almost as an examination of conscience: "Have I begged the question? Have I shown the logical connections among my sentences?"

When the student papers come to me for grading, the *Elements of Style* makes itself immensely and practically useful, because I can refer to the rules by number when grading. If a student has attempted to prove a certain equality by starting with the equality and deriving " $0 = 0$," I merely write "Rule 16," assign an appropriate number of points, and move swiftly to the next problem. Upon receiving his homework back, the student then looks up "Rule 16: Don't write the proof backwards" and reads a thorough explanation of what he did wrong, why it is wrong, and how to do better next time. This allows me to grade each problem much more quickly (a similar approach using "feedback codes" is described in Tolle [12]), and consequently I can grade more student proofs more frequently, but it also, paradoxically, permits more depth of feedback. When I used to come across such a "proof" at the top of a stack of homework papers, I would write at most "This proof is backwards! You started with the conclusion and derived a true but highly uninteresting statement"; by the time I reached the twenty-eighth paper in the stack, my commentary would have shrunk to a weary, laconic scrawl: "Proof is backwards!" Using the *Elements of Style*, the phrase "Rule 16" leads every student to a detailed explanation of the error with an unhurried patience that I could not otherwise provide.

2.3 Tracking Student Progress

My students typically turn in three ten-point homework assignments per week, each including four to six proofs. I grade selected proofs from each assignment, marking logical errors and indicating violations of the rules. I sometimes take off points for rule violations, especially for logical errors and egregious stylistic problems, but for less severe errors of style I usually do not take off points. At the top of each homework assignment, however, I write not only the grade out of ten points for mathematical content, but also a list of which rules from the *Elements of Style* the student has violated. I keep track of all this information in my grading spreadsheet.

The accumulated points on all homework assignments count for about 20% of a student's course grade, but an additional 7% or so is determined by their *progress* in implementing the rules listed in the *Elements of Style*. I naturally expect many rules to be broken at the beginning of the semester, when the students begin writing proofs for the first time, but I want my students to make fewer and fewer stylistic errors as they learn. At present, I determine progress in a fairly subjective way, looking at my spreadsheet of errors committed and asking myself whether the student's record shows clear progress. (I have not found a fair way to compute this grade formulaically—as a function of the number of rule violations, for example—because there are too many confounding factors. For example, a poor student may break no errors on an assignment by writing a disorganized mishmash of facts, guesses, and conjectures too vague in content to violate any specific rules. Some students make no errors because they turn in no assignments at all.)

The grading process helps students notice the specific mistakes they make, keep track of patterns of mistakes from assignment to assignment, and so measure their progress. Whereas a student facing a sea of red ink and an overall grade of “6 out of 10” may feel too overwhelmed to read my comments, resolving instead simply to “do better next time,” when that same student sees “6/10; Rules 2, 11, 14,” he is immediately drawn to look up the rules in his *Elements of Style* handout and leaf through his paper to see where he violated them. He can set himself a specific goal of obeying Rule 14 next time, and then achieve it by consulting the handout while preparing the next homework assignment. The ability to make measurable progress on several small goals can take the sting out of the generally lower homework scores that so often disappoint students in their first proof-writing courses.

For the professor too, using the rules in the *Elements of Style* makes tracking the progress of individual students and of the class as a whole much easier. A glance at my spreadsheet of rule violations suffices to show which students have mastered the basic structure, grammar, and style of proof-writing. Likewise if I find myself writing “Rule 20” on the papers of many students, it is easier to recognize that I should review the use of quantifiers in the next class period.

3 Outcomes

The costs of using the *Elements of Style* are negligible. There are some printing expenses, and a few minutes of class time early in the term should be devoted to discussing the rules. Recording rule violations in a spreadsheet also takes a few extra seconds per homework assignment.

The benefits, on the other hand, are very great. The amount of time saved by using the rules as shorthand while grading student proofs is significant, especially if the class size is large. More importantly, the document helps establish a common vocabulary for discussing proofs and what makes them good or bad. When talking with students about their graded homework in office hours, I find that they no longer treat their errors as isolated mistakes tied to this proof or that proof; they see them instead as patterns of errors that they are in danger of committing on future assignments, but can prevent with diligence. I have overheard students say to one another in the study lounge, “Man, I broke Rule 19 again! I’ve been trying to avoid that one.”

Moreover, in a transition course like Concordia’s Linear Algebra, the *Elements of Style* can act as a bridge between the old world of calculations and the new world of proofs. For years many math majors exulted in the “one right answer” that made mathematics so appealing to them; they succeeded in math precisely by being very careful and attentive to detail. Now, however, they are faced with the double challenge of discovering valid logical proofs—a creative act partly subject to the whims of the muses—and then writing them in a formal, rigorous manner satisfactory to their professors. Nothing ever can or should make undergraduate proof-writing as systematic or even routine as the computations of Calculus I, but the *Elements of Style* can offer students clear expectations and guidelines against which to check their work. Before turning in their homework they can verify that every variable was introduced and appropriately quantified; they can double-check that every “=” links two mathematical objects that are actually equal; they can eliminate ambiguous pronouns from their pages. Discovering a proof may always be a mysterious act, but writing it and checking its style can be made more routine, restoring to students a measure of confidence in their work.

I have used my *Elements of Style* document for seven semesters in three courses: three times in Linear Algebra at Concordia, three times in an upper-level abstract algebra course at Concordia, and once in an upper-level linear algebra course at St. Norbert College. In each case the results have been successful. Because I started using the *Elements of Style* so early in my career, I do not have control groups against which to compare the efficacy of my handout. Speaking anecdotally, however, I have seen students begin the semester deriving hypotheses from the conclusions and end the term writing grammatically flawless proofs and handling multiple quantifiers with ease. A colleague teaching real analysis commented that students who used my *Elements of Style* handout impressed her with strong proof-writing skills beyond her expectations.

At least five colleagues at other institutions have adopted or adapted the handout for their own classes; those who used it regularly report that it was helpful for them and for their students.

There are some numerical data as well. I first kept track of which rules individual students violated in a fall semester abstract algebra course. In September the number of rules violated per student per assignment ranged from 0 to 4, with an average of 1.2, but by the end of the term it was exceedingly rare for a student to violate more than one rule and the average had dropped to 0.4. I have observed similar results in linear algebra courses as well. There is no proof that this improvement is entirely due to the *Elements of Style*, but it seems likely to be a strong factor, and without the *Elements of Style* the rates could not have been measured at all.

Finally, part of the goal of the *Elements of Style* is to help students know the criteria distinguishing good proofs from bad proofs, so that their grades become less mysterious to them. It therefore makes sense to ask how students perceive the handout. At the end of one semester, in addition to the college-wide course evaluation form, I gave my abstract algebra students a questionnaire which asked, “Did the *Elements of Style* handout help you? If so, how? If it was not helpful, why not?” Eight of the eleven students in the course responded. Two students gave neutral responses, stating that they did not use the handout “personally,” but that I should “still hand it out though.” The other six responded positively, stating that the handout was “helpful.” One said, “That handout shows your class the expectations you have for their homework in general,” and another wrote, “I think I quickly mastered the basic format of proof making thanks to this handout.” Other students focused on the handout’s use in grading, commenting, “Especially easy to find out exactly what you had points taken off for,” and another noted, “I think it made it easier for us to realize if we kept making the same type of mistake.”

4 Extending the Method

Using an *Elements of Style* document is an excellent way to help students learn to write clear, valid proofs, whether in a first proofs course for sophomores or in an abstract algebra course for seniors. No great adjustment should be needed to handle an especially large or small class. If a class is large enough that one or more student graders are employed, the document would give them criteria for judging proofs fairly, and it could help ensure consistency from grader to grader.

Readers of this chapter will probably want to customize the document for their classrooms, perhaps adding rules to reflect their tastes or adjusting examples to fit the subject of the course. They may also find new ways to integrate the *Elements of Style* with their pedagogy; for example, students doing peer reviews could use the rules to help them critique one another’s proofs. Students will internalize the rules best, I believe, if they are invoked frequently by name or number, and so the *Elements of Style* should be used consistently every time a proof is assessed, graded, or even discussed, whether in homework, on tests, or at the chalkboard. In this way the professor and students together can use a common, documented, well-defined vocabulary to discuss the stylistic merits of proofs, which can inform and promote every other effort to foster good proof-writing.

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Appendix

A The *Elements of Style* Handout

Years of elementary school math taught us incorrectly that the answer to a math problem is just a single number, “the right answer.” It is time to unlearn those lessons: those days are over. From here on out, mathematics is about discovering proofs and writing them clearly and compellingly.

The following rules apply whenever you write a proof. I will refer to them, by number, in my comments on your homework and tests. You will find them summarized on the last page. Take them to heart; repeat them to yourself before breakfast; bind them to your forehead; ponder their wisdom with your friends over lunch.

1. **The burden of communication lies on you, not on your reader.** It is your job to explain your thoughts; it is not your reader’s job to guess them from a few hints. You are trying to convince a skeptical reader who doesn’t believe you, so you need to argue with airtight logic in crystal clear language; otherwise she will continue to doubt. If you didn’t write something on the paper, then (a) you didn’t communicate it, (b) the reader didn’t learn it, and (c) the grader has to assume you didn’t know it in the first place.
2. **Tell the reader what you’re proving.** The reader doesn’t necessarily know or remember what “Problem 5c” is. Even a professor grading a stack of papers might lose track from time to time. Therefore the statement you are proving should be on the same page as the beginning of your proof. For an exam this won’t be a problem, of course, but on your homework, recopy the claim you are proving. This has the additional advantage that when you study for tests by reviewing your homework, you won’t have to flip back in the textbook to know what you were proving.
3. **Use English words.** Although there will usually be equations or symbolic statements in your proofs, use English sentences to connect them and display their logical relationships. If you look in your textbook, you’ll see that each proof consists mostly of English words.

In particular, use a word, not a symbol, after each punctuation mark and at the beginning of each sentence; otherwise the punctuation mark looks too much like part of the mathematical formulas. The

sentence “For all $x \in \mathbb{Z}$, $x^2 \in \mathbb{Z}$,” for example, is slightly harder to parse than the equivalent sentence “For all $x \in \mathbb{Z}$, its square $x^2 \in \mathbb{Z}$.”

4. **Use complete sentences.** If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use grammatically correct sentences, complete with verbs, to convey your logical train of thought. A good way to test whether your proof has complete sentences is to read the proof aloud.

Some complete sentences can be written purely in mathematical symbols, such as equations (like $a^3 = b^{-1}$), inequalities (like $o(a) < 5$), and other relations (like $5 \mid 10$ or $7 \in \mathbb{Z}$). These statements usually express a relationship between two mathematical *objects*, like numbers (e.g., 7), vectors (e.g., \vec{v} and \vec{w}), or sets (e.g., V and \mathbb{R}).

If your verb is a symbol, the objects it joins together must be given in symbols too, not in words. For example, “Thus $x \in A \cap B$ ” and “Thus x lies in the intersection of A and B ” are good style, but “Thus $x \in$ the intersection of A and B ” looks funny.

5. **Show the logical connections among your sentences.** Use phrases like “Therefore” or “because” or “if... then...” or “if and only if” or “we see that” to connect your sentences.
6. **Know the difference between statements and objects.** A mathematical object is a *thing*, a noun, such as a group, an element, a vector space, a number, an ordered pair, etc. Objects either exist or don’t exist. Statements, on the other hand, are mathematical *sentences*: they can be true or false.

When you see or write a cluster of math symbols, be sure you know whether it’s an object (like “ $x^2 + 3$ ”) or a statement (like “ $x^2 + 3 < 7$ ”). One way to tell is that every mathematical statement includes a verb, such as $=$, \implies , \iff , $<$, $>$, \equiv , \in , \subseteq , $|$, \approx , \cong , and \triangleleft .

7. **Don’t interchange $=$ and \implies .** The equals sign connects two *objects*, as in “ $x^2 = b$ ”; the symbol “ \implies ” connects two *statements*, as in “ $ab = a \implies b = 1$.” And please, please don’t just use a generic “ \rightarrow ” to connect two lines. That symbol has no meaning in that context; it doesn’t tell the reader anything.
8. **Use whitespace.** Don’t cram your proof into a few lines of the paper, filled from left margin to right margin. Let your proof breathe! When you start a new thought, start a new line. Use indentation to organize your sentences. This helps the reader understand your thought much better, and it also encourages you to be more clear.
9. **Use multiple sheets of paper.** Some people write tiny, trying to cram everything onto a single sheet of paper, with the result that their proofs are so terse as to be incomprehensible. An extra tree will gladly sacrifice its life to help your proofs be legible and understandable.
10. **Use scratch paper.** Finding your proof will be a long, messy process, full of false starts and dead ends. Do all that on scratch paper until you find a real proof, and only then break out your clean paper to write your final proof carefully. *Do not hand in your scratch work!*

Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a “brain dump,” throwing everything you know onto the paper before trying to find logical steps that prove the conclusion. *That is what scratch paper is for.*

11. **“ $=$ ” means equals.** Don’t write $A = B$ unless you mean that A actually equals B . This rule seems obvious, but there is a great temptation to be sloppy. In linear algebra, for example, some people might

write

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 6 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$$

(which is obviously false), when they really mean that the first matrix can be row-reduced to the second. Likewise they might write “ $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$,” when they really mean that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V .

12. **Say exactly what you mean.** Just as the symbol “=” is sometimes abused, so too people sometimes write $A \in B$ when they mean $A \subseteq B$, or write $a_{ij} \in A$ when they mean that a_{ij} is an entry in matrix A . Mathematics is a very precise language, and there is a way to say exactly what you mean; find it and use it.
13. **Don’t write anything unproven.** Every statement on your paper should be something you *know* to be true. The reader expects your proof to be a series of statements, each proven by the statements that came before it. If you ever need to write something you don’t yet know is true, you *must* preface it with words like “assume,” “suppose,” or “if” (if you are temporarily assuming it), or with words like “we need to show that” or “we claim that” (if it is your goal). Otherwise the reader will think he’s missed part of your proof.
14. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like

$$A = B \leq C = D,$$

she expects to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and she expects the *point* of the entire line to be the more complicated fact that $A \leq D$. For example, if you were computing the distance d of the point $(12, 5)$ from the origin, you could write

$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean Theorem, the second is just arithmetic, and the *point* is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write “ $\sqrt{12^2 + 5^2} = 13 = d$,” your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2 + 5^2} = d$.

15. **Don’t beg the question.** Be sure that no step in your proof makes use of the conclusion! That is called “begging the question” or “circular logic,” and it makes your proof invalid.

For example, consider this student’s “proof” that $\vec{0} + \vec{0} = \vec{0}$:

<p>Claim: $\vec{0} + \vec{0} = \vec{0}$.</p> <p>Proof: $\vec{0} + \vec{0} = 1\vec{0} + 1\vec{0}$ $= 1(\vec{0} + \vec{0})$ $= 1\vec{0}$ $= \vec{0}. \quad \square$</p>

To move from the second line of his proof to the third, the student replaced $\vec{0} + \vec{0}$ with $\vec{0}$ because they’re equal—but *that is exactly what needs to be proven!* This argument only shows that $\vec{0} + \vec{0} = \vec{0}$ is true if we *already* know it to be true, so it’s a worthless argument.

16. **Don't write the proof backwards.** For some reason, beginning students tend to write “proofs” like this “proof” that $\tan^2 x = \sec^2 x - 1$:

Claim:	$\tan^2 x = \sec^2 x - 1.$
Proof:	$\tan^2 x = \sec^2 x - 1$
	$\left(\frac{\sin x}{\cos x}\right)^2 = \frac{1}{\cos^2 x} - 1$
	$\frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x}$
	$\sin^2 x = 1 - \cos^2 x$
	$\sin^2 x + \cos^2 x = 1$
	$1 = 1$

Notice what has happened here: the student *started* with the conclusion, and deduced the true statement “ $1 = 1$.” In other words, he has proved “If $\tan^2 x = \sec^2 x - 1$, then $1 = 1$,” which is true but highly uninteresting.

Now this isn't a bad way of *finding* a proof. Working backwards from your goal often is a good strategy *on your scratch paper*, but when it's time to *write* your proof, you have to start with the hypotheses and work to the conclusion.

17. **Be concise.** Most students err by writing their proofs too short, so that the reader can't understand their logic. It is nevertheless quite possible to be too wordy, and if you find yourself writing a full-page essay, it's probably because you don't really have a proof, but just an intuition. When you find a way to turn that intuition into a formal proof, it will be much shorter.
18. **Avoid weasel words.** There are some notorious phrases that advertise that you don't really understand the logic you need. Be wary of phrases like “clearly,” “obviously,” and “the only way this can happen is . . .”
19. **Introduce every symbol you use.** If you use the letter “ k ,” the reader should know exactly what k is. Good phrases for introducing symbols include “Let $x \in G$,” “Let k be the least integer such that . . .,” “For every real number a . . .,” and “Suppose that X is a counterexample.”
20. **Use appropriate quantifiers, once.** When you introduce a variable $x \in S$, it must be clear to your reader whether you mean “for all $x \in S$ ” or merely “for some $x \in S$.” If you just say something like “ $y = x^2$ where $x \in S$,” the word “where” doesn't indicate which of the two you mean.
- Phrases indicating the quantifier “for all” (\forall) include “Let $x \in S$ ”; “for all $x \in S$ ”; “for every $x \in S$ ”; “for each $x \in S$ ”; etc. Phrases indicating the quantifier “there exists” (\exists) include “for some $x \in S$ ”; “there exists an $x \in S$ ”; “for a suitable choice of $x \in S$ ”; etc.
- On the other hand, don't introduce a variable more than once! Once you have said “Let $x \in S$,” the letter x has its meaning defined. You don't *need* to say “for all $x \in S$ ” again, and you definitely should *not* say “let $x \in S$ ” again.
21. **Use a symbol to mean only one thing.** Once you use the letter x once, its meaning is fixed for the duration of your proof. You cannot use x to mean anything else.
22. **Don't “prove by example.”** Most problems ask you to prove that something is true “for all”—for all $x \in G$, say, or for all vector spaces V . You *cannot* prove this by giving a single example. Your answer will need to be a logical argument that holds for *every example there could possibly be*.

23. **Write “Let $x = \dots$,” not “Let $\dots = x$.”** When you have an existing expression, say $\vec{u} - r\vec{v}$, and you want to give it a new, simpler name like \vec{x} , you should write “Let $\vec{x} = \vec{u} - r\vec{v}$,” which means, “Let the new symbol \vec{x} mean $\vec{u} - r\vec{v}$.” This convention makes it clear to the reader that \vec{x} is the brand-new symbol and $\vec{u} - r\vec{v}$ is the old expression she already understands.
- If you were to write it backwards, saying “Let $\vec{u} - r\vec{v} = \vec{x}$,” then your startled reader would ask, “What if $\vec{u} - r\vec{v} \neq \vec{x}$? And for that matter, what is \vec{x} ?”
24. **Make your counterexamples concrete and specific.** Proofs need to be entirely general, but disproofs—counterexamples—should be absolutely concrete. When you provide an example or counterexample, make it as specific as possible. For a set, for example, you must name its elements, and for a function you must give its rule. Do not say things like “ θ could be one-to-one but not onto”; instead, produce an actual function θ that *is* one-to-one but not onto.
25. **Shun pronouns, especially “it.”** Pronouns do have a proper place in the English language, of course. When you write a proof, however, you are usually juggling several mathematical objects at once, and when you use a pronoun like “it,” too often the reader won’t be able to tell which of them is the antecedent.
- Never use a pronoun unless the antecedent is crystal clear from the grammar itself.*
26. **Don’t include examples in proofs.** Including an example very rarely adds anything to your proof. If your logic is sound, then it doesn’t need an example to back it up. If your logic is bad, a dozen examples won’t help it (see rule 22). There are only two legitimate reasons to include an example in a proof: if it is a *counterexample* disproving something, or if you are performing complicated manipulations in a general setting and the example is just to help the reader understand what you are saying.

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Learning Proofs via Composition Instruction Techniques

Sharon Strickland and Betseygail Rand

Abstract

Inspired by the drafting and revision process often employed in rhetoric and composition courses, this paper reports on adapting these techniques when teaching novice students how to write mathematical proofs. Rather than assign many proofs as homework throughout the semester, the instructors assigned approximately ten proofs with the stipulation that students could revise each proof up to two more times (for a total of three submissions), with careful instructor feedback provided between submissions.

Difficulty Level: Medium; Course Level: Transitional

1 Background and Context

Years ago, Betseygail Rand was assigned to teach her university's Freshman Experience course, a general introductory course to the humanities (and far outside her comfort zone.) In the course of preparation, a colleague recommended *Engaging Ideas: The Professor's Guide to Integrating Writing, Critical Thinking, and Active Learning in the Classroom*, by John Bean [2]. Rand read it with one ear cocked towards her math courses—could its ideas apply to mathematical thinking?

The premise of the book is that critical thinking is best taught by having students make deep, structural revisions to their writing. As the student analyzes and restructures her paper over and over, she deepens her understanding of the topic. Rand and Sharon Strickland decided to apply the idea to math students encountering proof-writing for the first time. We applied it at two schools, Texas Lutheran University, a small liberal arts college of roughly 1400 students, and Texas State University, an emerging research university of 31,000 students, both in central Texas. At the former, Linear Algebra was chosen, and the technique was tried over two semesters, by two instructors. Class size was under 20 students in both classes. Linear Algebra was (at that point) the course that served to introduce math majors to proof-writing. At the latter institution, the course was Modern Geometry, composed of 35 preservice secondary teachers. In it, students are first exposed to proof-writing, but the goal is a more basic familiarity with writing proofs. At both institutions, two semesters of calculus were required as a prerequisite.

2 Description and Implementation

The instructors created distinct proof-writing assignments, as opposed to embedding proof problems in existing homework assignments. Each proof was to be formally typed and submitted for feedback. Students were expected to use Geometer's Sketchpad or Equation Editor in Microsoft Word, or an equivalent program. (LaTeX was not required, but recommended if a student was considering continuing on in graduate school.) Students were expected to turn in three drafts of the same proof, and received instructor feedback after each draft. (While proofs were modeled and discussed during class time, the assignment was specifically intended not to intrude on class meeting times.) Each class had 10 stand-alone assignments over the course of the semester. The authors were interested in creating a trade-off: would students learn to write better proofs if the instructor chose to assign fewer proofs, but spend more time on each one?

As given to the student, the assignment was rather bare—merely “Prove or disprove the following statement:”. Students were given dates that each draft would be due. Ideally, the instructor returned each draft by the subsequent class period, and so a cycle of three drafts and two rounds of feedback took a minimum of five class periods. Often a second or third draft of a proof fell on the same due date as the first draft of the next proof.

Instructors assigned grades for each draft, and replaced each grade with the grade on the subsequent draft. If a student was happy with her grade on a first or second draft, she could stop there and take that grade for the assignment. We recommend requiring students to resubmit all previous drafts along with the current draft, so that the instructor can trace the evolution of the student’s thought process. (An unintelligible statement may make more sense if the instructor can see what the student was trying to fix.) We also recommend that instructors make the first draft mandatory. If the first (or second) draft is optional, there is a grave temptation for a student to regard the third draft deadline as the “real” deadline, and to skip the redrafting process altogether.

Students were asked to compose the first draft without help from tutors, instructor office hours, or collaboration. After the first draft, they were allowed to seek help from tutors or instructors, and some did. We wanted to convey the message to students that they were capable of tackling a proof using their own faculties, and were welcome to assistance only after they’d made a genuine attempt at writing the proof.

The heart of the assignment lay in the thoughtfulness required to construct appropriate instructor feedback. As instructors are inevitably pressed for time, we were concerned about this imposition. The time constraints will be discussed in more detail later.

The pedagogical method described in Bean is specific about the concept of revisions. In particular, revising does not mean editing—the focus is not on spelling or grammar, or even poor phrasing [2]. Rather, the idea is to get students to restructure the entire paper—to essentially begin to see the topic from a bird’s-eye view, and understand its organization and internal connections in increasingly complex layers. (Bean notes that students’ grammar will deteriorate as they wrestle with a difficult topic, and that they will automatically catch most grammatical mistakes as they grow to understand the topic through the revision process.)

The instructor’s role, therefore, is to teach the students how to conduct their own revisions. For example, an instructor might point out that an entire section is murky, but would not explicitly describe how the student might strengthen it and clarify the argument. The instructor should draw the students’ attention to weaknesses, but let *them* wrestle with the nature of the fix. (In addition, the authors attended English composition classes in order to observe the techniques in the classroom.)

In designing our proof-writing method, we sought to replicate this phenomenon—could we teach students to find their own mistakes? To this end, we were quite prescriptive about what form feedback should take. Feedback ought to locate a problem in the proof, but not solve it. Here is an example, hypothetically submitted by a student:

Prove or disprove that a real, symmetric 2x2 matrix has real eigenvalues.

Student proof: Let

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

be a matrix. Then we do

$$\begin{pmatrix} a - \lambda & b \\ b & d - \lambda \end{pmatrix}.$$

Taking the determinant and setting it equal to zero, $(a - \lambda)(d - \lambda) - b^2 = 0$. That becomes $(a - \lambda)(d - \lambda) = b^2$. Since b^2 is never imaginary, it makes sense that λ will never be imaginary.

□

The final sentence is incorrect, and in addition the student has omitted several key steps. The instructor should resist the urge to provide the missing steps—a more productive response is to highlight the sentence

and write “faulty implication”. The student is then left to wrestle with the mathematical concepts before submitting the next draft. (This proof is one of the shorter assignments taken from the Linear Algebra class. While the language is rocky—“we do this, then we do that”—our instructors would have resisted drawing the student’s attention to that until the more glaring errors were addressed. Often students will polish their writing without prompting, as the drafts progress. However, sometimes rocky language is actually incorrect—for example, a student who uses “therefore” instead of “and” when linking clauses in the hypothesis. In this case, instructors would note the rhetorical error.)

Over the course of the classes, our feedback improved as some observations emerged. Student errors tended to fall into three categories: first, students might have basic misunderstandings of what a proof is. What does it mean to prove a result? We called these errors *fundamental errors*. A typical case would be a student who provides an example in lieu of a proof. These errors tended to cluster at the beginning of the semester, when students found proof-writing clunky and awkward, and may not have understood why they were being required to write out “obvious” facts. These are the most egregious errors.

Next, students made errors reflecting the content of the course, which we called *content errors*. In a linear algebra course, they could misuse a definition. (The example above, concerning the real eigenvalues of symmetric matrices, also has a content error.) In a geometry course, consider a statement to be proved involving a quadrilateral. If a student uses properties of a parallelogram in proving the result, the student has committed a content error. We felt these errors revealed the struggle to master the underlying mathematical material. (These are the type of errors that never go away—skilled mathematicians are essentially struggling with the underlying content.)

Finally, students made errors indicating a lack of familiarity with mathematical language. We called them *rhetorical errors*. Often a student might grasp the mathematical reasoning, but communicate it awkwardly: “The matrix A has only a trivial solution.” The instructor might correctly infer that the student meant “The equation $Ax = 0$ has only a trivial solution” but had not accurately stated what she wanted to state.

Recognizing the three broad categories of errors informed the feedback that instructors were able to give. Sometimes letting a student know that her error was rhetorical and not a content error helps her gain confidence in her understanding of the math, while focusing on her communication of the proof. A student who is making fundamental errors after a month of practice can be flagged, and pulled into office hours for closer instruction. (It is not always clear whether the student has made a fundamental error, content error, or rhetorical error. In addition, they can be compounded simultaneously! Clever students. See Appendices for further examples of feedback on ambiguous proofs.)

Feedback was fine-tuned by each instructor, according to the sophistication of the students in the course. A prospective middle school teacher (in the geometry course) may not benefit from the minimalist style that works well for a strong math major (in the linear algebra course). In these cases, the instructors fleshed out feedback by providing additional scaffolding, without compromising the student’s role in the revision process. Typically it was done in three steps: tell the student what she’s written, remind the student of the goal, and draw her attention to the gap in between. For example, consider a hypothetical student who has submitted the following geometry proof:

Prove or disprove that the triangles have the same area.

Excerpt from student proof: Since we already notice that triangles have congruent angles, therefore the triangles have SAS (which is a way to find congruence of triangles). So we have proved that all triangles are congruent, thus have the same area.

Instructor feedback: Although you correctly identify the respective congruent angles, you falsely claim that this leads to SAS congruence (false implication). You were asked to show that the triangles had the same area. Your strategy was to first show that they are congruent and therefore have equal area. By making a mistake in the congruence aspect, you overall fail to show that they have the same area. Reconsider triangle congruence. Are they actually congruent? If

so, how can you show that correctly? And if they are not, what can you do to show they nevertheless have the same area? Can you come up with an example of two triangles that are not congruent and yet have the same area?

The instructor has still not provided a solution for the student, but has given support towards a solution by expanding the feedback, and posing some questions that the student might consider. (Editors' note: for a similar approach at a more advanced level, with less instructor feedback and more student attempts, see DeLong's article in this volume [3].)

In the course of coding student errors, we developed a list of common categories of errors. This list was compiled from our coding data, as well as work by Selden and Selden [5], and Andrew [1]. This proved useful in later semesters, as a reference sheet to be used when reviewing drafts of proofs. It is included at the end of this chapter, in Appendix A.

3 Outcomes

The technique was implemented formally, once by each author. In addition, both authors and one other instructor have imported aspects of the technique into other classes.

It appears that this technique reduces the fear and anxiety some students feel about writing proofs. As this was part of a pilot study, we had embedded proof questions on the final exam of a control class and an experimental class. As final exams are not generally returned to students, the control and experimental sections took the same final exam. While our n was too small for rigorous statistical analysis, the experimental class did not perform better on the embedded questions than the control class did. (Nor did they perform any worse.) We did, however, see a larger number of students who attempted the proof questions on the exam in the experimental class — the control group left incomplete four times as many proof questions on the final exam, compared to the experimental group. In the Content category, the control group made 31 errors on proofs, compared to 24 in the experimental group. In the Rhetoric category, the control group made 7 errors, compared to 6 made by the experimental group. Both groups performed identically on fundamental errors.

Reducing the number of students who skip the proof questions altogether is noteworthy—perhaps they did in fact have less anxiety and writer's block about attempting to draft a proof. We hypothesize that this measurement — embedded questions on a final exam—does not well capture the full benefit of the drafting assignments. Essentially the students have been asked to write their first draft on the exam, and so the conclusion should be “Students did not show improvement in their ability to construct first drafts, but were more willing to try.” Furthermore, unsolicited, we received both verbal affirmation and strong positive comments on the semester evaluations. Students very much enjoyed the process, providing further support for the claim that the technique reduced anxiety related to learning to prove.

Second, and similar to the first point, we had a higher rate of attempting proof assignments under this technique. Prior to implementation, students often skipped the homework problems that asked them to write a proof, whereas this ongoing, distinct assignment was completed by a much higher portion of the class. (This is an informal assessment. The proof assignment was completed at the same rate by the experimental class as the combined computation-and-proof assignments were completed by the control class. Both the grader and the instructor anecdotally commented that they commonly saw control students only completing the computation portion. We do not have hard data on how often that occurred, though.)

Third, both instructors have reflected on the quality of the proofs during the experiment in relation to those from either previous semesters or semesters since then. One, due to an odd departmental scenario, could not utilize the technique in a recent semester. She noticed that the quality of the proofs for the recent linear algebra course was far below those from the experiment reported here. On the midterm, several students submitted a “proof by example”. By contrast, the drafting/revision class eradicated fundamental errors like “proof by example” by the second round of assignments. The other instructor has reviewed proofs by

students in prior semesters and has noticed that more students struggled longer into the semester overall than did students in the drafting/revisions version of the same course. Overall, we believe that because the drafting/revision process fosters intensive attention to students' proofs, we achieve better early support for students new to learning how to write proofs.

This seems to help them understand more quickly the role and purpose of proofs so that their attention can be focused less on the structure and purpose of proving (less in that they understand this sooner in the semester; not that it is less important) and more on the relationships and key content ideas needed in proving.

Finally, the instructors report a close and individualized knowledge of student understanding. By giving feedback on repeated drafts of the same proof, instructors can see whether or not the student learned from her initial mistakes.

The major shortcoming of the method is the imposition on instructor time. Prior to using the method, all instructors used student graders for the homework assignments. When using the method, homework was split—computational assignments were sent to the student grader, and the proof drafts were graded by the instructor. As instructors often find themselves impossibly crunched for time, this could be a real cost.

We return to the original tradeoff: would students learn to write better proofs if the instructor chose to assign fewer proofs during the semester, but spend more time on each one? In practice, the instructors have informally both landed somewhere in the middle. Rather than recreate the exact formal environment in a non-study classroom, they have both chosen to incorporate aspects of the revision process. Rand often invokes the revision process on a case-by-case basis, in response to individual struggling students, or a proof that was largely misunderstood by all students. Strickland tends to offer more opportunity for revisions early in the semester and wean students off by the end.

4 Extending the Method

As is, the method works well in small classes, with all levels of familiarity with proof-writing. We believe it could be extended to larger classes. The key limited resource is instructor time; therefore we provide two possible modifications to get around this bottleneck.

The first is peer review: students can provide feedback to each other. For this to be effective, students need to be trained in how to give structural feedback, and not just editing. (To facilitate this, we recommend giving to students the coding list of typical errors found at the end of this chapter.) At the beginning of the semester, the instructor would need to monitor the process closely, and help train effective peer reviewing. However, as the semester progresses, the instructor can reduce her supervision and free up her time. (Editors' note: for more information in this volume on training students to peer review effectively, see Hitchman [4].)

Secondly, the students can work in groups, and submit one proof per group. While groups need their own supervision, this would significantly reduce the number of proof drafts to grade in each round.

Finally, the methods could be blended: students in groups peer review each other's drafts, and then pick one to represent the group. (We strongly recommend that the weaker drafts also be turned in to the instructor, so that less sure students are discouraged from hiding out altogether. They could be spot-checked, so that weaker students are receiving a bit more help from the instructor.)

Technology is useful insofar as students may write and submit their proofs electronically, and instructors may comment and return the proofs online. Strickland used the file-sharing features of a course management system to organize student drafts and feedback. Students uploaded their proofs to appropriate folders (e.g., Proof 3 Draft 2) and she returned the feedback files to them in the same way. Furthermore, she had them submit their work in a file format that was easily modified (e.g., Word docs or Geometer's Sketchpad) so that she could type directly on their submissions with her comments. While this did not cut down on the time needed to provide careful feedback, it did assist her in organizing which students had completed which proofs and which drafts, which in a roundabout way slightly decreased the overall time spent grading.

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- [5] Annie Selden and John Selden, “Errors and Misconceptions in College Level Theorem Proving,” in *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics*, Vol III, 457-470, Joseph D. Novak, ed., Cornell University, 1987. Available as Tennessee Technological University Tech Report No. 2003-3 at <http://www.math.tntech.edu/techreports/techreports.html>.

5 Appendix A: Common Errors

The following is a list of common types of errors. These are distinct from the categories detailed earlier, of *fundamental*, *content*, and *rhetorical errors*, and most errors could occur in all three categories. Consider a student who misuses a definition, a vocabulary and grammar error. Possibly a weak student might not understand how to invoke a definition and use it precisely, and may have committed a fundamental error. Another student might understand in general how to use a definition, but has not fully grasped the mathematical content defined, and has committed a content error. Finally, students may fully understand the definition and its role in the proof, but have written a proof using rather rocky language, and the instructor may draw their attention to their rhetorical error. Thus in the list below, a given type of error has many different presentations, and it is up to the instructor to surmise whether a fundamental, content, or rhetorical error has been revealed—or if the instructor cannot tell.

In practice, the instructor used these errors to give feedback to the students. When coding proofs for the study, we coded them with both error type and whether they were fundamental, content, rhetorical, or ambiguous in quality.

Basic Problems

- Wrong problem
- Wrong method (a method was specified in the assignment, but this method was not used)
- Scratchwork (as submitted, proof was insufficiently formal)

Applying known material

- Misusing a theorem or result
- Misusing vocabulary or grammar

Flow of Proof

- Logical order (ideas were not presented in a logical order)
- Circular argument

- Faulty implication (A does not imply B, regardless of the truth of A and B)
- Extraneous detail

Omissions

- Error-caused omission (an error caused large, key steps of the proof to be omitted. The error itself should be coded with a second, accompanying code.)
- Assertions (a correct statement which requires more justification, according to context)
- Omitted sections (did not address one or more sections of the proof)
- Local omission (omitted a single, isolated step)

Miscellaneous

- Locally unintelligible (a single line is incomprehensible)
- Proof by example
- Notation (notation is awkward, confusing, or misleading)
- Weakened result (hypotheses have been strengthened, or full strength of statement has not yet been proved)
- False statement

We do not claim that this list is exhaustive or that categories are non-overlapping. But we have found it to be useful for ourselves and our students in clarifying and guiding the revision process.

6 Appendix B: Sample Coded Proofs

For the following three proofs, we display instructor feedback using italics, within the proof. We also provide our coding of the errors, and any additional comments, after each proof.

1. Determine whether or not $W = \{\vec{x} : x_1 + x_2 \geq 0\}$ is a subspace of \mathbb{R}^2 , and prove your answer.

Proof:

- (a) Contains the zero vector $0 + 0 \geq 0$

$$0 \geq 0$$

$$0 = 0 \checkmark$$

- (b) Closed under addition $u_1 + u_2 \geq 0$ and $v_1 + v_2 \geq 0$

$$\text{so } u_1 + u_2 + v_1 + v_2 \geq 0.$$

Omitted Step. Communicate to the reader where to find the components of the summation vector.

- (c) Closed under addition $u_1 + u_2 \geq 0$ $a(u_1 + u_2) \geq a \cdot 0 = 0$

Faulty implication

Therefore W is a subspace. \square

In the first example, our coding matches the feedback fairly closely. This one would be coded as two errors: local omission (rhetorical) and faulty implication (content).

2. Prove or disprove: In Diagram 1, E is the midpoint of segment \overline{AB}

Proof: The circles share the same radius AB and BA and the circles intersect at points C and D , and the intersection between segment AB and CD is at the point E , and so E is the midpoint of AB .

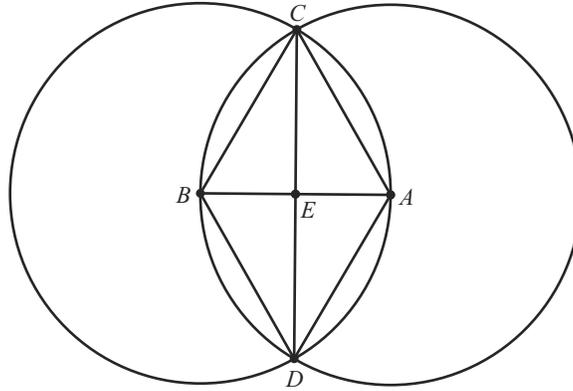


Figure 1: A and B are the centers of the respective circles

You've pretty much just claimed that it's true, without a proof or argument. What does it mean to be a midpoint? How can you show that E meets that definition?

In this example, we would code this as an assertion (fundamental). The student does not appear to grasp what it means to construct an argument, and has just asserted the conclusion after describing the picture.

3. Prove that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ are not similar matrices.

Proof:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix}$$

$$(1 - \lambda)(1 - \lambda) = 0$$

$$\lambda = 1, \lambda = 1$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} 4 - \lambda & 0 \\ 0 & 4 - \lambda \end{pmatrix}$$

$$(4 - \lambda)(4 - \lambda) = 0$$

$$\lambda = 4, \lambda = 4$$

□ *Omitted conclusion*

In this example, the student has computed eigenvalues for both matrices, but has failed to draw the conclusion that, since similar matrices have the same eigenvalues, the matrices must not be similar. Does she consider this self-evident and obvious, belying a rhetorical omission? Or does she remember that we often compute eigenvalues, but isn't clear how to resolve the computation to a conclusion, belying a content omission? We code this error as local omission (ambiguous). Ultimately we are always speculating when it comes to the interior thought process of a student, and sometimes it is wise to refrain from speculation altogether.

Abstract Algebra, Part of Writing Across the Curriculum

Jill Shahverdian

Abstract

Abstract Algebra is a required course for the mathematics degree at Quinnipiac University, and the first proofs course for most mathematics majors. This article presents a critical-thinking framework called “concentric thinking” that provides a way to scaffold writing prompts that helps students to construct proofs. Developed by Quinnipiac’s Writing Across the Curriculum program, concentric thinking is composed of three cognitively linked tasks—prioritization, translation, and analogization (PTA). The author uses the PTA framework to describe a sequence of tasks that culminate in a proof of the division algorithm.

Difficulty Level: High; Course Level: Advanced

1 Background and Context

Quinnipiac University is a private university with 6000 undergraduate students in liberal arts, business, and health sciences, and 2000 graduate students in business, law, and health sciences. All undergraduate students complete 46 credits of a general-education curriculum that includes two semesters of first-year English and three semesters of university seminars. Both sets of required courses focus on reading, writing, and discussion. Writing assignments and discussions in English are linked to reading, while writing assignments and discussions in the university seminars are topic driven. The three seminar courses explore the relationship between the individual and the community. At the 100-level, students explore questions of the local community; at the 200-level students explore questions of the national community; at the 300-level students explore questions of the global community.

The Department of Mathematics and Computer Science graduates approximately 20 mathematics majors each year. Half to three-quarters of the graduates continue in the School of Education Master of Arts in Teaching program. Each year, one or two students go on to attend graduate school in mathematics or statistics. The remaining graduates take jobs mostly in insurance or finance.

The required proofs classes for mathematics majors are Abstract Algebra (one semester) and Advanced Calculus (one semester). I teach Abstract Algebra every year, usually in the spring semester, with 22-28 students. Any section of Abstract Algebra will have 8-10 seniors in their final semester who have experience with proofs from Advanced Calculus and 14-16 juniors who have no or limited experience with proofs. The prerequisite for Abstract Algebra is Linear Algebra.

The textbook for the course is *Abstract Algebra* by John Beachy and William Blair [3]. Since the course is, for most students, a first course in proof-writing, the structure of the textbook allows me to start slowly and concretely and then gradually increase the level of sophistication and abstraction. The examples provided in this paper come from the first weeks of the semester, so the topics (integers and divisors) are ones that might be found in a transitions-to-proofs course or a first number theory course.

2 Description and Implementation

2.1 Writing Across the Curriculum

My approach to teaching proofs in Abstract Algebra relies on standard methods: reading the textbook, emphasizing the importance of vocabulary, and construction of proofs. The meta-narrative of the course is Writing Across the Curriculum (WAC). Quinnipiac University has a strong WAC program, and approximately 70% of faculty have participated in WAC training. The key idea of WAC is that writing facilitates thinking. Courses are not designated as writing courses. Instead, faculty are encouraged to incorporate their WAC training into courses, not with the idea that writing more will lead to better writing, but with the idea that WAC pedagogies improve critical thinking, which leads to better writing. In particular, the two required English composition courses and the three required university seminars rely on WAC methods. Therefore, all undergraduate students have experience in WAC methods. By explicitly presenting Abstract Algebra as the disciplinary extension of WAC, students view it as a continuation of both their mathematics courses and their writing experiences.

In Abstract Algebra, the pedagogical WAC components I use are concentric thinking, informal writing, and peer review. Concentric thinking is a critical-thinking framework [4] developed by Writing Across the Curriculum at Quinnipiac University. Sometimes called the PTA model, it is represented by three related cognitive activities—to Prioritize, to Translate, and to make an Analogy (analogize). The first step is to distinguish between most important and least important information. This prioritization includes an explanation of why these are the most or least important pieces and requires an understanding of context. The second step requires students to write given information in their own words. The translation requires a student to understand difficult information. The third step is for students to transfer their understanding to a new situation. This depends on relating new information to previously learned material and identifying how the pieces of information are similar or different.

Informal writing, also called exploratory writing or expressive writing, is writing for the self, as opposed to formal writing which is writing for an audience. (Both formal and informal writing are important to the WAC narrative, and both types of writing are important in Abstract Algebra. I emphasize informal writing here because it is one of the pedagogies I use to help students improve their formal writing.) Annotation, journaling, and in-class writing are examples of informal writing and are intended to help students think in ways that will lead to better formal writing. Informal writing may be ungraded or graded for only a small portion of a final grade. As a low-stakes assignment, informal writing provides students with an opportunity for exploration and risk-taking.

Annotation is an informal writing technique used extensively in our required English and university seminars. Students annotate their texts according to course questions or objectives. The act of annotation slows their reading and encourages critical engagement with the text. A marginal note might be a note to self or a response to the author, but either results in more productive reading.

Peer review (reading each other's writing) may be designed to assist students in their critical thinking or in their writing skills. In English composition, faculty primarily use it for the former. The goal is for students to engage with the writing on the level of ideas. In providing feedback to the author, the reviewer is also learning to respond to an idea and to articulate expectations for effective writing.

2.2 Implementing PTA in Abstract Algebra

If Writing Across the Curriculum is the meta-narrative of Abstract Algebra, then PTA (Prioritize, Translate, Analogize) is the cognitive framework. Using the PTA framework, I constructed a sequence of informal-writing, peer-review, and formal-writing tasks focused on the proof of the division algorithm.

The first step in the PTA division algorithm sequence, and many of the informal writing prompts throughout the course, are modeled after the reading technique described by Matt Boelkins and Tommy Ratliff in

“How We Get Our Students to Read the Text Before Class” [2]. I assign reading questions for each class. They range from straightforward comprehension questions (“List the divisors of 10”), to comparison questions (“Explain the difference between a group and a field”), to questions requiring the student to connect several ideas (“Why isn’t 1 a prime number?”). Students are required to email their responses before class, or in the case of annotation, email a picture of their annotation before class. I grade the response as 2 if correct, 1 if attempted but incorrect, or 0 if no attempt is made.

The PTA sequence for the division algorithm occurs in the second week of the semester. Thus, when we begin the following sequence of linked prompts, students have at least a minimum exposure to the idea and process of proof-writing. For instance, we have discussed the definitions of even and odd and I have modeled the process of constructing a proof from the idea stage to the final draft stage. Students have practiced writing proofs both independently and collaboratively.

The first three steps for the division algorithm sequence focus on prioritization and translation.

1. In Beachy and Blair, read §1.1 up to Theorem 1.1.4. Annotate the definition of “divisor” by identifying the three most important words. Why are these the most important words? Be prepared to discuss your prioritization in class. Email your response, including a picture of your annotation, by 9:00 am on Wednesday.

The reading assignment includes the definition of “divisor,” the well-ordering principle, and a statement and proof of the division algorithm. The reading question asks the students to engage in prioritization, one of the fundamental tasks of critical thinking. By asking them to identify the most important words, I seek evidence that they are thinking about the definition and the importance of language. Since they have used annotation in English and the university seminars, an assignment that directs students to read their algebra textbooks before class makes more sense when the assignment directs them to annotate the reading according to particular questions. With the widespread use of smartphones, a student can email a picture of the annotated page. This is easier than collecting and checking each book and allows me to see the annotations before class.

In class we discuss the annotations and students are asked to justify their prioritization to each other. Then I provide the following two informal writing tasks and ask students to respond in their class notes.

2. Given the definition of “divisor,” determine what happens if two integers are not divisors. Write a definition for what happens in this situation.
3. Given a statement of the division algorithm, use examples to convince yourself that the statement is true.

Both prompts are examples of the critical-thinking task of translation. In step 2, having prioritized the key words, the student must reframe a definition to create a new definition. In step 3, the student must translate the general definition to a specific example. A nice follow-up question for class discussion is “How many examples did you need to do before you believed the statement? Are your examples sufficient or do we need to prove the statement?”

The next step in the division algorithm sequence (writing the proof) focuses on peer review.

4. I have typed a draft of the proof of the division algorithm, except I omitted several steps and justifications. Your job is to peer review my proof, just as you peer-reviewed essays in your English composition classes.

Students know they are looking for gaps in the proof and flaws in the logic. By framing this as peer review, students can rely on questions used in previous writing courses, such as

- Has the writer considered all complexities and nuances?

- Has the writer avoided drawing conclusions too quickly?
- Is the organization effective?

After students peer review my proof, I write the proof on the board and ask the class to fill in the missing steps and justifications. This is the reward—the informal writing tasks have provided students with time and space to think. They have explored nuances in the definition of divisor and the statement of the division algorithm, nuances that are often overlooked in a cursory reading of the text. Students have practiced putting mathematical thoughts related to the division algorithm proof into writing before I ask them to share their thoughts with the class. The PTA model results in better thinking, which ultimately results in a better proof.

The final step in the division algorithm sequence:

5. Now that you have a complete proof, construct an outline of the proof. One way to do this is to annotate the proof by identifying the key steps.

This requires the student to prioritize by distinguishing between the most important and the least important elements of the proof. It also forces the student to review the statement of the division algorithm and the proof.

After the division algorithm sequence, when students have seen that writing and analyzing (and prioritizing and translating) have helped them construct a proof, I can use the PTA model to create other informal writing tasks that culminate in a proof. Students have the opportunity to analogize when they are asked to transfer knowledge. The transfer may come in reusing a proof technique. Sometimes the transfer comes when students realize that they can use a previously proved theorem to help construct another proof.

I also use the PTA model to help students outside of class. When they come to my office with questions about the material, I may guide them with the following questions:

- Have you written down the definitions that relate to this problem? (prioritize, translate)
- Have you written a numerical example? (translate)
- What is the most important word or phrase in this question? (prioritize)
- What is this question asking? (translate)
- Does this problem remind you of other problems? (analogize)

In using WAC techniques in Abstract Algebra, I aim to increase the effectiveness of what I already do in the classroom instead of adding work. Additional preparation was required on my part, but the amount of grading remained the same. For instance, I reduced the weekly graded homework assignment by one problem in exchange for adding reading questions. In this way, I also avoided adding to the student work load. Developing reading questions is time-intensive but reading and grading the student responses is quick. For a class of 25 students, I can read responses and record grades in 20 minutes.

I needed to proceed at a slower pace in the first half of the semester to allow for plenty of informal writing assignments. Since the students are productively reading the text and engaging in cognitively-linked writing assignments, I found them prepared to move faster in the second half of the semester. Thus, I did not eliminate topics or reduce the amount of material covered in a normal semester of Abstract Algebra.

3 Outcomes

I have used reading questions for nine semesters, annotation for six semesters, and peer review for two semesters. The PTA sequence of cognitively-linked assignments is a natural extension of my pedagogical techniques and my participation in Writing Across the Curriculum workshops.

Informal writing prompts can, of course, be designed without a theoretical framework. However, I found two advantages to using the PTA model as a framework:

- it provides me with a trail of writing that offers insight into a student's thinking
- it is a familiar framework for my students, given their experiences in the two required English courses, the three required university seminars, and other courses taken with WAC-trained faculty.

In determining effectiveness of the PTA model in Abstract Algebra, I focused on the idea that better thinking leads to better writing. I saw evidence of better thinking in students' productive reading and their ability to "get unstuck" while thinking about the steps of a proof.

3.1 Productive Reading

The email-based approach to get students to read and annotate the textbook works well in my Abstract Algebra course. As Boelkins and Ratliff write, the reading questions provide a first exposure to terminology and ideas, allowing for a second and deeper exposure during class [2]. By emailing a response or annotation, the student is required to translate mathematics into prose and avoid symbols. Because I receive and read the responses before class I can adjust my lesson plan to begin discussion with the understanding (or misunderstandings) of the students.

Most importantly, the reading questions require students to be active readers. As they engage with the text to answer the questions, they see a connection between reading and writing, and how both connect to thinking. In the same way that they would bring an annotated and dog-eared copy of the current novel to a discussion in English class, when they annotate, dog-ear and use the mathematics textbook in conjunction with class notes and discussions, I know they are seeing the reading-writing-thinking connections.

On the first and last day of the semester, I have asked students to respond to survey questions, including, "Rate your comfort level with reading a mathematics textbook." The answer choices are "no way, can fake it, decent, expert". For four semesters, with 81 out of 92 students responding, 63% of students rated their comfort level as "decent" or "expert" on the first day. On the last day, 89% rate themselves as "decent" or "expert". One student commented, "I thought I knew how to read math but it is harder than I expected." Another student wrote, "I realize that I never read the book in other math classes—I just read the examples."

3.2 "Getting Unstuck"

I often find that my students get stuck while writing proofs and either give up, google the proof, or ask to read a peer's proof. The problem is that students don't know how to "get unstuck". Two components of the PTA model can help students push themselves (or a peer) an additional step or two along the path of constructing proofs. First, after using PTA questions in class and hearing me ask the same questions in office hours, students begin to use the same PTA questions when working without the instructor. I observed this about four weeks into the course while students were working in groups. When one student didn't understand the definition of "onto function," and another student suggested the group create an example, I knew that the second student had begun to internalize the PTA critical-thinking framework. As the semester progressed, I overheard this type of peer-to-peer conversation more frequently.

The second component of the PTA model that helps students "get unstuck" is peer review. I encourage students to work together, but want them to collaborate instead of splitting up the assignment. Peer review techniques emphasize the role of a peer as an asker of probing questions and a source for finding gaps, instead of a peer as a source for the answer. As one student said, "Groups allowed us to each be the teacher at times, and the student at others, making proof-writing a very manageable task."

3.3 Additional Outcomes

Two additional benefits of the PTA model are a greater level of student engagement and an increased awareness of the process of proof-writing.

When students have an opportunity to reflect and organize their thinking before being asked to participate, the amount and quality of participation improves. This is most noticeable in the classroom, but also holds true for questions and discussions that occur in office hours.

The PTA framework emphasizes the idea that a mathematical proof is not only a product; it is the final step of a process whereby a student assembles logic and ideas into a paragraph that is submitted for a grade. In the language of WAC, “writing is both a process of doing critical thinking and a product that communicates the result of critical thinking” [1, p. 4].

The PTA model creates an explicit connection between the writing in non-mathematics classes and the writing expected in Abstract Algebra. A future study might examine if seeing the critical-thinking model in multiple contexts encourages students to apply it in other learning situations.

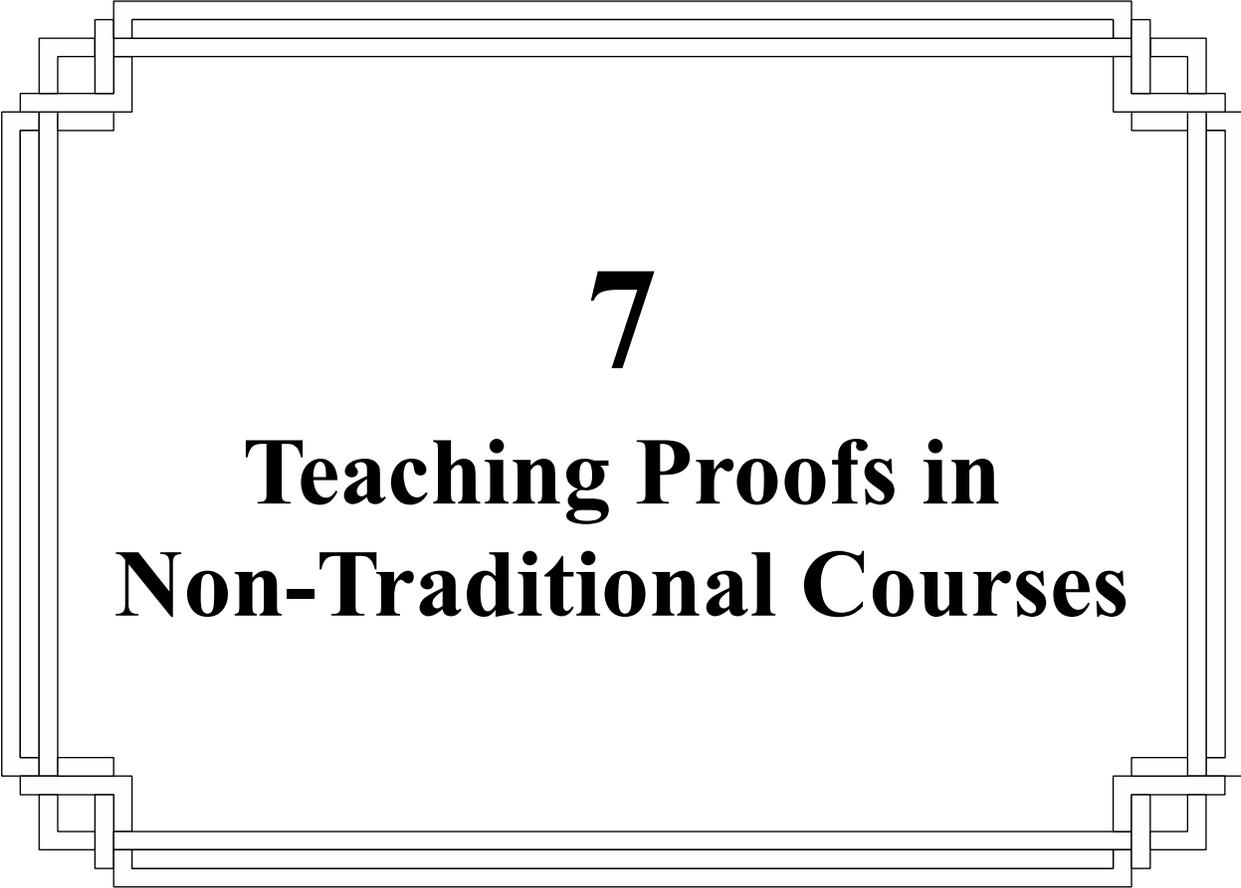
4 Extending the Method

In looking to extend the method, there are three components to consider—a cognitive framework, writing techniques, and common learning experiences. For my students, the common learning experiences provide a cognitive framework and Writing Across the Curriculum offers writing techniques. These allow me to leverage thinking and writing habits that students have practiced prior to Abstract Algebra. Without a cognitive framework, there is no common language for practicing critical thinking. Novice proof-writers would need to be introduced to a critical-thinking model and this would take up time in the mathematics class but should be seen as a good investment. Without Writing Across the Curriculum, a proof-writing course can still incorporate the informal writing techniques used in non-mathematics classes. Almost all schools have a common writing experience for students, either English composition or a writing-intensive course. By explicitly using the language of the writing course, the instructor will provide a familiar context for them learning to write proofs and thus help students see proof-writing as an extension of writing in non-mathematics courses.

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7

Teaching Proofs in Non-Traditional Courses

Discovering the Art of Mathematics: Proof as Sense-Making

Julian Fleron, Christine von Renesse, and Volker Ecke *

Abstract

We describe a model for mathematics for liberal arts courses where students are active participants in an authentic mathematical experience in which proof as sense-making plays a natural and essential role. At the center of the model is the pedagogical cycle: guided discovery, group collaboration, whole-class discussion, and synthesis into proof. With this cycle as the basis for the course, students begin to see proof and proof-writing as key components of sense-making, as extensions of narrative and rhetorical structures that are fundamental to human communication, and as complements to how we come to know and understand things in other areas of study. Having reframed proof in this broader context, we also share the benefits we have found in its use with mathematics majors.

Difficulty Level: High; Course Level: Non-traditional

1 Background and Context

Described here is an approach to proof that has grown out of the tradition of inquiry-based learning [2]. Originally used more prominently in Mathematical Explorations, one of our university's two mathematics for liberal arts (MLA) courses, where we have assessment data to support its success, it has grown to deeply inform our approach to proof in our courses for mathematics majors as well. Our description here encompasses both audiences.

1.1 Proof as Sense-Making

Proof is fundamental to mathematics. Yet its nature and role continue to be actively debated. (See, e.g., Jaffe and Quinn [8] and Zeilberger [19].) While this volume focuses on pedagogical aspects of proof, the fundamental, philosophical substance of the debates about proof can help us reconceptualize our approach to proof in the classroom.

In his paper [7] of the same name, Reuben Hersh argues that “Proving is Convincing and Explaining.” We believe the active, process-oriented nature of this phrasing is valuable. Our purpose in this paper is to develop an expanded conception of proof: **proof as sense-making**.

The term sense-making has been used in mathematics education research for quite some time. (See, e.g., Schoenfeld [17].) It has achieved a higher profile with the recent publication of “Focus in High School Mathematics: Reasoning and Sense Making” [12] by the National Council of Teachers of Mathematics. This publication advocates directing significant attention to a primary focus on “curricular emphases and instructional approaches that make reasoning and sense making foundational to the content that is taught and learned.” **Sense-making** is defined as “developing understanding of a situation, context, or concept by

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connecting it with existing knowledge.” [12, p. 4] The authors of this publication note, “In practice, reasoning and sense making are intertwined across the continuum from informal observations to formal deductions, . . . despite the common perception that identifies sense making with the informal end of the continuum and reasoning, especially proof, with the more formal end.”

In journals, in mathematical autobiographies, and while participating in whole-class discussions, students regularly report that other subjects make sense to them but mathematics never has. We can build on their abilities to make sense in other subjects, which often have important connections to explaining and convincing, to help encourage sense-making in mathematics. By reframing the notion of proof, students’ opportunities for sense-making are expanded and deepened. In settings where it has been lost or underutilized, which includes many MLA classrooms, proof can become a tool for reclaiming sense-making in mathematics classrooms.

1.2 Proof and the Liberal Arts

An elegantly executed proof is a poem in all but the form in which it is written. — Morris Kline [10, p. 470].

In his beautiful response “On Proof and Progress in Mathematics” to Jaffe and Quinn [18], William P. Thurston repeatedly uses the phrase “human understanding” and emphasizes psychological and social dimensions of proof, saying, “The measure of our success is whether what we do enables *people* to understand and think more clearly and effectively about mathematics.” He declares “human language” to be the number one factor “important for mathematical thinking.”

Communication in human language is one arena in which our liberal arts students are most comfortable. We lose critical connections between mathematics and the other disciplines if we only compare them in terms of what is generally referred to as “content,” ignoring the issues of process, communication, and differing burdens of proof.

Instead, we suggest using the critical connections between mathematics and the liberal arts in the area of both communication and process to positively empower traditionally disenfranchised students to succeed in mathematics. These connections are particularly powerful when we consider the notion of proof in an inquiry-based setting [2].

1.3 Proof as Discourse

In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. . . Deductivist style hides the struggle, hides the adventure. The whole story vanishes. — Imre Lakatos [11, p. 142].

How can mathematicians describe what proofs are and what they look like to general education students? The landscape of proof is fraught with many different methods, types, and styles; underlying mathematics that may not be well enough understood; and a style of communication that is dense and often full of technical notation. While this style of communication is typical of working mathematicians, it need not be what we teach our students, especially not our general education students. Instead, we can rely on their working knowledge of several types of discourse.

In “Narrative Structure in Inquiry-Based Learning” [9], C. Kinsey and T. Moore beautifully describe how narrative structure provides a valuable context for mathematics students involved in inquiry-based learning:

The students learned to look at a new topic as a new cast and setting and to look at new questions as new plot points. . . They expected to be involved in the conflict of the story. They looked at the given examples as foreshadowing and hints to resolve the conflict and clarify the complications. Most importantly, they expected a resolution that made sense.

We can expand the role of narrative in proof-writing via narrative's dramatic arc. In inquiry-based learning there is exploration; this is the rising action in the dramatic arc. In the dramatic arc there is a climax and then falling action. In an inquiry-based mathematics classroom this is the student discovery, the "a-ha moment," the statement of a conjecture. In the dramatic arc the final stage is the denouement, the resolution. The parallel in mathematics is the explanation of human sense-making, the communication of justification, proof. The arc compares nicely with the technique in 2.2, which is the focus of this paper.

Proof-writing is, to borrow language from our colleagues in composition, a rhetorical situation - authors convincing an audience of the legitimacy of their message. In an inquiry-based setting the students are clearly cast as the authors. At each stage we can help them learn to see their discoveries, their "a ha moments," and their conjectures as the analogues of thesis statements in expository or persuasive papers. Students are used to framing and then defending a thesis statement. By repeatedly reminding them of the connection, we help them understand the process of doing mathematics. They begin to see that the use of rhetoric as a key part of the process is not so different than it is in many other areas of human discourse.

These aren't the only areas of discourse where we find important context for student proof-writing and for the pedagogical approach at the heart of this paper. A. Szbaó and A. Ungar argue [14] that mathematics and the transition to formal axiomatic foundations "grew out of the more ancient subject of dialectic." As a counterpart of rhetoric, dialectic arose from the Socratic dialogues that can be seen as a precursor to inquiry-based learning. We see that the technique in 2.2 is re-forming the natural and important connections between proof, inquiry, and discourse.

1.4 Setting

The approach described in this paper was developed by the authors and our colleague Philip K. Hotchkiss for the MLA course Mathematical Explorations at Westfield State University as well as for the National Science Foundation funded project *Discovering the Art of Mathematics* for which the same group serves as principal investigators. Our university is a comprehensive university of about 5,000 full-time undergraduate students and 700 graduate students. Mathematical Explorations is one of several courses offered to satisfy the two-course general-education requirement in mathematics. Class size is typically 30 – 35 with some smaller honors sections. More than 60 sections of this course have been taught by us using the approach described here. *Discovering the Art of Mathematics* provides freely available curriculum materials (see Section 2.2) to support inquiry-based learning in MLA [4]. The materials have been used by a number of faculty members at other institutions and are available for both classroom use and beta-testing.

2 Description and Implementation

Mathematics — this may surprise or shock some — is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof . . . The deductive stage, writing the result down, and writing its rigorous proof are relatively trivial once the real insight arrives; it is more like the draftsman's work not the architect's. — Paul Halmos [6, pp. 380–1]

Proof is an essential part of the complete cycle of mathematical sense-making: begin with an exploration of a mathematical object or pattern; collect data and observe connections; use informal and/or inductive reasoning to make conjectures about them; verbalize definitions, assumptions and/or axioms; develop logical, rigorous arguments to establish the validity of the result. As described above, and illustrated below, for MLA the proofs are expressed in natural language to a much greater degree than the ways mathematicians typically communicate. But they are no less proofs.

2.1 Our Technique: A Look Into Our Classroom

While there are differences in our teaching, we are united in our use of the technique to be described. We describe it here in first-person narrative as scenes like those described are typical of our classrooms each time we meet with our students.

It's still seven minutes before class starts when I walk into the classroom. Each of the nine tables in the classroom has two or three students seated at it, most with their notebooks open already doing mathematics. I begin to tour the room to see what the different groups of students are collaborating on. It is generally quite a mix. They all started from the same guided-discovery investigations designed to help them solve some larger problem or (re-)discover a significant mathematical result. Some groups remain in earlier phases – collecting data, finding patterns, and recording ideas. Some are in slightly later phases – making guesses, formulating conjectures, and articulating connections. All of the activities are part of the continuum of sense-making.

By the time class is supposed to start, every student is doing mathematics. I've said nothing to the class as a whole, given no direction for them to start working, yet the students and groups have picked up their mathematical explorations where they previously left off, either from the last class or from subsequent work outside of class. I may make a few announcements. I may help organize a whole-class discussion. Many students and groups will ask questions, to which I will generally respond with my own questions. I will not lecture. My role is to facilitate the students' work.

The focus of student work is the big investigation posed the previous class:

Determine all possible values generated by the Diophantine equation $3a + 5b$ when $a, b \geq 0$.

The students also have a short sequence of investigations as finer prompts:

1. *Choose whole number values for $a, b \geq 0$. For them, evaluate the expression $3a + 5b$.*
2. *Repeat for a different pair of values for a, b .*
3. *Repeat for another dozen different pairs of values for a, b .*
4. *Can every positive whole number be obtained as the result of evaluating $3a + 5b$, or are there some numbers that cannot be made from this expression?*
5. *Continue investigating until you feel comfortable making a conjecture that identifies exactly which, all infinitely many of them, positive whole numbers can be made by evaluating the expression $3a + 5b$ for $a, b \geq 0$ being whole numbers.*
6. *Prove your conjecture.*

A typical interaction with a group of students follows. As I observe I realize Student A and Student B have been working together on one strategy while their group-mates Student C and Student D have collaborated on a different approach. Student A begins explaining:

Student A: I think you can get any positive number. You just take away a multiple of 3 and then it ends in zero. That's easy.

Student B: How would I do that for 86?

Student A: 86 minus 6 is 80, so um, $a = 2$ and, um, $b = 16$.

Student B: (computes) OK. So ending in zero means that five goes into it, right?

Student D: I don't get what you're doing. Why are you looking at 86? Did you make the whole list up to 86?

Student C: We've only gotten to 24 so far. (She points to a list which is a precursor of Figure 1 in Appendix A.)

Student A: You don't need the whole list.

Student D: Yes, you do. You have to find all solutions.

Professor: Student D, Student C, do you agree with Student A and Student B's conjecture that you would get any positive whole number?

Student C: There are some missing. You can't get 4.

Student A: So for 4, we would subtract ... What's a multiple of 3 that ends in 4? Um, 24 works. So $4 - 24$ is negative 20, which ends in zero, so it's 4 times 5.

Student B: No, negative 5.

Student D: But negative numbers are not allowed.

Student B: Oh, ok. (*thinking*)

Professor: So it sounds like Student A and Student B may want to adjust their conjecture.

Professor: (*turning to Student C and Student D*) Could you convince me that 86 is going to be in your list?

Student C: That's too much to write all of them down.

Professor: Could you convince me that 86 would eventually appear on your list without writing them all down?

Student C, Student D: Hmmm ... (*thinking*)

To gain a richer appreciation of students working on this proof, and to see the final results of their work, see our blog at www.artofmathematics.org/blogs/jfleron/3a5b-proofs that contains written proofs and classroom videos. The proof by Students C and D is included in Figure 1 in Appendix A and as "Proof 4" online, while Student A's idea appears online in the video "3a+5b-proof-0006-III-CvR".

2.2 Our Technique

We hope that this vignette has given you a picture of our technique for actively involving MLA students in a sense-making exploration of mathematics. The focus of this technique is the repeated application of the cycle

1. Student exploration via guided-discovery.
2. Group collaboration and discussion for sense-making.
3. Whole-class discussion focusing on big ideas, strategies, and finding misconceptions.
4. Synthesis of results into proof.

Communication is essential in each of these areas, from students' reading and interpreting the prompts, which spark the investigations, through the final reporting via proof. Details about each of the stages include: **Guided-Discovery:** We utilize the inquiry-based materials of the National Science Foundation supported project *Discovering the Art of Mathematics* [4]. A central component of this project is a library of 11 freely available guided-discovery learning texts, each with sufficient material to support a semester-long, three-credit MLA course.¹

Group Collaboration: The students work in groups on the guiding questions, while the teacher circulates around the room observing, encouraging, and supporting the students. We find it helpful not to answer student questions directly but to respond instead with questions that will deepen their thinking; see von Renesse and Ecke [15].

Whole-Class Discussion: We facilitate whole-class discussions to make students aware of other strategies, possible misconceptions, connections between ideas, and to talk about communal standards of rigor. We rely

¹All materials are available online at <http://www.artofmathematics.org/>. This project also provides faculty resources, videos illustrating inquiry-based learning in action, professional development workshops, and a host of other resources.

heavily on *Classroom Discussions: Using Math Talk to Help Students Learn, Grades K-6* [1]. Such whole-class discussion can take place at various stages of the students' explorations. We like to use it to compare student conjectures, refine the precision of students' claims, and to work out the details of a mathematical explanation.

Synthesis: As the students clarify their thinking, they are asked to write, explaining their results and the validity. The proof-writing tasks vary. Sometimes they are fairly narrow, providing justified answers to a series of shorter investigations. Some tasks result in proofs that are more narrative in nature, describing the students' investigations as well as their results. Other tasks result in proofs that more closely resemble typical mathematical writing with conjectures followed by proofs. While in each phase the language and style used is more informal than what one typically finds in mathematical writing, the level of rigor remains high.

Student proofs and classroom videos that highlight the technique described here, in several different curricular areas, are available at

<http://www.artofmathematics.org/classroom/proof-as-sense-making>.

3 Outcomes

Our discussion above suggests sense-making is necessary for successful proof-writing. For MLA students, key ingredients for sense-making that lead to proof include: context for what proofs are and how they relate to other forms of writing, communication, collaboration, active engagement, exploration and discovery, creativity, enjoyment, and personal responsibility. Without these key ingredients there is little substance upon which proofs can be built and, subsequently, no possibility for meaningful proof-writing to take place.

Preliminary data from the *Discovering the Art of Mathematics* project provide positive evidence for the effectiveness of our approach. The study is based on pre- and post-survey data from several sections taught by four different professors between 2010 and 2012 at Westfield State University.² (Unless indicated otherwise, results are based on a total of $n = 192$ matched pre/post responses.)

Specifically, students showed increased agreement with the statement *A proof is something you have to construct based on your own understanding* ($p < .01, n = 41$).³ They reported increased enjoyment in *using rigorous reasoning in a math problem* ($p < .01$) and in *discovering new mathematical ideas* ($p < .01$). The essential roles of the group collaboration and whole-class discussion phases in our technique are validated by students who report learning mathematics best when *I explain ideas to other students* ($p < .001$) and when *I work on problems in a small group* ($p < .01$).

Many statistically significant, positive changes in affect and meta-cognitive awareness have also been found. While these changes may not correlate precisely to proof and proof-writing, we think that they correlate with the more active involvement of students in the learning process of which proof is part. These results can be found online with those described above.

Since the fall semester of 2009 our team has taught 37 sections of our MLA course to over 900 students. Overwhelmingly we have seen positive changes in our students' abilities to make logical arguments and craft meaningful proofs. Their initial anxiety of searching for the "proof the professor wants" is assuaged by the whole-class discussions where the students see multiple approaches validated. They begin to think independently. They think creatively. Their "proof stories" richly illustrate their thinking and provide a

²A summary of the results will appear soon at <http://www.artofmathematics.org/about/goals-and-evaluation>.

³Our evaluation instruments and statistical measures are based on the large scale study "Evaluation of the IBL Mathematics Project: Student and Instructor Outcomes of Inquiry-Based Learning in College Mathematics" by S. Laursen, M. Hassi, A. Hunter and T. Weston, which you can find online at http://www.colorado.edu/eer/research/documents/IBLmathReportALL_050211.pdf. The effect sizes in our study compare favorably to those reported in this larger study by Laursen et al. While Laursen et al. view these effect sizes as relatively small, we note that the changes are statistically significant and occur in the desired direction.

context for rigorous reasoning. We are often surprised by the novel proofs of our students, approaches that we would not have thought of on our own and that enrich our understanding of the mathematics at hand.

Extended to mathematics majors, as described below, our technique has also helped our major students think more critically and with greater reflection. They are driven less by procedural thinking, especially in proof situations, and more by sense-making. The disjointed and incomplete proof attempts to which we have become accustomed have been replaced by much richer “proof stories.” The students’ thought processes are much clearer, and their ability to reason without gaps has improved dramatically.

4 Extending the Method

Our focus so far in this paper has been MLA courses. We now indicate how mathematics majors may benefit from this approach.

A typical introduction-to-proof course for majors is often structured around a number of different proof techniques. Students often see these methods as a checklist to be gone through in serial order as they try to find the “correct” one to “apply” when asked to prove something. That is, a formal structure precedes any exploration, investigation or attempt at sense-making. For example, consider proof by induction as taught to our major students. We find it important to involve students in looking at data, finding patterns and making conjectures. Explicitly considering the role of inductive reasoning and its limitations provides important context for the necessity of the deductive stage and the power of proof by induction in this stage. Without these opportunities, students often view proof by induction mechanically as just two cases (base and general) to be checked rather than the critical proof method for establishing a general result that it is.⁴ (Editors’ note: For a technique that helps students explore the process of proof by induction, see Ensley’s paper in this volume [3].)

We suggest reframing the role of proof as described here at least in our first- and second-year courses for majors where the developmental work on proof is done. In these early stages of their undergraduate careers, our majors have read and created far more narrative and expository pieces outside of mathematics than they have read or created mathematical proofs. They too can benefit from understanding proof in the broader framework described here. This is not to preclude apprenticeship in typical methods of proof and proof techniques, but to provide a broader base with deeper connections to already-developed skills.

We can illustrate this concretely with a calculus activity we have used successfully. In our experience calculus students are unable to decipher the proof of the first fundamental theorem of calculus presented in typical textbooks. However well-intentioned our lecture on this topic may be, it seems to develop little understanding in our students. In contrast, we have found that the cycle we’ve described enables students to develop their own proofs of this central result. The earlier trips around the cycle involve investigating the derivative as slope and linear approximation by tangent lines (in several different contexts including speed and distance) as well as approximation of areas by Riemann sums. Once the sense-making is developed to a high level, the stage is set to have the students investigate the general problem of finding the area under the curve when the function has an antiderivative. While somewhat informal, the proofs the students develop of this essential result illustrate deep understanding and ownership.

Proof is sense-making and we expect students to sense-make at each stage — not just when a final, somewhat formal proof is called for. By developing expectations in our major students that they should be careful in their reasoning always — when communicating verbally with each other and with us, when they work on homework, when they write answers on exams, when they submit written work that is not so explicitly in a proof setting — we are developing their sense-making abilities. These efforts have resulted in marked increase in our major students’ abilities to develop and write proofs.

⁴For inquiry-based materials that allow students to explore the differences between inductive and deductive reasoning, see Chapter 4 of our project’s book on reasoning [5]. For more on the essential role of inductive reasoning in problem-solving and the development of proof ideas, see Polya [13].

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Appendix

A One Student Proof for $3a + 5b$

2
2
2
1
2

7/10

Group Proof

Conjecture: The numbers that result from evaluating $3a+5b$ when $a,b \geq 0$ are all positive integers with the exception on 1,2,4 and 7.

Proof

This shows that you cannot get the numbers 1,2,4 and 7

- $3(1) + 5(0)$

$3+0=3$ this is the smallest possible number you can get

- $3(0) + 5(1)$

$0+5=5$ which is the closest number to 4 that you can get

- $5(1) + 3(1)$

$5+3=8$ this is the closest number to 7 that you can get

Proof

It's your proof

In every single set of tables, the solutions to the equations increase by 3. In each set of tables, you have to substitute "B" for either 1, 2 or 3. For each separate table, your "B" has to remain constant to either 1,2 or 3. While "B" stays consistent, your "A" must increase by one digit in each equation. The solutions in each separate table increase by 3. As a result of them starting at different numbers and increasing by 3, that is how you can get the infinite amount of numbers as your solution.

Thus shows

$3(1) + 5(3) = 18$	$3(1) + 5(2) = 13$	$3(1) + 5(1) = 8$
$3(2) + 5(3) = 21$	$3(2) + 5(2) = 16$	$3(2) + 5(1) = 11$
$3(3) + 5(3) = 24$	$3(3) + 5(2) = 19$	$3(3) + 5(1) = 14$
$3(4) + 5(3) = 27$	$3(4) + 5(2) = 22$	$3(4) + 5(1) = 17$
$3(5) + 5(3) = 30$	$3(5) + 5(2) = 25$	$3(5) + 5(1) = 20$
$3(6) + 5(3) = 33$	$3(6) + 5(2) = 28$	$3(6) + 5(1) = 23$
$3(7) + 5(3) = 36$	$3(7) + 5(2) = 31$	$3(7) + 5(1) = 26$
$3(8) + 5(3) = 39$	$3(8) + 5(2) = 34$	$3(8) + 5(1) = 29$
$3(9) + 5(3) = 42$	$3(9) + 5(2) = 37$	$3(9) + 5(1) = 32$
∞	∞	∞

Figure 1: One way students prove which numbers can be generated by the Diophantine equation $3a + 5b$ when $a, b \geq 0$.

B Sample Learning Cycle and Proof

In this section we illustrate the approach concretely with a focus on the latter part of the cycle. The investigations which set the context are directly from the book *Discovering the Art of Mathematics - Music* [16]. We encourage you to try the problems on your own before reading the solution.

Previously, guided investigations prompted students to find the number of ways you can create a rhythm with k beats on a drum if there are n counts in a measure. Students worked collaboratively trying to find organizational structures that allowed them to make sense of the data; having found the three rhythms with 2 beats on a measure of 3 counts (110, 101, and 011), the students searched to find connections between this new piece of data and the earlier data. As part of a whole-class discussion more than one group's organizational strategy resembled Pascal's triangle and this approach was adopted as the class's method of organizing their data. (A sequence of guided investigations was available in reserve had this not been raised by students as part of the discussion.) Subsequently, the learning cycle began again with students working on the sequence of investigations below.

Independent Investigation: *In your group, study Pascal's triangle and find at least three different patterns. Be creative! Then share the patterns with your class.*

One of the patterns you probably found is the addition pattern: when adding two adjacent numbers the result will be right beneath the two numbers that you added. Mathematicians love finding patterns but they also wonder why the patterns occur and how you can be sure that they will continue. Our goal now is to understand why the addition pattern in Pascal's triangle occurs and to make sure that it will always happen.

1. *In Figure 2, fill all possible rhythms in the respective boxes.*

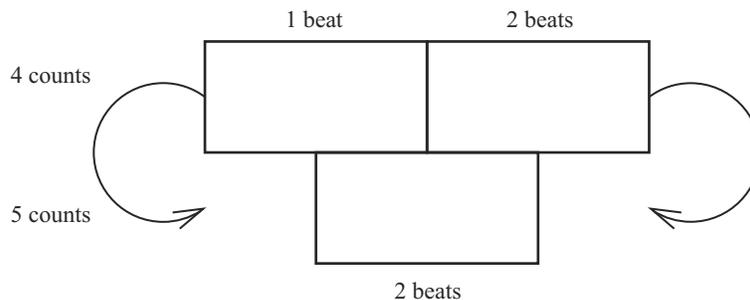


Figure 2: Part of Pascal's Triangle.

2. *Now look at the rhythms you filled into Figure 2; can you see any structure that suggests how the rhythms in the upper boxes are connected to the rhythms in the box below? Explain the structure you found.*
3. *Go to another place in Pascal's triangle and choose three similarly positioned boxes. Fill them with rhythms and see if your structure applies here as well.*
4. *Does your structure apply to the top of the triangle?*
5. *Explain in your own words why the addition pattern in Pascal's triangle occurs, using the structure you found. Be specific in your arguments.*

The cycle of student exploration, group collaboration, and whole-class discussion continued as the students developed increasingly robust ways of making sense of the underlying structure. For the final synthesis the expectation is not an algebraic proof but rather organized sense-making expressed in the students' own words. The following proof is written in a language that is typical for our students.

We focused on an example to see why the pattern worked — 4 counts and 1 beat, and 4 counts and 2 beats. There are 4 rhythms with 4 counts and 1 beat: 1000, 0100, 0010, 0001. There are 6 rhythms with 4 counts and 2 beats: 1100, 0110, 0011, 1001, 1010, 0101. We had to understand why it makes sense that there are exactly $4 + 6 = 10$ rhythms with 5 counts and 2 beats.

What we found is a method to create all the rhythms with 5 counts and 2 beats from the row above. The key idea is: *Any rhythm has to start with either a 0 or a 1.*

Here's how we created all of the rhythms that have 5 counts and 2 beats and start with a 0. The last 4 counts of each of the rhythms will exactly come from the box on the top right, since those are *all* rhythms with 4 counts and 2 beats. We cannot use rhythms with 4 counts and 1 beat or more than 2 beats, since we need to have exactly 2 beats. So only one box in the upper row provides what we need. So now we have 6 rhythms.

Then we created the rhythms with 5 counts and 2 beats starting with a 1. Again, there is only one box that provides what we need, since we now want rhythms with 4 counts and 1 beat. The desired box is on the top left. This adds the remaining 4 rhythms.

This explains why we get *all* rhythms, but we need to make sure we don't get anything twice. The rhythms coming from different boxes start on different digits (0 or 1), so they're different. Rhythms from the same box already had to be different somewhere so the new rhythms formed are also different.

If we pick any other pair of adjacent boxes our method will work in the same way.

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Teaching Writing and Proof-Writing Together

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Abstract

This article describes the synergy obtained by teaching proof-writing in the more general context of teaching how to write well. It describes a course, “Reasoning about Reasoning,” that was offered at Cornell University as a first-year writing seminar. The course was taught in an inquiry-based format, in which students worked on problems in class in groups and then wrote up their results as formal papers that went through a series of revisions.

Difficulty Level: High; Course Level: Non-traditional

1 Background and Context

This article describes a course, “Reasoning about Reasoning,” that was taught in the mathematics department at Cornell University as a first-year writing seminar. Cornell University is a private Ivy League university, and at the same time is the public land-grant institution of New York State. It is highly selective and has about 14,000 undergraduate students. Cornell has a writing-across-the-curriculum initiative in which first-year writing seminars are offered by many departments across the university. Most students take two such courses during their first year at Cornell, which they choose from the diverse collection of courses offered each semester. This course marked the first time that a first-year writing seminar was offered by the mathematics department, and it was developed and offered four times by the author as a graduate student.

Although first-year writing seminars are offered on many subjects and in many departments at Cornell, they all adhere to a set of common guidelines. Each course is organized around a sequence of six to ten formal essays that form a logical sequence. At least three of these essays are required to undergo a series of guided revisions. Half of the classroom time in each seminar is expected to be spent on work directly related to writing, and in order to allow for sustained interaction between the instructors and the students around their writing, each seminar is capped at 17 students. As part of the interaction, instructors are expected to meet in individual conferences with students at least twice during the semester. The goal of the writing program is to develop students’ abilities to write good analytical prose, and its philosophy is that this is a goal that is best achieved through a sustained experience writing within a particular discipline.

Because students get to pick the seminars in which they want to enroll through a lottery system, the students in this class came from a wide range of backgrounds. Some were intending to major in mathematics or other sciences and had exceptionally strong mathematical backgrounds, while others had picked the course on a whim because they thought the topic looked interesting, and were not intending to take any other math courses at all at Cornell. So, the course materials had to be flexible enough to accommodate many kinds of students.

Although writing and proof-writing are typically taught in different contexts, there are a lot of gains from teaching them together. The goal of most analytical writing classes is to help students acquire the skill of writing clear, convincing arguments. It is, of course, rare for a writing class to have mathematics as the subject of the writing, but mathematics is, in fact, remarkably well suited to the goal, since making clear and correct arguments is a central part of what mathematics is about. Furthermore, mathematics is virtually the only domain of inquiry in which there is wide agreement as to what constitutes a correct and coherent

argument. Many writing classes focus on analyzing works of fiction, presumably because most writing classes are taught in English departments, by teachers whose expertise is in this area. However, it seems to me that mathematics is actually a better setting for teaching analytical writing because mathematical writing focuses on deductive reasoning, and there is therefore less ambiguity about what can be considered a correct argument.

At the same time, a writing seminar is a great setting for teaching mathematics in general, and proof-writing in particular. Students tend to enter mathematics classes, especially lower-level mathematics classes, expecting to be told facts and shown procedures to be mimicked. By contrast, they tend to enter a writing seminar expecting to think about some of the most important issues we would like them to think about in writing proofs: how to write clear and convincing arguments, and what makes an argument clear and convincing. Most mathematicians would agree that mathematics is much more about reasoning and making arguments than it is about memorizing procedures and facts. This is particularly true of proof-writing. So, in a strange way, a writing seminar might be an even better venue for teaching proof-writing than a mathematics class.

2 Description and implementation

2.1 The Structure of the Course

The mathematical content of the course, as I taught it, focused on examining how we know what is (mathematically) true. Students read excerpts from a variety of writings on the subject of reasoning, including Douglas Hofstadter's *Gödel, Escher, Bach* [9], Martin Gardner's *Logic Machines and Diagrams* [6], and Imre Lakatos's *Proofs and Refutations* [10]; writings that provided examples of mathematical reasoning, such as Euclid's *Elements* [5]; and writings that posed problems for students to reason about, such as Lewis Carroll's *A Tangled Tale* [2], David Henderson and Daina Taimiņa's *Experiencing Geometry* [7], and Raymond Smullyan's books *The Lady or the Tiger* [15] and *What is the Name of this Book?* [16]. Some of them were used as starting points for class discussions about reasoning that were independent of any specific assignment we were working on, while others, particularly those that posed problems, were used as the basis for written assignments.

The main focus of the class was on these papers that the students wrote. They formed a logical progression of interrelated topics that asked the students to reason about mathematical problems and to explain their reasoning, and then to reflect on and write about the process of mathematical reasoning. Several sample assignments are given in Appendix A. The problems that we looked at came mainly from logic and geometry, and particularly non-Euclidean geometry. These were fruitful areas to look at because they had relatively accessible problems that students were unlikely to have previously encountered.

A typical progression as we were working on a problem went as follows: first, I would pose a problem about an unfamiliar topic, and give the students some time to work on this problem in small groups in class. (For example, on the very first day of class, I would give out the Babylonian tablets assignment reproduced in Appendix A.1 and ask students to start working on it in groups, with no preamble or background beyond what is found in the assignment.) While the students were working in groups, I would move from group to group, discuss their progress in making sense of and solving the problem, and try to steer each group in a fruitful direction if they seemed to be stuck. Periodically, I would stop the groups and facilitate a whole-class discussion about the progress that the groups were making in order to share ideas between groups and to make any suggestions or comments that I wanted everyone to hear. If there were readings that were relevant to the problem, I would most often assign them to be read shortly after we had already started working on the problem in class, so that the students would get to start thinking about the problem with no preconceived ideas, and would hopefully then view the readings as support for what they were trying to understand rather than just as an assignment to be checked off a list. Once I felt that enough time had been spent working on

the problem in class, I would assign a due date for a written first draft of the paper. After they had turned it in, I made comments on the papers, led a class discussion about common problems with the papers, and then gave students the opportunity to revise them one or two more times (depending on the assignment). I tried to make comments on the papers that would show students which parts were incorrect, incomplete, or unclear, without giving them specific instructions on how to revise them. As a result, many of my comments were phrased as questions.

The writing assignments all required students to write mathematical explanations. In some cases, these were proofs, while in other cases they were other kinds of explanations, such as how to carry out a procedure, why a definition made sense, or what reasoning the author had used to come to a conclusion. Within our class culture, proof was just one form of mathematical discourse we were trying to master, and our aims in writing proofs were the same as our aims in the other kinds of mathematical writing we were doing: to provide as clear and as convincing arguments as we could. This goal was emphasized again and again throughout the course: when we were discussing how to go about writing each paper; in the comments on each paper; and in our class discussions about how to revise them. I noted that coming up with convincing arguments about mathematical problems and communicating them clearly in writing is what professional mathematicians do, and is what makes someone a mathematician at any level. It sometimes took students a long time, a third of the course or more, to really believe that this was what I was expecting from them and that a big part of what I was grading them on was their ability to write an explanation that would be readable by, and convincing to, one of their peers. Once they understood this, however, the change in their writing and how they approached writing was dramatically noticeable.

2.2 Useful Lessons from Teaching Writing

There were a number of aspects of the course that were natural in the context of a writing seminar that turned out to be surprisingly useful in teaching proof-writing. One was that most of the assignments went through a series of revisions after I had a chance to make comments on them and discuss common problems with the class. This was a requirement of all writing seminars at Cornell, but I found that it made a big difference in how the students learned the material. Even when they didn't make many changes between drafts, the act of thinking about what they had written and how they could make it better really changed the way they approached the writing assignments in the first place, and emphasized that the goal of the class was to improve their reasoning and writing skills. Students wrote more carefully because they knew they were going to have to go back and provide justification for anything that was unclear. I think that this is the aspect of the course that made the biggest change in the students' mathematical writing, including proof writing, and the way they approached thinking about mathematics.

Another thing that we did was to pay close attention to the small details of how things were worded—in other words, to writing style. I think that this was also surprisingly helpful for the mathematical skills they were building. It has always been my experience that the more clearly a student can explain something, the better he or she understands it, and I found while teaching this class that this rule even extends to small details of sentence structure. As a mathematician with limited training in teaching writing, I found Joseph Williams's book *Style: Ten Lessons in Clarity and Grace* [17] to be extremely helpful in figuring out how to think about working with my students around issues of stylistic control and sentence structure. In the writing seminar, we spent a significant amount of class time looking at individual sentences from students' papers and discussing how to improve them. However, when I give writing assignments in regular mathematics classes, I spend much less time on this—not because it wouldn't be valuable, but because students are less open to doing this outside of a writing class.

A third aspect of the writing seminar format that I found beneficial in teaching proof-writing was the requirement that all students set up appointments to meet one-on-one with the instructor outside of class at least two times during the semester. This was a general requirement of the writing seminar program, but

it turned out to be a valuable opportunity to make sure that all students were getting sufficient individual feedback, not just those students who chose to attend office hours.

Two other aspects of the course were slightly unusual, and worth mentioning because they seemed to be beneficial. One is that because we weren't following a textbook, I had each student sign up for one or two days during the semester when he or she would be responsible for taking notes on whatever happened in class that day, writing up them as html files, and then posting them on our class web page after I approved them. This worked out really well: it was a good experience for the students taking the notes, and we then had a record of what had happened in class each day which the students and I could refer back to later on. The notes often included summaries of the discussions of student proofs that had occurred in class. Another aspect of the course that seemed to be useful was that I set up a discussion board to which all students were required to post each week. Sometimes I gave a specific prompt, while other times they were just left to post about whatever aspects of the course they were currently thinking about. This led to some really nice online discussions outside of class time, which tended to be focused on areas of disagreement between students or places where they felt stuck on assignments.

3 Outcomes

In the end, this course was extremely successful. It was clear to me that the students' writing and reasoning skills significantly improved over the course of the semester. They had a better understanding of what a clear explanation required, were better able to judge for themselves if they had succeeded in giving one, and they were better at continuing to work on a problem until they understood it and could explain it well. Furthermore, student comments made it clear that they were aware of the changes. Here are several examples of student comments, taken from course evaluations and other writing that they did for the course, which demonstrate this awareness and also highlight some of the other strengths of a course like this:

- My writing has definitely improved—I am a much more focused and organized writer now ... and this is definitely attributable to my writing seminar. ... [The written comments on papers] were just enough to give me a hint at whatever I needed to do to improve a statement, but not too much so that my abstract thinking was inhibited.
- Thinking about revisions for the vertical angle theorem brings up lots of ideas on different ways to prove theorems. I did not realize I would have to be quite so detailed and specific. ... [S]eeing so many different approaches helped me realize there are different ways of thinking about the same concepts.
- One of the primary aspects of the course that I enjoyed was the instructor's constructive criticisms on our formal essays. Even though during the course my feelings toward these comments were anything but joyous, I really thought they helped me learn to write better. They taught me to make absolutely sure that I justified everything I wrote about extremely clearly.
- [I]t was very helpful that we could revise each paper multiple times. I believe my writing improves best this way. Overall, I would definitely recommend this course to future computer science and math majors, because it actually explains what it means to prove an argument, something which I never really understood until now.

The course also demonstrated to the math department and the university writing program that a mathematical first-year writing seminar was not only possible, but was a great venue for teaching both mathematics and writing skills. Although Cornell has one of the oldest writing-across-the-curriculum initiatives in the country, dating back to 1966, the math department had never offered a first-year writing seminar prior to this one. The first time I taught the course in 1999, it was as an experiment to see if it would be possible.

It turned out to be so successful that I was asked to teach it three more times before I graduated and left the university in 2001. Since then, Cornell has continued to offer first-year writing seminars in the math department on a regular basis, although their mathematical content has varied greatly depending on who was teaching them. Many other people have taught these seminars since I left, under titles such as “To Infinity and Beyond,” “The Dementia of Dimension,” “Pictures in 1000 words or less,” “Experiencing Mathematics Through Writing,” and “Certainty and Ambiguity: Exploring Mathematical Concepts Through Writing.”

4 Extending the method

Teaching this writing seminar has also had a fairly significant impact on my subsequent teaching. Since leaving Cornell, I have been at the University of Northern Colorado, a much less selective school whose primary mission is preparing future teachers. When teaching math courses, I have continued to use many of the techniques described here for teaching proof-writing, and even some of the same assignments, such as using the assignments reproduced in Appendices A.2 and A.3 in geometry classes. I still often use the course structure described in Section 2.1 as the basis for many of my classes. I still routinely assign formal writing assignments that go through a series of revisions and are graded on the basis of their mathematical exposition in addition to their correctness, and I still feel that this is one of the most effective techniques I know for improving my students’ mathematical communication and proof-writing skills. I have used the assignments in a wide variety of classes for undergraduate pre-service elementary and secondary teachers, Master’s-level in-service teachers, and Ph.D. students in mathematics education. These classes have ranged in size from 7 to 37 students. However, the writing-intensive teaching methods are definitely best suited to classes of 15 to 25 students. For descriptions of some of the mathematics courses in which I have used these methods, see the detailed course notes for my Modern Geometry I [11], Modern Geometry II [12], and Mathematical Modeling [13] courses published in the online *Journal of Inquiry-Based Learning in Mathematics*. (Editors’ note: also, for more information in this volume on the logistics of inquiry-based approaches in different courses, see Ellis-Monaghan [3], Ernst and Hodge [4], and Rault [14].) Thus, the ideas presented here certainly can be used outside of the setting of a writing seminar, and work well with a wide range of kinds of students.

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Appendix

A Sample Writing Assignments

This appendix includes several sample formal writing assignments used. They are not all of the assignments we used, and, in any case, the assignments varied from semester to semester.

A.1 Babylonian Tablets

Figure 1 is a copy of the front (obverse) and back (reverse) of a clay tablet found by an archeological dig of an ancient Mesopotamian city.¹ The tablet probably dates from around 1350 BC. The symbols on the tablet are called cuneiform, i.e., wedge-shaped, because they are made up of single wedge-shaped marks that were impressed with a stylus upon the tablet while it was still wet.

Explain the meaning of the tablet. Draw as many conclusions as you can about its contents, but be sure to explain all of your reasoning as clearly as possible. You might find it helpful to imagine that you are writing an explanation to be placed in a museum with the tablet.

¹I always gave this assignment to groups to work on on the very first day of class. The tablet, which, as students soon discover, shows the multiplication table for nines in the base 60 babylonian system, is from an archeological dig of the Temple Library of Nippur conducted by the University of Pennsylvania. This drawing of the tablet originally appeared in the subsequent report [8], and is reproduced with a very accessible commentary and explanation in *Episodes from Early Mathematics* [1].

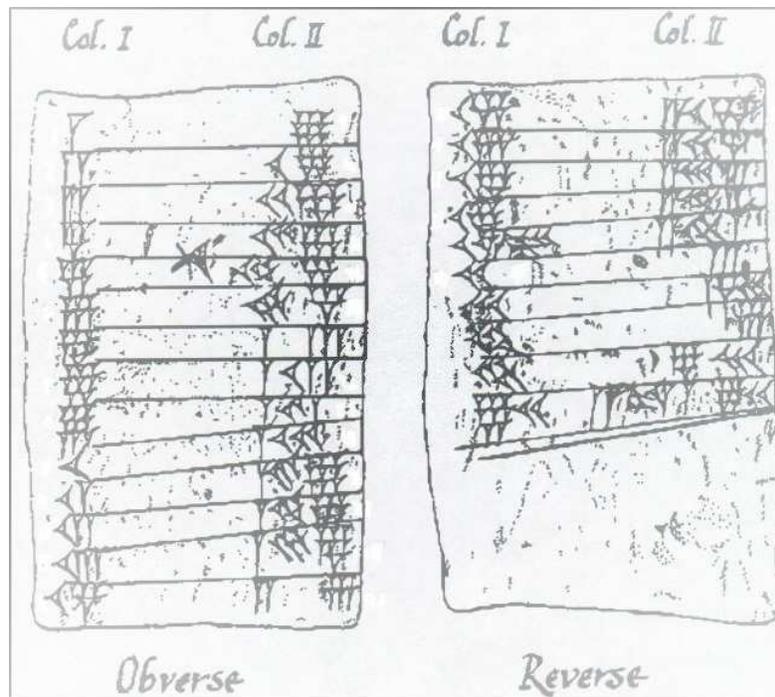


Figure 1: An ancient Babylonian tablet

A.2 Vertical Angle Theorem

For this assignment, you should define what you think an angle is, and use your definition to prove the vertical angle theorem.

In doing this, you can use Problems 3.1 and 3.2 from the reading² as a guide, but you need to integrate your answers to the questions into a unified, coherent whole. You should try to make your proof as clear and convincing as possible. The best proofs are those that not only convince the readers that something is true, but also allow them to understand why it is true. You should bear this in mind when you're deciding what assumptions to make and what kind of a proof to give. This is one reason that you want your assumptions to be simpler than the fact that you're trying to prove, and as obvious as possible. (It's also one reason that different people will give different proofs if they think of angles in different ways. Explaining why two angles have the same shape is different than explaining why they have the same degree measure.) If you decide to give a proof using degree measure, be careful not to confuse an angle (which is some kind of geometric object) with its degree measure (which is a number). If you want to talk about measuring angles, you'll need to explain how you want to assign a measurement, and also why angle measurements under your definition have any properties that you need for your proof. This is one reason that the authors of the reading suggest that proofs using symmetries are "generally simpler" than proofs involving measuring angles.

²The reading this refers to is an excerpt from *Experiencing Geometry* [7]. Problems 3.1 and 3.2, which appear in the excerpt along with quite a bit of supplementary explanation and suggestions of how to approach them, are as follows:

Problem 3.1: What is an Angle? Give some possible definitions of the term "angle." Do all of these definitions apply to the plane as well as to spheres? What are the advantages and disadvantages of each?

For each definition, what does it mean for two angles to be congruent? How can we check?

Problem 3.2: Vertical Angle Theorem (VAT) Prove: opposite angles formed by two intersecting straight lines are congruent. (Note: [Such] angles . . . are called **vertical angles**.) What properties of straight lines and/or the plane are you using in your proof? Does your proof also work on a sphere? Why? Which definitions from Problem 3.1 are you using in your proof?

A.3 Side-Angle-Side on a Sphere

Recall the following fact about planar triangles:³

(SAS) If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

This is Euclid's Proposition 4.

Decide whether or not SAS is true on the sphere. If you think that it is true on the sphere, then give a convincing argument that shows that it is true. If you think that it isn't true, explain why it is false, show how to modify the statement to make it true on the sphere, and then prove your modified statement. In either case, explain and justify any decisions that you had to make in order to come to your conclusion. Try to write explanations that would convince a reasonable skeptic. You will probably find it helpful to thoroughly understand Euclid's planar proof before trying to decide what happens on the sphere.

A.4 Minesweeper

In a three-to-five page paper, explain what a strategy is and discuss the strategies that you developed to play the game Minesweeper.⁴ Give a set of criteria for evaluating whether or not a given strategy is a good strategy. Try to be as concrete as possible, so that an independent observer could decide if a strategy meets your criteria with making any subjective decisions. Evaluate your strategies according to the criteria you developed. How are your criteria related to the kinds of reasoning that you were doing while you were playing the game?

A.5 Final Written Assignment in Lieu of a Final Exam

Complete both of the following:

A. Write an essay that connects two or more of the following topics discussed in class this semester: syntax vs. semantics; formal systems; the game of minesweeper; soundness and completeness; does $.9999\dots = 1$?; form vs. meaning in *Gödel, Escher, Bach*; the process of Proofs and Refutations; truth vs. provability; or any other topic discussed in class, subject to my approval.

B. We have read a lot of dialogues this semester.⁵ Write your own 3–5 page dialogue illustrating ideas that we studied this semester that you think are particularly interesting. (Take this as an opportunity to show me something that you learned this semester.)

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³This assignment was given after we had read Euclid's Proposition 4, and also after we had done a lot of work deciding which lines are straight on the sphere and which of Euclid's postulates were true on the sphere. I still use this assignment along with the previous assignments in my Modern Geometry I class; for more details about how I use them, see my class notes in the *Journal of Inquiry-Based Learning in Mathematics* [11]. This assignment was also related to readings we did from Lakatos's *Proofs and Refutations* [10] about the role of finding counterexamples to refine conjectured theorems. The version of Proposition 4 given here is Sir Thomas Heath's literal translation of Euclid [5], and the language is a bit unconventional to modern ears; this is something we discussed in class.

⁴Prior to working on this, the students had been asked to spend time playing minesweeper while trying to be self-aware of the reasoning that they were doing. It relates directly to ideas of soundness and completeness of formal systems that we were reading about in *Gödel, Escher, Bach* [9].

⁵All of *Proofs and Refutations* [10] and large parts of *Gödel, Escher, Bach* [9] are written as dialogues.

Teaching Reasoning Skills in Calculus I and Below

Jane Friedman

Abstract

This paper discusses exercises that help students in calculus and other lower-division courses develop skills that are prerequisites for proof-writing. The exercises are designed to help students learn to read and understand definitions, understand the logic of theorems, and to critique arguments. In general, calculus textbooks do not provide sufficient problems of this sort. Examples are provided.

Difficulty Level: Medium; Course Level: Non-traditional

1 Background and Context

Even students who are successful in calculus often have difficulty making the transition to upper-division mathematics courses. When I teach courses where proof-writing is required, I see students making very basic mistakes, such as trying to prove a universal statement by example, or using the converse of a theorem in a proof. Most students have had a previous course introducing them to proof, yet they still make these errors. There are many reasons, but fundamentally it is because the kind of reasoning required is different from reasoning in non-mathematical contexts [1, 2]. This kind of reasoning is not usually emphasized in lower-division mathematics courses. Most students need time to become comfortable with the precision and rigor required to write proofs. In order to make proof-writing more accessible to students in upper-division courses, it is desirable to introduce aspects of mathematical reasoning into every mathematics course in college.

This paper describes exercises that I use in classes at the level of calculus I and below to help students build the skills that are prerequisites for proof-writing. These skills include understanding and exploring definitions, understanding the logic of conditional statements, and critiquing arguments. This paper will focus on calculus I, but the techniques described can be used in classes below calculus I, such as college algebra, as well as courses above calculus I.

I teach at the University of San Diego, (USD), an unaffiliated Catholic university, in the College of Arts and Sciences. Of the approximately 4,000 undergraduate students, between 30 and 50 are typically math majors. Each year USD offers about fifteen sections of a four-unit calculus course for math and science majors; it is this course on which I focus. Students intending to major in mathematics are typically a minority in the course.

I have been influenced by the wonderful book *An Accompaniment to Higher Mathematics* by George Exner [3]. In this book, written for students taking upper-division courses such as abstract algebra, Exner provides basic instruction to students in how to read mathematics for conceptual understanding. Exner details an active approach to reading mathematics that requires students to read with a pencil in hand, stopping to explore definitions and theorems. I have found that Exner's method can be used to write exercises for classes below the level he targeted, including calculus and college algebra and mathematics content courses for future elementary school teachers. I enjoy writing the supplementary problems, and with experience they become quite simple to write.

2 Description and Implementation

The exercises I use are not found in typical calculus textbooks so I create my own handouts with exercises and spaces to write answers. A portion of almost every class meeting is devoted to working through these handouts, which are neither collected nor graded. Typically I will introduce a topic in a brief lecture, perhaps doing an example, introducing a definition, or discussing a theorem. Then I give the students time to work in groups or individually on a handout, while I move around the classroom observing and answering questions. If it seems that many students are having the same difficulty, I will interrupt their work to make a clarifying comment to the class as a whole. Towards the end of the class period, we hold a whole-class discussion of the material. At the beginning of the semester I may walk the students through the beginning problems on a handout, before letting them finish the work on their own. As the semester goes on they become accustomed to this way of working and need less guidance. I put questions of the same type on quizzes and tests to ensure that the students recognize the importance of this work and to hold them accountable for it.

There is a change in what I emphasize and in how class time is spent, but I have found there is only a minimal change in what topics are covered in the course. More specifically, the work on the handouts replaces much of the time that I used to spend lecturing. Instead of watching me work out examples, the students are creating their own examples and counterexamples. Instead of watching me at the board, they are actively working to understand the meaning of important definitions and theorems. There is more emphasis on developing reasoning skills and less emphasis on computation. Although there is somewhat less time for applications, I am able to cover all of the essential topics. Since the emphasis is on reasoning and conceptual understanding, the approach benefits all students in the class, those who never take another mathematics class as well as mathematics majors.

2.1 Understanding and Exploring Definitions

Mathematics is built on clear and precise definitions, which must be understood deeply and completely. Many elementary proofs are simple applications of relevant definitions. Deep understanding of a definition comes through a process of questioning and exploration. Exner suggests that each new definition should be explored by creating examples and non-examples, the latter of which is an instance of something that does not satisfy a given definition. They are often helpful in understanding concepts, since they help illuminate the boundaries of the concept. Ideally this becomes habitual for the students.

I begin this work with definitions early in the semester as we review functions. During the first or second class meeting the topic of “function” will be introduced, as well as “increasing” and “decreasing” functions. I provide a handout, the exploration and subsequent discussion of which serve as an introduction to these topics. Before having the students begin their work, I will sketch increasing and decreasing functions on the board and use my sketches to illustrate the definitions of increasing and decreasing given in the handout. I will also model for the class how to use the definition to show that a particular function is increasing. In the excerpt from this handout below, students begin the work of learning to construct their own examples and non-examples.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

Answer the following based on these definitions. Your justifications and explanations must be based on these definitions.

1. Give an example of a function that is increasing. Explain in detail how you know that the function is increasing.

2. Give an example of a function that is neither increasing nor decreasing on an interval I , if possible. If it is not possible, explain in detail how you know that it is not possible. If it is possible, explain in detail how you know that your example is neither increasing nor decreasing.

3. Give an example of a function that is both increasing and decreasing on an interval I , if possible. If not possible, explain in detail how you know it is not possible. If it is possible, explain in detail how you know that the function is both increasing and decreasing.

Another mode of exploring definitions is to ponder the reasons for specific details of a definition. The question below could be used in a college algebra course or as part of a review in a calculus course.

Definition: An **exponential function** is a function of the form

$$f(x) = a^x,$$

for some positive constant a .

Why do you think a is restricted to be a positive number in the definition?

For concepts with multi-part definitions it is illuminating to explore situations where one part of the definition is satisfied but another is not. The concept of continuity is a natural setting for this type of exploration.

Definition: A function f is **continuous at a number** a if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

1) a) Is it possible for $\lim_{x \rightarrow a} f(x)$ to exist if a is not in the domain of $f(x)$? If possible, sketch an example. If not possible, explain why it is not possible. Justify your answer.

b) Is it possible for $f(x)$ to be continuous at $x = a$ if a is not in the domain of $f(x)$? If possible, sketch an example. If not possible, explain why it is not possible. Justify your answer.

c) Is it possible for a to be in the domain of $f(x)$ and $\lim_{x \rightarrow a} f(x)$ not to exist? If possible, sketch an example. If not possible, explain why it is not possible. Justify your answer.

d) Is it possible for a to be in the domain of $f(x)$ and $\lim_{x \rightarrow a} f(x)$ to exist, but $\lim_{x \rightarrow a} f(x) \neq f(a)$? If possible, sketch an example. If not possible, explain why it is not possible. Justify your answer.

2.2 Understanding the Logic of Conditional Statements

Students often have difficulty understanding how to interpret and apply conditional statements. Early in the semester I discuss the meaning of conditional statements and the relationship between a conditional statement and its converse and contrapositive. Given a theorem whose structure is “if A then B”, students often make the error of assuming that the converse is true, essentially understanding the conditional statement as the (un)implied biconditional statement “A if and only if B.” This is how we often understand these kinds of statements in ordinary English [2]. Textbooks compound the issue by providing mostly exercises which require students to apply a theorem in a situation where the hypotheses of the theorem hold. The exercises rarely require students to determine whether or not a theorem applies in given circumstances. Those few textbook exercises where students are presented with a situation where a theorem cannot be applied, usually explicitly tell the students to show that the hypotheses of the theorem are violated. If these are the only exercises to which students are exposed, they can come to believe that they can always apply theorems, unless explicitly told to check if the hypotheses hold. Fortunately it is easy to write exercises that give students practice in identifying when a theorem can or cannot be applied. If we are studying the theorem “if A then B”, I ask the students to tell me what the theorem says about a variety of different situations, some where A holds, some where A doesn’t hold, some where B holds and some where B doesn’t hold. In each case I ask them what the theorem says about the situation. I emphasize that if A does not hold, then the result doesn’t follow and B may or may not be true.

The intermediate value theorem (IVT) is a good place to practice this kind of work. In the questions below, 1a), b), and d) violate the hypotheses of the IVT in different ways; only in 1c) can the theorem be applied.

The Intermediate Value Theorem (IVT) *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*

1) *Can the IVT be applied to the following situations? Why or why not? If the IVT can be applied, what does it allow you to conclude? Justify your answers in detail.*

a) $f(x) = x^2$ on the interval $(0, 2)$.

b) $f(x) = x^2$ on the interval $[-1, 1]$.

c) $f(x) = x^2$ on the interval $[0, 2]$.

d) $g(x) = \tan x$ on the interval $[0, \pi]$.

One way to help students overcome the erroneous tendency of assuming the equivalence of a conditional statement and its converse is to teach them the difference between:

What does the theorem say about this situation?

and

Is the conclusion of the theorem true in this situation?

The questions below, again in the context of the IVT, are designed to help students understand this crucial distinction:

2) Consider the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 2 \end{cases}$$

on the interval $[0, 2]$.

a) Can the IVT be applied? Why or why not? Justify your answer.

b) Is it true that for all N between $f(0)$ and $f(2)$ there exists a number c in $(0, 2)$ where $f(c) = N$? Justify your answer in detail.

c) Explain why your answers to a) and b) don't represent a counterexample to the IVT.

3) Consider the function

$$g(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 + x & \text{if } 1 < x \leq 2 \end{cases}$$

on the interval $[0, 2]$.

a) Can the IVT be applied to this situation? Why or why not? Justify your answer.

b) Is it true that for all N between $g(0)$ and $g(2)$ there exists a number c in $(0, 2)$ where $g(c) = N$? Justify your answer in detail.

c) Explain why your answers to a) and b) don't represent a counterexample to the IVT.

The functions $f(x)$ and $g(x)$ are discontinuous on the interval $[0, 2]$, so the IVT cannot be applied. However, for $f(x)$ the conclusion of the theorem is true, while for $g(x)$ the conclusion is false. Neither one is a counterexample to the IVT, and together they illuminate the relationship between a conditional statement and its converse. In one case the hypotheses of the statement are false and the conclusion is true, while in the other, both the hypotheses and conclusion are false.

2.3 Critiquing Arguments

In order for students to become good at writing proofs, they need to be able to recognize valid and invalid reasoning. They need to be able to read their own work and recognize and correct faulty arguments. Throughout the course I give students the opportunity to recognize the common error of trying to prove a general statement with one or two specific examples. The handout introducing the concepts of increasing and decreasing functions also contains the following:

Consider the function $y = |x|$ on the interval $[-10, 1]$. Is the following sentence correct? Why or why not? Justify your answer in detail.

The function $y = f(x) = |x|$ is decreasing on the interval $[-10, 1]$, since $-3 < -2$ and $f(-3) = 3 > 2 = f(-2)$.

Mistakes students make provide a good source of answers to critique. For example, on an exam I asked students to compute $\lim_{x \rightarrow 0} \cos(1/x)$ or show that the limit did not exist. The question below is based on two of the more common wrong answers to the exam question. Students who made these errors were thus given an opportunity to reflect on their own mistakes, and the other students were given an equally valuable opportunity to critique the work of others.

Suppose that you are a tutor in the mathematics tutoring center helping your friends Angelo and Brenda. Angelo and Brenda are working on the problem

Find $\lim_{x \rightarrow 0} \cos(1/x)$ or show that this limit does not exist.

Angelo says that $\lim_{x \rightarrow 0} \cos(1/x)$ does not exist because $\cos(1/x)$ is undefined at $x = 0$.

Brenda says that since $-1 \leq \cos(x) \leq 1$, we know that for all $x \neq 0$, $-1 \leq \cos(1/x) \leq 1$, so we can find the limit using the squeeze theorem.

Both Angelo and Brenda have errors in their reasoning. Explain the mistakes each student is making and explain in detail how to investigate this limit.

In this case, I did tell the students that neither Angelo nor Brenda had used valid reasoning. In other problems I do not give students this information. It can be valuable to give them situations where there are two valid or two invalid arguments to critique. The goal is for students to think carefully about the substance of the arguments and not use faulty heuristics such as “one of these must be right; so let me figure out which seems more likely to be correct.”

3 Outcomes

This approach to teaching lower-division courses has been informed by my experiences in teaching upper-division courses involving extensive proof-writing. It is difficult for most undergraduate students to understand and write proofs. I believe introducing mathematical reasoning to students from the beginning of their college career can help them develop the skills needed to understand upper-division courses. (Editors’ note: For another approach for incorporating proofs into a calculus setting, see Oh [4], in this volume.)

When I use the methods in lower-division courses I see over the course of a semester noticeable improvements in student performance. At the end of the semester most students understand that in order to apply a theorem, one must verify that the conditions of the theorem hold. They understand that sometimes a theorem tells you absolutely nothing and that this is independent of whether or not the conclusion of the theorem is true. They are much more proficient at answering the kinds of questions I have shown in this article. This work enables students to understand calculus concepts more deeply and provides a good foundation for later proof-based courses.

Below are some results from a final exam, given to a class composed of 18 first-year students. These questions simultaneously test knowledge of calculus and the reasoning skills that were infused throughout the course.

Consider the following true theorem:

If f has a local maximum or minimum at c , then c is a critical number of f .

a) State the converse to the true theorem above.

b) Is the converse stated in a) true or false? Justify your answer completely.

All 18 students correctly stated the converse, and 14 correctly stated the converse was false. Six students gave entirely correct answers to b), justifying their answers with a specific counterexample, while three more students attempted to provide a counterexample but had errors in their answers.

Another question stated the mean value theorem (MVT) and asked students to determine what, if anything, the MVT says about the following situations, justifying their answers.

i) $f(x) = \cos x$ on the interval $[\pi/2, \pi]$.

ii) $g(x) = x^2 + |x|$ on $[-1, 2]$.

All students recognized that the MVT could be applied in i), and three gave perfect solutions, clearly checking the conditions of the theorem and drawing the correct conclusion. Four other students gave answers marred only by errors made when calculating

$$\frac{f(\pi) - f(\pi/2)}{\pi - (\pi/2)}.$$

All but three students recognized that the MVT could not be applied in ii). Eight of the students correctly stated that the reason the MVT does not apply is that $g(x)$ is not differentiable at $0 \in [-1, 2]$. These are much better results than these students would have been capable of at the beginning of the semester, and much better results than I would expect without providing the students with the opportunity to work on the kind of problems described in this paper.

At USD my colleagues have recognized the importance of teaching reasoning in calculus I. We have formed a calculus working group (CWG), that meets regularly to discuss issues of pedagogy. It works towards aligning instruction to four common learning outcomes, which are consistent with the approach I have described here. (See Appendix A.)

4 Extending the Method

All of the types of questions discussed here could be used throughout the mathematics curriculum, including in proof-based courses. They support students' development as proof-writers at all levels. Although in this paper the focus is on calculus I, these types of questions can also be used in lower-level courses. Appendix B contains some examples from precalculus. The approach could be used at institutions where calculus is taught in large lectures, where the professor could pose questions, allow students some time to think about them, and then lead a discussion of the answer, or have TA's lead a discussion in section meetings.

References

- [1] L. Alcote and A. Simpson, "Dealing with categories mathematically," *For the Learning of Mathematics*, 22 (2002) 28-34.
- [2] S. Epp, "The role of logic in teaching proof," *American Mathematical Monthly*, 110 (2003) 886-899.

[3] G. Exner, *An Accompaniment to Higher Mathematics*, Springer-Verlag, New York, 1996.

[4] M. Oh, "Introducing Proofs to Calculus Students", in this volume.

Appendix

A USD's Common Learning Outcomes for Calculus I

The four common learning outcomes for USD's calculus I courses are:

Upon successful completion of the course, the student will be able to:

(1) *Clearly communicate complete solutions to problems verbally and in writing. This involves using complete sentences to explain individual steps in the solutions, correct notation and proper units.*

(2) *Explain, interpret, and correctly apply definitions. Provide examples and non-examples to illustrate definitions.*

(3) *Use valid reasoning (be able to provide a logical sequence of statements that follow each other) and be able to identify invalid reasoning. Provide counterexamples to disprove statements that are not always true.*

(4) *Determine and explain when theorems apply to a situation and apply them correctly.*

B Examples of Handouts

This appendix contains several examples of handouts (with the space to write answers removed). The first few are for calculus I.

1. Find an example of each of the following, if possible; if not possible explain why it is not possible.

(a) A function that is neither even nor odd.

(b) A function that is both even and odd.

2. Is the function $y = |x|$, odd or even or neither?

3. Susie says

The function $y = x^4 + x^2$ is even. Here is how I can prove that it is true. $f(-1) = (-1)^4 + (-1)^2 = 1 + 1 = 2$ and $f(1) = 1^4 + 1^2 = 1 + 1 = 2$, since I have shown that $f(-1) = f(1)$, I have shown $f(x)$ is even.

(a) Is Susie's reasoning correct? Why or why not?

(b) Is it true or false that $f(x) = x^4 + x^2$ is even? How would you justify your answer?

(c) Explain the difference between the question asked in (a) and the one in (b).

Definition:

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

A function f is called **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

4. Show that $f(x) = 2x$ is increasing for all real numbers.
5. Show that $f(x) = x^2$ is not increasing on the interval $(-1, 0.5)$.
6. Consider $f(x) = x^2$, where is this function increasing? Where is it decreasing?

Direct Substitution Property. If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

1. (a) What is a polynomial?
 (b) What is the domain of a polynomial?
 (c) What is a rational function?
 (d) What is the domain of a rational function?
2. Let $f(x) = 3x^2 - 2x + 1$, and $g(x) = \frac{x^2 - 4}{x + 1}$. What, if anything, does the direct substitution property tell us about the following? Justify your answers.

(a) $\lim_{x \rightarrow 1/2} f(x)$

(b) $\lim_{x \rightarrow 2} g(x)$

(c) $\lim_{x \rightarrow -1} g(x)$

Fact. If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

3. Use the definition of a limit to justify the above fact.
4. What, if anything, does the above fact tell you about the following limits? Justify your answer.

$\lim_{x \rightarrow 0} f(x)$, where $f(x) = \frac{1}{x}$ and $\lim_{x \rightarrow 0} g(x)$, where

$$g(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

5. Use the fact above to find

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}.$$

Justify your reasoning.

Definition Let c be a number in the domain D of a function f . Then $f(c)$ is the

* **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D .

* **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D .

The absolute maximum and minimum values of f are called extreme values of f .

1. Find the absolute minimum values of $f(x) = |x^2 + x - 2|$ on $[-3, 2]$.
2. Let $f(x) = 2e^{-x}$ on $[0, 1]$. Find the extreme values of f .

Definition The number $f(c)$ is a

* **local maximum** value of f if $f(c) \geq f(x)$ when x is near c .

* **local minimum** value of f if $f(c) \leq f(x)$ when x is near c .

3. Sketch the graph of a continuous function on the interval $[-4, 4]$ that has a local minimum that is not an absolute minimum on that interval and a local maximum which is not an absolute maximum on that interval, if possible. If not possible, explain why it is not possible. If it is possible, label the local maximum and the local minimum.
4. Sketch the graph of a function, if possible, with an absolute minimum which is not a local minimum. If not possible, explain why it is not possible. If it is possible, label the local minimum.

Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

5. For each of the following functions on the given intervals determine i) what the extreme value theorem says about that function on that interval and ii) what extreme values, if any, that function has on that interval.

(a) $f(x) = x^2$ on $(-3, 5)$.

(b)

$$f(x) = \begin{cases} e^x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{on } [-2, 2]$$

(c) $f(x) = \ln x$ on the interval $[1, 3]$.

The following examples are from pre-calculus.

1. Here is a theorem:

Theorem 1. If A and B are any positive real numbers then

$$\log(A \cdot B) = \log A + \log B.$$

What, if anything, does Theorem 1 tell you about each of the following? Explain your answers.

- (a) $\log(10 + 20)$
- (b) $\log(10) \cdot \log(100)$
- (c) $\log(10 \cdot 3)$
- (d) $\log((-20) \cdot (-30))$
- (e) $\log 3 + \log 5$
- (f) $\log(10^5)$

2. Here is a theorem:

Theorem 2. If A and B are any positive real numbers then

$$\log(A/B) = \log A - \log B.$$

What, if anything, does Theorem 2 tell you about each of the following? Explain your answers.

- (a) $\log(1000 - 100)$
- (b) $\log(100) - \log(1000)$
- (c) $\log(1/20)$
- (d) $\log(20 \cdot 30)$
- (e) $\log(45 - 23)$
- (f) $\log(25)/\log(45)$
- (g) $\log(3/(-2))$

1. Suppose that $f(x) = x^2 - 1$.

Find each of the following:

- (a) $f(y)$
- (b) $f(x - 1)$
- (c) $f(x) + f(2)$
- (d) $(f(x))^2$
- (e) $f(x^2)$
- (f) $f(f(x))$

2. Suppose that $g(x) = x^2 + 1$. Determine whether the following are true or false. Prove your answers.

- (a) For all real numbers x , $g(x - 1) = g(x) - g(1)$.

- (b) For all real numbers x and y , $g(x + y) = g(x) + g(y)$.
- (c) For some real number x , $g(x + 1) = g(x) + g(1)$.

The absolute value of x is the distance between x and zero on the number line. In symbols we denote the absolute value of x by $|x|$. We can define $|x|$ as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

1. Use the above definition to find

- (a) $|-2|$
- (b) $|\sqrt{3}|$
- (c) $|0|$

2. Solve the following equations:

- (a) $|x| = 3$
- (b) $|x| = -4$
- (c) $|x + 1| = 5$

3. Theorem. If a, b, c are real numbers and $a \neq 0$ then the solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- (a) What, if anything, does the above theorem tell you about $bx^2 - cx + a = 0$ (assume $b \neq 0$)?
- (b) What, if anything, does the above theorem tell you about $10x - 0x^2 + 4 = 0$?

4. The Zero Product Property.

$a \cdot b = 0$ if $a = 0$ or $b = 0$. And if $a = 0$ or $b = 0$ then $a \cdot b = 0$.

- (a) Suppose that $(x - 1)(2 + x) = 0$, what can you conclude? Justify your answer.
- (b) Use the zero product property to solve: $9 - x^2 = 0$.

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Introducing Proofs to Calculus Students

Minah Oh

Abstract

In this article, I describe a method that I used to introduce mathematical proofs to Calculus I students. Every other week, the students were required to prove a given mathematical result by following a worksheet that explained each step they needed to take. Throughout the semester, each student turned in his/her own proof after working on the worksheet in groups.

Difficulty Level: Medium; Course Level: Non-traditional

1 Background and Context

James Madison University (JMU) is a master's-level institution with approximately 18,000 students. The Department of Mathematics and Statistics offers B.A. and B.S. degrees with a major in mathematics and B.S. degree with a major in statistics. A Masters of Education in Mathematics is also offered that prepares secondary school teachers for positions of instructional leadership as teachers. At JMU, approximately 10 sections of Calculus I are offered in the fall semester while approximately five sections are offered in the spring. Each section has an average of 28 students, and most of them are freshmen. The course consists of three 50-minute meetings and a longer 75-minute meeting each week.

Though the mathematics department offers an Introduction to Proofs course typically taken after the calculus sequence, I have found it beneficial to expose students to proof-writing early in their education. In this article, I describe a technique that I used to introduce proofs to my Calculus I students at JMU. The goal of the experiment was to help the students learn what a proof is, get used to writing proofs, and get rid of the fear of proving something.

2 Description and Implementation

2.1 Description

Once every two weeks, the students were given 75 minutes to prove a proposition by following a worksheet that explained each required step. The topics chosen were the standard proofs that I would previously have done in lecture. The students organized themselves into groups of three or four. After I explained to them what needed to be proved, they discussed the worksheet among themselves and occasionally asked me questions. They were required to write up a complete proof after the group discussion was over.

Throughout the semester, I slowly reduced the worksheets' instructions. The last worksheet said only "Prove the fundamental theorem of calculus part 1" and provided one hint. (See Appendix D for details.) Although this is an unusually difficult task for calculus students, many of them were able to write a mathematical proof with only minor errors, after I provided the outline of the proof verbally. (See Section 3 for a sample proof of the fundamental theorem of calculus part 1 by an average student and Section 2.4 for the verbal explanation that I gave.)

Students often have a difficult time understanding and writing proofs, and we are accustomed to their not understanding the proofs that we demonstrate on the board. This method is different. The students'

mindset changes from that of a passive listener to one of an active prover. They learn that a proof consists of a sequence of mathematical steps, and they try to understand them on their own and with their peers by following the worksheet.

2.2 Role of the Instructor

The purpose of the worksheet was to break up a proof into manageable pieces or statements, so that the students could explain to themselves and to their peers why each was true. This was a task the students were likely to accomplish. It was my responsibility to create effective worksheets. See Section 2.4 for details about creating effective worksheets and the Appendix for samples. The solution (i.e., complete proof) was given to the students shortly after they turned in their own proofs. Most of the students took their time to read through the complete solution carefully to understand how they could have improved their own proofs.

In the classroom, students asked me numerous questions during group work. One rule I followed was not to write anything on paper or on the board when replying. This was to prevent students from simply copying my proof, which would be equivalent to teaching the proof via lecture. This was also helpful in keeping the students focused on my verbal explanations. I had to repeat myself numerous times during group work, since often the students were not be able to understand my logic the first time. After having a group discussion, with my help, the students were able to resolve their problems and move on to the next step of the proof.

My role was especially important in the beginning of the semester when many students were confused. I made sure that I spoke with all groups multiple times during each class.

2.3 The Method and the Curriculum

Each student's proof counted as a bonus quiz. Such positive reinforcement was helpful in keeping the students' morale high. It also helped students overcome the fear of writing proofs, since they were not afraid of attempting a proof and failing. Students were given a 75-minute proof-writing bonus quiz every other week. I found myself writing many fewer proofs on the board during lecture (since the students were doing them in groups with the worksheet), which gave me extra time for a longer quiz. The proof-writing bonus quiz took place during the longer class meetings. The quizzes can be made into 50-minute ones if necessary. (See Section 4 for details.)

2.4 Creating the Worksheets

The worksheets needed to outline the mathematical steps of the proof at a level that students could understand. The first step I took in creating the worksheet was to write the complete proof of the chosen proposition. Then, I wrote in multiple steps what I would have said during lecture when presenting the proof. The steps to include on the worksheet were chosen based on their mathematical importance and on the students' proof-writing abilities. As the semester progressed and the students became better at writing proofs, I provided fewer instructions. Students were expected to write proofs with complete mathematical rigor, and so it was important for me to select proofs that involved mathematics that the students could understand completely.

For example, proving that differentiability implies continuity was one of the first proofs that I assigned. (See Appendix A for the corresponding worksheet.) The first step given in the worksheet was to write down exactly what they were trying to prove. Once the students were familiar with proofs, they did not need to be told this. In the later worksheets (see Appendices B, C, D), this step was not included. In Appendix A, the complete algebraic manipulation required in the proof was given. The students were simply required to discuss in groups why such algebraic manipulation was correct. In a later worksheet (see Appendix B), however, the students had to perform the algebraic manipulations themselves.

The proof of the mean value theorem was done after the proofs of the product rule and the quotient rule. Although similar templates of instructions were given in each of the corresponding worksheets (compare Appendix B and C), the proof of the mean value theorem was longer and more difficult. This shows that the students were making progress.

The last proof of the semester was the fundamental theorem of calculus part 1. As seen in Appendix D, the only instruction given in the worksheet was to use the hint. The verbal outline of the proof that I gave to the class before they started working in groups was:

Your goal is to start with the left-hand side $F'(x)$ and eventually show that it is equal to $f(x)$. Write the limit definition of $F'(x)$. Then substitute the function $F(x)$ and $F(x + h)$ by its definition. Make some manipulations so that you can apply the given hint.

When the students had further questions on how to complete the proof after following my instructions, I asked them:

What do we know about c ? Can we apply the squeeze theorem to complete the proof?

(Editors' note: for another approach that incorporates proofs into a calculus class, see Friedman [1], in this volume.)

3 Outcomes

The following two proofs were written by the same student. I am presenting her work verbatim. The first proof was written at the beginning of the semester, while the second was written at the end. The first topic was to prove that differentiability implies continuity, while the second topic was to prove the Mean Value Theorem. (See Appendices A and C respectively for the corresponding worksheet that she followed.)

1. Prove that if f is differentiable at $x = c$, then f is continuous at $x = c$.

Student's Proof:

Given f is differentiable at $x = c$ then $\lim_{x \rightarrow c} f(x) = f(c)$.

$$\lim_{h \rightarrow 0} f(h + c)$$

$$\lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \cdot h + f(c)$$

$$\lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \rightarrow 0} h + \lim_{h \rightarrow 0} f(c)$$

$$f'(c) \cdot 0 + f(c)$$

$$\lim_{h \rightarrow 0} f(c + h) = f(c)$$

Therefore, $f(x)$ is continuous at $x = c$.

The algebraic manipulation $f(c + h) = \frac{f(c + h) - f(c)}{h} \cdot h + f(c)$ was given in the worksheet. Although the student appears to know what had to be proved (see Step 1 of the worksheet: i.e., $\lim_{h \rightarrow 0} f(c + h) = f(c)$), the proof is simply a collection of mathematical symbols. Though only a short proof is required, and the worksheet includes most of it, the student plainly does not understand how to write a proof, or even what a proof is.

2. (Mean Value Theorem) Suppose $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) . Prove that there exists at least one number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Student's Proof:

Find the secant line of $f(x)$ between a and b and get $g(x)$.

$$g(x) = y - f(a) = \frac{f(b) - f(a)}{b - a} \cdot (x - a) \quad (1)$$

$$g(x) = y = \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a)$$

$$\text{Let } h(x) = f(x) - g(x) = f(x) - \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a).$$

$h(x)$ is continuous because $f(x)$ is stated to be differentiable and $g(x)$ is just a line.

$$h(a) = f(a) - \frac{f(b) - f(a)}{b - a} \cdot (a - a) + f(a) = 0 \quad (2)$$

$$h(b) = f(b) - \left(\frac{f(b) - f(a)}{b - a} \cdot (b - a) + f(a) \right) = 0$$

$$h(a) = h(b).$$

Because of Rolle's Theorem, there exists a point c where

$$f'(c) = 0 \quad (3)$$

since $h(a) = h(b)$.

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Although the proof contains mathematical errors, they are all minor. For example, in (1), $g(x)$ is not $y - f(a)$ but y , and (2) should be $h(a) = f(a) - \frac{f(b) - f(a)}{b - a} \cdot (a - a) - f(a) = 0$. Also, (3) should be $h'(c) = 0$. These minor errors aside, the proof is written in a clear and logical fashion with complete sentences. The student now understands that proof-writing is a process of explaining to the reader why a certain mathematical result is true. Furthermore, proving the Mean Value Theorem is much more difficult than proving that differentiability implies continuity. It is evident that the student became stronger at understanding and writing proofs. Such improvement was clear in all students' work.

It was also noticeable that, as the semester progressed, many students were handing in flawless proofs well before the 75-minute period was over. One interesting observation is that stronger students and weaker students (in taking exams) performed similarly in writing proofs. This shows that even the students who do not perform well in class may still be able to think logically and mathematically.

The following is the proof of the fundamental theorem of calculus part 1 written by an average student. The worksheet provided very few steps for this proof. (See Appendix D to see the corresponding worksheet

that she followed.) Recall that I explained the outline of the proof verbally for this topic. (See subsection 2.4 for details.)

(Fundamental Theorem of Calculus Part 1) Prove that if f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and $F'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$.

Student's Proof:

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(t)dt}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t)dt \\
 &= \lim_{h \rightarrow 0} \frac{1}{(x+h) - x} \int_x^{x+h} f(t)dt \\
 &= \lim_{h \rightarrow 0} f(c)
 \end{aligned}$$

c is in the closed interval $[x, x+h]$.

$$x \leq c \leq x+h$$

$$\lim_{h \rightarrow 0} x \leq \lim_{h \rightarrow 0} c \leq \lim_{h \rightarrow 0} x+h$$

Thus by squeeze theorem,

$$\lim_{h \rightarrow 0} c = x.$$

And because we know that f is a continuous function,

$$\lim_{h \rightarrow 0} f(c) = f(\lim_{h \rightarrow 0} c) = f(x).$$

Overall, not only did the students improve greatly in writing mathematical proofs, they also started to think more mathematically and logically in general. In addition to the improvements mentioned earlier, they became better at justifying each step of their solutions on exams and also at forming clear questions in class and during office hours.

Here are some quotations from students regarding this method, presented in the exit survey:

I like the proofs that we have to do in class. It helps me to learn math in a different way.

I liked the amount of quizzes as well as the proofs we did on Thursdays. They were hard, but I never felt tons of pressure and was comfortable asking for help.

It was tough, but I learned a lot.

One thing that the instructor should be mindful of is that the students will resist strongly at first. They might have a difficult time understanding what they are supposed to do with the worksheet. During the first month of the semester, it may seem like the method is not working and the students dislike the activity. It is important that the instructor does not give up. I trusted in my students' ability to reach the next level of logical and mathematical thinking. At some point, I noticed that the students were actively engaged in group discussions, and the proofs they turned in came to resemble proofs that I would write. It was also impressive to see how carefully they wrote each line of a proof in order to avoid mathematical errors.

4 Extending the Method

If necessary, the method can be used during a 50-minute period instead of a 75-minute one by combining group work with traditional lectures. For example, the first few steps in the worksheet could be completed in lecture, and the students could discuss the remaining steps in groups and then write up their own complete proofs.

Since students are working in groups, the method can be used in smaller or larger classes as long as the instructor can interact with all groups. Although it may not be as effective, the method can also be used for an online course if the instructor can provide a worksheet that has extremely clear instructions and the students can have extensive online discussions with their group members and instructor.

Furthermore, this method can be applied to other courses, such as Introduction to Proofs or Linear Algebra, where the students are still learning how to think step by step to write a proof. For these courses, the worksheets could give less information due to the higher mathematical abilities of the students. It would also be a good assignment for the students to create these worksheets themselves at the end of the semester.

References

[1] Jane Friedman, "Teaching Reasoning Skills in Calculus I and Below," in this volume.

Appendix

A Sample Worksheet 1: Differentiability and Continuity

1. Prove that if f is differentiable at $x = c$, then f is continuous at $x = c$.

- Step 1. How do you show that f is continuous at $x = c$? Write down what you have to prove.
- Step 2. Rewrite the limit appearing in Step 1 by using a new variable, $h = x - c$.
- Step 3. Discuss with your group why the following is true: if $h \neq 0$, then

$$f(c + h) = f(c + h) - f(c) + f(c) = \frac{f(c + h) - f(c)}{h} \cdot h + f(c).$$

- Step 4. By using Step 3, find $\lim_{h \rightarrow 0} f(c + h)$. Discuss where you are using the given information that f has a derivative at $x = c$.
- Put all of the above steps together and write down a complete mathematical proof.

B Sample Worksheet 2: Product Rule and Quotient Rule

1. Prove the product rule: if f and g are differentiable, then so is their product $P(x) = f(x)g(x)$, and

$$P'(x) = f'(x)g(x) + f(x)g'(x).$$

- Step 1. Write the limit definition of $P'(x)$ in terms of f and g .
- Step 2. Subtract $f(x+h)g(x)$ from the first term of the numerator and add $f(x+h)g(x)$ to the second term of the numerator. Convince yourself that you are simply rewriting your answer in Step 1 by doing so without changing anything.
- Step 3. Make manipulations in step 2 to reach

$$\lim_{h \rightarrow 0} \left(f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right).$$

- Step 4. We would like to split the limit in step 3 into separate limits of each term appearing there, but in order to do so, we know that the limits $\lim_{h \rightarrow 0} f(x+h)$, $\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$, $\lim_{h \rightarrow 0} g(x)$, and $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ must all exist. Verify that each of these limits do exist. Then, write down what each of those limits is, and conclude your argument. To verify that $\lim_{h \rightarrow 0} f(x+h)$ exists and it is in fact equal to $f(x)$, explain why f is continuous, and why continuity of f implies $\lim_{h \rightarrow 0} f(x+h) = f(x)$.
- By putting the above steps together, write a complete mathematical proof.

2. Prove the Quotient Rule: if f and g are differentiable, then so is their quotient $Q(x) = \frac{f(x)}{g(x)}$, and

$$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

- Step 1. Write down the limit definition of $Q'(x)$ in terms of f and g .
- Step 2. Rewrite the answer in step 1 so that you do not see fractions in the numerator.
- Step 3. Make manipulations in step 2 to reach

$$\lim_{h \rightarrow 0} \frac{1}{g(x+h)g(x)} \left(g(x) \cdot \frac{f(x+h) - f(x)}{h} - f(x) \cdot \frac{g(x+h) - g(x)}{h} \right).$$

- Step 4. Verify, as in step 4 of the proof of the Product Rule, that we can split the limit in Step 3 into separate limits of each term appearing there and conclude your argument.
- Step 5. By definition of $Q(x)$, $g(x) \neq 0$ for all x . Discuss with your group why this implies that $g(x+h) \neq 0$ for all h with small enough $|h|$. (Hint: Use the fact that g is continuous.)
- By putting the steps together, write down a complete mathematical proof.

C Sample Worksheet 3: Mean Value Theorem

1. Prove the mean value theorem: Suppose $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) .

Then there exists at least one number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

- Step 1. Consider the line joining $(a, f(a))$ and $(b, f(b))$. If this line is the graph of the function $y = g(x)$, find a formula for $g(x)$.
- Step 2. Define $h(x) = f(x) - g(x)$. Why is h continuous on $[a, b]$? Why is h differentiable on (a, b) ?
- Step 3. Find $h(a)$ and $h(b)$.
- Step 4. Notice that we can apply Rolle's theorem to h by Step 2 and Step 3. What does Rolle's Theorem tell you?
- Step 5. What is $h'(x)$? Conclude that there exists a c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- By putting the above steps together, write a complete mathematical proof.

D Sample Worksheet 4: Fundamental Theorem of Calculus Part 1

1. (The fundamental theorem of calculus, part 1) Prove that if f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative $F'(x)$ is $f(x)$, i.e.,

$$F'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

(Hint: If f is continuous on $[a, b]$, then at some point c in $[a, b]$, $f(c) = \frac{1}{b-a} \int_a^b f(x)dx$.)

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A decorative border consisting of multiple parallel lines forming a rectangular frame. The corners are rounded and feature a complex, overlapping geometric pattern that resembles a stylized knot or a series of interlocking lines.

8

Long-Term Activities

Mastery Problems Promote Polished Proofs in Abstract Algebra

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Abstract

This article describes a mastery-oriented assessment of proof-writing in an abstract algebra course. It is based on the assumptions that all students can learn to write proofs, if given sufficient time, and that demonstrating mastery in proof-writing should be a core outcome of a course in abstract algebra. This method only gives credit for proofs that demonstrate complete mastery, while allowing repeated opportunities for resubmissions.

Difficulty Level: Low; Course Level: Advanced

1 Background and Context

Recently, my department made some changes to its typical course assignments, and for the first time I inherited, eagerly I might add, abstract algebra into my load. In preparing to teach the course, I consulted with a colleague who had taught it several times. He expressed some frustration at the inability of the middle-to-lower students to write completely correct and polished proofs. Thanks to the blessing of partial credit, many students survived the course with passing, if not exemplary, grades via a semester-long collection of only partially correct proofs, along with strong effort and solid performances on the more computational and conceptual assessments. Proof-writing is both a central tool in and key outcome of an undergraduate abstract algebra course, and yet, as we were observing, it often proves difficult for many students. What, we wondered together, might I do differently to help them improve their proof-writing facilities?

I teach at Taylor University, a liberal arts college of about 2000 residential, traditional-aged undergraduates. Because of our size we are not able to offer different abstract algebra courses specialized for different student populations, such as those in mathematics education or those headed to graduate school in mathematics. Rather, we have students with a wide variety of abilities and backgrounds all by necessity in the same course. Our typical abstract algebra course consists of a mix of second-year, third-year, and fourth-year students. Because the course is only offered every other year, only about half of the students will have already had a linear algebra course. Moreover, roughly 20% of the students will be planning to go to graduate school in mathematics or another subject, about 40% will be planning to go into secondary education, and approximately 40% will be planning to go into some other career. The course typically has between 12 and 18 students. It has a prerequisite of Justifications in Mathematics, which is what we call our “bridge course,” and so students going into the course would have been taught the mechanics of proof-writing. The only class for which Abstract Algebra is a prerequisite is Advanced Algebra, which typically only about one-fourth of the students go on to take.

After some contemplation, and reading of various ideas in the literature on teaching and learning, I instituted a requirement that I call “Mastery Proofs.” This was in order to try to meet the specific goal of ensuring that *all* students leave the course having demonstrated mastery of the skill of constructing and writing correct and polished proofs. As is rather obvious from the name of the method, the Mastery Proof assignment takes a mastery approach to proof-writing, through the opportunity of resubmission without penalty. The expectation, and the requirement, is that proofs be “perfect,” or nearly so, in order to receive

any credit. In other words, to encourage students to move beyond only partially correct proofs, I only accept those that are completely correct.

2 Description and Implementation

2.1 Motivation

I first looked for ideas in the MAA *Notes* volume, *Innovations in Teaching Abstract Algebra*. Chapter 1 had several relevant suggestions [3]. One of its co-authors advocated allowing homework rewrites, with a slight penalty for resubmission. Another encouraged copious feedback on homework assignments. Both ideas were compelling, but weren't completely satisfactory in my context for addressing the issue at hand. My weaker students often lack confidence in their proof-writing abilities, hence I thought that the copious comments might be demoralizing. And although I thought that the resubmission idea would be well received by them, I wasn't convinced that only a second submission would suffice.

At about that time, I had been reading *How Learning Works: Seven Research-Based Principles for Smart Teaching*, by Ambrose, et al. [1]. Among other principles, the book discusses mastery learning, which brought to mind my experience in the Michigan Calculus Project with gateway tests, which are mastery tests on differentiation and integration. Research on the method indicates a strong correlation between students' having passed a gateway and having acquired the skills being tested on it [7]. I began to wonder whether proof-writing should be a "gateway" skill in Abstract Algebra, in other words a skill in which all students who pass the class should demonstrate at least some mastery.

The final impetus for the idea was Ken Bain's book, *What the Best College Teachers Do* [2]. In his study he found that the best teachers expect more, even from their less skilled students. The best teachers also use evaluation schemes that stress learning over performance. A standard homework assignment that is evaluated in the typical way with a score and some comments can be seen to put focus on the student's performance on the task. On the other hand, assignments for which resubmission without penalty is allowed can begin to shift the students' focus onto their learning. Moreover, although the standard of perfection might at first glance seem to emphasize performance rather than learning, combined with the process of feedback and resubmission it communicates to students the expectation that they can learn the material.

Putting the pieces together, I decided on the rather simple idea of the Mastery Proofs assignment.

2.2 Details of the Method

Each week I assign one reasonably challenging proof from the exercises in the course textbook [6] for the students to do as their Mastery Proof for that week. Each Mastery Proof has an initial due date within a couple of class periods of being assigned. I grade and return them by the following class period. Each submission is classified as M (for "Mastered") or NM (for "Not Mastered"). Students know that to receive a designation of M , their submission must be perfect, or nearly so. In other words, their submission must be both mathematically correct and well written, with good grammar and style, to receive the desired M . I give very little feedback on papers that receive the NM designation, merely indicating with an underline or circle where things first went awry. I intentionally keep such feedback to a minimum, because I want the students to be responsible for finding and correcting the flaws in their work. (Editors' note: for a similar approach at a less advanced level, with more instructor feedback and fewer student attempts, see Strickland and Rand's article in this volume [8].)

A few times I have given a student an M for a proof that was not quite perfect, but that was very close and on which the student had made several attempts, each showing improvement. This is one reason why I call the assignment "Mastery Proofs" instead of the name that I originally contemplated: "Perfect Proofs."

If a student receives an M on a proof, then no further work is required on it. However, if a student receives an NM , he or she is expected to resubmit an improved version. To make my assessment of the submissions easier, and to remind students of how their thinking has progressed, I require that each submission be turned in stapled to all previously submitted attempts on that proof. Each Mastery Proof is ultimately due at the time of the exam covering that material. Hence the proofs assigned during the first half of the semester are due by midterm time, and those assigned during the second half of the semester are due by the final exam. In theory, each student is allowed to resubmit each Mastery Proof as many times as desired; however, in practice the number of resubmissions is bounded by the time constraint of the final deadline and the time that it takes me to grade and return the submissions, which I always have managed to do by the next class period. Most students recognize that timely resubmissions are potentially to their benefit, and so promptly submit revisions. Only occasionally do students gamble with submissions very close to the final deadline.

Students are allowed to consult their course notes and textbook, but no other printed or online resources, when completing the Mastery Proofs. In addition, I require the Mastery Proofs to be done individually, without any conversation with fellow classmates or any other person. However, I do allow, and encourage, students to come to my office hours to discuss their Mastery Proofs with me. The originality requirements are pretty easy to enforce because we have a small mathematics department with a tight-knit community of mathematics majors and faculty members. Due to our level of familiarity, plagiarism is rare and pretty quickly detectable. However, in larger departments or other contexts where community honor norms may not be as strong, an instructor using the method would need to consider ways of enforcing the requirements. One option could be to use an online submission tool that can detect plagiarism. Another would be to periodically give follow-up quizzes on selected Mastery Proofs. This could help ensure that student work is original and have the added benefit of encouraging retention of the material.

The Mastery Proofs count for 15% of the final course grade, with the score equal to the percentage of assigned Mastery Proofs for which the student eventually earns a designation of M . The only exception to this all-or-nothing approach is that a proof that is eventually mastered but for which a first attempt was not submitted by the initial deadline will only count as three-quarters of a proof when calculating the percentage of mastered proofs.

The Mastery Proofs requirement is not the only homework assignment that I give my students. I use a parcel of assignments consisting of Daily Exercises, Team Homework, and Mastery Proofs. The Daily Exercises are basic conceptual problems, as well as some shorter and easier proofs, taken from the exercises in the textbook [6]. Some are discussed at the beginning of each class period but are not collected. The Team Homework is done in teams of three or four students and is collected once per week. The problems for the Team Homework assignments are more challenging than the Daily Exercises. Like the Mastery Proofs, the Team Homework exercises are expected to be written well, and so are graded for both correctness and quality of presentation. The Team Homework exercises are often, but not exclusively, proofs.

The Daily Exercises, Team Homework, and Mastery Proofs provide students with both proofs and computations, individual work and team work, and problems of varying difficulty. The Daily Exercises are meant to help students build the basic skills and understanding that are foundational for the more difficult problems and the proofs. The Team Homework allows students to work together to extend their abilities and deepen their understanding, and it also gives students the opportunity to develop and write proofs in a collaborative environment. The Mastery Proofs then hold students accountable for being able to write proofs of their own.

The Mastery Proofs require some extra grading and bookkeeping on my part. Because I grade the Mastery Proofs and the Team Homework, I do not attempt to collect and grade the Daily Exercises. Regrettably, I think that this diminishes the time and attention that some students pay to working on them.

The Mastery Proofs also increase office hours traffic, as many students want to discuss their not-yet-mastered proofs, although I count this as a good thing. When working with a student in office hours, I try to get the student to do as much of the thinking as possible. This is particularly important when helping students with the Mastery Proofs, because I want to provide useful help without simply telling them where

they went wrong and how to fix it. I do this primarily by encouraging them to explain their thinking, asking questions that probe for understanding, summarizing what I've heard, correcting their vocabulary as needed, and praising them when appropriate. This is explained in more detail in DeLong and Winter [4, Ch. 8].

3 Outcomes

3.1 Student Performance

I have used the Mastery Proofs approach the last three times that I taught Abstract Algebra. In the three sections, I had a total of 36 students. Of them, 25 got credit for mastering all of the assigned Mastery Proofs. Of the remaining 11, eight of them got credit for either all but one or all but two of the Mastery Proofs. There were two students who mastered fewer, but they mastered at least half of the assigned proofs. The remaining student did not master any, but she withdrew from the course at mid-semester due to health issues.

Course grades over the semesters were roughly 40% A, 40% B, and 20% C. Although this distribution may seem high, it does not indicate a lack of rigor. The course typically covers the standard chapters on groups and rings in our textbook (Chs. 0–19) [6]. Our university is classified as “more selective” by *US News*, so we are working with strong students. The course distribution is skewed slightly higher than would be typical for a proofs-based course in our major. This was obviously due in part by having so many students get full credit on 15% of the course grade. But the correlation between completion and grade is interesting. Almost every C student mastered all or almost all of the proofs. Hence, even those students are leaving the course knowing how to write perfect proofs, albeit with limited instructor feedback and multiple attempts. In other words, this method has met the goal of ensuring that *all* students show eventual mastery in writing correct proofs through a revise and resubmit process.

3.2 Student Feedback

At the end of the three semesters in which I used the Mastery Proof method, I included a true-false question on a supplementary online course evaluation asking the students whether or not, in their opinion, they had learned to write proofs in abstract algebra better because of the Mastery Proof requirement. Of the students who responded, 87.5% of them agreed that the requirement improved their proof-writing skills. Hence on this rough measure, a high majority of the students saw the benefit of the method.

On the supplementary evaluation, I also included a free-response question in which I asked the students to comment on the Mastery Proof requirement. The large majority of the comments were positive. Many students commented on the fact that the assignment built their proof-writing skills and confidence. Some specific reasons noted were that the requirement forced their independence, encouraged thoroughness and precision, and helped them deepen their understanding of the main ideas. Two people specifically commented that they thought that the Mastery Proofs were the most beneficial aspect of the course. The few negative comments generally revolved around the concern that the assignment was difficult and stressful.

3.3 Strengths of the Method

Students do write perfect proofs, and several of them. Even those students who are weaker and less confident in their proof-writing abilities are able to complete most of the assigned problems perfectly, thereby increasing both their facility in proof-writing and their confidence. Moreover, by focusing on, and expecting, mastery of proof-writing, the Mastery Proof method encourages students to adopt a learning-goal orientation. That is, students are increasing their competency in, overcoming difficulties in, and mastering a given task. Students with a learning-goal orientation have been shown to seek out more challenging tasks and give persistent effort when faced with failure [5].

The method also often improves the presentation of student work, because it gives students a natural incentive to typeset their submissions. Because resubmissions often involve modification of partly correct submissions, students quickly realize that editing an electronic file is much more efficient than rewriting a handwritten submission.

The method also makes grading quicker and simpler in the following sense: the grading of Mastery Proofs is almost completely objective. It is true that the line between an M and an NM is subjective, but that is the only such decision that needs to be made. No partial credit needs to be debated, either in my own mind when marking papers or by a student's advocating for a higher score. Nor do extensive comments need to be given. In addition, another strength of the method is that it does not use any class time, other than for collecting and returning papers.

3.4 Potential Shortcomings of the Method

There is a danger that the Mastery Proofs may not provide sufficient challenge to the best students. Although the Mastery Proofs are decidedly non-trivial, in order to expect that everyone in the course be able to master *every* proof, they have to be reasonable in difficulty. Hence, each semester there were some students who could, and did, receive an M designation on every, or almost every, Mastery Proof on the first attempt. This was addressed, in part, by the fact that the Team Homework assignments had problems that were often more challenging than the Mastery Proofs.

On the other hand, because students can resubmit (almost) as often as they desired, there is little extrinsic incentive for them to make really strong first attempts. That is, a student can game the system by turning in something incomplete or perfunctory for an initial submission, counting on the availability of resubmissions. In spite of this possibility, I believe that it is important to give full credit for all proofs that are mastered by the final deadline, regardless of the number of attempts, in order to emphasize that the goal is mastery and not speed. In practice, I have not found that students try to take advantage of the system in this way.

4 Extending the Method

The Mastery Proof method is clearly best suited to smaller class sizes, because of the grading and book-keeping load. For larger classes, an instructor may need to find ways to modify the method to make it more manageable. If teaching assistants are unavailable to help, technology might be used to simplify some of the exchange of papers and bookkeeping. If an instructor felt the class size precluded collecting one Mastery Proof every week, the method could be modified so that a new Mastery Proof was assigned every other week instead. Requiring students to write six "perfect" proofs in a semester is surely better than requiring them to write none!

Many instructors who allow resubmissions of tests or assignments use a decreasing scale, whereby the maximum score given on the assignment is a decreasing function of the number of attempts. In other words, an instructor may wish to modify the method so that the earlier mastery is rewarded more than later mastery.

When I assigned a Mastery Proof in Abstract Algebra, I gave the whole class the same proof to work on that week. However, an instructor may find it beneficial to assign different problems to different students. If the class were small enough, students or groups of students could be assigned different proofs. For example, proofs could be assigned according to current course performance. If different proofs were assigned, the instructor could set aside class time to allow students who had mastered their proofs to share their submissions, for example on an overhead projector, with the class. Thus, students could benefit from seeing more proofs than just the ones that they were assigned.

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From Formal to Expository: Using the Proof-Checking Word Processor *Lurch* to Teach Proof-Writing

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Abstract

Lurch is a piece of mathematical software that can help students learn to write mathematical proofs, by providing frequent and immediate feedback to each step in their work. Herein we discuss how that software was applied in an introduction-to-proofs course, and how it helps students make the transition from procedural to proof-oriented mathematics. We also mention resources available for easing the integration of *Lurch* into other instructors' courses.

Difficulty Level: High; Course Level: Transitional
Technology Based

1 Background and Content

Lurch is a free word processor that can check the mathematical reasoning in a user's document.¹ Although it would be easy to make a word processor that could check simple arithmetic, or even algebra, *Lurch* aims to do much more than that. It can check the steps of a mathematical proof, even a proof not written in a formal style. The current version (0.8, released April 3, 2014) works best for introduction-to-proof courses, covering topics such as logic, introductory set theory, algebra, and number theory. We frequently add new features, including support for additional mathematics topics.

Lurch has been tested in two semester-long courses, one taught by each of the authors. The first author tested an early version of the software in a formal logic course at Bentley University, a business university of 4,000 undergraduates in Massachusetts with a strong focus on the liberal arts. Bentley's mathematics majors go into careers applying mathematics in the business world, often actuarial science. Bentley students all have institution-issued Windows laptops. The course was an elective for students in the Bentley honors program, and thus was populated with freshmen through seniors.² We will call this "Course B."

The second author tested a recent version in an introduction-to-proofs course at the University of Scranton, a Jesuit, liberal-arts university in Pennsylvania of about the same size as Bentley. Scranton's mathematics majors frequently go into mathematics education, graduate school in pure mathematics, or jobs in industry in computing or finance. They are not required to own laptops. The course is a bridge course required for mathematics and mathematics education majors, most at the end of their sophomore year. This chapter concentrates more on this course, because it is closer to the main target audience and it used a far newer (and therefore more sophisticated) release of *Lurch*. We will call it "Course S."

The following section overviews the software, as well as the philosophy and pedagogy we associate with it. Outcomes of testing and possible future directions of work appear in the remainder of the chapter.

¹*Lurch* was developed by the authors, with support in part from the National Science Foundation's Division of Undergraduate Education, in the Course, Curriculum, and Laboratory Improvement program (grant #0736644). Views expressed herein are not necessarily those of the National Science Foundation.

²A third course using *Lurch* is ongoing at the time of this writing, and thus no survey data are yet available. But the instructor's experience is consistent with the ideas herein.

2 Description and Implementation

2.1 Project Goals

The *Lurch* project primarily targets students in their first proof-based courses, giving frequent, immediate, and clear feedback on the steps in their work. Existing research on educational technology suggests that our goals make sense and are achievable: investigations into automated assessment systems [7, 12] show the value of computers' giving high-frequency, individualized, and immediate feedback [10, 13, 14, 15]. The value of such feedback is documented in reviews of educational research [6], which validates our common-sense assumption that more feedback, in immediate response to the student's actions, is better for learning.

Lurch is a free, mathematical word processor that works on any platform and has a significantly lower learning curve for typing mathematics than \LaTeX . Because *Lurch* documents are word-processed, they are more legible than typical written assignments; neat, organized homework is easier to grade [8]. We find this to be even more true of work done in *Lurch*, because the validity of each individual step in many cases has been verified.

2.2 Proofs in *Lurch*

Unlike other mathematical word processors that merely typeset mathematics, *Lurch* gives mathematical expressions actual semantics. That is, the software does not just see the symbols the user types, but knows enough of their meanings to check the user's work. The screenshot of our application in Figure 1 shows red, yellow, and green icons interspersed throughout the document, which give the user feedback on the correctness of each step of his or her work. (The feedback can be disabled to obtain a clean view or printout.) Green thumbs-up icons follow a correct step of work that is correctly justified, red thumbs-down icons follow a step of work that contains an error or is supported by the wrong reason, and yellow lights follow

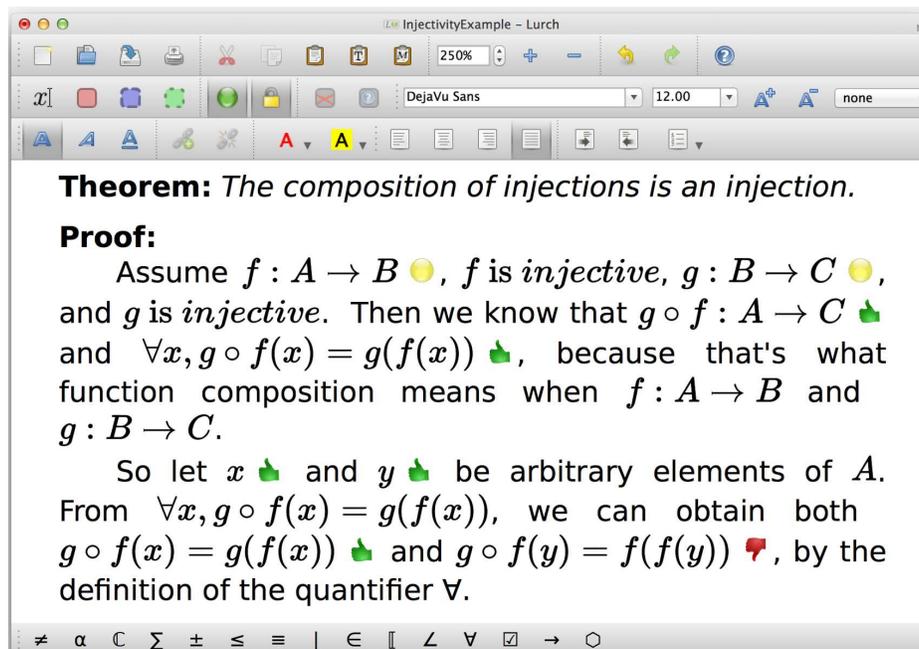


Figure 1: A *Lurch* screenshot explained in Section 2.2. Although it was captured on a Mac, *Lurch* is free and open-source for Windows and Unix as well. By default, *Lurch* uses colored traffic-light icons rather than the thumbs shown here; the thumbs are an option useful for colorblind users and in black-and-white printing.

mathematical expressions that are used as premises to justify later steps of work, but are not themselves justified. Yellow lights therefore indicate the hypotheses in the proof.

Lurch knows the meanings of these mathematical expressions, in the sense that *Lurch* is aware of the rules defined elsewhere in the same document (or documents on which the document depends) that govern the use of each symbol in mathematical expressions. It allows the user to use the rules to draw new conclusions from existing mathematical expressions, verifying that the steps that he or she took to do so follow the available rules. The rules can be anything from low-level rules of logic (such as modus ponens), so users can create new logical systems from scratch in *Lurch*, up to high-level mathematical axioms (such as definitions of algebraic structures).

2.3 A Minimal Learning Curve for All Users

The user interface through which users tell *Lurch* the meaning of the mathematics in their documents must be as simple as possible so that users are not distracted from the mathematics. To this end, we employ a user interface paradigm we call *bubbles*.

In order for a section of text to be treated as a mathematical expression, the user must mark it as such with a single click of a toolbar button shown in Figure 2 (or the corresponding keyboard shortcut). The user selects a portion of text that he or she wishes *Lurch* to treat as “meaningful mathematics” (equations, definitions, proof statements, etc. as opposed to labels, citations, and commentary)³, and then clicks the “Meaningful expression” button on the toolbar. *Lurch* then wraps the text in a bubble and reports the understood meaning in a tag atop the bubble, as in Figure 3.

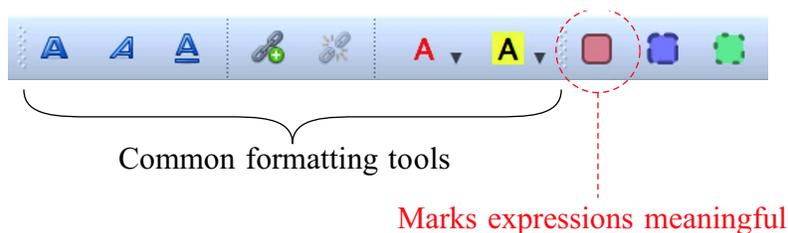


Figure 2: One of the *Lurch* toolbars, shown here on Mac OS X. Some of the buttons are the typical formatting buttons one expects to see in any word processor, such as bold, italics, and underline. The most important new one to note here is the red button for marking text as meaningful.

Lurch draws a bubble around a mathematical expression if and only if the user’s cursor is inside it, so when the cursor is not in a mathematical expression, there is no extra visual clutter. But when the cursor is in one, the interface makes this fact clear. Right-clicking a bubble exposes a menu of actions relevant to the mathematical expression in the bubble. The interface for editing mathematics is similar to that of word processors like *Word* and *LyX*. It also brings a pedagogical benefit. Student users, just learning what a proof is, are forced to indicate which parts of their own proofs are meaningful and which are not. *Lurch* gives feedback on only those portions of the document marked as meaningful.

2.4 Course Integration

Course S met twice per week for roughly two hours per session over the fifteen-week semester. (It is a four-credit course.) All students had personal laptops and agreed to bring them to class, which was held in a technologically enhanced classroom with large tables where the students could work. (If the students had not had their own laptops, a computer lab could have served as the classroom.)

³The user uses bubbles to cite reasons and premises as well. This chapter does not describe every aspect of the user interface, focusing instead on the impact of the software on student learning. For more information, see Carter and Monks [3, 4].

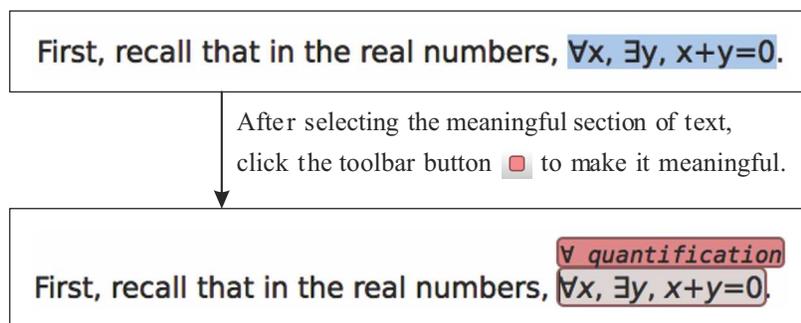


Figure 3: How a sentence within a *Lurch* document changes when a user applies the button shown in Figure 2 to a section of text. In the top of the figure, the user has selected a section of text. After telling *Lurch* to mark it as meaningful, the red bubble shown on the bottom of the figure appears, indicating that *Lurch* has read the expression, and knows that it is a universal quantification.

The course satisfies a University of Scranton general-education requirement as a writing-intensive course. The goal is to train students to read and write mathematical proofs, covering elementary topics in logic, set theory, functions, algebra, inequalities, relations, number theory, and combinatorics. Elementary definitions and rules for the topics are provided with *Lurch*, and the students did not have to do anything beyond installing the software to have them available.

The first few classes of the course were devoted to introducing the students to the nature of mathematical proof primarily via the *Toy Proofs* software discussed in Section 2.6. At that point *Lurch* was used both in lecture and for assignments for approximately the next 9–10 weeks, with the remainder of the course assignments and lectures done using \LaTeX .

In class, the instructor and students typically would start with a blank *Lurch* document and type and validate proofs together. The instructor used a computer connected to a projector and the students followed along on laptops, also participating in the construction of the proof interactively. This had the distinct pedagogical advantage over the blackboard, pencil, and paper of allowing proofs to be constructed “from the outside in”: starting with just the premises and conclusion as an initial outline of the proof, the user can then insert lines to justify the conclusion, then insert other lines to justify those lines, and so on, so that the proof grows from the ends towards the middle. A medium like the chalkboard does not as easily allow for inserting lines, and therefore tends to force writing a proof from top to bottom.

The second advantage to using *Lurch* to take notes in class in this manner is that the students quickly became familiar with the software. To learn the interface, students were asked to take a ten-page tutorial that comes built into *Lurch*, and were led through their first few *Lurch* proofs by their instructor in class. We estimate the total course time spent teaching *Lurch* exclusively to have been about one hour for the entire semester. The first in-class exam in Course S was given entirely in *Lurch*, as was the in-class portion of the final exam. The results of the exams are discussed in Section 3.

The students were assigned to construct, type, and submit approximately eight to twelve proofs per week as graded homework. They were required to type their proofs in *Lurch*, *LyX*, or \LaTeX . They were not required to use the bubbling interface in *Lurch* to give their proofs meaning (to *Lurch*), nor to have *Lurch* validate the steps of their work. Nevertheless, all the students chose to use *Lurch* voluntarily, and most chose to have *Lurch* validate their proofs, especially in the beginning of the course while their concept of mathematical proof was still in the formative stage. Rather than finding *Lurch* to be an additional obstacle to writing their proofs, the students found it quite easy to use.

Having *Lurch* available, especially in the early weeks of the semester, to give students feedback on their proofs outside of class, while they were constructing them, made it possible to teach proofs effectively, using a transitional approach described in the following section.

2.5 What is a Proof?

Any introduction-to-proof course faces this obvious question. It must be answered succinctly at first, to foreshadow what is to come, and in more detail as the course unfolds. The succinct initial answer given in Course S was that a *proof* is an explanation of why a particular mathematical claim must be correct. This motivates two key goals of a mathematical proof: *objectivity* (why the claim “must be correct”) and *exposition* (a proof is an “explanation”).

These two goals are not always easy to achieve simultaneously. Proofs that include every detail in a rigid syntax (such as those in a formal system) are easy to verify mechanically, in principle even by readers with little mathematical sophistication. They provide objectivity. We will call these “formal proofs” in this document.

The proofs typically found in books and journals are usually more effective at imparting understanding to the reader. They are tailored to a particular target audience, which affects such things as the language in which they’re written and the quantity of detail provided. Skipping details that the target audience already knows or believes can better emphasize the main ideas. We will call these “informal proofs” or “expository proofs” in this document.

2.6 The Students’ Journey: From Formal Proof to Semiformal Proof to Expository Proof

A Gentle Introduction

We can begin to build the bridge from what students know coming into the course (mainly calculations from algebra and calculus) to proofs by illustrating that a calculation satisfies our criteria for a proof if we can justify each step of the work with a reason that explains why the step is correct, as in Figure 4.

$$\begin{aligned} a \cdot b + a &= a \cdot b + a \cdot 1 && \text{by multiplicative identity} \\ &= a \cdot (b + 1) && \text{by distributivity} \end{aligned}$$

Figure 4: A calculation a calculus or algebra student might do as part of a larger problem becomes a “formal” calculation if we provide a justification for each step.

Figure 4 assumes students have not only heard of the multiplicative identity and distributive laws, but also has a hint of what it means for a cited rule to justify a step of work! To prepare them for this, the course introduced students to some of the basic features of a formal mathematical reasoning system through some fun, non-mathematical systems.

The *Toy Proofs* software, developed and maintained as part of the *Lurch* project, implements these game-like systems and is available online [5]. Three example systems in the *Toy Proofs* software emphasize that the purely mechanical nature of the systems provides the objectivity we want in proofs. The systems are so mechanical that the “statements” in the *Toy Proofs* games (e.g., sequences of colors or shapes) have no obvious meaning! Yet the computer can verify the “proofs” (the players’ sequence of moves) anyway.

From Toy Proofs to Formal Proofs

After a few assignments on the *Toy Proofs* systems, the students were then presented with a formal system for propositional logic. Simultaneously, they graduated from the style of validation in *Toy Proofs* (where the software prevents the user from making a mistake) to the style used in *Lurch*. Since *Lurch* is a word processor, a student can type anything he or she likes in a document, including incorrect statements, unjustified statements, and incorrectly justified statements. *Lurch* merely informs the user which statements are correctly justified and which are not; it does not prevent the user from making errors.

The logical system used is a variation of Gentzen's natural deduction [9, pp. 395–515]. A formal proof in this system consists of a list of lines containing (a) a line number used as the label for the line, (b) a statement (at a given level of indentation), (c) a reason justifying that statement when necessary, and (d) the line numbers of zero or more premises. An example of such a proof is shown in Figure 5.

Theorem: $P \text{ and } Q \Rightarrow P \text{ or } Q$

Proof:

- | | | | |
|----|--|---|--------------------------|
| 1. | Assume $P \text{ and } Q$ | - | |
| 2. | Q | | by and-; 1 |
| 3. | $P \text{ or } Q$ | | by or+; 2 |
| 4. | $P \text{ and } Q \Rightarrow P \text{ or } Q$ | | by \Rightarrow +; 1, 3 |

Figure 5: An example of a formal proof in propositional logic validated by *Lurch*. It uses the style of formal proof used in Course S, but *Lurch* is not restricted to this or any other particular style. (See, for example, Figures 1 and 6.)

After propositional logic, the students learn predicate logic and rules for equality, in the same formal style. At this point, the overwhelming level of detail required in a formal proof starts to become tedious, and the students are ready to advance to the next level of sophistication.

The Journey Continues

This pattern permeates the course. Once the students master a concept to the point where it becomes tedious or repetitive, the tedious parts are eliminated in favor of a more advanced proof style. Such parts are eliminated by two mechanisms, both in *Lurch* and on the chalkboard: shortcuts and smart rules.

Shortcuts are well-defined ways to omit portions of a proof without losing the objectivity of the proof's correctness. Students accumulate many shortcuts throughout the semester as the instructor deems appropriate. For example, it is nearly always the case that the statement at line n in a proof is used as a premise on line $n + 1$, so one shortcut is that premises don't have to be cited if they immediately precede the line being justified. Another basic shortcut is that most lines in a proof do not need to be numbered or labeled, excepting those that need to be referred to frequently in the rest of the proof.

A *smart rule* is a high-level rule of inference that can be used to justify a statement using a more sophisticated algorithm than simple pattern-matching. For example, once students have mastered propositional logic and moved on to other mathematical topics like set theory or number theory, they can use a smart rule called "logic" to justify a single step of work that could otherwise be justified by a whole proof in propositional logic. In *Lurch*, this smart rule simply checks, for premises P_1, P_2, \dots, P_k and conclusion C , if the statement " $P_1 \text{ and } P_2 \text{ and } \dots \text{ and } P_k \Rightarrow C$ " is a tautology.

As shortcuts and smart rules are presented in the course, the style of proof slowly transitions from the fully formal proofs that provide objectivity and rigor, to expository proofs common in mathematics, that communicate ideas more clearly. This transition passes through an intermediate style of proof that we call *semiformal proof*, illustrated in Figure 6.

The main restrictions that define a semiformal proof in the course are that every statement must be on a separate line and subproofs must be indented. It should be easy to convert a semiformal proof into a formal one, while on the other hand, if you combine the individual lines into a few paragraphs, a semiformal proof becomes very like a traditional, expository proof. Thus, semiformal proofs serve as an intermediate style of proof that bridges the gap between formal proofs and expository proofs.

<p>Theorem: If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.</p> <p>Proof: Let x be arbitrary. Assume $x \in A$. $A \subseteq B$ is given. So $x \in B$ by the subset. But $B \subseteq C$ is also given. So $x \in C$ by definition of subset as well. Thus, $A \subseteq C$ by the definition of subset (since any x such that $x \in A$ is also in C). ■</p>
--

Figure 6: An example of a semiformal proof in elementary set theory validated by *Lurch*. Semiformal proofs still use only one statement per line, but permit omitting line numbers, not specifying immediately preceding premises, and more natural spacing and punctuation, making them a natural waypoint on the road from formal proofs to expository proofs.

Journey's End

The transition to informal proofs begins with a review of some common typesetting and writing conventions in mathematics. Due to the shortcuts and smart rules mentioned earlier, *Lurch* can handle a small subset of informal proofs; one example (with an intentional error) appears in Figure 1. But mathematicians use many shortcuts that *Lurch* does not yet support, and so as the students made the transition to informal proofs, the instructor required them to switch to writing their assignments in \LaTeX .

This produced high-quality typesetting for the students, but \LaTeX cannot validate their proofs for them. Thus we were unexpectedly pleased to notice that some students, on their own initiative, wrote proofs in *Lurch* (using existing shortcuts and smart rules) prior to writing a more succinct version in \LaTeX for submission. This was excellent confirmation that students valued *Lurch* for exactly the purpose we intended. The fact that *Lurch* can validate all three styles of proof makes it possible for students to answer what is arguably the most common question they ask about proofs: “How do I know if I am right?”⁴

Lurch is still limited, in that it cannot understand all types of proof. Some proofs the students did near the end of the semester were beyond *Lurch*'s capabilities, and so they had an opportunity to do proofs without the help of its constant feedback. Yet several students continued to use *Lurch* at that point and in a future course by their own choice.

3 Outcomes

At the conclusion of Course S, the students were surveyed about the use of *Lurch* in the course, and whether they feel that the software achieves our goals. The survey was anonymous and not given to the instructor until after the course grades were already submitted and the course was over. Of eleven students in the course, ten responded to the survey.

Free-form comments on the survey cited the value of frequent, immediate feedback, and that the software helped boost the confidence of some intimidated students. They also mentioned that it helped them learn proof structure, and the mathematical rules and definitions whose correct use the software checked. This was confirmed by the students' ability to correctly respond to questions that were asked in class by the instructor.

⁴A natural concern is that having *Lurch* tell students when their work is right might hinder them from learning to manage self-confidence and self-doubt. The authors' experience with students and *Lurch* revealed no such problems. Students who learned proofs using *Lurch* transitioned well to proofs without *Lurch*.

The first exam in Course S was given in class entirely in *Lurch*. The students brought laptops to class and were given a *Lurch* file containing the exam to fill out. One question on the exam provided a 17-line, formal proof of one of De Morgan's Laws for quantifiers, and asked the students to provide reasons for each statement. This question was designed to test understanding of proof concepts without the difficulties of crafting a proof from scratch. If the correct reasons and premises were cited for each line, the students could ask *Lurch* to validate their proof. Given the enormous number of possible reason and premise combinations available it is practically impossible to succeed on this question by random guessing. Incredibly, *every student in the class answered this question correctly.*

This important understanding of reasons in a proof, even for students who struggle with constructing their own proofs from scratch, seems to have been gained through the use of *Lurch* in the initial assignments leading up to the exam. While no control data are available from the same semester, the instructor's observation is that this was a vast improvement compared with two classes in previous semesters that did not use *Lurch* but were given quite similar assignments and exams.

The final exam in Course S was given in two parts. The first part, a take-home portion, asked students to write advanced, informal proofs using \LaTeX . Even though some steps and certain aspects of the notation in those proofs were beyond *Lurch*'s ability, several students worked on semiformal drafts of their proofs in *Lurch* before writing them up in expository form in \LaTeX .

The second portion of the final exam was taken in class and consisted of proofs that had to be done in *Lurch* on their personal laptops, but whether or not the students wanted to have *Lurch* validate the proofs was left up to them. Every student but one chose to add bubbles to their proofs and have *Lurch* validate them before handing them in.

Thus students did not seem to perceive the extra effort and time required for bubbling and validating their proofs as a hindrance, even on a timed, in-class portion of the final exam. In fact, even students who did not ultimately receive an A grade in the course outscored an A student on the portion of the final exam where the A student (perhaps overconfidently) chose not to use *Lurch* for validation.

As mentioned in Section 2.3, student users, just learning what a proof is, are forced to indicate which parts of their own proofs are meaningful and which are not. Experience from testing shows that this is perhaps the clearest lesson students learn from *Lurch*, compared to our experience in courses without *Lurch*. Students come away from a *Lurch*-integrated course knowing very solidly how a proof is structured, because they were forced to point out to *Lurch*, before it would check their work, which portions of text are meaningful mathematics, and which cite reasons or premises. In fact, it was not until after this learning had fully sunk in that students perceived the bubbling process as a burden.

Students in both Courses B and S reported moderate to strong agreement with the following three statements that even our current version (which is missing some major planned features) is meeting our goals. The first two suggest that students heeded *Lurch*'s feedback, and worked harder to create valid steps of reasoning after initial errors. Furthermore, they saw this as a valuable feature of the software.

I spent a lot more time on homework because *Lurch* showed me my mistakes.
The constant feedback *Lurch* provides about my work is valuable.
Lurch contributed to how much I learned in this course.

Even the current incomplete version was, according to the students, not an extra burden to master, as they reported disagreement with the following two statements.

This early version of *Lurch* is not yet beneficial to students.
Doing homework in *Lurch* is more difficult than doing it on paper.

While repeated guesswork alone could not possibly create most proofs in *Lurch*, a natural concern is whether, with enough feedback, students lose the need to be creative, and can just experiment until a proof

is done, having learned nothing. Analogous concerns have long existed about the extent to which calculators are used in mathematics classes. Thus our surveys also asked for agreement or disagreement with the sentence “It is possible to do a proof in *Lurch* by experimental clicking and typing, without thinking.” In the early version of *Lurch* used in Course B (which was not a word processor and did a lot more work on the students’ behalf) the average response was neutral. The more recent version of *Lurch* used in Course S has significantly improved this score to moderate-to-strong disagreement.

4 Extending the Method

4.1 Customizing *Lurch* for Other Approaches and Courses

This chapter describes one particular method and style of teaching students mathematical proof. *Lurch* can be customized beyond just the rules and definitions that ship with the program, to match the lecture notes or textbook used in any particular course. On the other hand, the method we have described can be applied without any significant changes to any class size if an instructor so desires, and with advances in technology, the ability of *Lurch* to give students feedback on their proofs will only improve with time.

... A function can also be *increasing* 🍀. Such functions satisfy the following property:

Definition of increasing: If f is *increasing* and $s < t$ then $f(s) < f(t)$. 🍀

We might use this rule in a proof as follows.

Example:

... Since we were given that g is *increasing* 🍀, and we know from arithmetic that $1 < 2$ 🍀, we can conclude that $g(1) < g(2)$ 🍀 by the definition of increasing. Thus ...

Figure 7: In this sample *Lurch* document, an instructor specifies a new definition and illustrates how it can be used. *Lurch* validates that the symbol *increasing* was not previously defined, certifies that it understands the instructor’s new rule, and validates the statement $g(1) < g(2)$ (which has been justified by citing the name of the new definition as a reason, and providing the two premises required by the rule).

All the rules and definitions that are available to students when doing assignments in *Lurch* are written in *Lurch* documents directly. Users who want to customize the mathematics in *Lurch* can inspect *Lurch*’s built-in, foundational documents, copy them, and alter them to suit their own style, or write new ones from scratch. A document containing rule definitions can be referenced by another document as a dependency, making the rules in the former available in the latter.

For example, Figure 7 shows a snippet of lecture notes that an instructor might type into a *Lurch* document to define a new rule. It first declares a new term, “increasing,” which *Lurch* verifies is not already defined. The instructor then writes the definition in the form and with the level of detail desired. The instructor then uses bubbles to communicate to *Lurch* the rule’s structure, according to the specification in [4].

For a student to use that rule to justify a statement, he or she must supply the two required premises and cite the rule by name (“increasing” in this case), as on the bottom of Figure 7. *Lurch* checks that for the appropriate values of f , s , and t in the rule definition, the required premises are available to justify the desired conclusion.

4.2 Improving the Software

The main challenge with using a new piece of software is getting both the instructor and students over the initial learning curve in the shortest amount of time. Learning *Lurch* itself ought to require the smallest possible percentage of the course, leaving nearly 100% to spend on the mathematical course content. We confront this challenge head-on in several important ways.

Each feature a piece of software provides and each control it presents on screen contributes to the learning curve. Thus the fewer such commands and controls (and the simpler each one is), the better. Our current user interface does well by this metric, but we have more advances planned for future implementation.

The simple bubble interface described in Section 2.2 could be even simpler. For example, *Lurch* might automatically find text to be placed in bubbles by detecting the names of currently defined reasons, or a valid mathematical expression surrounding the cursor. An enhanced graphical interface could draw arrows from the bubble containing the cursor to each meaningful expression it uses as a premise, so that users can immediately see the premise-conclusion relationships they have set up.

Customizable notation and more sophisticated shortcuts and smart rules will allow *Lurch* to be more useful for teaching mathematics beyond an introduction-to-proof course. Textbooks, rule libraries, help documents, and other instructional materials are being developed to make *Lurch* easy to integrate into courses at more institutions; one is already available [11]. These are just a few examples of how we will continue to reduce the overhead necessary to using *Lurch* in a mathematics course.

Interested readers are encouraged to visit the *Lurch* website [1] to download the free software, find instructional materials, introductory videos, future development plans, the project's email list, and more. More details about the *Lurch* project have been published elsewhere [2, 3].

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Growing Proof-Writing Skills Throughout the Undergraduate (Majors') Curriculum

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Abstract

Successfully shepherding a significant number of mathematics majors to graduation is a challenge at a university in the middle of the college-skills demographic. Monmouth University does this with a carefully structured curriculum that starts with a transition-to-proof course in the freshman year. Subsequent courses are then organized to make use of, and extend, skills developed in earlier courses.

Difficulty Level: High; Course Level: Transitional, Advanced

1 Background and Context

Monmouth University is a medium-sized (roughly 4500 undergraduates, 1500 graduate students) private comprehensive university, offering master's degrees mostly in applied fields and one doctoral program in nursing education. We have approximately 125 mathematics majors distributed over the four years, most of whom come to us planning to teach at the secondary level. Our mathematics majors range from graduate-school-bound to very weak. We have had majors graduate with a mathematics SAT score of 480 (though certainly not our best students). We want to have as many majors as possible since it is a good preparation for almost any career; however, we find it a challenge to bring the lower half of our incoming students to a level appropriate to graduate as mathematics majors.

Many of our majors come to us with no prior experience in mathematical reasoning or proof. (Numerous local high schools over-reacted to the NCTM's recommendation to de-emphasize two-column proofs, and removed proofs entirely from their geometry curriculum.) Our current curriculum structure was developed from the top down. A long-time faculty member, Tom Smith, was teaching Real Analysis, and found that he simply couldn't complete the material he wanted to cover given the proof skills of the students at the beginning of the course. He proposed that we completely revise our Discrete Mathematics course, which was taken by mathematics, computer science, and software engineering majors, replacing it with two different courses. One, for computer scientists and software engineers, would continue to be called Discrete Mathematics. The other, Introduction to Mathematical Reasoning (hereafter referred to as IMR), would start our freshman mathematics majors on the right path to success as seniors in Real Analysis. As we started teaching IMR in a more uniform fashion, we realized that, to bring the students to where we wanted them to be, we needed to build, in their sophomore and junior years, on what they had learned in IMR. Now that we do this, by the time they are in Real Analysis they are really ready for it.

This sequence of courses — Introduction to Mathematical Reasoning in the freshman year, Linear Algebra in the sophomore year, Number Theory and Modern Algebra in the junior year, and Real Analysis in the senior year — forms the core of the theoretical part of the major. To complement the proof courses, all majors take the standard calculus with differential equations and statistics sequences. Many students take one or two additional proof courses as electives. Most take two mathematics courses per semester to complete their degree (a total of 42 credits in mathematics).

I will describe the flow of the courses: what we do in the introductory course to get them started and how we continue that growth in the later courses. I will discuss how we have adjusted the courses — the content, the sequencing, the pedagogy — over the years to ensure that they build successfully on each other.

2 Description and Implementation

2.1 Introduction to Mathematical Reasoning

At Monmouth, unlike at many schools, students are strongly encouraged to choose a major as they enter, even if they may end up changing majors partway through their studies. We encourage students to take IMR as soon as possible, normally sometime in their first year, along with calculus or even precalculus. Its only prerequisite is placing at or above the precalculus level on our placement examination. I tell the students that the course is “math boot camp.” Just as, in the army, one enters boot camp as a civilian, but after finishing it one is a soldier, students enter IMR as civilians, but when they finish, they are (beginning) mathematicians. They start learning to use the language of mathematics and the ways in which we reason in mathematics. As with boot camp, there is a certain amount of indoctrination built in. Of course, we do this in a less authoritarian manner, always open to question or discussion. Nonetheless, through repetition, an attitude is being taught. When they start the course, most of them implicitly believe “the example proves the rule.” That is, if they check an assertion on one small numerical example and it works, the assertion must always be true. To counter this, on the first day of class they consider statements such as

- “if x is a real number, then $x^2 \geq x$ ” (which most of them are sure is true, since the examples they naturally try don’t include numbers between 0 and 1)
- “all non-negative numbers of the form $n^2 + n + 11$, where n is an integer, are prime.” (They don’t have to look very far to find a counterexample, but most just try $n = 1$, and maybe 2 and 3, all of which do yield primes.)

These and similar statements are intended to “disequilibrate” them, make them rethink their assumptions.

IMR classes are kept small — a maximum of 20 students. This allows the instructor to check in with students individually at least a few times per period as they work on developing proofs. Most days, about one-third of class time is spent with students working (individually or in pairs) on proofs, while the instructor circulates helping those who need it. The rest of the time is spent in interactive lecture. The content of the course is quite similar to a standard introduction-to-proof or discrete mathematics course. We start with logic, first propositional then predicate. Next we introduce elementary number theory ($a|b$, an introduction to congruence) to give them an opportunity to try extremely simple real proofs. We introduce induction toward the end of this followed by chapters on set theory and functions. There is also a bit of combinatorics and graph theory both for practice with induction and to give students a bit of relief from the heavy dose of proof.

Where our course differs is the level of the students, and thus the amount of attention we must pay to mathematical language. To teach such a course to first-semester freshmen, many of whom have never written a proof, we cannot use a standard text aimed at second-semester sophomores or juniors. Our students are hearing mathematical language used properly for the first time, and need to learn it. We introduce truth tables very briefly, just to explain the difference between several connectives in mathematics and in common language. “Or”, for example, is always inclusive in mathematics, but usually exclusive in English. “Implies” is different from “if and only if”, whereas in casual conversation that distinction is usually not made. Then we introduce a somewhat redundant system of axioms that are chosen to correspond to the way we reason mathematically. (See the Appendix for the full list of axioms.) This differs from a standard logic course, where axioms are chosen to be as independent as possible. Having some redundancy teaches students very early that there is often more than one correct proof for a theorem.

The axiom schemas are pretty standard properties: commutativity and associativity of \wedge and \vee , De Morgan's laws and modus ponens (which we call "detachment"), for example. Students start with two-column proofs (statement on the left, justification, such as axiom or rule of deduction used, on the right). We realize that later in their mathematical careers they normally will suppress these logical details. However, this structure teaches them that every step of a mathematical proof must be justifiable. We make the point very strongly that, while proofs are written beginning from the hypotheses, they are usually developed in large part by working backwards from the conclusion. The formal proofs can essentially only be developed this way. I find that, to get students started on this process of working backwards from the conclusion, it's not adequate simply to have them do some examples in class. What does work is to meet with every student in the class, in pairs, in my office. When I do this, every student who then does the homework succeeds with the proofs. After a few days of doing proofs in this propositional system (i.e., without quantifiers), we introduce axioms for quantification.

As we start "real" proofs (that is, about mathematics, rather than pure logic) in baby number theory, we emphasize working both down from the definitions of the terms involved in the hypotheses, and up from the definitions of the terms in the conclusion. We expect students to continue to do primarily two-column proofs throughout the semester, to ensure that they get in the habit of being able to justify each step. However, we also have them do roughly one paragraph proof per day to help them start learning that style. Our locally-developed text [1] also includes "find the error in this proof" exercises, which were taken from actual student errors on homework or exams. This helps them focus on correct use of the rules and definitions. Largely because our students' algebraic skills need work, they find induction proofs the most challenging. Even when they are using an appropriate approach, they are likely to get lost in the symbolic manipulation. Thus again with the first induction proofs I have students meet with me in my office in pairs to get them started.

To document students' growth, I have them keep a proof portfolio, where they put their first night's homework (determining whether assorted statements are true or false) and the proofs of which they are most proud from the six basic types they do: logic proof, number theory, set theory, etc. At the end of the course, I ask them to write a short (three-page) paper summarizing what they have learned about proof and the use of examples in mathematics, using their portfolio items to illustrate their assertions. In addition to developing proofs, some time is spent emphasizing the importance of examples for purposes other than justifying that a statement is true — as potential counterexamples, as an aid to understanding what a theorem means, and sometimes as a guide to why a theorem is true.

2.2 Linear Algebra

Linear Algebra, normally taken in the sophomore year, renews students' acquaintance with the proof techniques learned in IMR as freshmen. Students primarily do "A is a B" proofs: that particular sets are vector spaces or subspaces, linearly independent sets, spanning sets, or bases, or that particular functions are linear transformations. They give students a chance to practice the skills they developed in IMR, particularly working both forwards and backwards from definitions, but with newly introduced mathematical objects. What is perhaps surprising is the difficulty students have with a question such as "Determine whether the set $S = \{(x, y) \in \mathbb{R}^2 \mid y = 2x + 1\}$ is a subspace of \mathbb{R}^2 ." Considerable time needs to be spent on how to unpack this definition into something they can use. Since linear independence is typically proven by contradiction, a technique only minimally covered in IMR, this is another area of growth for our students in Linear Algebra. They also get more experience figuring out, from a statement of a theorem, what it is asking them to prove. They see a few more sophisticated proofs, including proofs by induction for determinants and the existence of bases, but are not generally expected to develop such proofs on their own. Currently we teach this course fairly traditionally, via interactive lecture interspersed with students working sample problems. Linear Algebra is a prerequisite for the later theory courses.

2.3 Number Theory

In their junior year we really start pushing their proof-development skills. Typically they take (elementary) Number Theory in the fall and Modern Algebra (for which Number Theory is a prerequisite) in the spring. We teach Number Theory via a modified Moore Method [5], also now called “inquiry-based learning” (IBL). For those unfamiliar with the Moore Method, students have to develop proofs of virtually all of the theorems on their own. The degree of “modified” is left up to the instructor. I have taught it several times by assigning groups of two or three students to give three presentations per week each. However, one semester I taught it in a more full-blown Moore Method, where students were not allowed to look at any books other than our textbook nor to discuss proofs with each other. Our text [4], an IBL book published by the MAA, has no proofs, just statements of definitions, examples, and theorems. Recent adaptations of the method to undergraduate courses involve choosing a careful ordering of the theorems to build on previous ones, virtually ensuring success. I was quite surprised by how well it worked, at least using this text. Following a procedure I learned when attending a conference on IBL, I made it clear to the class that I would first ask students who had not presented recently if they could prove the next theorem. I encouraged those who felt at a loss to see me during office hours, where I would nudge them toward developing a proof of an upcoming theorem. With one exception, the whole class became able to develop their own proofs, with about half the class really blooming, developing, on occasion, interesting proofs that I had not considered. (Editors’ note: for more details about running an IBL course, see Ernst and Hodge’s paper [2] and Rault’s paper [6] in this volume.)

Students are expected to transition quickly from the two-column style of IMR to the standard paragraph format. I generally do not find this to be a problem, although some colleagues do. I encourage students, when stuck, to return to the methods from IMR. After all, basically it was just presenting the standard logic of mathematical reasoning and a general guide for what to do when stuck: unpack the definitions, try working both forward from the hypotheses and backwards from the desired conclusion.

There are several ways we are trying to get them to grow in this course. The theorems in the text are sequenced carefully so that theorems proved earlier that day are often used in the upcoming theorems. Hence one new proof technique they learn is to use previously proved theorems, rather than going back to the definitions every time. Second, IMR discourages students from using examples as proofs. Unfortunately, a side effect is that it discourages them from trying examples at all. However, we all know that a sufficiently rich example often gives one an understanding of why the result is true. That is, proof development and consideration of possible counterexamples are two sides of the same coin. The text is interspersed with appropriate examples for students to work, and then conjectures to form as a result of them. Thus, a second goal for this course is that students gain an understanding of how well chosen examples can help proof development. Third, Number Theory is the first course in which students aren’t seeing a particular type of proof and then practicing several of the same type. In IMR, they did “divides” proofs, “subset” proofs, etc.; in linear algebra, they did “subspace” proofs and “linear transformation” proofs. They now are simply proving mathematical facts about integers, not always tied to a particular style. For some this is liberating, but others feel that they have been thrown into the deep end. However, after a few weeks most make at least some progress. They start gaining confidence that, when not under the time pressure of a test, they can sometimes develop their own proofs.

2.4 Modern Algebra and Real Analysis

Modern Algebra introduces more abstract systems and ways to think about abstraction. Since most of our students are planning to teach at the secondary level, we have chosen (using Hungerford [3]) to start with rings and rings of polynomials, the algebraic objects within which high school algebra lives. We do not get to groups until the second-semester course, which is an elective. Modern Algebra builds on skills developed in both prerequisites, Linear Algebra and Number Theory. Many of the proofs (that an object is a ring, ideal,

homomorphism) are just slightly more abstract versions of proofs from Linear Algebra. However, now that our students have had experience proving theorems on their own in Number Theory, *they* are ready to prove straightforward theorems about rings and homomorphisms, analogues of which we proved *for* them in Linear Algebra. Their biggest problems in this course tend to be with understanding the abstract concepts, and what counts as an example of a proposed theorem. The rings of integers modulo n that, essentially, students worked with in Number Theory now become new mathematical objects that form examples in Modern Algebra. This is a conceptual change, from simply a different way to add and multiply, in Number Theory, to a totally new object, \mathbb{Z}_n . Mathematical objects are no longer single numbers. Now rings or ideals are also objects that may exemplify conjectures.

Real Analysis is normally taken in the first semester of the senior year. Number Theory is a prerequisite, and students are encouraged to take Modern Algebra also prior to Real Analysis. Our students have always found it to be the hardest of our courses. Due in part to the alternation of quantifiers in many definitions, Real Analysis pushes students' proof-writing skills to a still higher level. Students bring their experience developing their own proofs in Number Theory and their enhanced ability to test conjectures on substantial examples in Modern Algebra. This lets them focus on the complicated structure of the real numbers, and of the definitions of limit and derivative, instead of developing all the skills in one course.

2.5 The Courses Evolve

We have been teaching IMR in this way since 2004. It undergoes minor annual modifications, but the locally developed textbook has been essentially the same since about 2006. The text has been used in all sections of the course since about 2008, sometimes supplemented by more standard textbooks. While some faculty started teaching Number Theory as an IBL course in 2003, we only started doing it consistently after our current text [4] was published by the MAA in 2007. Linear Algebra has only in the last few years moved toward having students apply the skills acquired in IMR to the basic proofs of linear algebra (that a set is a vector space, a subspace, a basis, that a function is a linear transformation, etc.).

3 Outcomes

In 1998 we had about 58 majors distributed through the four years. We had so few majors that we could only offer our junior- and senior-level courses every two years. As a result, students would often take Number Theory after Modern Algebra. The gradual increase in our majors has allowed us to make Number Theory a prerequisite for Modern Algebra, allowing us to build students' proof skills over the courses. For the last five or so years we have had over 120 majors. The course sequence I have just described is not the only change we have made over the years in our major. Thus, it is not clear how much of the increase is due to IMR and the sequencing of these courses. However, one advantage of IMR in the first year is that students know from the start that a mathematics major involves proofs. At Wabash College, where I taught for twenty years, each year there would be students who bailed out of the mathematics major as sophomores or juniors when they were suddenly faced with proofs. Thanks to IMR, essentially every student who is willing to do regular homework develops at least minimally adequate proof-writing skills. Due to the care we take to ensure that students learn the needed skills, the only students we lose are those who are unwilling to work hard, or those who realize they don't really like mathematics — they just like to compute. They become accountants.

I can see the effect of IMR on students in Number Theory and Modern Algebra. At the beginning of Number Theory, students now have little problem doing their first proofs involving divisibility and congruence, even though it has been at least a year since they learned the concepts in IMR. Complaints such as "I don't even know where to start this proof" are much less frequent. When they occur, I can reply by reminding students of procedures they learned in IMR: "List what is given at the top. List what is to be proved

at the bottom. Unpack the definitions. Then look for a relationship.” They nod, go off to try this, and only return for more help when they encounter a conceptual problem.

Our modifications to our program have been quite successful. Real Analysis is still the hardest course for most of our majors, but we can cover the material we believe should be in the course. In the past, we had a significant number of withdrawals and Ds. Now, students have grown enough in their proof-writing skills by the time they take Real Analysis that virtually all of them earn at least a C in the course. We have just introduced a statistics concentration in the mathematics major. These students are required to take IMR, Linear Algebra and Number Theory, but not Modern Algebra. It remains to be seen how well these students, with one fewer proof course, will do in Real Analysis.

One problem we have been wrestling with is that some faculty find that students really want to hang onto the two-column format as a comfort blanket into their later courses. They want to include excessive detail, such as “Now we add two to both sides of the equation.” I haven’t found this a problem myself: it takes a couple of days of transition in Number Theory. Some of my colleagues are currently experimenting with other introduction-to-proof books that do not use two-column proofs. My past experience with such books is that most are written with second-semester sophomores, or juniors, in mind, and are too sophisticated for our freshmen.

4 Extending the Method

Our system would not be scalable for larger class sizes without having TAs in the classroom and grading the homework. Indeed, even though we keep IMR capped at 20, grading a half dozen proofs from 20 students three times a week is really quite a burden. Students need this regular feedback as well as meetings during office hours as soon as they have trouble. We already do use some technology. We use our course management system for students to submit writing assignments in Number Theory and Real Analysis, for example. Certainly much of the material could be submitted online, or adapted to virtual classrooms. Using inquiry-based learning for Number Theory, however, needs synchronous interactions (though one could imagine this being done with adequately sophisticated technology via some form of live video chat).

One issue we still have is that we don’t really have space in the curriculum to offer a course in which students explore mathematics, coming up with their own conjectures, even simple ones. This isn’t necessarily due to our sequence of courses, but does seem to be a gap in the experience of our majors. Number theory would seem a natural opportunity for this experience. However, we use our IBL course to have students develop the ability to construct their own proofs. For this we need a book in which the theorems are so carefully sequenced that, with the exception of a few challenging proofs for the best students, every student should be able to prove every theorem. Finding an alternative conjecture experience is an issue the department plans to discuss in the near future.

The biggest challenge for other institutions desiring to adopt our sequence is identifying mathematics majors in the freshman year so that they can start in a course such as our IMR. Monmouth may be somewhat peculiar in asking applicants to indicate, on their application forms, their prospective majors. One can, of course, encourage advisors to put freshmen who say they are thinking about majoring in mathematics in such a course at the same time as calculus. We do get a certain number of transfer students: perhaps one-third of our majors come here after taking some courses at another institution. It is possible for students transferring in calculus, linear algebra, and differential equations to finish here in two years by taking IMR in their first semester, Number Theory in the second semester, then Modern Algebra and Real Analysis in the second year. Similarly it would be possible for students to take IMR at other institutions as sophomores. However, we find that for our students the early introduction to proof makes a big difference in their understanding of what mathematics is about. It takes time for students to develop the necessary mathematical sophistication. I am skeptical that this can be done in less than three years with the type of students we get. The careful sequencing of their development of proof skills requires departmental discussion and agreement. It can

mean the difference for the weaker half of our majors, however, between finishing a mathematics major and switching to another major. While most of us take particular joy in the bright student who is going on to graduate work in mathematics, those weaker students have important roles to play in our society also. Helping them succeed rather than leave in frustration can be very satisfying.

References

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Appendix

A Axiom Schemas

Excluded Middle	$P \vee \sim P$
Double Negation	$\sim(\sim P) \Leftrightarrow P$
Commutative	$(P \wedge Q) \Leftrightarrow (Q \wedge P)$ $(P \vee Q) \Leftrightarrow (Q \vee P)$
Associative	$((P \wedge Q) \wedge R) \Leftrightarrow (P \wedge (Q \wedge R))$ $((P \vee Q) \vee R) \Leftrightarrow (P \vee (Q \vee R))$
Distributive	$(P \wedge (Q \vee R)) \Leftrightarrow ((P \wedge Q) \vee (P \wedge R))$ $(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$
De Morgan	$(\sim(P \wedge Q)) \Leftrightarrow (\sim P \vee \sim Q)$ $(\sim(P \vee Q)) \Leftrightarrow (\sim P \wedge \sim Q)$ $(\sim(P \vee \sim Q)) \Leftrightarrow (\sim P \wedge Q)$
Contrapositive	$(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$ $(\sim P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow P)$ $(P \Rightarrow \sim Q) \Leftrightarrow (Q \Rightarrow \sim P)$
Alternate Implies	$(P \Rightarrow Q) \Leftrightarrow (\sim P \vee Q)$ $(\sim P \Rightarrow Q) \Leftrightarrow (P \vee Q)$
Equivalence	$(P \Leftrightarrow Q) \Leftrightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$
Addition	$P \Rightarrow (P \vee Q)$
Simplification	$(P \wedge Q) \Rightarrow P$
Transitive	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$

B Rules of Inference

1. If P (respectively Q) appears on one line of a proof, and $P \Leftrightarrow Q$ is an instance of an axiom schema, then Q (respectively P) may be written on a later line of the proof, citing the name of the axiom schema and the line number for P (respectively Q).
2. (Detachment) If P appears on one line of a proof, and $P \Rightarrow Q$ appears on another line of the proof, then Q may be written on a later line of the proof, citing Detachment and the line numbers for P and $P \Rightarrow Q$.
3. (Conjunction) If P appears on one line of a proof and Q appears on another line of the proof, then $P \wedge Q$ may be written on a later line of the proof, citing Conjunction and the line numbers for P and Q .
4. (Implies) One may put P as a given on one line of a proof. If one then arrives at Q on some later line, then $P \Rightarrow Q$ may be written on the next line, citing the line numbers starting with that of P and ending with that of Q .

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