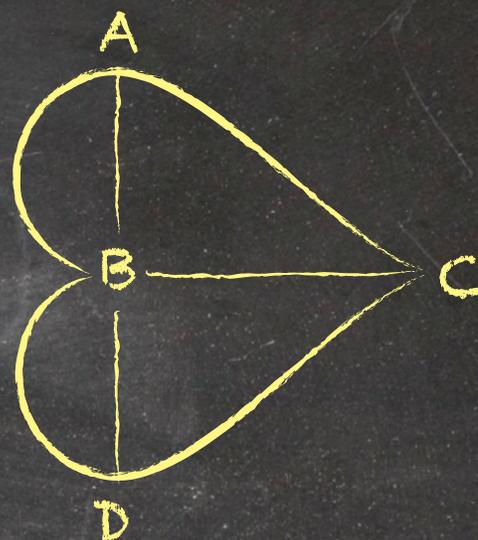


Teaching Mathematics with Classroom Voting With and Without Clickers

Starting on any one of the four land masses A-D, is it possible to cross all seven of Königsberg's bridges exactly twice?

1. Yes
2. No, it's impossible
3. Sometimes
4. I can't tell



Edited by Kelly Cline and Holly Zullo

Teaching Mathematics with Classroom Voting

With and Without Clickers

© 2011 by
The Mathematical Association of America (Incorporated)

Library of Congress Control Number 2011932376

Print ISBN 978-0-88385-189-0

Electronic ISBN 978-1-61444-301-8

Printed in the United States of America

Current Printing (last digit):

10 9 8 7 6 5 4 3 2 1

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and

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Published and Distributed by
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**An Introduction to
Teaching Mathematics
with Classroom Voting**

Introduction

We started working on this volume after having lunch during the Joint Meetings with a varied group of people, each using classroom voting in different ways. Each of us had developed our own techniques for using this teaching method successfully. Sometimes we had arrived at the same strategies, while other times we were using quite different approaches. We realized that all this collective experience should be gathered together in one place, so that it would be easily accessible to those wanting to begin using or extend their use of classroom voting. In this volume we have brought together perspectives from a diverse group of people with expertise in classroom voting, so that an instructor can quickly learn how to be successful with this pedagogy. Two of the papers published in this volume are reprints or updates of what has previously been published elsewhere, while most papers were written specifically for this volume, discussing the issues that someone new to the pedagogy will grapple with. The end result is a vibrant collection of papers illustrating the use of classroom voting in nearly every mathematics service course, as well as many courses taken by mathematics majors.

This volume is divided into three sections, plus a combined, single collection of references for all papers. The first section provides background information on classroom voting, including a thorough description of the pedagogy, suggestions for running a class-wide discussion, and contrasting viewpoints on issues such as whether voting should be graded and whether technology improves or detracts from the pedagogy. This section is essential for newcomers to classroom voting, and even experienced users will likely find new ideas to consider. The second section contains three papers that present results of studies carried out to determine the effects of classroom voting in mathematics. The first paper presents results from a large study at Cornell University suggesting that classroom voting is effective at improving student learning in mathematics when it is used to motivate students to participate in small group discussions. The second paper concerns a study of attempts to structure the small group discussions in order to make them more effective, and the third paper provides results from a large survey of student reactions to classroom voting in mathematics.

The heart of the volume is the third section, which addresses the use of classroom voting in several specific courses, spanning most service courses plus many courses for mathematics majors. Many of the authors of these articles have made their question banks available for public use, and these can be found at mathquest.carroll.edu/resources.html. While each paper describes the use of classroom voting in a particular course, each author also contributes their own style and pieces of wisdom that are transferrable to other courses. Thus, readers interested in using classroom voting in statistics may turn first to the papers dealing directly with that course, but they will still find valuable information in the other papers. In what follows, we provide a brief note on the specific contributions of each paper, beyond the course being discussed.

McGivney and McGivney-Burelle describe a project at the University of Hartford to write questions for a Math for Liberal Arts course. They discuss the question-writing process, including a paradigm shift after their first trial.

There are four papers on statistics, each presenting a slightly different view. Murphy, et al, report on their NSF-funded project at the University of Oklahoma. They present four different lesson plans for statistics courses directed towards a variety of audiences, and they discuss question-writing with a multi-disciplinary team. Gunderson and McGowan discuss three ways in which they have used clickers: as part of a regular lecture, for an exam review, and to gather data in a lab setting. Bruff touches on a variety of aspects of his use of voting, including results from two-cycle voting and ways to improve questions that don't pinpoint the desired concept. Peck discusses question-writing and presents results from an experiment with teaching two sections of statistics, one with voting and one without.

The fastest way to spread a new pedagogy is to teach teachers. The next two papers discuss using voting with pre-service and in-service teachers. Ernie, et al., present a series of voting questions designed to help pre-service teachers

think deeply about their conceptualization of probability. Serros, et al., discuss how they used clickers to promote discussion in a workshop for in-service teachers.

The next group of papers focuses on the use of voting in college algebra and precalculus. Schlatter discusses how he uses classroom voting, conducted with index cards, to help students use different modes of expression, make valid inferences, and interpret mathematical data. He also presents questions used to motivate new material. Gibson presents a multi-faceted lesson plan for slopes and average rates of change, which he uses in a large-lecture college algebra course. Lomen discusses a variety of questions that he has used in college algebra to introduce a new topic, check student understanding of a topic just presented, and review a topic. He also shares his implementation of voting without technology. Finally, Hofacker, et al., present questions they have used in precalculus, emphasizing the student responses and various strategies for leading discussions.

In her paper on calculus, VonEpps discusses in depth how to plan a lesson by writing a few of her own questions to supplement questions available in existing databases. Sharp presents a variety of questions that she has used in a large-lecture calculus course. Terrell uses vector calculus as a vehicle for illustrating how she uses voting to draw out and build on students' own preconceptions.

In "Integrating Classroom Voting into Your Lectures," Storm shares his lesson-planning technique, illustrated with two lessons from differential equations. He gives tips for maintaining content while integrating voting. Cline, et al., use statistical analysis of past voting patterns to identify the most challenging questions on fundamental concepts in differential equations, finding that these tend to produce the richest student discussions.

The two papers on linear algebra each present a complete lesson plan. Cline's paper emphasizes the student discussions following each vote. Zullo's paper focuses on the lesson-planning process, with particular attention to maintaining flexibility.

Generally classroom voting has been used in lower-level courses. In the final paper in this volume, Lock extends classroom voting well into the reaches of upper-division courses, including an introduction to proof course and abstract algebra. Lock demonstrates that multiple choice questions can have significant learning value in teaching proof and advanced concepts.

This volume would not have been possible without the contributions of the authors. We extend our heartfelt gratitude to all of the authors who contributed papers for this volume. Their willingness to share their experiences with classroom voting will be greatly appreciated by faculty just beginning to experiment with this pedagogy. We also appreciate the careful reviews by Steven Maurer and the members of the Editorial Board of the MAA Notes series. Their helpful suggestions have significantly improved this volume.

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1

Teaching Mathematics with Classroom Voting

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1.1 What is classroom voting?

Classroom voting is a powerful new pedagogy that has developed an impressive record of success in mathematics, the sciences, and engineering. In this teaching technique, the instructor poses a multiple-choice question to the class, then gives them a few minutes to work through the question and discuss it in small groups before each student votes on the correct answer, either using a hand-held electronic “clicker,” by a show of hands, or by raising a colored index card (A = red, B = blue, etc.). After the vote, the instructor goes around the class, asking students to explain their vote. The vote gives the instructor immediate feedback as to the state of the students’ understanding from each individual in the class. More importantly, the vote requires every single student to play an active role, to grapple with some mathematical issue, to discuss it in a small group, and to register an opinion.

Education research shows that classroom voting and other teaching techniques which require students to actively engage in the material during class produce substantial improvements in student comprehension and retention of the concepts when compared to presentation methods that allow the majority of the students to remain as passive observers who are simply taking notes (see, e.g., [11, 19]¹).

1.2 History and Evidence of Effectiveness

The pedagogy of classroom voting was developed for the physics classroom by Harvard University’s Eric Mazur in his influential book *Peer Instruction: A User’s Manual* [15]. In this book he described a specific classroom voting technique, called “Peer Instruction,” in which students vote on each question twice, once after an individual consideration of the question, and then again after a small group discussion. In the years that followed, this teaching technique spread throughout the S.T.E.M. disciplines (science, technology, engineering, and mathematics), and a growing number of studies have successfully documented the effectiveness of this technique. Many surveys have reported very positive student attitudes towards the use of this teaching method (see, e.g., [9, 10, 25]). Some studies have successfully gathered evidence that classroom voting really does produce improvements in student learning [5, 6, 20, 23]. Further, Lasry [12] shows that the learning gains are not dependent on a specific technology, but that the same results are achieved whether electronic clickers or colored cards are used.

¹All references for this volume are in a combined list in a separate file. If you have a print version of the volume, the combined references are in the back of the book.

In the years that followed Mazur's book, this teaching method was adapted to mathematics, and classroom voting questions (often called ConcepTests) were written by many groups, first by Cornell University's "GoodQuestions" project for differential calculus and by the Harvard Calculus Consortium to accompany the Hughes-Hallett et al. text [18], then later by the University of Oklahoma's "Classroom Response Systems in Statistics Courses" project for statistics, and by our own efforts as part of Carroll College's "Project MathQUEST" to develop questions for differential equations and linear algebra. To access these libraries, see the links at

mathquest.carroll.edu/resources.html.

There have been numerous papers written about the use of this teaching method in collegiate mathematics [1, 2, 3, 4, 7, 8, 13, 14, 17, 21, 22, 24]. These papers overwhelmingly report how much students enjoy this teaching method and how it can create a positive and engaging learning environment. Many report increased student attendance, as well as increased enthusiasm for mathematics and the recruitment of more mathematics majors. The most significant evidence for the improvement of student learning in mathematics comes from the study conducted by the Cornell GoodQuestions Project, which not only demonstrates the effectiveness of classroom voting, but also provides important insight into why this teaching method works, and what in particular makes it so powerful [16]. See Section 2 of this volume, where we have reprinted this key paper.

1.3 Our Experience

We began using classroom voting in calculus and multivariable calculus at Carroll College starting in the fall of 2004, incorporating questions produced by the Cornell GoodQuestions project, those written for the Hughes-Hallett et al text [18], as well as the collection written by Mark Schlatter [22]. Rather than formal Peer Instruction, with two cycles of votes, we simply asked students to both work through the question and discuss their thinking with at least one other person before casting a single vote. We were consistently impressed with the power of this teaching method to engage students and create a more active learning environment, while at the same time students reported that voting made mathematics class more enjoyable. In our discussions, we identified five key advantages of this pedagogy [7, 8]:

1. The act of voting requires every single student to consider a question and form an opinion, thus actively engaging in the material.
2. Voting provides immediate feedback to the instructor, who can then modify the lesson based on student responses.
3. Voting provides feedback to the students when the correct answer is revealed, in a much faster process than waiting for a homework-correction cycle.
4. Voting is a powerful way to create very fruitful student discussions, both in small groups before the vote, and class-wide discussions after the vote.
5. Students consistently indicate that they have more fun doing mathematics with voting than they do in a traditional class. By creating an environment in which the students take pleasure in doing substantial mathematics, we are creating an effective learning environment.

1.4 Our Advice

We have found that things work best if on the first day of class we emphasize to the students that the purpose of the voting is to get them to participate in small-group discussions. No one is allowed to vote unless they have discussed their thinking with at least one other person. In our classes we have not found it necessary to give points or penalties for right/wrong votes. The discussions are an essential part of the learning process and our purpose is to help them learn and discover the important ideas. After the vote, we call on various students by name, Socratically asking them to explain their vote. If a strong majority votes correctly and the first couple of students we call on can provide good explanations, we quickly move on. However it is often the conceptually more difficult questions that are the most effective teaching tools, so we usually look for questions that will provoke a more complex result. In the post-vote discussion, we do our best not to give away the right answer too quickly, as this brings the discussion to an immediate halt. Instead, we try to project the expectation that the students must identify the key points for themselves. Thus, in

the discussion we try to stay very poker-faced, neither confirming nor denying the accuracy of the various students' statements. When the discussion works out well and the students converge on the correct understanding in a reasonable period of time, we only announce the right answer and clarify things afterwards. During this discussion all answers are acceptable, right or wrong: The only unacceptable response is: "I just guessed." We tell the students that if they don't have any ideas, they should be talking to the people around them: They must be prepared to offer something to the discussion. After the discussion has reached a sense of resolution, then we summarize the main points, giving the students a much-needed sense of closure.

Some of the most fruitful discussions result when only a very small percentage of of students vote correctly. The question may have provoked a common misconception, allowing us to deal with the issue right in class. Students are usually quite shocked to find out that a majority of them is wrong, provoking interest and curiosity when this is revealed. We usually initiate the class discussion in the standard way. The students start to verbalize their thinking, and sometimes as we write this on the board, the class will be able to sort things out on their own. Other times, however, this doesn't work. In that case, we might ask for the person (or couple of people) who voted for the correct answer to explain their thinking. This, too, can work or not, depending on whether that student has now been swayed by the majority's reasoning. If the key facts do not come out of the discussion naturally, then we play a more active role and start asking leading questions. Another option is to simply tell the class that the majority is wrong, and then to call for another round of small group discussion and voting.

We usually intersperse the questions throughout each class period, rather than regularly grouping them together at either the end or the beginning. In particular, we have had good success with using questions to lead the students through the material, to provoke issues, to challenge them, and to pose problems involving material that they have not yet mastered. It is impressive how often they can reason things out for themselves. Then after they have struggled through a challenging question or two, they are ready for a short lecture segment resolving the confusion and generalizing the concepts.

One of the greatest challenges in successfully using classroom voting is time management. In a typical class period we often use four or more voting questions, and all the resulting discussion may occupy more than half of the class time. How do we cover the necessary material, when all this time is spent on discussions? It is important to view the voting questions not as something extra to be squeezed into an already very busy lecture period in addition to everything else; the time spent on the questions must replace sections of the lecture. We have found that we can maintain the same pace in our courses, covering the same material and giving the same exams, by replacing many of the instructor-led examples and parts of lecture with voting and student discussions. We are not primarily using voting as an assessment tool, to quiz the students about how well they understood what we just presented. We are using voting as a method of instruction, a way to present new ideas to the students. Further, using the feedback from the voting, when we see that the students understand a new concept quickly, we can immediately move on, thus gaining some efficiency.

1.5 Resources and Conclusions

So how do you get started? You may want to pilot test classroom voting in a class period or two, just to see how things go. However, in order to really see the results of this method, we recommend trying to incorporate a classroom voting question or two into almost every period of a course for a complete semester. If clickers are available on your campus, talk to your IT department. If not, hand out colored cards. Clickers have their advantages, but they are certainly not necessary.

Next, you'll need voting questions that are appropriate for your class and students. Writing all of these questions by yourself is rarely practical, so we have created a web page linking to all of the publicly available libraries of classroom voting questions for mathematics that we have located, including the libraries used by us and the other authors in this volume (mathquest.carroll.edu/resources.html). We currently have links to collections of classroom voting questions for a wide range of classes, including:

- Liberal arts mathematics
- Statistics
- College algebra

- Precalculus
- Calculus
- Multivariable calculus
- Differential equations
- Linear algebra
- Bridge to higher mathematics
- Group theory

Many faculty start by using questions written by others, then gradually add increasing numbers of original questions as they develop more experience. If you develop your own questions, please share them. By contributing to a public body of classroom voting questions, we can learn from each other and make our lives a little bit easier.

Classroom voting is worth the effort. It takes some time, energy, and a willingness to run your classroom in a slightly different way. In the end, this teaching method really makes a difference, because it requires every student in the class to actively participate in a small group discussion, to form an opinion, and to register this opinion. The students engage with mathematics, the instructor gets feedback, the students get feedback, and the students have more fun learning mathematics.

Acknowledgements This paper is based on work supported by National Science Foundation Grants DUE 0536077 and 0836775. Any opinions, findings, and conclusions, or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

2

Key Issues in Classroom Voting

Holly Zullo Kathy Gniadek Derek Bruff Kelly Cline
Carroll College Sacred Heart Academy Vanderbilt University Carroll College

2.1 To Use Clickers or Not to Use Clickers

Many people think of classroom voting as being synonymous with clickers, but in fact voting does not have to be conducted electronically. Many faculty have students vote without clickers, either because they do not have clickers available, or because the faculty member prefers clicker-less voting. Clicker-less voting can be conducted by a simple show of hands, with colored index cards (red = a, blue = b, etc.), or by having students hold fingers in front of their chests to indicate the option number for which they are voting. Further, a recent study in physics [49] suggests that the same results are achieved whether we use electronic clickers or vote with colored cards. For a list of vendors who sell clickers, please see mathquest.carroll.edu/resources.html.

An Argument for Using Clickers — Kelly Cline and Holly Zullo, Carroll College

We have found that clickers improve our voting process. Clickers are easy to use, fun for the students, and the votes are anonymous during the voting process. The anonymity leads to independent voting. Since students do not see other votes, they vote for what they think and are not swayed by seeing most of the class raise their hands for the first answer. We appreciate the fast and accurate data collection, as we keep track of the voting results for further study and use in lesson planning.

Perhaps most importantly, however, clickers allow the students to vote as they are ready. When we ask a question, we allow the software to begin collecting votes immediately. We can watch as votes are submitted, and when 80–90% of the class has voted, we call for the remaining votes. Without the technology, we would have to decide when to call for votes based on the amount of conversation in the room and how much of that conversation seems to be focused on the question at hand. Furthermore, we use the clock on our software to note how long it has taken to collect all of the votes on a question, and we record this information to use for future lesson planning.

An Argument Against Using Clickers — Kathy Gniadek, Sacred Heart Academy

I decided to use the “clicker-less” voting method in all of my classes. When I observed the system after it was initially purchased for our high school, I realized that although the responses were anonymous, the students were able to see on screen the rapidity of other students’ responses. This displaying of the casting of the votes, even without indicating specifically what has been voted for, is very common in clicker usage. In math classes below calculus, I see this component as distracting and pedagogically unsound for two reasons: when students are thinking about a math

problem they should not be distracted by other students' visual responses. Secondly, math is not a spectator sport. Students need time for pencil and paper activities without being visually bombarded by other students' voting results. This method can cause a lack of confidence and basic insecurities in students who frequently are already less than comfortable being in math class. Although the issue could be dealt with by turning off the projector, I find it simpler to use a clicker-less voting method.

The format for the clicker-less environment that I use in class is as follows: I wait until I have an important concept for which I want to check the level of understanding. I tell the students that I am going to give them a problem they have to think about on their own and answer in their notebooks without any consultation with their peers.

While the students spend about 5 minutes thinking about this problem, I put 4 possible answers up on the board labeled a, b, c, or d. Then I ask them to all close their eyes or put their heads down and raise their hand if they answered "a." I tally their responses manually and then call for the votes for answer "b," etc. After all responses are tallied, eyes open (or heads come up) and then I call a "check." They immediately start talking with their peers about their answer choices. I let this go on for about 3–5 minutes. Then all eyes close (or heads down) and I tally once again. After this second tally, I put all the results on the board so the students can see how peer instruction affected the tally results. Then we all discuss the possible answer as a class, with me providing input and direction as a teacher. Finally, as a class, they have one more chance to choose the right answer before I reveal the solution. If most of the class is still way off base, I ask one of the voters who chose the right answer to explain their reasoning.

At Sacred Heart Academy, my students have had experience with the "clickers" in Spanish and Chemistry. After polling my students and collecting data regarding their preference for clickers in any classroom versus a clicker-less environment, 12/40 students favored the clicker environment because they loved the timed questions and because of the fact that having a clicker in their hand kept them focused. They also sheepishly admitted that a good chunk of teaching time is used up by the teacher fiddling with the e-instruction technology in class which affords them some visiting time with their friends. This is one reason why many teachers opt for the clicker-less environment which is certain, swift and incredibly time efficient. Interestingly, 28/40 students favored the clicker-less environment, which accounts for 70% of my students this past year. The reasons they gave for *not wanting to use the clickers* included:

1. We hate the timed environment.
2. We disliked the grading pressure, which was highlighted by activity right on the screen.
3. Second guessing happens, along with loss of confidence, when you see all the other answer buttons from the other students.

I have surveyed several different groups of students regarding clicker use, and the results have been very consistent from year to year.

2.2 Two-Cycle or One-Cycle Voting

The modern conception of classroom voting can be attributed to Eric Mazur [56], who pioneered the idea in physics classes. His model is to have two rounds of voting: first the students vote individually, then they discuss the question with their peers (peer instruction) and vote again. The results from the first vote may or may not be shown to the class, but the instructor can see the results from both votes and can see if the votes converge towards the correct answer. A contrasting model is to conduct only one round of voting, allowing peer instruction to begin immediately.

An Argument for Two-Cycle Voting — *Derek Bruff, Vanderbilt University*

I feel that it is important in most instances that students respond to clicker questions individually before discussing questions with their peers. Given the opportunity to discuss a question with peers prior to responding to the question, some students are likely to postpone any serious consideration of a question until they hear what their peers think about it. Some of these students may not think hard about a question at all, knowing they can rely on their peers to do their thinking for them. By having students vote on a clicker question prior to discussing the question with their peers, I give all students the opportunity to consider a question on their own. I believe this sets the stage for deeper learning since students must assess their own understanding of a concept before entering into a discussion of it. It also sets the stage for more productive peer discussions since all students are given the chance to bring something to that discussion.

Similarly, it also prepares students to engage more deeply in the class-wide discussion that follows the second vote. By this time students have thought about the question on their own and discussed it with their peers. They are ready to share their thoughts with the class and find out what the correct answer is. I find these reasons personally compelling, but I look forward to research that determines the effects on student learning of the choice to have students respond to clicker questions individually prior to group discussion.

An Argument for One-Cycle Voting — *Kelly Cline and Holly Zullo, Carroll College*

We have adopted a one-cycle voting model where we pose each question once and students are allowed to discuss a question with their peers at any time before they each cast their own vote. Many of our students will pause and think about the question on their own before discussing with their peers, while others prefer to jump straight into peer discussion. In this way we are able to accommodate a variety of learning styles and personalities. We hold students accountable for their own thoughts in the post-vote, class-wide discussion, when we call on individual students and ask them to share their reasoning with the class. If a student admits to simply voting the way a peer recommended, without understanding the underlying ideas, we remind them that this is unacceptable, and that each person must be able to offer some insight into the problem.

Allowing immediate peer interaction allows us to ask more challenging questions, perhaps with less introduction, than if we were requiring an individual to vote first. If each student is first required to vote individually, the instructor should be sure that a good majority of the class will be able to make significant progress with the question in the time allotted. However, with peer interaction allowed immediately, we can ask questions which we anticipate only a small percentage of students will be able to figure out, instead expecting students to learn from each other in their discussions. Often these more difficult questions provoke the richest discussions and are the most powerful teaching tools, as they require students to work together.

Finally, we save time by voting on each question only once. This increased efficiency is valuable and allows more time for discussion.

2.3 To Grade or Not to Grade

When voting is conducted with clickers, it is easy to assign grades for voting based on participation and/or accuracy of answers. Many of the articles in Section III of this volume mention the issue of grading — authors either explain how they assign grading, or they give a brief explanation of why they do not grade voting. Here we present short arguments for each position.

An Argument for Grading Voting — *Derek Bruff, Vanderbilt University*

The first two times I taught my statistics course using clickers, I did not include the clicker questions in the students' course grades. However, rates of participation in clicker questions dropped to around sixty-five percent during my second offering of the course. Students who choose not to respond to clicker questions during class are only hurting themselves by not participating and not letting their responses influence my "agile teaching" choices. One might argue that motivating these students by including clicker questions in course grades is not an instructor's responsibility. However, instructors frequently assign grades as a way of motivating students to participate in learning activities. For instance, most instructors grade students' homework, at least in part, to motivate students to complete that homework under the assumption that students are best able to learn mathematics by working mathematics problems.

Furthermore, since each student's participation in peer instruction and class-wide discussion has an impact on all of my students' learning, students who do not participate are detracting, in a small way, from the learning experience of the other students in the class. This provides an additional reason to include clicker questions in students' grades, in much the same way that participation grades in discussion-based classes in the humanities are used to enhance the quality of class discussions and thus provide a more significant learning experience for all students.

For these reasons I decided to include clicker questions in my students' course grades in a minimal way during my third offering of the course. Students were given participation grades for each class session in which clicker questions were asked. These grades were based on effort, not accuracy, so students were given full credit whether or not they answered a question correctly. For instance, if four questions were asked on a particular day and a student answered

three of them (correctly or not), then that student would receive a 75 for his or her participation grade that day. I dropped the lowest four participation grades and averaged the rest to contribute five percent of the students' overall course grades. This grading system was sufficient to motivate almost all of my students to attend class and respond to clicker questions regularly.

An Argument Against Grading Voting — *Kelly Cline and Holly Zullo, Carroll College*

The main motivation for grading classroom voting is to encourage participation. However, we have found that we can effectively encourage participation by calling on students by name during the post-vote, class-wide discussion. Since students know they are likely to be called on, they are motivated to take the voting seriously. This has been an effective means of generating full participation in all of our classes at Carroll, which may have up to 40 students. We acknowledge that the power of calling on individuals would decrease as class size increased beyond this number.

Aside from grading simply not being necessary to motivate student participation in our classes, we have been very happy with the classroom atmosphere created by ungraded voting. This emphasizes to the students that classroom voting is not primarily an assessment tool, but instead is a teaching method. We are doing this because they will learn the key mathematical ideas through the process of working through and discussing these questions. Several studies have documented how students view classroom voting positively when they see that it is being used primarily for their benefit, rather than for the instructor's convenience [27, 36, 81]. In general, it is good when we can persuade students to participate in learning activities without constantly offering an immediate grade-based incentive for every action on their part, instead helping them to focus more on the long-range learning goals.



**Studies of Classroom
Voting in Mathematics**

3

Can Good Questions and Peer Discussion Improve Calculus Instruction?¹

Robyn L. Miller, Everilis Santana-Vega, and Maria S. Terrell
Cornell University

Abstract Preliminary report of the results of a project to introduce peer instruction into a multi-section first semester calculus course taught largely by novice instructors. This paper summarizes the instructional approaches instructors chose to use, and the subsequent results of student performance on common exams throughout the course of the term.

Originally published in *PRIMUS*, vol. 16, #3 (2006), pp. 193–203, © Taylor & Francis. *PRIMUS* is available online at www.informaworld.com/.

The full text of this article is available online at

<http://www.tandfonline.com/doi/abs/10.1080/10511970608984146>.

The complete article is reprinted, with permission, in the print edition of this book.

¹Support for the Good Questions project was provided by the National Science Foundation's Course, Curriculum, and Laboratory Improvement Program under grant DUE-0231154. Opinions expressed are those of the authors and not necessarily those of the Foundation.

4

Using Peer Instruction and i-clickers to Enhance Student Participation

Adam Lucas¹

Saint Mary's College of California

Abstract In my Calculus classes I encourage my students to actively reflect on course material, to work collaboratively, and to generate diverse solutions to questions. To facilitate this I use peer instruction (PI), a structured questioning process, and i-clickers, a radio frequency classroom response system enabling students to vote anonymously. This paper concludes that PI and i-clickers enhance student participation and comprehension. It is important however that students write down their reasoning during PI so as not to be led astray by dominant group members.

Originally published in *PRIMUS*, Vol. 19, Issue 3, May 2009, pp. 219–231, © Taylor & Francis. *PRIMUS* is available online at www.informaworld.com/.

The full text of this article is available online at

<http://www.tandfonline.com/doi/abs/10.1080/10511970701643970>.

The complete article is reprinted, with permission, in the print edition of this book.

¹Supported by the Mills Scholars program and Computing and Technology Services at Saint Mary's College of California.

5

Student Surveys: What Do They Think?

Holly Zullo <i>Carroll College</i>	Kelly Cline <i>Carroll College</i>	Mark Parker <i>Carroll College</i>	Ron Buckmire <i>Occidental College</i>
John George <i>Helena High School</i>	Katharine Gurski <i>Howard University</i>	Jakob Juul Larsen <i>Engineering College of Aarhus, Denmark</i>	
Blake Mellor <i>Loyola Marymount University</i>	Jack Oberweiser <i>Carroll College</i>	Dennis Peterson <i>Capital High School</i>	
Richard Spindler	Ann Stewart <i>Hood College</i>	Christopher Storm <i>Adelphi University</i>	

5.1 Introduction

Many individual faculty have surveyed their students about classroom voting, and they generally report positive results. How robust are these results across a wide variety of students, campuses, instructors, and courses? In this study, a total of 513 students in 26 classes were surveyed regarding the use of classroom voting in their classes. (See Appendix A for the survey form.) Fourteen instructors from ten different schools participated. The classes surveyed were primarily freshman and sophomore level courses in calculus, multivariable calculus, linear algebra, and differential equations. While several questions show the variation in response that one might expect, other questions generate consistent results, showing that student opinion in these areas is uniform across many variables.

5.2 Aggregate Results

The aggregate results are overwhelmingly positive. 93% of the students surveyed say that voting makes the class more fun. While having fun certainly does not equate to learning, it is a good first step and tends to encourage attendance. 90% of the students say that voting helps them engage in the material, and 84% say it helps them learn.

Students love examples and always seem to be clamoring for more. About half of the students surveyed (48%) say that they would be better prepared for the homework and exams if the instructor did more examples on the board and less voting. Given that students are so enamored of examples, this response is not unexpected. Indeed, the fact that half of the students *don't* think that more examples would prepare them better than the voting does seems to be a very positive result. Supporting this idea, 74% of the students said that the amount of voting in their class should remain the same or be increased.

Question	Max % voting Yes	Min % voting Yes	Mean % voting Yes	Standard Deviation %
1. Does the voting help make the class more fun?	100	67	92.5	8.4
2. Does the voting help you engage in the material?	100	70	89.3	9.4

Table 5.1. Questions with smallest variation in survey results

Classroom voting is frequently conducted by having the students discuss a question in a small group, vote, and then participate in a Socratic discussion where the instructor asks individual students to explain their vote to the rest of the class. Some instructors hesitate to put students on the spot by calling on individuals, but 74% of the students indicate that they feel comfortable being called on to explain their vote. Furthermore, the discussion is frequently allowed to continue for several minutes, providing the students with the opportunity to converge on the correct answer themselves, before the instructor confirms the correct answer. Only 24% of the students said they would learn better if the teacher just explained what the right answer was, instead of spending time with discussions after each vote. The value of that discussion has been clearly demonstrated [58], and students recognize the importance as well.

Perhaps most telling of all, 77% of the students surveyed say that given the opportunity, they would choose a voting section of a mathematics class over a non-voting section.

5.3 Course-by-Course Results

When studying the survey results for individual courses, we find that the responses to some questions vary little from course to course, while others vary significantly, perhaps indicating that the response is sensitive to the instructor's teaching style, experience, or some other aspect of the course. Summary statistics are presented in this section, with supporting details in Appendix B.

Table 5.1 presents the results from questions 1 and 2. Here, the "Mean" column is the simple average of the proportion of "yes" votes among the 26 individual classes, and the "Standard Deviation" column is the standard deviation of those proportions. Thus, because we have not weighted these means by class size, they are slightly different than the overall proportions of students discussed in the preceding section.

Questions 1 and 2 show the least variation in results. Regardless of the setting, students universally find that voting makes class more fun and helps them engage in the material. Thus, anyone looking to liven up their class and engage students would likely benefit from incorporating classroom voting.

Questions 4, 6, and 7 show moderate variation in the results (Table 5.2). Here we see that while usually more than half of the students give a positive response, some classes yield a much better outcome than others. (Note that in question 7, a vote of Yes is actually, in our eyes, a negative response, and so we have italicized this row.) These results indicate that something specific to the course or instructor may be impacting the students' satisfaction with classroom voting. The answers to question 4 would be expected to vary based on how the voting is used in class. Answers to questions 6 and 7 likely vary based on the instructor's style and personality.

Question	Max % voting Yes	Min % voting Yes	Mean % voting Yes	Standard Deviation %
4. Does the classroom voting help you learn?	100	57	84.0	11.5
6. Do you feel comfortable being called on to explain your vote to the rest of the class?	94	48	74.5	11.8
7. Would you learn better if the teacher just explained what the right answer was, instead of spending time with discussions after each vote? (<i>Note that a Yes vote does not support the use of clickers.</i>)	48	0	23.8	10.8

Table 5.2. Questions with moderate variation in survey results

The remaining questions, 3, 5, and 8, show wide variation in the responses (Table 5.3). Note that in question 3, a vote of Yes is actually a negative response, and so we have italicized this row as well. Here differences in instructor style, implementation of classroom voting, and other course specifics are certainly at play, impacting whether students view classroom voting positively or negatively. The widely varying responses to all three of these questions indicate that there is much work to be done in understanding the best practices of classroom voting pedagogy and in training faculty to execute it to its greatest potential.

Clearly the responses to question 5 critically depends on how much voting is currently being used, and as this varied considerably among the 26 classes in this study, it is not surprising to see so much variation in student responses to this question. Further, the responses to question 3 depend on this issue as well: If voting is being used extensively, then students will be likely to see fewer examples done on the board, and thus will vote yes on this question. Yet it is still interesting to note that even those classes where majorities of students asked for more examples in question 3 and for less voting in question 7, there were still strong majorities agreeing that voting is fun and helps them engage in the material as indicated by questions 1 and 2.

Question 8 may be the most powerful, in that it asks the students to evaluate voting overall, and decide whether they would prefer a class with voting over one without. The mean response to this question is very positive with 74% saying that they would choose voting, although there is strong variation on this point. There were only three classes that had fewer than 60% of students respond favorably to question 8. Interestingly, we find no common characteristics, as they were three different courses taught at three different schools by instructors with varying degrees of experience using voting.

Question	Max % voting Yes	Min % voting Yes	Mean % voting Yes	Standard Deviation %
3. Would you be better prepared for the homework and exams if the instructor did more examples on the board and less voting? (<i>Note that a Yes vote does not support the use of clickers.</i>)	90	12	50.2	21.7
5. How do you think that the amount of voting used in this class should change? (Yes = Increase/remain same)	100	33	73.8	16.2
8. Suppose that two sections of a math class were offered and one would have classroom voting, while the other would not. Which would you choose? (Yes = Voting)	100	20	74.4	17.7

Table 5.3. Questions with wide variation in survey results

5.4 Student Focus Groups

In an attempt to gain deeper insight into student reactions to classroom voting, several focus groups were conducted with students at Carroll College. Student comments in these small-group sessions support the findings from above and bring up other issues as well.

In all of the classes at Carroll College that use voting, the professors call on students by name to participate in the class-wide discussion instead of asking a question of the class as a whole and waiting for volunteers. Students indicate that this practice does cause some stress, but acknowledge this has a positive effect. They say they are motivated by this stress to take the voting seriously and put effort into discussing the question at their tables so that they are prepared to contribute to the class-wide discussion. Part of making students comfortable with this approach is not judging them based on their answers. Says one student, you can “give your opinion and it’s OK” to be wrong.

The focus groups provided some insight into the variation in responses to question 7, regarding the benefit of the post-vote discussion. Students favor short discussions to long, drawn-out discussions. Several said that if none of the

first three or so students called on gives the right answer, then the instructor should just tell the class the right answer. They believe that long discussions waste time and can be confusing, particularly if they go in the wrong direction for a long time before getting on the right track. So the response to question 7 on the survey likely depends on how long the individual instructor tends to let discussions run, and how quickly the instructor helps point the discussion in the right direction. While professors see some value to letting students wander with an idea for a while and sorting it out on their own, there is a limit to how much time students are willing to do this before they become frustrated and confused.

Students also discussed their opinions on how the questions were used in class and whether that usage was effective. They feel the voting questions are the most useful when asked after a short lecture has been given on a new topic. They say they like the feedback they get from the questions, and after voting they are ready to hear any additional information provided by the instructor. Most of the students in the focus groups expressed frustration when the instructor used voting questions to introduce a new topic. They felt unprepared to answer these questions and indicated that those discussions provided the least benefit. The very strongest students, however, tend to like these questions the most, saying that they like the opportunity to think on their own about a new topic before hearing about it from the instructor. The majority of students said that the best class days are ones where about half of the time is spent on lecture and half on voting.

5.5 Conclusion

There are some important limitations to this study. Although the study encompassed 513 students in 26 classes taught by 14 instructors at 10 different institutions, it is important to note that the largest of these classes had only 33 responses to the survey, and thus our results are only applicable to fairly small classes. Further, 14 of the 26 classes were taught at Carroll College, and the majority of the remaining classes were taught at other private liberal arts colleges. Thus we do not view this study as being definitive, but rather as an initial starting point that we hope will inspire further work and greater collaboration as we continue to study these issues.

Acknowledging these limitations, this study shows that the overall positive reaction of students to classroom voting in individual classes is robust across a wide variety of courses and instructors. Students simply have more fun and are more engaged in a course that incorporates voting. The majority of students also indicate that the voting questions and resulting discussion help them to learn, although there is significant variability in the size of that majority. The challenge, then, is to discover the best practices for this pedagogy to maximize learning for all students.

Acknowledgements This paper is based on work supported by a National Science Foundation Grant (DUE-0536077). Any opinions, findings, and conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Appendix A: Student Survey

Project MathQUEST Student Post-Course Survey

1. Does the voting make the class more fun?
Yes / No
2. Does the voting help you engage in the material?
Yes / No
3. Would you be better prepared for the homework and exams if the instructor did more examples on the board and less voting? **Yes / No**
4. Does the classroom voting help you learn?
Yes / No
5. How do you think that the amount of voting used in this class should change?
increased / decreased / remain the same
6. Do you feel comfortable being called on to explain your vote to the rest of the class?
Yes / No
7. Would you learn better if the teacher just explained what the right answer was, instead of spending time with discussions after each vote?
Yes / No
8. Suppose that two sections of a math class were offered and one would have classroom voting, while the other would not. Which would you choose?
Voting / Non-voting
9. What do you like best or least about classroom voting?

10. How could we make classroom voting better? If we wanted to improve the process, what could we do?

Appendix B: Class-by-Class Survey Results

The number shown in the table is the percentage of students from that class voting “Yes” on that question. For question #5, votes of both “increase” and “remain the same” count as a vote of “Yes.” For question #6, a vote of “Voting” is counted as “Yes.”

	Question:	1	2	3	4	5	6	7	8
Class	# Responses								
1	27	92.6	77.8	63.0	74.1	59.3	81.5	14.8	66.7
2	21	90.5	95.2	42.9	90.5	66.7	85.7	23.8	76.2
3	33	93.9	100.0	12.1	93.9	87.9	63.6	27.3	78.8
4	26	88.5	88.5	65.4	80.8	50.0	76.9	38.5	76.9
5	23	95.7	73.9	56.5	56.5	73.9	73.9	47.8	65.2
6	27	92.6	70.4	77.8	59.3	33.3	77.8	25.9	66.7
7	19	100.0	100.0	63.2	100.0	57.9	89.5	15.8	100.0
8	18	100.0	93.8	47.1	93.8	70.6	94.1	11.8	76.5
9	20	85.0	90.0	50.0	80.0	85.0	55.0	30.0	70.0
10	9	100.0	88.9	55.6	77.8	66.7	66.7	22.2	55.6
11	18	94.4	100.0	50.0	88.9	94.4	61.1	27.8	83.3
12	23	100.0	91.3	56.5	87.0	65.2	91.3	17.4	82.6
13	24	87.5	95.8	25.0	83.3	83.3	75.0	16.7	83.3
14	32	96.9	93.8	37.5	90.6	90.6	68.8	28.1	90.6
15	21	100.0	100.0	14.3	100.0	81.0	85.7	14.3	95.2
16	18	83.3	94.4	64.7	77.8	77.8	76.5	44.4	72.2
17	18	88.9	94.4	38.9	94.4	83.3	88.9	11.1	88.9
18	11	72.7	90.9	45.5	81.8	72.7	81.8	36.4	63.6
19	17	100.0	88.2	17.6	94.1	94.1	88.2	0.0	94.1
20	16	87.5	81.3	68.8	81.3	81.3	68.8	31.3	75.0
21	15	66.7	86.7	80.0	73.3	73.3	66.7	26.7	66.7
22	12	100.0	91.7	50.0	91.7	58.3	66.7	33.3	66.7
23	17	94.1	70.6	88.2	64.7	52.9	64.7	23.5	35.3
24	21	95.2	95.2	23.8	95.2	100.0	47.6	19.0	90.5
25	18	100.0	100.0	22.2	94.4	100.0	61.1	11.1	94.4
26	10	100.0	70.0	90.0	80.0	60.0	80.0	20.0	20.0



**Classroom Voting in
Specific Mathematics Classes**

6

Questions to Engage Students in Discussion (Q.E.D): Using Clickers in a Mathematics for Liberal Arts Course

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6.1 The Audience

Contemporary Mathematics (M116) draws more students than any other math course at the University of Hartford because it is generally viewed as the least onerous path to satisfying our university's mathematics graduation requirement. Most students are future musicians or artists and frankly would rather be in a recording session or art studio than giving up three hours a week for mathematics. A significant number have not taken a math course in three years. Furthermore, in a paper we assign on the first day of class, most students describe their mathematical history as a series of repeated frustrations, failures, and occasional humiliations. Interestingly, on the first day of class we also ask students to respond to two questions using a Likert scale of 1(low) – 5 (high): “What is your ability in math?” and “What is your interest level in math?” The vast majority of responses are 1 and 2. (We also ask them what grade they expect and their answers are generally A's and B's. Go figure.)

6.2 The Course

We designed M116 several years ago believing that any remediation of high school math would be a waste of the students' time and ours. More constructively, we wanted to show students another, more useful and up-to-date, side of mathematics that was essentially independent of the high school curriculum. To do so we chose four topics from discrete mathematics:

1. *Voting Methods* — Plurality, Instant Run-off, Hare, Borda, Condorcet and Approval together with Arrow's Theorem.
2. *Simulation* — Using the random integer generator on the TI-83+/84 graphing calculator.
3. *Recursion* — With an emphasis on finance problems using the application APPS on the TI.
4. *Graphs* — The Konigsberg Bridge Problem, The Traveling Salesman Problem, graph coloring, and minimal cost spanning trees.

These topics are pair-wise independent in terms of content; each requires no knowledge of algebra, much less statistics or calculus; interesting problems can be introduced on day one to motivate each topic; and each covers material that students see as being relevant to their lives. The text [56] we use was written by one of the authors of this article.

6.3 A Problem

For many years all twenty-plus sections of M116 each year had been taught via a lecture format. Student evaluations of the course had been positive for the most part and students frequently commented that for the first time they could see “what this stuff was good for,” although a few did wonder if it really was mathematics. However, about three years ago, several of us became increasingly restless with the passive classroom environment. About this time we became familiar with the *Seven Principles for Good Practice in Undergraduate Education* [14] and the work of Eric Mazur at Harvard with peer instruction [56]. Both highlighted the importance of students working collaboratively. In particular, we wanted students to be working actively on mathematics for (most of) the seventy-five minutes that we have them twice a week. And we wanted to have a way of knowing if they were “getting it.” As most teachers do, we would, of course, punctuate our lectures with questions along the lines of “Are you with me?” but because of the fragile nature of their confidence in mathematics only a very small percentage gave a clear sign that they were, which was generally sufficient for us to push onwards.

Fortunately in the fall of 2005 we also learned that a colleague in our College of Education was successfully using Option Power student response devices or “clickers” with PowerPoint successfully in his classes. He gave us a highly enthusiastic introduction to his clickers and our university followed with a generous grant to purchase a classroom set. Over the winter break that year we took a crash course in PowerPoint (especially its animation features), and designed a series of voting slides for the spring semester.

6.4 First Day with Clickers

For those of us who are used to the lecture method the first class was daunting. How would students react? Would everything work? Do we have enough material for the full 75 minutes? All needless concerns as things turned out. After 30 seconds of instructions (i.e., mentioning how to overwrite answers, what to expect on the screen, telling them that the clickers had no “street value”) we asked a series of informational questions about class demographics and student interests which raised their interest in clickers. (None of our students had used them before). The class reaction was immediate and electric. But this was only a trial run. The serious business was about to begin.

6.5 Typical Class

Our classroom is a converted physics lab with long, stationary tables and high metal lab chairs making it not at all conducive to small group gatherings. We have found that the easiest way to put students in groups is to arrange the chairs in groups of three with one at the narrow edge of the table and two along the table itself creating 8–9 triangles of students. Sometimes students then become groups of two and others of four and one or two prefer to work alone, which is fine with us. Although we encourage group discussion we want each student to think out a solution for themselves and vote separately after the discussion. We must frequently remind students that each individual is responsible for defending his/her answer.

Occasionally we begin a class with a voting question to introduce the material at hand, but more often we introduce the voting slides after an initial discussion. When the voting slides appear we let them speak for themselves attempting to avoid the PowerPoint pitfall of reading what students can read for themselves.

What happens next is crucial. The better students want to answer immediately. Consequently, on the first day we often see the counter on the screen beginning to tally votes with not a word being spoken. We quickly stop and introduce what has become a course mantra: “Think — Murmur — Vote.” A bit of quiet time must be followed by audible discussion before anyone votes. It takes a question or two for this to pattern to sink in (and some periodic reminders are necessary), but eventually virtually everyone is thinking, discussing, and voting on mathematics for a

good part of the class. While these discussions are taking place we, the instructors, are often circulating around the room listening to conversations or offering feedback.

We don’t want students to simply follow the lead of a group member they perceive to be a strong math student. Students are responsible for their own votes, and we’ll ask students randomly for explanations of their vote. If some students just don’t “get” the problem (even after help from their partners) we tell them not to vote. If most students have the correct answer once the class results are shown we’ll often ask “Can you see why someone might have voted differently?”

As a side note, some faculty members use clickers to take attendance and give quizzes and tests. We have not done so since collecting this data would strongly suggest to students that we know how each of them is voting all the time. This would greatly undermine the sense of confidentiality that we are trying to develop. Of course, this prevents us doing early intervention for those who might be floundering, but by visiting the groups while they’re working on the problems we generally get a good sense of who is having trouble.

6.6 A First Try: “Vertical” Questions

Our first version of voting slides consisted almost entirely of “vertical” questions; that is, questions that would elicit a histogram with a single spike indicating that the entire class had the correct answer. In simplest terms we would describe a topic, give an example or two of how it’s applied, and then present a voting slide that was a slight variation of the examples we had just given. It was our way of checking whether the class was “with us” or not.

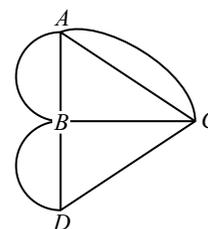
These kinds of questions have the advantage of bolstering the confidence of the majority who had the correct answer while at the same time giving us a chance to correct those few minor misconceptions that persisted.

One example of a vertical question that we used during our first semester with clickers is shown below. We posed this question during the unit on graph theory right after we had discussed Euler’s solution to the Königsberg bridge problem, which outlined a strategy for determining whether a graph had an Eulerian path, an Eulerian circuit, or was non-traversable.

Clicker Question #1

Starting at any land mass (A – D) determine if this modified Königsberg bridge problem with 4 land masses and now 8 bridges

1. Has an Euler circuit.
2. Has an Euler path.*
3. Is non-traversable.



The question was a simple variation of the original Königsberg bridge problem and virtually every student voted for #2. In a follow-up discussion students mentioned that since exactly two vertices, namely B and D , were odd the graph had an Euler path. The vertical histogram which resulted let us know that students understood Euler’s solution to the Königsberg bridge problem; however, this question and the responses students gave did little to open up the conversation, challenge students’ thinking, or reveal any misconceptions students might hold.

6.7 Questions to Engage students in Discussion (Q.E.D)

We spent the summer and fall semester of 2006 revising the 700+ PowerPoint slides (of which 150 or so were voting slides) that we had designed and we were feeling pretty good about our work when a chance meeting set us off in a very different direction. We gave a talk at the 20th Annual ICTCM conference in February, 2007 and afterwards happened to meet another speaker, Holly Zullo, of Carroll College (a co-editor of this MAA volume) who was using clickers in a differential equations course. Explaining our approach with “vertical” questions, Holly mentioned that the Carroll College group was more interested in designing “horizontal” questions; that is, questions that would produce a more uniformly distributed bar chart indicating that the class was divided on the answers to the question. We have since renamed these types of questions QED’s — “Questions that Engage Discussion.” (Examples of other QEDs can

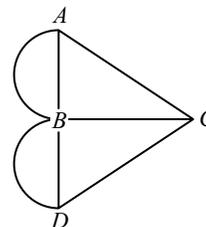
be found at mathquest.carroll.edu/qed.html.) One example of a horizontal or QED that we developed is presented next.

After discussing and solving the famous Königsberg bridge problem we posed the following question:

Clicker Question #2

Starting on any one of the four land masses A – D is it possible to cross all seven of Königsberg's bridges exactly twice?

1. Yes.*
2. No, it's impossible.
3. Sometimes.
4. I can't tell.



This question frequently results in a bar graph with several bars of roughly equal height. With the horizontal graph displayed students are expected to explain and defend their answers. A few students see how to apply Euler's theorem directly to this QED and answer "yes." They often include in their explanation that allowing each edge to be crossed twice is analogous to doubling the number of edges at each vertex. Doubling an odd or even number, they point out, results in an even number so all vertices will have even degree. However, most students immediately put pencil to paper and try to trace the required route. A few find one and so there are a few more "yes's," but most become frustrated trying to keep track of which bridges have been crossed and how many times. Consequently, there are always several "no's" or "I can't tell's." Another group will not find a route the first time but do find one with a second effort and will vote "sometimes." Others give up and vote "I can't tell." A fair number of students fail to pay attention to the word twice or don't seriously consider the implications of this added condition and vote for #2 based on the fact that the graph, as drawn above, has four odd vertices. These types of questions lend themselves to the most fruitful follow-up discussions. Still later we generalize this question by asking whether one can traverse all the edges on *any* connected graph twice and only twice. On a test we replace "twice" by "three times." These questions invite students to think hard about how one word might change the answer to the problem.

Another example of a "vertical" question that was rewritten as a QED to illicit a more "horizontal" bar graph was posed during the Voting Methods unit. In this unit we study 6 voting methods including Plurality, Run Off, Hare, Borda, Condorcet, and Approval. In our first semester with the clickers we would introduce a voting method and then test students' understanding by asking them to determine the winner of an election using the indicated method. A sample question is shown below.

Clicker Question #3

In the preference schedule shown below which candidate wins the election using the (3-2-1-0) Borda Count Method?

1. A*
2. B
3. C
4. D

16	10	20	8
A	B	C	C
B	A	A	A
D	D	D	B
C	C	B	D

Again, as is true with most "vertical" questions, nearly all of the students performed the required computations and determined that A was the winner by the Borda Count method. (Note: Each of the candidates earned the following number of points: A: 124 pts; B: 60 pts; C: 84 pts; D: 46 pts). The few students who did not vote for the correct answer had made computational mistakes. Again, this type of question did little to engage students in a rich discussion about mathematics.

We extended clicker question #3 to create the following QED:

Clicker Question #4

Suppose there is a majority winner in an election, will the Borda Count method always pick that winner?

1. Yes.
 2. No, and here is a counter example.*
 3. I don't know.
-

This QED often yields a bar graph with each bar roughly the same height. The students who vote “yes” have often incorrectly generalized results from the previous class which showed that Plurality, Run-Off, and Hare will always pick the majority winner if such a winner exists. Several students correctly vote “no” and are able to create appropriate counterexamples (see Clicker Question #3 to see a counterexample) others vote “no” but create inappropriate counterexamples. For instance, they create a preference schedule and go on to explain that the winner found by the Borda Count Method is not a majority winner—that is, they thought they needed to show that the Borda Count Method always picked a majority winner. Finally, some students who vote #3 have admitted they are confused about what it means to win be a “majority” winner while others don't know how to go about creating a counterexample to this question. Like other QEDs the quality of the question is evidenced by the complexity of the follow-up discussion. Discussions of this question have wandered into the nature of “if-then” statements in mathematics; the use of counterexamples to disprove conjectures; and criteria for determining the “fairness” of voting systems.

6.8 What Students Say

At the end of each semester we have asked students to complete an anonymous questionnaire regarding their experiences with clickers. The results (listed in decreasing order of the numbers of responses) have been virtually identical on each occasion. What students enjoy most about clickers is:

1. *Anonymity* — Privacy is the key. Students value being able to answer questions without the potential embarrassment of being wrong.
2. *Immediate feedback* — In parallel fashion they appreciate knowing in real time if they are correct. When students get the right answer their confidence level increases and they are more likely to be engaged with what follows.
3. *See what others think* — Even when their answers are incorrect they generally see that they are not the only ones who didn't know the answer, and they have an opportunity to hear (or take part in) a discussion of why another answer is correct.
4. *Fun* — Clickers draw on students' use of other electronic media (cell phones, remote controls, . . .). And friendly competition often develops once students become more comfortable in their groups.
5. *Encourages collaboration* — By discussing a problem in their groups students hear other points of view. The give-and-take discussion that we encourage helps them better understand the material.
6. *Simple to use* — Those who are technophobes need not fear. It takes ten seconds to learn how to use a clicker and they are virtually indestructible.

6.9 What Faculty Say

There are two full-time faculty and one adjunct faculty who are currently teaching M116 using clickers. Our collective impressions are:

1. *Immediate feedback* — Student responses allow us to continue with new material with a high degree of confidence that they are “getting it” or provide fertile ground for discussion. In either case, we have a much better sense of where our class is at any moment.
2. *Richer course* — Student responses allow us to think about writing sharper questions (and answers) that get more to the heart of the material and also require students to think more deeply about the content. In fact,

student scores on variations of exams we used several years ago have improved significantly since we started using QEDs.

3. *Much more interesting class for us* — Using the QEDs and clickers we have noticed that class discussions are more electric. Classroom energy is a two-way street and we have all felt energized by the level of student engagement with the clickers.
4. *Forces us to think more deeply about the course* — With less time devoted to “lecture” we have to think hard about the essential topics in each section and build our QEDs around these topics. There is always the concern that we will not “cover” as much material as before. After two full years of using the clickers we have not had to trim the content in any way. What we have done within each unit is remove repeated examples of the same type which are aimed at assessing isolated skills and replaced them with one QED that covers multiple ideas.

6.10 Pros and Cons

With any technology there are some occasional problems. Initially, our computers had difficulty locating the OptionPower receiver and the program occasionally froze or some slide didn’t register those who had voted. However, these annoyances happen less frequently now. In addition, one of us has recently made the transition to Turning Point clickers and they have also been found to be completely reliable.

In conversations with colleagues the biggest impediment to teaching with clickers is the investment of time required to create interesting questions in a PowerPoint format. It is pointless to simply transcribe one’s notes in PowerPoint and then add a voting question here or there. QED-like questions, which we are convinced is the way to go, require time and thought to create.

Finally, there is a cost factor — \$1,500 is large outlay of funds for many schools. But if they are used in multiple sections over the course of several years, the cost per student is remarkably low.

With all this in mind, however, we simply wouldn’t teach M116 without clickers because of the close connection to the students that they encourage and the richer course they enable us to present.

6.11 The Future

There are several initiatives we have for the immediate future. We plan to revise once again our QED’s using a variety of questioning formats such as:

1. Which of the following is not . . .
2. Which of the following could not be . . . ?
3. True / False (A nice change of pace)
4. Which formula best models the problem posed above?
5. Which of several diagrams is correct or which doesn’t belong?
6. This answer could be the solution of which problem?

Second, we plan to videotape selected groups in the class to record the discussions students have in response to our QED’s believing that will give us more insight into developing further QED’s. Finally we plan on using clickers in more advanced math classes including real analysis. One manufacturer (Turning Point Inc) has introduced software (Turning Point Anywhere) that allows questions to be posed in any open application. Because of the heavily symbolic nature of upper level math courses, PowerPoint has not proved to be a useful medium in which to teach these courses. Rather we have used Microsoft Journal on a Tablet PC which allows us to present cursively written notes. Turning Point Anywhere can be introduced on the fly at any point in a class and thus allow for the same kind of results that we have enjoyed in M116.

7

Clickers in Introductory Statistics Courses

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Michael B. Richman School of Meteorology <i>University of Oklahoma</i>	Robert Terry Department of Psychology <i>University of Oklahoma</i>

7.1 Introduction

In 2003, the American Statistical Association (ASA) funded the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Project College Report* [33]. This report makes recommendations for the teaching of statistics. Examining the evolution of enrollment in statistics courses, the report notes that statistics courses now serve much larger numbers of students with a more diverse set of backgrounds, goals, interests, and attitudes. Courses are now offered in a wide variety of departments including business, economics, educational psychology, engineering, mathematics, psychology, sociology, and statistics. The content and teaching of statistics courses have also evolved in response to the availability of technology as well as advancements in statistics as a field of study. Building on recommendations put forth in Cobb [19], the GAISE report recommends that statistics education (verbatim from p. 1 of [33]):

1. Emphasize statistical literacy and develop statistical thinking;
2. Use real data;
3. Stress conceptual understanding rather than mere knowledge of procedures;
4. Foster active learning in the classroom;
5. Use technology for developing conceptual understanding and analyzing data;
6. Use assessments to improve and evaluate student learning.

At the same time that the GAISE report was being completed, the authors of this chapter were teaching introductory statistics courses at the University of Oklahoma and had started talking about how useful it would be to have a set of clicker questions for real-time assessment of student understanding. We realized that such questions could help address (at least) GAISE *recommendations 1. statistical literacy and statistical thinking, 3. conceptual understanding, and 4. active learning*. (The three other GAISE recommendations were addressed during our grant project years at the discretion of individual instructors, but were not the focus of our project and are not addressed in this paper.) Our three

courses used different textbooks, from three different publishers, and we couldn't find clicker questions that we would be willing to use in our classes. Thus, we decided to write our own, knowing that "composing these questions from scratch constitutes perhaps the largest effort required to convert from a conventional lecture presentation to a *Peer Instruction* format" [56, p. 26]. In addition, we thought that our efforts could be useful to other statistics instructors because it is much quicker to shop through a set of existing questions than it is to write questions from scratch. To that end, we sought and were awarded funding from the National Science Foundation's Course, Curriculum, and Laboratory Improvement Program (NSF CCLI-0535894).

We wrote, vetted, and piloted a starter set of annotated set of multiple choice questions (available at www.ou.edu/statsclickers), distributed among eight broad topics: *general concepts, descriptive statistics, probability, probability distributions, confidence intervals and hypothesis testing, sampling, tables, correlation, and regression*. We chose this topic structure to organize the items because we determined that these topics are what we, as instructors, would look for if we were shopping for questions to use in our classes. This set of topics is similar to the list used to organize the items in the ARTIST database (Assessment Resource Tools for Improving Statistical Thinking, <https://ore.gen.umn.edu/artist/>, NSF CCLI ASA-0206571).

Like most efforts to write good multiple choice questions, we drew on the expertise of experienced instructors and on the literature addressing student misconceptions. Statistics education is a young research area [3]: the first volume of the *Statistics Education Research Journal* appeared in 2002, replacing the *Statistics Education Research Newsletter* that started in 2000 [64]. As noted above, statistics educators have documented that students lack conceptual understanding. However, at this point, publications that identify specific misconceptions are limited in number (see, for example, the CAUSE *readings and publications* list, www.causeweb.org). As examples, a few articles [13, 31, 32] identify the following common misconceptions (among others): some students

- do not understand that a sampling distribution is a distribution for a sample statistic;
- believe that correlation implies causation;
- believe that groups must be the same size in order to compare them;
- believe that good samples must represent a high percentage of the population;
- believe that the average is the most common number;
- confuse mean and median and fail to consider outliers.

Moore [60] contains papers related to the teaching of statistics at the undergraduate level, including (among others) a section about active learning, a section about assessment, and relevant bibliographies. Several references included in the assessment section bibliography discuss the construction and use of multiple choice items.

Beatty, Gerace, Leonard, and Dufresne [5] identify effective designs for clicker questions: *directing attention and raising awareness, stimulating cognitive processes, formative use of response data, and promoting articulation discussion*. While many of the chapters in this volume emphasize the last design, our use of clicker questions also emphasized *stimulating cognitive processes* and *formative use of response data*. To stimulate cognitive processes, Beatty, et al. recommend using "questions that cannot be answered without exercising the desired habits of mind" (p. 35). About formative use of response data (via the bar graph of students' responses), Beatty, et al, state: "To provide maximally useful information to the instructor [and students], questions should be designed so that *answer choices reveal likely student difficulties* [italics in original]" (p. 35). In almost every instance in our use of clicker questions, we had the goal of identifying student difficulties so that we, as instructors, could determine how to proceed.

In the next four sections, we describe in more detail how we each used a question or question set to implement those design objectives in our classes:

- Box-and-whisker Plot Lesson Plan — McKnight
- Hypothesis Testing Lesson Plan — Murphy
- Expected Value Lesson Plan — Richman
- Methods for Reporting Statistical Results Lesson Plan — Terry

Each lesson includes an indication of how we would classify the question or question set in the framework proposed by Beatty, et al. [5]; each lesson also explains how the instructor used the response distribution to determine subsequent pedagogical moves. For the example questions described in each lesson, we use an asterisk (*) to indicate the intended correct response to a clicker question.

7.2 Box-and-Whiskers Plots Lesson Plan — McKnight

Even after students have learned the mechanics of making a box-and-whiskers plot, a topic included in many statistics texts these days, they still have little intuition about what these plots tell them about the distributions of data sets or random samples.

Context The students in this course are primarily juniors and seniors majoring in engineering, mathematics, secondary mathematics education, or a science (e.g., physics, meteorology). Calculus is a prerequisite for the course. The course counts towards a minor in mathematics.

Objective Students will be able to determine what box-and-whisker plots reveal about the skew and spread of distributions or data sets.

Sequence This lesson is used after box-and-whisker plots are introduced but before continuous random variables and their probability density functions (pdf's) have been introduced. It foreshadows continuous pdf's and a demonstration of the utility of box-and-whisker plots.

Lesson content The lesson uses a set of pdf's for normal distributions, student's t distributions, chi square distributions, and F distributions, drawn using Mathematica. For each distribution, the pdf is graphed and the graphs of the different distributions are compared. I do this with a *Mathematica* notebook made into a slide show projected on screen to the whole class. Figure 7.1(a) is an example of a pdf graph from the notebook (a chi square distribution in this case); Figure 7.1(b) is another slide, comparing a normal, a student's t, and a chi square distribution. The graphs in the slides are color coded and labeled in the notebook although that is not shown here.

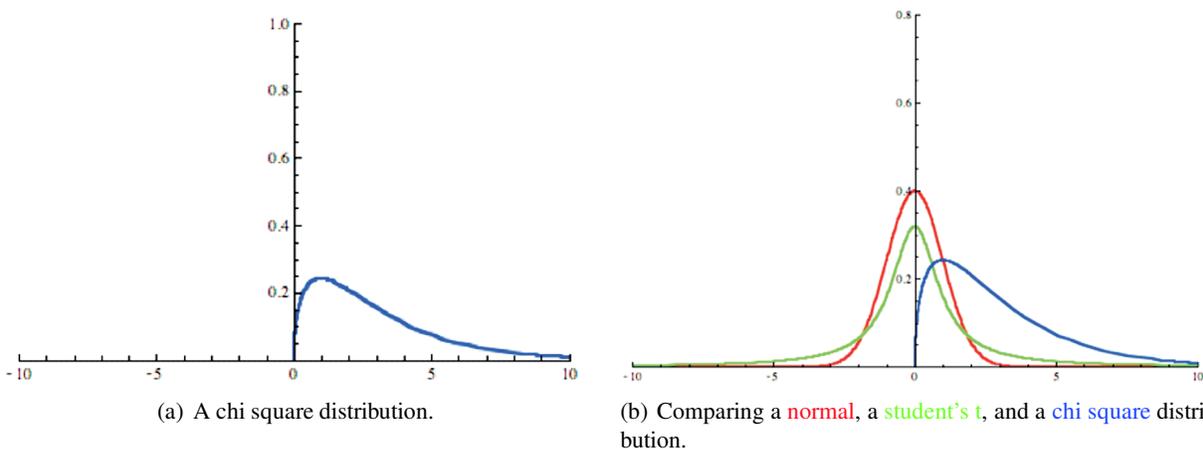


Figure 7.1.

After briefly discussing this content, I show a modified box-and-whisker plot for each of the distributions above the graph of the pdf for that distribution. Since these continuous distributions do not really have endpoints and outliers, I use modified box-and-whisker plots with the whiskers extending to the 5th and 95th percentiles so that the whiskers include the middle 90 percent of the distribution and the box includes the middle 50 percent of the distribution. Figure 7.2(a) is an example of a normal distribution and its box plot; Figure 7.2(b) is an example of a Chi-square distribution and its box plot.

After introducing the box plots for each distribution and noting its skew and spread, I show a new series of graphs with the pdf's for two distributions graphed on the same set of axes and the box plots for each pdf shown above the graphs of the pdf's. Figure 7.3(a) is an example of a normal distribution compared to a Student's t distribution; Figure 7.3(b) compares a normal distribution, a chi-square distribution, and an F distribution with their respective box plots. From these graphs we can see how the spread and skew of the box and the whiskers can help us to picture which pdf is involved.

Clicker Questions In the Beatty, et al., framework, the question set I used (Figure 7.4) can be classified as *stimulate cognitive processes: interpret representations*. I have used this set of questions in the same course over several semesters. The clicker questions and resulting discussions take about twenty to thirty minutes in a typical class. Be-

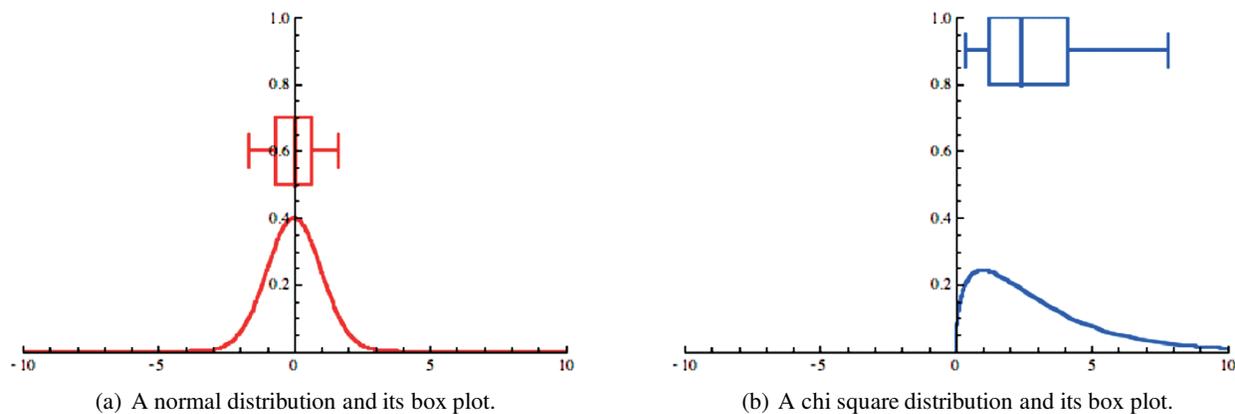


Figure 7.2.

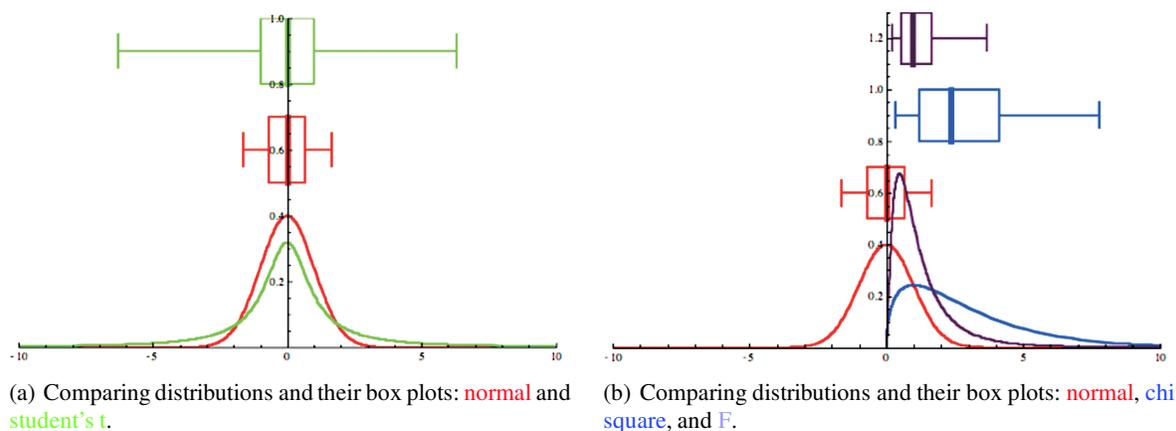


Figure 7.3.

tween one fourth and one half of the class chose one of the incorrect answers, which generates discussion to address the thus-identified difficulties. Progress through the set of questions typically shows increasing proportions of the class getting the later questions right, regardless of the ordering of the questions in the set. If there have been persistent misunderstandings among about one-third of the class, I typically generate some pdf curves on the board and ask students to suggest the appearance of their box-and-whiskers plots (and at times to draw them on the board). I have occasionally drawn two pdf's on the board and asked students to discuss differences in the box-and whisker plots for the two graphs. When there has been little persistent misunderstanding by students after the clicker questions, I typically regard this discussion as finished and move on to the next topic.

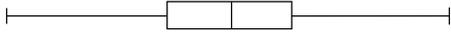
(The clicker question shown in Figure 7.4(a) is an example of the annotations included in the questions we produced through our grant project. In the interests of space, the additional examples of questions appearing in this paper do not include the complete annotation. All of our questions with their annotations are available at www.ou.edu/statsclickers.)

7.3 Expected Value Lesson Plan — Richman

Generally, students do not have an intuitive grasp of expected value, although they have the ability to merge two key ideas to accomplish this task. By using the frequency definition of probability and adding the value of a number, the probability-weighted values can be calculated. Examining how expected value changes as the values and the probabilities vary is the key goal of this lesson.

Context The students in this course are primarily juniors and seniors majoring in Meteorology. Calculus is a prerequisite for the course.

If a large sample were drawn from a normal distribution and accurately represented the population, which of the following is most likely to be a box plot of that sample?

(A) 

(B) 

(C) 

(D) 

(E)* Two from (A)–(D) are correct.
 (F) Three from (A)–(D) are correct.
 (G) All from (A)–(D) are correct.

Explanations

(A) This response is one of two appropriate box plots; the other is (D). This box plot might represent a normal distribution because it is symmetrical both within the box and with the length of the whiskers. The box in (A) is wider than that of (D) indicating a larger standard deviation.

(B) A box plot for a normal distribution would be completely symmetrical. In this box plot, the median is not symmetrical within the box although the whiskers are of the same length.

(C) The whiskers are of different lengths so this represents a skewed rather than a symmetrical distribution.

(D) This response is one of two appropriate box plots; the other is (A). This box plot might represent a normal distribution because it is symmetrical both within the box and with the length of the whiskers. The box in (D) is narrower than that of (A) indicating a smaller standard deviation.

(E) Correct — (A) and (D) are both box plots for normal distributions.
 (F), (G) Only (A) and (D) are box plots for normal distributions.

(a) An example of a clicker question and its complete annotation.

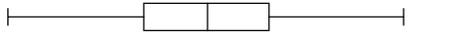
This box plot is for a sample that accurately represents a normal distribution:

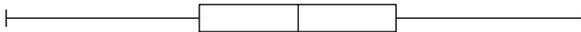


Which of the following box plots is for a sample that represents a student's t-distribution with the same standard deviation and sample size as the normal distribution above?

(A) 

(B) 

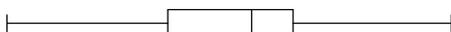
(C) 

(D)* 

(E) Two from (A)–(D) are correct.
 (F) Three from (A)–(D) are correct.
 (G) All from (A)–(D) are correct.

If a large sample were drawn from a chi-square (χ^2) distribution (with degrees of freedom ≤ 10) and accurately represented the population, which of the following is most likely to be a box plot of that sample?

(A) 

(B) 

(C) 

(D)* 

(E) Two from (A)–(D) are correct.
 (F) Three from (A)–(D) are correct.
 (G) All from (A)–(D) are correct.

(b) A set — along with the question shown in (a) — intended to demonstrate what box plots reveal about distributions (specifically normal, student's t, and chi square).

Figure 7.4.

Objective Students will use strategically engineered calculations to determine how these the expected value and variance change under additive and multiplicative constants.

Sequence After I introduce expected value for the mean and variance, the equations, and the idea that these are population estimates, I use a sequence of clicker questions that replaces what was formerly an instructor-centered lecture of worked examples.

Lesson content This description and the results shown in Figure 7.5 are from Fall 2007 use. The lesson integrates ideas about the mean, the variance, frequency, probability, and scale. Most of the students preferred to answer the questions intuitively rather than using formulas, which led to some interesting results. Once the students proceeded beyond the intuitive nature of the first pair of problems, logical faults in their perceptions emerged on some questions and the distributions of the answers to the clicker questions widened.

Clicker questions The question set I use can be classified as stimulate cognitive processes: extend the context [5]. In fact, many of the distracters in these items were drawn from actual student work on analogous open-ended problems. The initial pair of clicker questions (Figure 7.5(1) and 7.5(2)) is straightforward and mirrors the example that I present early in the lesson. Subsequent questions employ additive constants, multiplication of the range, and

<p>1) Suppose that a random variable x has only two values, 0 and 1. If $\Pr(x = 0) = 0.5$ then what can we say about $E(x)$?</p> <p>(A) $E(x) = 0$ 4%</p> <p>(B)* $E(x) = 0.5$ 86%</p> <p>(C) $E(x) = 1$</p> <p>(D) Either (A) or (C) is possible. 10%</p> <p>(E) Both (A) and (C).</p> <p>(F) insufficient information</p>	<p>2) Suppose that a random variable x has only two values, 0 and 1. If $\Pr(x = 0) = 0.5$ then what can we say about $\text{Var}(x)$?</p> <p>(A) $\text{Var}(x) = -0.25$</p> <p>(B) $\text{Var}(x) = 0$</p> <p>(C)* $\text{Var}(x) = 0.25$ 100%</p> <p>(D) $\text{Var}(x) = 0.5$</p> <p>(E) $\text{Var}(x) = 1$</p> <p>(F) insufficient information</p>
<p>3) Suppose that a random variable x has only two values, 3 and 4. If $\Pr(x = 3) = 0.5$ then what can we say about $E(x)$?</p> <p>(A) $E(x) = 0.5$</p> <p>(B) $E(x) = 1$</p> <p>(C) $E(x) = 3$ 5%</p> <p>(D)* $E(x) = 3.5$ 95%</p> <p>(E) $E(x) = 4$</p>	<p>4) Suppose that a random variable x has only two values, 3 and 4. If $\Pr(x = 3) = 0.5$ then what can we say about $\text{Var}(x)$?</p> <p>(A)* $\text{Var}(x) = 0.25$ 100%</p> <p>(B) $\text{Var}(x) = 0.5$</p> <p>(C) $\text{Var}(x) = 0.75$</p> <p>(D) $\text{Var}(x) = 1.0$</p> <p>(E) $\text{Var}(x) = 3.25$</p> <p>(F) $\text{Var}(x) = 3.5$</p>
<p>5) Suppose that a random variable x has only two values, 0 and 2. If $\Pr(x = 0) = 0.5$ then what can we say about $E(x)$?</p> <p>(A) $E(x) = 0$</p> <p>(B)* $E(x) = 1$ 100%</p> <p>(C) $E(x) = 2$</p> <p>(D) Either (A) or (B) is possible.</p> <p>(E) Both (A) and (B).</p> <p>(F) insufficient information</p>	<p>6) Suppose that a random variable x has only two values, 0 and 2. If $\Pr(x = 0) = 0.5$ then what can we say about $\text{Var}(x)$?</p> <p>(A) $\text{Var}(x) = 0$</p> <p>(B) $\text{Var}(x) = 0.25$ 5%</p> <p>(C) $\text{Var}(x) = 0.5$ 33%</p> <p>(D)* $\text{Var}(x) = 1$ 62%</p> <p>(E) $\text{Var}(x) = 2$</p>
<p>7) Suppose that a random variable x has only two values, 0 and 1. If $\Pr(x = 0) = 0.4$ then what can we say about $E(x)$?</p> <p>(A) $E(x) = 0$</p> <p>(B) $E(x) = 0.4$</p> <p>(C) $E(x) = 0.5$</p> <p>(D)* $E(x) = 0.6$ 100%</p> <p>(E) $E(x) = 1$</p> <p>(F) insufficient information</p>	<p>8) Suppose that a random variable x has only two values, 0 and 1. If $\Pr(x = 0) = 0.4$ then what can we say about $\text{Var}(x)$?</p> <p>(A) $\text{Var}(x) = 0$</p> <p>(B) $\text{Var}(x) = 0.16$ 14%</p> <p>(C)* $\text{Var}(x) = 0.24$ 86%</p> <p>(D) $\text{Var}(x) = 0.36$</p> <p>(E) $\text{Var}(x) = 0.6$</p> <p>(F) $\text{Var}(x) = 1$</p>

Figure 7.5. A set of questions intended to help students practice with calculating expected values.

probability weighting. Through this sequencing of questions, students can determine that the expected value is scale dependent and that adding a constant changes the mean by a magnitude equal to that constant. Some results are not as intuitive to some students as others and the sequencing of the questions can push students to think deeply about relationships and connections. Students who gave answers that were not correct discussed their thinking with other students and/or with me. In some cases, I presented a new example in response to the identified difficulties. By the end of the question set, students discovered, for example, that the mean is the proportion of non-zero values.

Closure We spent the last few minutes of class creating a table of how scale impacts the mean and the variance. This closing activity reinforced the idea that the mean is sensitive to scale changes under addition and multiplication, whereas the variance is unchanged under additive constants but changed under multiplicative constants.

7.4 Hypothesis Testing Lesson Plan — Murphy

I teach with a set of notes typed into a course packet. The course packet includes example problems with space for the students to work on them in groups during class. Clicker questions are not included in the notes but are used as an additional in-class activity. One of my favorite sets of clicker questions (Figure 7.6) developed during the grant period is intended to guide students through the process of conducting a hypothesis test. The questions in this set also provide opportunities to compare and contrast one- vs. two-tailed tests and the z-distribution vs. the t-distribution.

Context. Same as McKnight’s box-and-whiskers plot lesson plan.

Objective. Students will make informed decisions about critical values based on problem context and the target alternative hypothesis.

Sequence. With the goal of identifying persistent misconceptions and encapsulating ideas for students, I use this set of questions during a class session after I have introduced hypothesis testing and students have worked together on some problems in the course notes.

Lesson content and clicker questions. This description and the results shown in Figure 7.6 are from Spring 2008 use. As students walked into class, I presented the first question in Figure 7.6. Having a clicker question posed even before class had officially started encouraged students to get immersed in content immediately — they got out their notes and clickers to answer the question (Beatty, et al., design objective *directing attention and raising awareness* [5]). Noting that none of the 16 students who responded to the question answered it correctly, but wanting to spend more class time on other concepts, I explained that this particular question was intended to emphasize the importance of distribution assumptions (Beatty, et al., *stimulate cognitive processes: omit necessary information* [5]). I pointed out that the question stem does not specify that the population distribution is approximately normal. I emphasized that in some statistical situations, it is acceptable to assume this property, but that assumption should be explicit and conscious rather than neglected.

The subsequent three questions in this set all describe the population distribution as approximately normal, thus allowing focus to shift to other ideas. In Figure 7.6(2), choices (A)–(D) are the critical values for the left and right tails in the z-distribution and the t-distribution. Thus, students need to make two decisions: one vs. two tails and using

<p>1) A random sample of 25 observations, with a mean of 44.4 and a sample standard deviation of 3.5, is drawn from a population. If one sets up a hypothesis test that the population mean is equal to 43 against an alternative that the population mean is not 43, using $\alpha = 0.05$, what is the 0.05 significance point (critical value) from the appropriate distribution?</p> <p>(A) 1.96 38%</p> <p>(B) 2.064 56%</p> <p>(C) -1.96</p> <p>(D) -2.064 6%</p> <p>(E)* None of the above</p>	<p>2) A random sample of 25 observations, with a mean of 44.4 and a sample standard deviation of 3.5, is drawn from a population that is approximately normally distributed. If one sets up a hypothesis test that the population mean is equal to 43 against an alternative that the population mean is not 43, using $\alpha = 0.01$, what is the 0.01 significance point (critical value) from the appropriate distribution?</p> <p>(A) 2.576 15%</p> <p>(B) 2.797 35%</p> <p>(C) -2.576</p> <p>(D) -2.797</p> <p>(E) Both (A) and (C) are correct. 5%</p> <p>(F)* Both (B) and (D) are correct. 45%</p>
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Figure 7.6. A set of clicker questions intended to lead students through hypothesis testing.

<p>3) A random sample of 25 observations, with a mean of 44.4 and a sample standard deviation of 3.5, is drawn from a population that is approximately normally distributed. If one sets up a hypothesis test with population mean equal to 43 against an alternative that the population mean is not 43, using $\alpha = 0.01$, what is the value of the test statistic?</p> <p>(A)* 2.000 76%</p> <p>(B) 2.576 10%</p> <p>(C) 2.797 14%</p> <p>(D) 2.857</p> <p>(E) 10.000</p>	<p>4) A random sample of 25 observations, with a mean of 44.4 and a sample standard deviation of 3.5, is drawn from a population that is approximately normally distributed. If one sets up a hypothesis test with population mean equal to 43 against an alternative that the population mean is not 43, using $\alpha = 0.01$, does one reject the null hypothesis and why?</p> <p>(A) Yes, the test statistic is larger than the tabled value.</p> <p>(B) No, the test statistic is larger than the tabled value. 5%</p> <p>(C) Yes, the test statistic is smaller than the tabled value. 10%</p> <p>(D)* No, the test statistic is smaller than the tabled value. 86%</p> <p>(E) insufficient information</p>
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Figure 7.6. cont. A set of clicker questions intended to lead students through hypothesis testing.

the z-distribution vs. using the t-distribution. That 35% of the 20 students responding chose (B) and another 45% chose (D) indicated that students were correctly opting to use the t-distribution (*correctly* based on the guidelines provided in their textbook, namely $n < 30$). I made that observation to the class and then asked them to discuss the difference between (B) and (D), giving students the opportunity to discuss the implications of the alternative hypothesis $H_A : \mu \neq 43$ (Beatty, et al., *stimulate cognitive processes: compare and contrast* [5]). The question in Figure 7.6(4) can help students distinguish between the *critical value* (or *critical point*) from the *test statistic*. Instructors tend to use these terms in explanations, but it isn't clear that students make the correct connections for those terms.

7.5 Methods for Reporting Statistical Results Lesson Plan — Terry

Once students have learned the basic elements of drawing inferences, they often remain confused regarding the appropriate method to be used for reporting statistical results. This may not be surprising as many disciplines often develop unique styles for representing data analysis results. In point of fact, the American Psychological Association (APA) Task Force on Statistical Inference [85] ultimately concluded that confidence intervals should be the preferred method for reporting statistical results rather than simply p -values and effect sizes. Because students are hopefully aware at the end of the course of these different ways of reporting statistical results, this lesson is intended to give a sense of which of three methods are preferable with regard to presenting the most information.

Context. The students in this course are primarily freshman and sophomores entering the course with an algebra background but without calculus. Whereas the course serves as a quantitative literacy requirement within a College of Arts and Sciences, the students are mostly health-related majors (pre-nursing, pre-physical-therapy, pre-med, etc.). The course focuses on conceptual understanding over mathematical rigor and computational solutions.

Objective. Students will consider the information conveyed by reporting p -values only, p -values + effect size estimates, and confidence intervals. Students will recognize which method of reporting statistical results is preferable in terms of information content and will understand why confidence intervals are often preferable to p -values.

Sequence. This lesson is used in the next-to-last class session of the semester. The students are introduced to confidence intervals within the first unit of the course on sampling. Next, students are introduced to p -values during the third unit of the course on experimental design, with the emphasis on testing for statistical significance and the p -value as a way to rule out sampling error as a possible cause of an observed difference between treatment and control groups. In the next three units, students are introduced to various effect size measures: the standardized mean difference in descriptive statistics, relative risks and odds ratios in tabular association, and r and R^2 measures in correlation and regression. Finally, the last unit attempts to integrate the process of making statistical inferences via p -values, confidence intervals, effect sizes, and sampling distributions.

Lesson content. The lesson begins with a presentation of the clicker question, which is designed with the Beatty, et al., objective of stimulate cognitive processes: compare and contrast [5]. Figure 7.7 reports the Spring 2008 results. Noting that there is a wide of variety of responses to this question, as expected and intended, I ask the students

strategically engineered questions to create a Socratic dialogue between myself and the students. The first of these questions attends to option I and asks: If you know there is a mean difference between the Aleve and Tylenol groups, and you know the variability within each of the groups, can you compute an effect size? Once the students convince one another that this can be done, I then ask the next question: Can you eliminate chance as a possible explanation for the observed effect? Once students are comfortable with the idea that only with sample size information can this be done, we move to questions about option (A).

The next question is then: If the result is statistically significant, is it necessarily the case that it must be clinically significant? I have found that it is important to develop a set of examples demonstrating that statistical significance can be obtained but with little clinical impact, preferably with real data, such as the relation between cell phone use and brain cancer. With the proper dataset (e.g., large sample size), the students begin to understand that statistical significance may not mean clinically important differences, especially with large sample sizes.

Finally, to examine option (B), I ask if a very small p -value (e.g., $p < .001$) implies a large effect size. This is always a difficult concept to get across to students because it intuitively seems that very improbable results must be important. Again, using real examples as recommended in the GAISE Report [33], I demonstrate how p -values are as much a function of sample size as of effect size.

By now most students have determined that none of the options (A)–(C) give them exactly what they want to know. Using graphical tools, the students — in groups — work on interpreting several actual studies that use confidence intervals to present results. By the end of the lesson, students should recognize that the use of confidence intervals present at least three important pieces of information: 1) whether sampling error can be eliminated as a plausible explanation for an observed effect; 2) a direct estimate of the effect size located in the middle of the interval; and 3) a visual representation of the uncertainty as found in the lower and upper bounds of the confidence interval.

Robert is asked to conduct a clinical trial on the comparative efficacy of Aleve versus Tylenol for relieving the pain associated with muscle strains. He creates a carefully controlled study and collects the relevant data. To be most informative in his presentation of the results, Robert should report

(A)	whether a statistically significant difference was found between the two drug effects.	16%
(B)	a p -value for the test of no drug effect.	21%
(C)	the mean difference and the variability associated with each drug's effect.	5%
(D)*	a confidence interval constructed around the observed difference between the two drugs.	58%

Figure 7.7. A clicker question that generates good class discussion.

7.6 Concluding Remarks

As with other groups of teachers talking about classroom practice, we each learned about our own teaching by having conversations about how we were using the clicker questions. We found that we were using the questions in ways that literature suggests that instructors across disciplines use it:

- Pre-testing prior to instruction.
- Pre-testing prior to instruction, followed by post-testing.
- Testing concurrent with instruction.
- Post-testing subsequent to instruction.
- Testing for the specific purpose of generating class discussion.
- Testing, followed by peer-discussion, followed by re-testing.

In addition, the multi-disciplinary aspect of the grant project (undergraduate mathematics education, meteorology, psychology) had an unpredicted, but positive, outcome: our own teaching evolved as a direct result of discussing such facets of statistics as definitions, approaches, procedures, decision-making, and discipline-specific culture. The project group met at least weekly to discuss the clicker questions — priority topics; typical misconceptions that could be turned into meaningful, informative distracters; experiences with class-testing the items. We ended up producing fewer items than we had intended because these conversations were richer and lasted longer than we had anticipated. Differences in perspective can be partially attributed to the department housing the course and the socio-cultural expectations for the course. Some observations that we considered include:

- Whereas in mathematics the level of absolute truth (i.e., a proof or a counterexample) is high, in statistics the level of discipline-specific judgment and personal preference is high. As an example, not all of the authors on this paper agree on the “correct answer” to the clicker question in Terry’s lesson plan above.
- Many mathematics problems can be applied to specific context, but can also be context-free (mathematics can be powerful specifically because of the level of abstraction), whereas in statistics the context for the data is critical for producing a valid, meaningful conclusion.
- Different disciplines emphasize different aspects of statistics: an introductory course in psychology may emphasize sampling ideas, such as random assignment to treatment groups, that are not necessarily appropriate in science, such as meteorology, but a course in meteorology may emphasize a number of probability distributions that are not needed at the introductory level in psychology.
- There is not always agreement among statistics practitioners about the meaning of certain terms or the importance of certain topics, as evidenced by the number of questions that students asked during class that were brought to project meetings and generated discussion among the project team members or the number of instances of questions that one person would write that another person would claim not to understand.

Just as students learn from discussing concepts with each other, so we also learned by discussing those same concepts with each other.

Acknowledgments

The authors would like to thank the graduate research assistants: Andrew (Luke) Brown, Krista Hands, Matthew Schuelke, and Xiaoqian Wang. We would also like to thank the students in our classes who participated in the project.

This material is based upon work supported by the National Science Foundation’s Course, Curriculum, and Laboratory Improvement (NSF CCLI) program under Grant No. 0535894. Any opinions, findings, and conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

8

Using Clickers in a Statistics Classroom

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This paper provides three lesson plans to demonstrate how clickers were used in an introductory statistical methods course. One of the lessons was used in the lecture setting, one was part of an exam review session, and the third shows an example of how clickers were used in a smaller lab setting. In each case advice relevant to implementing the lesson will be provided.

8.1 Background about the Course

Statistics 350 is a 4-credit course taught every semester (14 week term) at the University of Michigan. Most students are undergraduates taking this course to fulfill a requirement for their major or to fulfill a quantitative reasoning requirement that is necessary for graduation from the University. Course topics include descriptive statistics (numerical and graphical summaries), probability and sampling distributions. Extensive attention is also given to inference procedures, including confidence intervals and hypothesis testing for proportions (one- and two-sample), means (one-sample, paired, independent, and one-way analysis of variance), simple linear regression, and chi-square analyses.

Students attend three hours of lecture and one 1.5 hour computer lab each week. There are six lecture sections that range in size from 60 students to over 400 students. The schedule of lectures also varies: There are sections offered each week as three one-hour sessions, two ninety-minute sessions, and one three-hour session. During any given week, however, the same basic material is covered in all lecture sections. Lab sections are more uniform in terms of size and structure. There are 50 lab sections, most of which have either 21 or 27 students. The same activities are covered during each 90-minute lab section. The goals of the labs are to reinforce concepts presented in lecture and provide hands-on data analysis experience using the statistical analysis package SPSS. Occasionally, some new material is covered in lab before it has been presented in detail during lecture. In a typical lab session students either work through data analysis activities or complete word problems with the guidance of their graduate student instructor (GSI). Despite the overall variation in size and structure in the course, we have successfully incorporated clickers into both lectures and labs.

8.2 Clickers in our Course: Historical to Current Use

Clickers have been used in Statistics 350 since September 2006. Initially the TurningPoint personal response system (www.turningtechnologies.com/) was used, which only allowed for multiple choice clicker questions. The department provided one small set of remotes which were used in the laboratory sections of the course just during exam

weeks for review. With this model of clicker use, the students did not need to purchase their own remote, however a system for the distribution and collection of the remotes did need to be developed.

Starting in May 2007, a new clicker system was introduced to Statistics 350, with the capability for students to input numeric responses as well as respond to multiple-choice questions. This system — Qwizdom® (www.qwizdom.com) — was adopted as the official personal response system of the College of Literature, Science, and the Arts at the University of Michigan. The college provides technical support to both instructors who choose to use clickers and to students, who may use the same clicker remote for several classes throughout their college years. Students were required to purchase and bring their own remote to class, which allowed clickers to be used in every lecture and lab session of the course.

There are many pedagogical goals that can be accomplished through a clicker question that have been described extensively elsewhere (e.g., [4, 5, 28]). The pedagogical goals of clicker questions most commonly focused on in Statistics 350 were:

- Checking student understanding of, or ability to apply, material recently covered;
- Addressing common misconceptions or pitfalls; and
- Collecting data about students for analysis, or to illustrate statistical phenomena.

For each of these goals, we provide an example lesson plan to demonstrate how clickers were used in our teaching of Statistics. When appropriate, the correct answer choice(s) for a clicker question is(are) shown in *italics*.

8.3 Lesson Plan I: Using Confidence Intervals to Test Hypotheses

Checking student understanding of, or ability to apply, material recently covered

The following is an example of a lecture lesson on using confidence intervals for hypothesis testing. For this lesson, the instructor presents several guidelines on the topic to students (see Figure 8.1). Immediately following the presentation, students are asked a series of four questions that required them to apply the principles. Each question had two parts that students responded to with clickers: The first part provided a specific hypothesis to test using a confidence interval and the second part asked students to explain their previous response. Note that for each question, the hypothesis and/or significance level changed while the confidence interval remained the same. The background information for the problem and the specific clicker questions are given in Figures 8.2 and 8.3. As can be seen, the first two questions (parts a and b) are straightforward applications of the lesson, while the second two questions (parts c and d) require students to extend the principles presented.

There was one semester where the second author had a particularly successful experience with this lesson. Most students in the class selected the correct answers to the clicker questions for parts a and b. However, when students

- A confidence interval (CI) provides a **range of plausible (reasonable) values** for the parameter.
- The null hypothesis gives a *null value* for the parameter.
- **If the null value is one of the reasonable values** found in the CI, the **null hypothesis would not be rejected.**
- **If the null value was not found in the CI of reasonable values** for the parameter, then the **null hypothesis would be rejected.**

Notes:

1. (1) The alternative hypothesis should be **two-sided**. Sometimes you can reason through the decision for a one-sided test.
2. (2) The **significance level coincides with the confidence level** (e.g., $\alpha = 0.05$ with a 95% confidence level). Sometimes you can reason through the decision if these don't exactly correspond.
3. (3) The relationship holds exactly for tests about a population mean or the difference between two population means. In most cases, it will hold for tests about a population proportion or the difference between two population proportions.

Figure 8.1. Summary of using confidence intervals for testing hypotheses.

<p>Try It! Cockroach Sugar Levels</p> <p>In a biology study the measured response was the sugar levels absorbed in the hindguts of cockroaches. A 95% t confidence interval for the population mean amount of sugar (in mg) was $4.2 \pm 2.3 \rightarrow (1.9, 6.5)$.</p> <p>Look at each set of hypotheses and determine whether you would reject H_0, fail to reject H_0, or you cannot tell.</p>

Figure 8.2. Background information for the cockroach example.

<p>Part a. Test $H_0 : \mu = 7$ vs $H_a : \mu \neq 7$ at $\alpha = 0.05$</p> <p>(A) <i>Reject H_0</i> (B) <i>Fail to reject H_0</i> (C) <i>Can't tell</i></p>
<p>Continuation of Part a. Why reject H_0?</p> <p>(A) <i>Since the value of 7 IS in 95% CI for μ</i> (B) <i>Since the value of 7 is NOT in 95% CI for μ</i></p>
<p>Part b. Test $H_0 : \mu = 6$ vs $H_a : \mu \neq 6$ at $\alpha = 0.05$</p> <p>(A) <i>Reject H_0</i> (B) <i>Fail to reject H_0</i> (C) <i>Can't tell</i></p>
<p>Continuation of Part b. Why fail to reject H_0?</p> <p>(A) <i>Since the value of 6 IS in 95% CI for μ</i> (B) <i>Since the value of 6 is NOT in 95% CI for μ</i></p>
<p>Part c. Test $H_0 : \mu = 6$ vs $H_a : \mu \neq 6$ at $\alpha = 0.01$</p> <p>(A) <i>Reject H_0</i> (B) <i>Fail to reject H_0</i> (C) <i>Can't tell</i></p>
<p>Continuation of Part c. Why fail to reject H_0?</p> <p>(A) <i>Since 99% CI would be wider</i> (B) <i>Since 99% CI would be narrower</i> (C) <i>Since 99% CI would still contain 6</i> (D) <i>Since 99% CI may not contain 6</i></p>
<p>Part d. Test $H_0 : \mu = 6$ vs $H_a : \mu \neq 6$ at $\alpha = 0.10$</p> <p>(A) <i>Reject H_0</i> (B) <i>Fail to reject H_0</i> (C) <i>Can't tell</i></p>
<p>Continuation of Part d. Why can't we tell?</p> <p>(A) <i>Since 90% CI would be wider</i> (B) <i>Since 90% CI would be narrower</i> (C) <i>Since 90% CI would still contain 6</i> (D) <i>Since 90% CI may not contain 6</i></p>

Figure 8.3. Eight clicker questions for the cockroach example

answered the first clicker question for part c, the class was evenly split between Fail to Reject H_0 and Can't Tell. Without any instruction from the teacher, and before seeing the second clicker question, students were told to discuss the answer with a neighbor. They were then given the opportunity to answer the clicker question again, and every student chose the correct answer. Students were obviously impacted by this shift in their collective responses — there was a murmur of surprise over the drastic change in responses, and one student even commented (from the back of the room) that seeing the change was “so cool” and that “this is so much fun.” On the follow-up clicker question for part c (where students were allowed to select more than one answer choice) and the similar clicker questions for part d, nearly all students selected the correct response(s).

The pedagogical technique of answer-discuss-answer (used for part c) is known as Peer Instruction [23, 56]. One of the primary benefits of using clicker technology is the ease and flexibility with which Peer Instruction can be implemented. An instructor can decide instantly whether the discussion and re-answer steps of Peer Instruction are

necessary. In the Cockroach Example, initial student answers to part c indicated confusion that was resolved by the subsequent small-group discussions, while answers to the remaining questions indicated that additional discussion was not needed. In cases where a large proportion of student answers are incorrect even after Peer Instruction, whole class discussion or additional instruction could be used to improve understanding. Viewing the instant tally of student responses provided by the clicker software can help direct the instructor to the most appropriate course of action.

8.4 Lesson Plan II: How to Look at your Data

Addressing common misconceptions or pitfalls

Many Statistics courses spend time discussing appropriate ways to look at and summarize data. For quantitative data, we often use a histogram to show the distribution and numerically summarize the center with a mean. Another aspect we emphasize in our class is that the data are often collected over time, in which case a time plot of the data would be important to examine. Students, however, often skip graphic exploration and immediately compute summary measures like a mean and standard deviation — even if there are features in the data (e.g., a trend over time or a bi-modal distribution) which might imply these would not be appropriate to compute.

To help students avoid this pitfall and get them thinking about the proper order of data exploration, the following clicker question was used in a review session for the first exam. The review took place a few weeks after material on how to turn data into information was covered in lectures and labs. As part of this review, the instructor recapped some of the main ideas covered in the course to date. One of the charts specifically addressed the tools for looking at and summarizing data (see Figure 8.4). After other concepts were recapped, a series of review questions were covered. Some of the questions were posed with clickers, others were not. For one question, the students were first given a brief background about some data that had been collected (see Figure 8.5). The slide shown in Figure 8.6 posed the clicker question asking students to determine the correct order in which to produce several graphs. This clicker question requires students to think about what they would do, but not to actually do it — a technique for writing effective clicker questions referred to as *strategize only* by Beatty et al. [5]. The correct order involves first examining a time plot, as this production data was gathered over 60 days. Then, only if the time plot showed evidence of stability, should the next tool be used — a histogram to look at the overall production distribution. Finally, if the histogram showed a

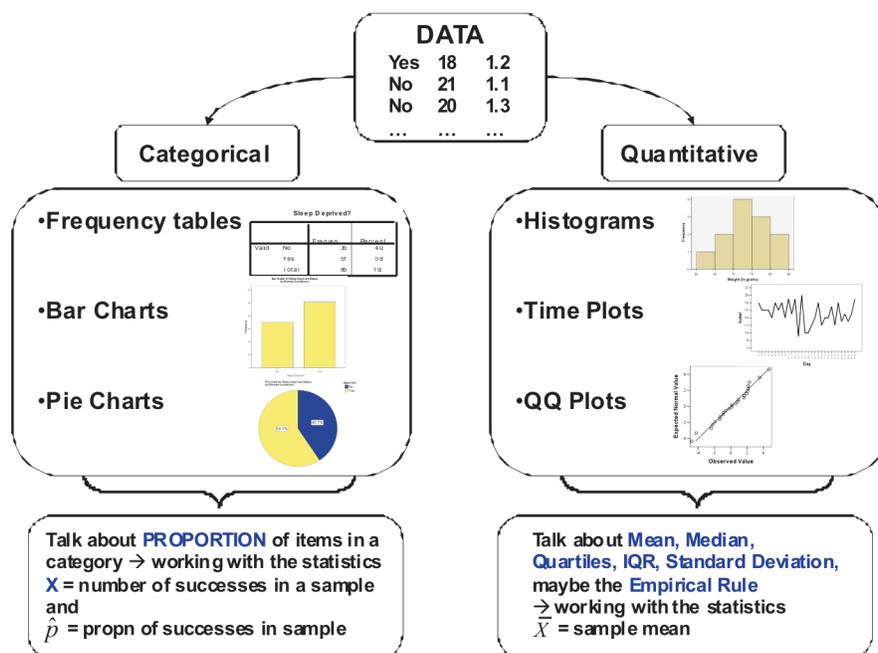


Figure 8.4. Summary of how to look at data.

A small manufacturer produces cups used by coffee store chain for serving coffee. Some changes to the current production process are proposed that might put the workers on an initial learning curve, but it would involve minor equipment changes at a very low cost, and could lead to a higher mean production level.

It was decided to set up this new production process and run an experiment.

Data for a 60-day production run under this new process have just come in (number of cups produced daily).

Figure 8.5. Background information for the “How to look at data” clicker question.

An employee has entered the data into SPSS and needs your assistance as to what to do next. These summaries should be performed in a particular order, moving to the next summary provided the previous summary indicates it is appropriate to do so. CLICK IN the correct order. . .

1. Compute the sample mean production level for 60 days
2. Make a time plot of the production levels
3. Make a histogram of the production levels

(Correct response: 2 3 1)

Figure 8.6. How to look at data clicker question.

reasonably homogeneous set of observations, with no strong skewness or outliers, then computing a mean production level would be reasonable. While some students selected each of the six possible combinations, just over 50% selected the correct sequence and about two-thirds correctly identified the need to examine the time plot first. After viewing these responses, the instructor was then able to lead students through the reasoning behind the correct sequence.

While the ability to enter response sequences may not be available with every clicker system, this question could easily have been converted to a multiple-choice format where the answer choices represented all possible sequences.

8.5 Lesson Plan III: Testing about a Population Proportion

Collecting data about students for analysis, or to illustrate statistical phenomena.

Many statistics courses spend time on inference procedures such as confidence interval estimation and hypothesis testing. For some courses, a smaller weekly lab section provides the opportunity for students to review and apply these procedures. Students often become more interested in these applications when the research question is something they can relate to, or that they generate themselves. In the lab following the lecture presentation on how to test hypotheses about a single population proportion, students were given the opportunity to come up with a theory regarding the UM student population and clickers were used to gather the data for testing this theory. An outline of this task was provided as a handout to the students in lab. The full handout is provided in the appendix.

The lab instructor first did a brief review of the steps for testing hypotheses about a population proportion and then distributed the handout. With the guidance of the instructor, students proposed and came to an agreement on a question of interest regarding the UM student population. The general form of the question was: “Do a majority of UM students _____?” Once the question of interest was set, the question was posed as a clicker question (see Figure 8.7). This session was run in the anonymous mode so the responses could not be traced back to particular students and students would feel more comfortable responding. The students in the lab used their clickers to respond and the results were immediately displayed, providing the total number of yes and no responses in this sample.

The students then formed small groups and carried out the analysis using the data just collected. The instructor moved around the room to address questions as the group work commenced. In some labs, small groups were formed first and each group came up with a question of interest. For these labs the clicker presentation had a set of slides, each

Yes or No

- Do you/Have you _____?

Figure 8.7. Collecting data clicker question for in-lab project.

like that in Figure 8.7, so the collection of data to address a number of theories was just a few clicks away.

With some clicker systems, numeric responses can be entered (e.g., height, number of text messages sent in a day, number of friends on Facebook), which would expand the types of data that could be collected and the types of analyses that could be conducted by the students on data that is of interest to them.

8.6 Final Thoughts

Several lesson plans involving clickers that were successfully implemented in an introductory statistics course were presented here. However, as with any new technology or pedagogical technique, clickers may not be effective if they are not used in a well-planned, purposeful manner. In particular, clicker questions need to be an integral part of the course that help students *think* about concepts and confront common misconceptions. Clickers can also be used for fun, to quickly and easily collect data about the students. This can be especially useful in a statistics course, for students to practice the procedures we teach.

8.7 Appendix: In-Lab Project Handout — Testing for a Population Proportion

Note: More space was provided in the actual handout for answering the questions but was condensed here for inclusion in this appendix.

Name: _____ Section: _____ Group: _____

In this project, we are going to investigate using the sample proportion to test a theory about the value of the population proportion. For the purpose of this activity, we are going to assume that the lab section is a representative random sample of the UM student body. Follow the steps below to practice using this test.

1. Determine a question to investigate by filling in the following question.

“Do a majority of UM students _____?”

Based on your question, write down the appropriate null and alternative hypotheses:

H_0 : _____ H_A : _____

where _____ represents: _____
(symbol) (verbal description of parameter)

2. Once the question has been determined, the instructor will pose the question to the class using the clicker system. Record the results below.

Number of students responding:	
Number of student who responded “yes”:	
Sample proportion of student who responded “yes” : (symbol and value)	

3. This test is often simplified by using a normal approximation for the binomial. Provide the checks necessary to see if this approximation may be made:
4. Now, calculate the appropriate test statistic. (Ask yourself if you are making an approximation or doing an exact binomial test.)

What is the distribution of the test statistic under the null hypothesis?

5. Use your test statistic in part (4) to calculate the corresponding p -value.
6. What is your decision at the 5% confidence level? Also write out your real world conclusion in the context of the problem.
7. Suppose instead of testing to see if there was a majority, you were testing to see if at least 90% of UM students _____ . Would you be able to use the normal approximation? Why or why not?

9

Engaging Statistics Students with Classroom Response Systems

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9.1 Introduction

In this paper, I describe my use of a classroom response system to engage students in the probability and statistics course I teach for undergraduate engineering students. The system makes it possible for me to expect each of my students to think about and answer the questions I pose to them in class, leading to greater participation and engagement. The system also provides me with immediate feedback on all of my students' learning, allowing me to tailor my class sessions to the learning needs of my students. Furthermore, I have found that asking multiple-choice questions of my students in this way helps them develop conceptual understanding of important ideas in statistics. Conceptual understanding is the primary learning goal in this course, and it is one that can be difficult to achieve with students who are often focused on procedures and computations. Thus, clickers help me to create an active, responsive learning environment during class in which students are engaged with important course content.

9.2 Course Overview

The one-semester probability and statistics course in which I use a classroom response system is designed to introduce undergraduate engineering students to fundamental concepts and procedures frequently used in engineering applications. The course typically enrolls between forty and sixty students, most of whom are juniors or seniors. Although many of them have taken a probability or statistics course in high school, few of them bring any great interest in probability or statistics to the course. As a teacher, I feel I have a responsibility to introduce my students to interesting and useful ideas and techniques and to do so in a way that engages them in the learning process.

To that end, in the syllabus I introduce students to the “enduring understanding” [83] of the course:

Few processes in manufacturing or other contexts produce consistently reliable results. Instead, we expect some level of variability in the results of these processes. Statistics is the science of quantifying this kind of variability, and it allows one to make informed decisions in the face of such variability.

The idea that variability can be quantified in such a way as to facilitate decision-making is one that we return to time and again in the course. I feel that students who develop an understanding of this idea will not only understand course content more deeply but will also have a framework with which to approach statistical phenomena in their future careers.

I find that “less is more” in this course and with these students. By focusing on a handful of important distributions and statistical methods, I am better able to develop the students’ conceptual understanding of these topics. I feel that students who understand, for example, two-sample hypothesis test methods as more than just “black boxes” that take in data and spit out P -values will be better able to use those and similar methods appropriately in future applications. The textbook I use, William Navidi’s *Statistics for Engineers and Scientists* [61], helps me take this kind of streamlined approach to this course.

In order to achieve these learning goals, I have my students spend about half of each day’s class responding to and discussing clicker questions as described below. In order to make class time available for this active learning, I ask my students to read their textbooks before class. In order to motivate them to do so, I have them complete short, online reading quizzes before most class sessions. Since these quizzes are due a few hours before class begins, I am often able to review student responses to the quiz questions before class and modify my lesson plans to respond to their expressed learning needs. Furthermore, since students encounter the course material in their textbooks before class, I am able to spend more class time helping students make sense of course content through clicker questions instead of introducing them to that content. Students are given the chance to refine and test their understanding of course content through problem sets due every other week.

Summative assessments in this course consist primarily of midterm and final exams. The first two times I offered this course, these exams consisted almost entirely of free-response, computationally-oriented questions. Since I wanted to know if my students could successfully apply the statistical techniques featured in the course and since these were the kinds of questions the students completed in their problem sets, it seemed sensible to ask these kinds of questions. The third time I offered the course, however, I changed the format of these exams. Enrollment in the course was higher than usual (fifty-six students up from thirty-six students the previous year) and I was worried about the time required to grade so many free-response exam problems. I decided to reduce the number of free-response, computational questions on the exams and add several multiple-choice, conceptual understanding questions to each exam.

Although this shift was initially made out of an interest to save myself time grading, it occurred to me as I prepared the first midterm exam that asking conceptual questions was consistent with my course goals and with the kinds of activities — conceptually-oriented clicker questions — in which the students participated during class each day. When my students performed very well on the computational questions on the second midterm exam but rather poorly on the conceptual questions on that exam, I was glad that I had included the conceptual questions. They helped me assess my students’ conceptual understanding independently of their ability to solve computational problems, and the results of the second midterm made clear to me that students could meet one of these learning goals without meeting the other. In this case, my students could compute confidence intervals and P -values but had trouble with the concepts behind those computations. By spending more time on those particular concepts during the remainder of the course, I was able to help the students improve their performance on the multiple-choice questions on the final exam.

I now devote approximately forty percent of my exams in this course to multiple-choice, conceptual questions. Often these exam questions are refined or enhanced versions of clicker questions asked earlier in the semester. To help students prepare for these conceptual exam questions, I post all clicker questions along with an answer key to the course Web site before each exam.

9.3 Generating Small-Group and Classwide Discussion

Following Mazur’s “peer instruction” teaching method [56], when using a classroom response system I typically pose a question and have my students “vote” their answers individually and independently. Depending on the question, this first vote can take anywhere from ten seconds to three minutes. I do not use the countdown timer feature of my classroom response system; instead I wait until most of the students have responded according to the response counter tool included in my system and then announce “Last call!” to the class, giving stragglers a few more seconds to submit their responses.

After the first, individual vote, I decide how to proceed based on the results visible in the bar chart the system produces in an effort to practice “agile teaching” [5]. On those occasions when most of my students answer a clicker question correctly on the first try, I will say a few words about the question, then quickly move on to the next topic. Doing so allows me to spend more class time discussing topics students find challenging. In most instances, however, I ask questions about which I expect there to be some confusion since such questions are often valuable learning

opportunities for students. For example, Figure 9.1 shows the results of such a clicker question. In this case only 17 percent of my students responded with the correct answer, choice 3. When writing clicker questions, I usually aim to have thirty to forty percent of my students answer them correctly on the first try. I find that this success rate is an indication that the question is pitched at a useful level of difficulty. In these cases, I instruct my students to discuss the question in groups of two or three. My engineering students usually participate well in this small-group discussion given their frequent participation in group work in out-of-class assignments in their engineering courses. However, occasionally I have to prompt a student to find a small group to join. I usually tell the students to discuss their reasons for their answers in their small groups and to try to come to a consensus within their small groups regarding the correct answer to the clicker question at hand. I often point out to my students that if they all agree on the answer to a question, they may all be wrong and so it is important for them to discuss their reasons regardless of their level of consensus.

After this peer instruction time, I have the students re-vote, possibly changing their answers based on the small group discussion. Often there is some convergence to the correct answer in this second round of voting. See, for example, the results in Figure 8.1 in which the percentage of students responding with the correct answer, choice C, for Sample Question 1 increased from 17 to 39 percent between the first and second votes. Sometimes there is even greater convergence toward the correct answer. Results such as these set the stage for a productive class-wide discussion of the question.

Sample Question 1 Consider the continuous random variable X = the weight in pounds of a randomly selected newborn baby born in the United States during 2007. Let F be the cumulative distribution function for X . It is probably safe to say that $P(X < 0) = 0$ and $P(X < 20) = 1$. Which of the following is *not* a justifiable conclusion about F given this information?

- A. $F(x) = 0$ for all $x < 0$.
- B. $F(x) = 1$ for all $x > 20$.
- C. The area under the graph of F between $x = 0$ and $x = 20$ is 1. [Correct]
- D. F is a non-decreasing function between $x = 0$ and $x = 20$.

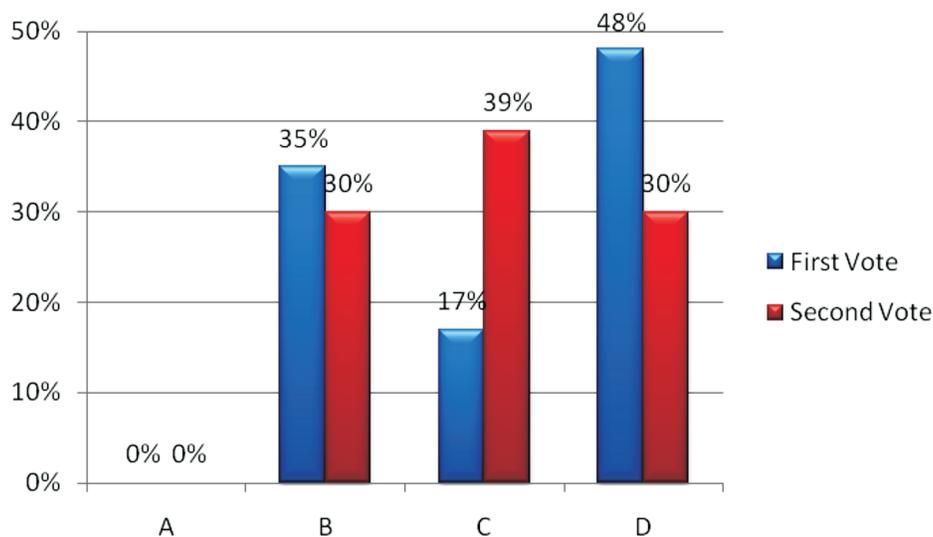


Figure 9.1. Results of two rounds of voting on sample question 1

Sometimes during the class-wide discussion that follows a second vote, I have volunteers provide reasons for choosing the more popular answer choices. Other times I will ask for students who changed their minds to explain why they did so since these students are often ones who resolved common misconceptions during the peer instruction time. I use the results of the second vote to guide the discussion. Popular incorrect answers show me where students' misconceptions lie, and I focus discussion on these misconceptions. In Figure 9.1, for example, the fact that no students selected choice A and 30 percent of students selected choice B indicates that the students are more confused about the right-hand side of the graph of a cumulative distribution function than they are about the left-hand side. Unpopular

correct answers, particularly on the second vote, show me that students likely need further instruction in the topic at hand. In Figure 9.1, for example, the fact that only 39 percent of students answered the question correctly, coupled with the popularity of answer choice B, indicates that a majority of students are likely confusing the definitions of a cumulative distribution function and a probability density function. This kind of analysis of results informs subsequent discussion of the question.

Sometimes the further instruction following a clicker question takes the form of a mini-lecture; other times a few hints are all that is necessary to help students make sense of the question. Occasionally, I will have the students discuss the question in their small groups and vote a third time after providing them with some hints. I typically keep a “poker face” about the correct answer during the class-wide discussion, preferring to let the correct answer become apparent to students as reasons for and against each of the answer choices are shared during the discussion.

9.4 Creating Times for Telling

One particularly compelling use of a classroom response system is to create “times for telling” [73] in which students are primed to listen to an explanation of a particular concept or technique. For instance, I adapted Sample Question 2 from Gelman and Nolan [34] and used it to create a time for telling. Since I asked my students this clicker question after they had read the section on Bayes’ Theorem in their textbook but before we had discussed the theorem during class, most students did not understand the theorem well enough to apply it to solve this problem. As a result, only 23 percent of the students answered the question correctly. The incorrect answers 90% and 75% were more popular, chosen by 28 and 38 percent of the students, respectively.

Sample Question 2 Through accounting procedures, it is known that about 10% of the employees in a store are stealing. The managers would like to fire the thieves, but their only tool in distinguishing them from the honest employees is a lie detector test that is only 90% accurate. That is, if an employee is a thief, he or she will fail the test with probability 0.9, and if an employee is not a thief, he or she will pass the test with probability 0.9. If an employee fails the test, what is the probability that he or she is a thief?

- A. 90%
- B. 75%
- C. $66\frac{2}{3}\%$
- D. 50% [Correct]

Following the suggestion of Gelman and Nolan, I then conducted a classroom experiment in which the students generated random numbers and simulated the situation described in the clicker question. Each student chose two numbers at random from a sheet of random numbers between 0 and 9. Students who chose a 0 for their first number were told they were thieves; other students were honest employees. Students who chose a 0 for their second number were told that the lie detector test gave an incorrect reading for them; other students had accurate lie detector test readings. I then asked for a show of hands from students who were reported as thieves by the lie detector test — those who were thieves and had accurate test readings along with those who were not thieves but had inaccurate test readings. I counted these students, then asked the students who were not actually thieves to put their hands down. Roughly half of the students put their hands down, demonstrating to the students that 50% was likely the correct answer to Sample Question 2. This experiment certainly cast doubt on the accuracy of the two most popular answers to this question, 90% and 75%.

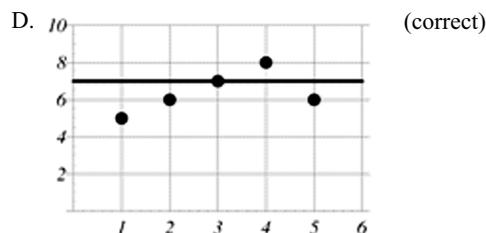
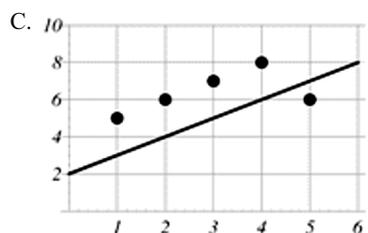
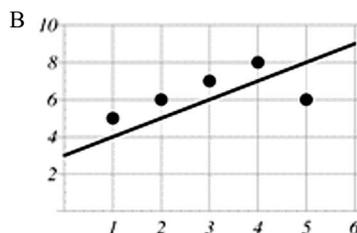
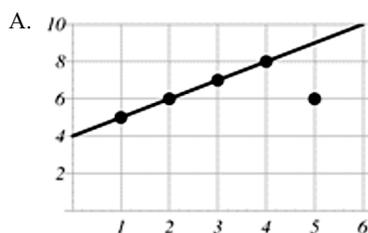
At this point, having seen that most of the students had likely submitted incorrect answers to Sample Question 2, the students were ready for me to explain the solution to this problem to them. The students were “ready” in a motivational sense because their curiosity about this problem was piqued and thus they wanted to listen to my explanation. The students were also “ready” in a cognitive sense since they had spent several minutes trying to understand the question at hand and thus they had thought enough about the question so that my explanation was likely to make sense to them.

Sample Question 3 is another example of a clicker question used to create a time for telling, one that does not involve a classroom experiment. I asked this question after the students had read the textbook section on least-squares lines but before we spent any class time discussing this topic. As a result, many students did not select the line that

best fits the data in the least-squares sense. Note that the question asks students to determine which of the four given lines is the best fit. None of the four given lines is the actual least-squares line, however.

After the students voted on this question, I asked several students to share reasons for their answers with the class. These students generally had sensible reasons for their answers, but the lack of consensus argued for the need for a definition of a line of best fit on which we could all agree. As a result, this clicker question and the discussion that followed helped ready students to hear the precise definition of a least-squares line. Note that the correct answer (in the least-squares sense) to Sample Question 3 is perhaps not the most intuitive answer. This, too, helps students focus on the definition of a line of best fit, particularly the sum of squares component of that definition.

Sample Question 3 Each of the following scatterplots represents the same set of bivariate data, but each features a different proposed line of best fit. Which of the proposed lines fits the data best? (Please pay attention to the scale on the vertical axes below.)



Using a classroom response system to create a time for telling in the ways described here is particularly effective for a number of reasons. Since clickers can be used to encourage all students to respond to a question — not only those who are willing and able to volunteer responses verbally during class — all students can be engaged in a process that prepares them for learning. Responding to such a clicker question requires students to commit to one response or another, and this act of commitment engages students more fully in the question at hand. This in turn means that they will be all the more surprised to find out that they are wrong, which helps motivate them to listen to an explanation of the correct answer. Furthermore, a classroom response system can be used to display the distribution of student responses to the students. Finding out that most of their classmates answered a question incorrectly can further increase students' interest in the question and its explanation.

9.5 Developing Conceptual Understanding

Since conceptual understanding is a key learning goal for my probability and statistics course, many of the clicker questions I use focus on this learning goal. One useful approach in writing such clicker questions is to isolate conceptual and procedural knowledge in distinct clicker questions. For instance, in a class on hypothesis testing, I worked an example for the students in which the P -value turned out to be 0.2743, indicating that there was little evidence that a new high school mathematics curriculum improved student performance on standardized tests. I then asked Sample Questions 4 and 5. Question 4 is designed to assess students' ability to interpret P -values, an example of procedural knowledge. Question 5, however, assesses students' conceptual knowledge of P -values, requiring students to understand the definition of a P -value and highlighting a common misconception about that definition: the misconception that the P -value is the probability that a null hypothesis is true. These questions allow me to assess my students' procedural and conceptual knowledge independently, which is important because the one does not imply the other.

Sample Question 4 In the previous example, suppose that one of the following P -values was calculated from the sample data instead of $P = 0.2743$. Which one would most strongly indicate that the new curriculum works?

- A. $P = 0.02$ [Correct]
- B. $P = 0.09$
- C. $P = 0.14$

Sample Question 5 Suppose that in the previous example, we had found that the P -value for the sample data was 0.04. What is the probability that the new curriculum has no positive effect?

- A. 2%
- B. 4%
- C. 8%
- D. We have insufficient information with which to answer this question. [Correct]

As another example of isolating conceptual and procedural understanding, consider Sample Questions 6 and 7 on the topic of computing probabilities associated with standard normal random variables. The first time I asked clicker questions on this topic, I only asked Question 7. Although students were somewhat confused on the first vote, after peer instruction time, every single student answered this question correctly. (That does not happen often!) However, I quickly realized that all this meant was that every student could accurately use the table of standard normal random variable values in the back of their textbook. Doing so is an important skill and one I was glad to know my students had mastered. However, equally as important is an understanding of the connection between probabilities such as these and areas under the standard normal curve. Question 7 alone did not allow me to assess my students' understanding of this connection. The second time I approached this topic, I first asked Question 6 and asked my students to refrain from using the table of values in their textbook when answering this question. This forced them to consider the graphical interpretation of this question, allowing me to assess their understanding of this interpretation. Only after discussing Question 6 did I move on to Question 7 to assess my students' ability to use the table of values.

Sample Question 6 Let Z be a standard normal random variable. Which of the following probabilities is the smallest?

- A. $P(-2 < Z < -1)$
- B. $P(0 < Z < 2)$
- C. $P(Z < 1)$
- D. $P(Z > 2)$ [Correct]

Sample Question 7 Let Z be a standard normal random variable. Which of the following probabilities is the smallest?

- A. $P(0 < Z < 2.07)$
- B. $P(-0.64 < Z < -0.11)$
- C. $P(Z > -1.06)$
- D. $P(Z < -0.88)$ [Correct]

Sample Questions 8, 9, and 10 focus on the definition of a confidence interval. They followed an example in which a 95% confidence interval for the mean of the population of birth weights of babies born in the United States in a particular year was found to be (6.85, 7.61). These questions address several misconceptions students commonly have about the meaning of confidence intervals. In one semester, the percentages of students answering these three questions correctly were 75%, 73%, and 67%. These data indicated to me that about a fourth of my students were definitely confused about the definition of a confidence interval. Since each of these three questions has only two answer choices, it is probable that a number of students answering the questions correctly did so by guessing, so the portion of students confused on this topic was likely even higher.

Sample Question 8 Given the confidence interval just constructed, is it correct to say that there is a 95% chance that μ is between 6.85 and 7.61? Is this statement correct?

- A. Yes
- B. No [Correct]

Sample Question 9 Given the confidence interval just constructed, is it correct to say the following? “If the process of selecting a sample of size 30 and then computing the corresponding 95% confidence interval is repeated 100 times, 95 of the resulting intervals will include μ .”

- A. Yes
- B. No [Correct]

Sample Question 10 Given the confidence interval just constructed, is it correct to say that 95% of all birth weights will be between 6.85 and 7.61 pounds?

- A. Yes
- B. No [Correct]

Independence is another concept with which students often struggle. Sample Question 11 assesses students’ understanding of this concept. This question also illustrates a particularly useful strategy for writing clicker questions. I posed this question as a free-response question on the pre-class reading quiz the students completed the night before the class session on independence. The morning before that class session, I reviewed my students’ answers to this free-response question and adapted three of their responses into answer choices for the clicker version of this question. These three responses were all incorrect to a degree, so I added a correct response as a fourth answer choice. Utilizing this strategy meant that the wrong answers to this clicker question were based on my students’ actual misconceptions about independence and not on my assumptions about their difficulties with this topic. It also allowed me to point out to the students that some of the answer choices for this clicker question were drawn from their responses to the pre-class reading quiz, which in turn increased their motivation to take future reading quizzes more seriously. This strategy works particularly well for questions in which students are challenged to explain ideas clearly and precisely.

Sample Question 11 Suppose A is the event that it rains today and B is the event that I brought my umbrella into work today. What is wrong with the following argument? “These events are independent because bringing an umbrella to work doesn’t affect whether or not it rains today.”

- A. These events are not independent, because one’s decision of bringing an umbrella is dependent on the likelihood of rain. (However, rain is definitely not dependent on one carrying an umbrella, although Murphy’s Law might prove the opposite.)
- B. Although bringing an umbrella to work doesn’t cause it to rain, given that you’ve brought your umbrella to work, the probability that it’s a rainy day is higher than the chance of rain on any random day. [Correct]
- C. These events are independent because the probability of bringing an umbrella to work doesn’t affect the probability of the event its rains today and vice versa.
- D. It is false because the fact that it is raining today means that it was probably predicted to rain. If you checked that prediction then you would be more likely to bring in an umbrella making the events linked.

Another type of conceptual understanding questions is what is sometimes called a ratio reasoning question. Sample Questions 12, 13, and 14 are examples of this type of question. Question 12 asks students to determine whether the relationship between two quantities is direct or indirect, assuming all other relevant quantities are held constant. Question 13 also asks about the relationship between two quantities, but this question asks students to be more precise about that relationship than Question 14. Question 14 goes a bit further and asks students to analyze the relationship between more than two quantities. Ratio reasoning questions such as these work particularly well in a statistics course since many topics lend themselves to such questions, answer choices for these questions are relatively easy to construct, and they help develop students’ intuitions about statistical relationships.

Sample Question 12 In the high school mathematics curriculum example, we found that if a sample of $n = 86$ students scored an average of 502, then the P -value was 0.2743. If a random sample of 860 students scored an average of 502, would this raise or lower our P -value?

- A. Raise
- B. Lower [Correct]

Sample Question 13 In the previous polling example, a sample size of 50 resulted in a particular margin of error. Approximately what sample size would you need if you wanted to cut the margin of error in half?

- A. 25
- B. 100
- C. 200 [Correct]
- D. 400

Sample Question 14 Suppose you construct a 95% confidence interval from a random sample of size $n = 20$ with sample mean 100 taken from a population with unknown mean μ and known standard deviation $\sigma = 10$, and the interval is fairly wide. Which of the following conditions would NOT lead to a narrower confidence interval?

- A. If you decreased your confidence level
- B. If you increased your sample size
- C. If the sample mean was smaller [Correct]
- D. If the population standard deviation was smaller

9.6 Gathering Data

Since it is often motivating to students to see statistical methods applied to data sets of personal relevance to them, I sometimes use my classroom response system's ability to pose free-response questions to my students to generate such data sets. For example, during our discussion of simple linear regression, I collected my students' shoe sizes and heights in inches using the classroom response system. Importing these data into a computer spreadsheet program allowed me to illustrate basic linear regression methods using these data. Without a classroom response system, I would have needed to gather these data before class through an online survey of some sort or by passing around a sheet of paper during a previous class. The clickers saved time and allowed for an immediacy to the activity that increased student engagement.

9.7 Conclusion

I find that using clickers to generate small-group and class-wide discussions about interesting and relevant questions and to create "times for telling" increases attendance, participation, and engagement. Since individual student responses to clicker questions can be tracked, clicker questions can be used as an objective class participation grade. Holding students accountable for their class participation in this way increases attendance. Classroom response systems also allow me to expect all of my students to respond to the questions I pose, not just those who are willing to volunteer to share their responses verbally in front of the class. As a result, more students participate regularly during class. Since I ask my students to respond individually to my clicker questions, students are given the chance to think independently about clicker questions prior to any group discussion. This helps engage them in the sometimes-difficult process of making sense of course content. Also, since classroom response systems are capable of displaying a summary of the responses to a question posed to students, I find these systems useful for practicing "agile teaching," an approach to teaching that responds during class to expressed student learning needs. Furthermore, displaying the results of a clicker question to my students can often motivate them to engage more deeply with a question. Knowing that their peers are split between two or more answers or that many of their peers answered a question incorrectly can interest students in a question and create a time for telling. These and other aspects of teaching with clickers are explored further in my book, *Teaching with Classroom Response Systems* [10].

Given my interest in developing my students' conceptual understanding of probability and statistics and in creating an active learning environment in my classroom, classroom response systems provide me with an effective way to engage my students.

10

Incorporating Clicker Technology in the Introductory Statistics Course

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10.1 Introduction

Most introductory statistics instructors consider understanding of important concepts to be one of the course learning goals. Teaching students how to blindly apply statistical methods to obtain results that they do not understand is not usually what instructors are striving for, although that is sometimes the unintended result! It is now understood that the development of conceptual understanding is facilitated by student engagement in class and by meaningful class discussions. This paper describes how the thoughtful use of clicker technology can be employed to enhance student engagement and inform instruction in an introductory statistics course.

This paper is organized into sections that address the following: a brief description of the context in which clickers were used (the university and the course), the motivation for incorporating clickers into the introductory statistics course, how the use of clickers was implemented in the course, what was learned about writing effective clicker questions, and the results of an informal experiment to assess the effect of clicker use on student learning and engagement.

10.2 Description of California Polytechnic State University and of the Course

California Polytechnic State University, located in San Luis Obispo, California, is a public, primarily undergraduate, comprehensive university with approximately 19,000 students. As a polytechnic university, there are large programs in engineering, agriculture, architecture, mathematics and the sciences, but as a comprehensive university there are also strong programs in business, education and the liberal arts.

The course in which clicker technology was integrated was Stat 217, Introduction to Statistical Concepts and Methods. This course is a 4-hour course that is taught over a 10-week quarter. Topics included in the catalog description are sampling and experimentation, descriptive statistics, confidence intervals, one and two-sample hypothesis tests for means and proportions, Chi-square tests, linear regression, and an introduction to multiple regression and to analysis of variance. There is only minimal coverage of probability in this course — just enough to support the ideas of sampling distributions and statistical inference. Students in this course are primarily majoring in one of the liberal arts or in kinesiology. This course also satisfies the university general education requirement in quantitative reasoning. Class size for this course is usually between 35 and 48 students.

10.3 How I Chose to Incorporate Clickers and Lessons Learned

After spending a quarter experimenting sporadically with the use of clickers, I decided to incorporate clickers into the Stat 217 course. (My question set is available at mathquest.carroll.edu/resources.html.) Because I wanted to be able to use the student responses to the clicker questions to inform instruction, it was important to me that students make an honest attempt to answer the questions posed. To ensure this, I decided that some of the clicker questions would be scored, allowing students to earn a “quiz point” for a correct answer. Even though these quiz points were not a large part of the overall course grade, they did motivate students to try to reason out the correct answer. In a typical class period (1 hour and 50 minutes), I incorporated 5 or 6 clicker questions, three of which were quiz questions. When a question was posed, students did not know whether or not it was a scored question until after they had responded. My hope was that this would motivate students to make a serious attempt at each question, so that I could make good instructional decisions based on what the responses told me about students’ understanding.

During the course of my work with clickers and also through observing others who were experimenting with this technology, I made three important observations. The most important is that it is critical to ask the right kind of questions and that writing good clicker questions is not an easy task. Writing good clicker questions is addressed in the following section. The second observation is that it is possible to overuse clickers. I found that incorporating about 3 clicker questions per hour was about right given the content of the introductory statistics course. The third observation is also one I think is very important to consider, and may seem obvious but is still worth mentioning: Don’t ask clicker questions unless you are willing to use the responses to inform instruction. You need to be willing to move on if the responses show that students understand the concept—don’t explain it again. And, even more important, don’t just move on if the responses reveal a lack of understanding. Students respect and appreciate an instructor who is willing and able to adapt instruction to their needs. And if that isn’t the case, using clickers will only make this even more obvious to students!

10.4 Writing Good Clicker Questions

Clicker questions are best able to inform classroom instruction if they can be used to assess conceptual understanding and if they lead to fruitful classroom discussions around student misconceptions and misunderstandings. As a consequence, clicker questions that merely ask for recall of facts, check on knowledge of definitions, or evaluate ability to perform computations are not of much use. These are things that can easily be assessed and reviewed in other ways and there are better uses of class time.

In writing a clicker question or choosing a clicker question from a question bank, you should begin with a clear idea of what the purpose of the question will be. In choosing the possible responses, it is equally important that you anticipate likely misunderstandings and points of confusion. The wrong answer choices should tell you something about where student thinking is going awry. I have found it useful to ask myself “What will I do as an instructor if students choose this response?” for each of the possible wrong responses.

For example, consider the following clicker question:

In a random sample of 2013 adults, 1283 indicated that they believe that rudeness is a more serious problem than in past years. Which of the test statistics shown below would be appropriate to determine if there is sufficient evidence to conclude that more than three-quarters of U.S. adults believe that rudeness is a worsening problem?

1. $\frac{p - .5}{\sqrt{\frac{(.5)(1-.5)}{2013}}}$
2. $\frac{p - .75}{\sqrt{\frac{(.75)(1-.75)}{2013}}}$
3. $\frac{\bar{x} - .75}{\sqrt{\frac{s}{2013}}}$

The correct answer is choice 2. If students choose answer 1, I know that they are falling into the trap of thinking that any test of hypothesis about a proportion has a null hypothesis of $H_0 : \pi = .5$, and we can address this through a class discussion. If students choose answer 3, I know that they were unable to distinguish between a question about a proportion and a question about a mean. I could then spend a bit more time talking about how to distinguish between such problems, focusing on numerical versus categorical data. In each case, I know what student error I am trying to correct.

As a second example, consider this clicker question:

Each individual in a random sample of 40 cell phone users was asked how many minutes of air time he or she used in a typical month. The data was then used to construct a 99% confidence interval for the mean monthly number of minutes of air time used. The confidence interval was (207, 293). Which of the following could be the 95% confidence interval constructed using this same sample?

1. (200, 300)
2. (218, 282)
3. (227, 313)

The correct answer is choice 2. If students choose answer 1, I know that they do not understand that the confidence interval will be narrower when the confidence level is lower. If they choose answer 3, I know that they do not understand that both the 99% confidence interval and the 95% confidence interval will be centered in the same place. In either case, I know how I will respond. And, of course, if nearly all students answer correctly, I can move on without spending further class time.

The best advice I can give about writing questions is to focus on the reasoning behind the answer choices that you select and to anticipate the way that students might reason — both correctly and incorrectly — when formulating questions. And, use this same kind of thinking to evaluate questions when choosing from a bank of existing questions. Ask “What will this question tell me about student understanding of something I think is important?” And if the answer is little or nothing, don’t waste class time with it! Think carefully and make deliberate choices when writing or selecting questions.

10.5 An Informal Experiment — Student Perception and Impact on Student Performance

During winter quarter 2007, I decided to conduct an informal experiment to gauge student opinion and to assess impact on student performance. Of course, in an educational setting it is rarely possible to do anything resembling a randomized controlled experiment, but useful information and insights are possible even from less than optimal experimental settings. In winter quarter, I taught two sections of Stat 217, one meeting on Tuesday and Thursday from 2:10 – 4:00 P.M. (the no clicker section) and the other meeting on Tuesday and Thursday from 5:10 to 7:00 P.M. (the clicker section). There were 49 students enrolled in the no clicker section and 44 students enrolled in the clicker section. The two sections were comparable in terms of gender and majors represented. Information on SAT/ACT scores or on students’ prior academic experience was not readily available, but it is not likely that the two sections differed substantially with respect to these variables.

In the clicker section, approximately 6 clicker questions were incorporated into each class meeting. Three of these were “quiz questions” for which the students could earn a quiz point for each correct answer. The other questions were not scored. Quiz questions and non-scored questions were interspersed throughout the class and students did not know whether or not a particular question was a scored question until after they had responded to the question. In the no clicker section, students received the same three quiz questions as a written quiz at the end of the class period, and quizzes were scored and returned at the next class meeting. The non-scored clicker questions used in the clicker section were also used in the no clicker section, but these were posed in a traditional format where the question was posed and then answered by a student in the class. At the end of the quarter, the students in the clicker section responded to a survey that asked about their opinion regarding clicker use. Student behavior and performance in the two sections was compared with respect to several variables, including a measure of attendance, quiz scores, final exam score and overall course grade.

Student opinion was quite favorable regarding the use of clickers. Eighty-nine percent reported that they enjoyed using the clickers in class, 93% said that they found the immediate feedback on quiz questions helpful, 68% said that they thought that use of the clickers influenced class attendance, and 100% recommended that the use of clickers in the course be continued. Some typical student comments were:

- “It was really easy to use and we could talk about the correct answers immediately.”
- “I liked them overall, and I think that they helped me pay attention more in class, because I knew I would have to respond eventually.”
- “Consider using them as part of the midterm exams. Using them for exams would be cool.”
- “They were fun to use and make you feel like you are on Who Wants to Be a Millionaire.”
- “They were interesting. Good to know immediately if your answer was right.”

Only one concern was expressed by two students who indicated that using clickers made them a bit nervous at first and that they felt some pressure to answer quickly when they saw other students responding.

Student attendance, as measured indirectly by the mean number of missed quizzes and in-class lab assignments and by the proportion of students with at least one missed lab or quiz, was higher for the clicker section than the no clicker section.

Number of Missed Quizzes and Labs per Student

	N	Mean	Standard Deviation
No Clickers	49	1.67	0.39
Clickers	44	1.05	0.26

P -value = .095

Proportion of Students with at Least One Missed Quiz or Lab

	N	Proportion
No Clickers	49	.59
Clickers	44	.41

P -value = .039

The clicker section also had a higher mean score on the final exam and final exam scores were less variable than for the no clicker section.

Final Exam Score

	N	Mean	Standard Deviation
No Clickers	49	68	19.14
Clickers	44	73	13.87

P -value = .079

The clicker section also had a higher mean score on the both midterms (P -values of .079 and .092 for midterms 1 and 2, respectively), as well as significantly higher mean quiz totals and lab totals (although this is partially due to the fact that students in the no clicker section had more missed quizzes and labs). Mean overall total points earned, which was used to determine course grade, was also higher for the clicker section (236.3 for the clicker section, 226.0 for the no clicker section, P -value = .077).

Of course, the results of this experiment must be interpreted with caution because it was not possible to assign students at random to one of the two experimental conditions (clickers, no clickers). However, the two sections were comparable in many respects. Both sections were taught by the same instructor, used the same textbook, had the same homework assignments, quiz questions and exams, were taught in the same computer classroom, and had the same meeting pattern of Tuesday and Thursday afternoon for 1 hour and 50 minutes. Time of day differed for the two

sections in order to keep the instructor the same for both sections. Because Stat 217 is a required course for particular majors, is typically taken in the sophomore year, and because California Polytechnic State University is a residential campus with primarily traditional-aged full-time students, student demographics (major, year in school, gender, age) were similar for the two sections. All things considered, I found the results regarding both student performance and student attitude supportive of using clickers in the classroom.

10.6 In Closing

My experience with clickers has been a positive one. This technology has changed the way I interact with students by allowing me to evaluate student understanding and adapt instruction accordingly. It has also changed the way that students interact with the course. Clicker use can facilitate student engagement and lead to more involvement in the learning process, as long as the questions are carefully constructed and the instructor is willing to modify instruction based on student responses and use the student responses to motivate class discussion. Over the course of my teaching career, I have experimented with a number of new technologies, finding that some were “keepers” and others were not. For me and the students in my introductory statistics course, clicker technology is definitely a keeper!

11

Using Clickers in Courses for Future K–8 Teachers

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11.1 Introduction

It is often the case that beginning K–8 teachers emphasize basic skills and procedures in their classrooms, but rarely probe more deeply for understanding and connections. This is not surprising, considering that this is what they probably experienced in their pre-college courses. Future teachers who themselves were students in classrooms that emphasized skills and procedures often lack deeper understanding or have developed basic misconceptions that make it difficult for them to probe or to answer questions from their own students about concepts and processes. The MAA Committee on the Undergraduate Program in Mathematics' *Curriculum Guide* [54] and the National Council of Teachers of Mathematics' *Principles and Standards* [62] recommended goals and objectives include that students develop the skill of understanding, representation, connections, and reasoning. This means that the curriculum should be seen as related ideas and concepts that can be used to solve a variety of problems, not as a set of unrelated facts and algorithms to be memorized.

Faculty teaching math content courses for teachers have a short window of opportunity to provide future teachers with experience in a conceptually rich environment. Faculty in these courses should model not only higher levels of questioning, but encourage higher level thinking. Unfortunately, often students find it difficult to know what they do not know. That is, students are not even aware that they do not understand the concept to which they have just been exposed or what they think they understand is actually a misconception. Listening to lectures or problem solving in groups provides important experiences, but does not provide a real-time way to interact personally with all the students. We have found that underprepared students working in groups do not recognize their misconceptions. Using misconceptions within voting item options focuses their attention on misconceptions, particularly when they must vote and reflect on their personal understanding. Teaching with personal response systems provides corrective feedback in a timely manner when it would be most beneficial to the students in analyzing their own understanding.

11.2 Our Experience

In winter 2008, we began the use of conceptual based items in math content courses for future elementary and middle school teachers. The model we implemented focused on the development and use of key conceptual items. We used

a multiple choice format for questioning, incorporating common misconceptions as the alternative responses. Usually one deep conceptual item was used at the beginning of class. This item was developed to help structure the major ideas and representations that were to be discussed that day.

“Representations are the means by which data are recorded and analyzed [62].” Students used representations to explain mathematical results. Tables, graphs, diagrams, lists, and equations are example representations that were used to represent mathematical ideas.

Personal response systems were used to collect and display student feedback anonymously and to provide instructor feedback as in the “just in time” teaching model. The class results from the first polling were discussed in small groups, often followed by re-voting, and then discussion by the full class. An important dialog began to emerge related to the misconceptions themselves. What was the reasoning of a student that selected response a? How can you explain that response c is correct? What mathematical tools/representations do we have to answer this question? Are there alternative methods of reasoning that may be used to arrive at the correct response? Why is this true?

This questioning and discussion methodology was used three to four times per week and often the discussion about representations lasted 20 minutes or more. Since this discussion enveloped the mathematical content for the class period, time actually may have been gained by addressing student misconceptions early. For the first time, in our combined 60 years of teaching experience, students began to tell us that they now understood what they did not know (due to the focus on misconceptions and voting with personal response systems for the first time in mathematics). Their metacognitive understanding and self-monitoring improved. In the past, these students thought that because they could watch others do mathematics and agreed with the processes, they understood the mathematical ideas themselves. By watching and following the same process in their work they could obtain a solution, but did not understand what led to that solution. Even worse, they were not even aware that they did not have this understanding. If they arrived at an incorrect solution, they often did not realize it was wrong or why they had arrived at this incorrect solution.

When students learn that they have voted differently than their peers in class, they now reflect on the question and their answers both individually and in small groups. Students use representations to model the concept or to organize and analyze information. They ask questions of each other and try to determine what may be causing the misconceptions.

A major challenge for future elementary and middle school teachers is evaluating reasoning presented by students. Unlike some of the recent literature on peer teaching [56], we are finding that sharing reasoning in small groups does not necessarily result in a more likely chance of voting for a correct response. Peer instruction often narrows the choices by removing a selection based on a misconception. For our future elementary and middle school teachers, part of the challenge is having a process for evaluating reasoning. Often in small group discussion if someone is persuasive and shares reasons that are intuitive, the vote may be swayed toward this response whether valid or not. Because our multiple choice format includes common misconceptions as the alternate choices and because each of these choices is eventually discussed with the whole class, future teachers are provided with a method for evaluating reasoning. This provides further evidence of the validity or not of the reasoning and also helps clarify the appropriate mathematical approach for a specific argument. This increased emphasis on representation of the mathematics lends itself to our students’ future teaching experiences. How will they help their future students explain and understand the mathematics they are using to solve problems? What follows is a sample lesson incorporating clicker items along with teacher annotations for a lesson focused on representations and reasoning in the analysis of probability experiments.

11.3 Sample Lesson with Instructor Annotations

Multicultural Games of Chance Applications of Multistage Experiments with Independent Events — Focus on Representations

Learning Objectives

- Students will select and create a representation appropriate for solving probability problems based on multiple independent events.
- Students will recognize multiple valid representations for solving such problems.

- Students will evaluate suggested representations for their appropriateness and validity for the given probability experiment.
- Students will find misconceptions and/or misrepresentations provided in examples of student thinking.

Prior Learning

Prior to this lesson, students have investigated experiments involving coins, dice, spinners, and marbles. They have utilized representations that include organized lists of outcomes, tables, tree diagrams, and have used Pascal's Triangle for binomial probability situations. New games are investigated as a new context for probability applications.

Context 1

The Game of Igba-Ita, a game of Nigeria, involves the throwing of cowrie shells [87]. Each cowrie shell looks like a small closed hand. One side is referred to as the open side and the other side is the back of the cowrie shell. The variation of the game that we will study uses three cowrie shells. Points are only awarded for the throw that lands with all three cowrie shells landing open side up.

The first item will be a springboard for discussion and analysis of all the provided responses as well as mathematical thinking and representations that may be used to support each argument. Not only is this intended to strengthen mathematical understanding in this math content course, but it is also scaffolding the foundation of math content teaching knowledge for their future teaching. Our focus will be on the number of equally likely outcomes in the sample space and the sample space itself to help answer questions like the following.

Clicker Item 1 The Nigerian game of Igba-Ita involves throwing 3 two sided shells, where each has an open side and a closed side that are equally likely. The probability that all 3 shells land open side up is

- A) $1/8$
- B) $1/6$
- C) $1/4$
- D) $1/3$
- E) $1/2$

Note on the item: Another possible response is the fraction $1/9$. Sometimes students develop a rote procedure of using the number of objects and the number of equally likely outcomes for each artifact to find the number of outcomes in the sample space without consideration of which would meaningfully be the base and which would be the exponent. In this case that process would result in three squared rather than two cubed.

Each of these results has been provided by former students in open format assessments. At this point students work on this problem individually for a few minutes, click on their answers, and the results are displayed for the entire class. Thus far when this item has been used, the votes are diverse. No option receives a majority of the votes and several options receive a significant number of votes. Students focus on the numbers inherent in the problem posed and may be recalling a multiplication principle, procedural knowledge, and then narrowing their selection.

Now just as Mazur [56] utilized in his peer teaching, students work in small groups to argue for their result and to provide mathematical support for their choice. Students then have an opportunity to revote [56], but often there is still some disagreement over the answer to this question. Part of the difficulty, is that many of their peers have plausible arguments at first glance. Students now share their reasoning with the full class. The focus of the discussion which supports their "mathematical knowledge for teaching" [2, 43] is answering the question, what might a student be thinking that selected choice A? B? C? D? and E?

The next clicker item summarizes the plausible reasons given for each of these results, even though there is only one correct result of $1/8$. The focus of the reasoning is to determine the number of equally likely outcomes in the sample space. The following may be left as an open discussion, but is very informative when voting takes place again.

Clicker Item 2 The probability that all 3 shells land open side up is

- A) $1/8$ There are 2 outcomes for each of the three shells, you find the product: $2 \times 2 \times 2$.
- B) $1/6$ Each of the 3 shells has 2 outcomes, 3×2 will provide the number of equally likely outcomes.
- C) $1/4$ The shells may land with 0, 1, 2, or 3 shells landing with open side up. This results in 4 outcomes, and only one of these has all 3 open side up.
- D) $1/3$ There are 3 game artifacts, this results in 3 equally likely results.
- E) $1/2$ For each of the game pieces there are only two ways a shell lands.

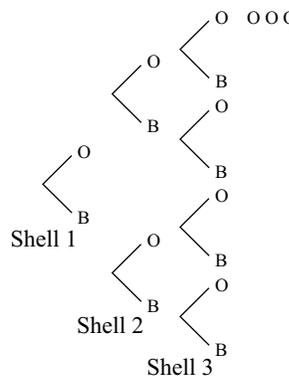
At this point, often students are confused as to what is the answer. The list of results often narrows to A, B, or C, realizing that both D and E are incomplete arguments. For many students it is difficult to go beyond the recognition of incomplete arguments. To many students the arguments for A, B, or C are very plausible, so we now move to evaluate sample representations for each of the five approaches for further analysis. This process relates to our students' experience in "appraising the mathematical validity of alternative solution methods" [43], and the ability to create and "use mathematical representations" [2].

Focus on Representations

Now that students have discussed and voted on this clicker item, they are challenged to create a representation for the argument they voted for. Doing this in small groups they are asked to develop representations for each of responses A–C (or A–E depending on the voting that takes place). Students are asked to share different representations on the board followed by one more vote. The instructor may directly ask specific students (groups) to share their result for one of the methods to lend to the follow-up discussion. Students may find it difficult to create a representation for one or more of the responses, so this follow-up clicker item may be used and modified utilizing the shared representations that are discussed in class — a clicker item informed by student work on the board.

Optional Clicker item 3 The probability that all 3 shells land open side up is

- A) $1/8$ To represent this I created a tree diagram. The first level I am labeling shell 1 and showing 2 paths/ways it may land, either open side up, O, or the closed side up on the back of the shell, B. No matter what happens to the first shell, the second shell has the same possibilities, O or B, now there are 4 paths or outcomes in my tree. Finally the 3rd shell may land either O or B. Now you can see there are $2 \times 2 \times 2$ or 8 outcomes for the 3 shells.

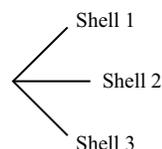


- B) $1/6$ To represent this I made a table of values as we did for cross products. Along the top I list shell 1, shell 2, and shell 3. Along the left-hand side I listed the two outcomes, open or closed. Now looking at the cross-product you have 6 results. You have shell 1 open or shell 1 closed, shell 2 open or shell 2 closed, etc. Each of the 3 shells has 2 outcomes, 3×2 will provide the number of equally likely outcomes.
- C) $1/4$ To represent this I created a list of results. The shells may land with 0, 1, 2, or 3 shells landing with open side up. This results in 4 outcomes, and only one of these has all 3 open side up.

	Shell 1	Shell 2	Shell 3
O	Shell 1, O	Shell 2, O	Shell 3, O
B	Shell 1, B	Shell 2, B	Shell 3, B

Outcomes	
All 3 shells land open sides up.	
2 Shells land open sides up.	
1 shell lands open side up.	
0 shells land open side up.	

D) $1/3$ I made a tree with the three shells listed. There are 3 game artifacts, this results in 3 equally likely results.



E) $1/2$ I made a listing of how shells may land. For each of the game pieces there are only two ways a shell lands.

Outcomes
Shells land open up.
Shells land back up.

In debriefing this analysis, the students share with the full class the answer to this question: what are the misconceptions or reasons that the representations for all but one of these answers are not valid? This elicits a variety of new questions. Why are you saying that D is incomplete? That C is looking at outcomes that are not equally likely? What other representations for A may be used to support this conclusion? Which of these representations for A would have meaning and show the result clearly?

Other representations that students have applied to this question are an outcomes list for part A or the row, 1 3 3 1, of Pascal's Triangle. This row of Pascal's Triangle corresponds to the 1 outcome in the sample space that has all three shells landing open side up, 3 outcomes that have two of the three shells landing open sides up, 3 outcomes that have one of the three shells landing open side up, and 1 outcome that has all three shells landing back side up. The sum of $1 + 3 + 3 + 1$ or 8 provides the total number of outcomes in the sample space and 1 of these outcomes results in all landing open sides up, so the probability value is $1/8$. Students using this representation are applying prior knowledge of binomial probability situations such as coin tosses.

Optional Activity After clicker item 1, discuss a simulation of this game with available materials thereby creating a probability experiment and use experimental probability to better understand the problem and perhaps narrow the number of alternative results. The cowrie shells were used as currency for a time in Nigeria so tossing three shells is very appropriate mathematically and as a different form of currency than we have available. Again assume the cowrie shells are fair game implements.

Often the in-depth discussion of representations, reasoning, and probability for this context will take a class period of 50 minutes. However, if time permits, start students thinking about a second context. For the second context use a variation of a game that uses two dice each with 3 possible outcomes. In part, this is to again move them away from automatic answers of multiplying the number of chance implements by the number of equally likely outcomes for each implement. Also, if students surmised the result from the first game as focusing on 2^3 equally likely outcomes will they reason carefully and not again say the same thing for this situation. Two medieval long dice of three-sides each works well here as chance artifacts or dice game variations such as the following Vietnamese game shared in class by a student with Vietnamese heritage. This game was played by her family for generations on Tet, New Year.

Context 2

In the Vietnamese Game, Bau cua ca cop, several six-sided dice are thrown. In this variation of the game, three different symbols appear on each of two similar dice with each image appearing twice on a die. The images are the fish, crab, and prawn. Find the probability that both dice land with the fish side up.

Optional Clicker Item 4 In this variation of the Vietnamese game of Bau cua ca cop, two six-sided dice are used. Each die has the symbols fish, crab, and prawn appearing twice, one symbol to a side. Assume each of the outcomes on a die is equally likely. The probability that both dice land with the image of the fish side up is

- A) $1/3$
- B) $1/6$
- C) $1/8$
- D) $1/9$
- E) $1/36$

The correct answer to item 4 is D) $1/9$. It is instructive to contrast the results, sample spaces, and the representations used for the Igba-Ita game to the Bau cua ca cop game.

Final Note

These items focused on the situation where all the game implements landed the same way. These could be easily adapted to other outcomes, particularly now that students have representations that may provide this information. Since we find that often the representation and the number of possible outcomes in the sample space confuses our students this was a fruitful beginning to these application problems.

Assignment Find the theoretical probability and explain your solution referring to a representation for each of these problems. Assume all chance implements are fair.

1. In the game of Igba-Ita, what is the probability that only 2 of the shells land open side up?
2. In the variation of the Vietnamese game of Bau cua ca cop, what is the probability that only 1 of the dice lands with the fish side up?
3. The ethnologist Stewart Culin studied hundreds of games played by Native Americans in the late 1800's. One of the games of chance he describes is the Zuni game, Thlaspatsa ananai. The chance game implements consisted of 5 two-sided wooden dice (1.5 inches long and $1/4$ inch thick), painted black with incised lines on one side, the other side left unpainted or marked. Dice are tossed and scores are: 10 points and a win for all 5 black sides up. 5 white up is 5 points. 4 white up is 4 points, 3 white up is 3 points, 2 white up is 2 points, one white up is 1 point. To win 10 points must be achieved [25]. Assuming these are fair game implements, find P (earning 1 point on a throw).
4. Culin also studied games of the Hawaiian Islands, describing the game of Lu-lu. Four two-sided flat disks are thrown. One side of each disk is blank. On the other side a different number of dots is painted. One disk has a single dot, a disk has 2 dots, another 3 dots, and finally one disk has 4 dots. Points are awarded based on the total number of dots on a throw of all 4 disks [26]. In this variation of the game, each player throws the chance implements once on each turn. Find P (earning 10 points on a throw).

11.4 Summary

Mathematical knowledge for teaching is an important emphasis of our coursework preparation in mathematics for future teachers. This knowledge encompasses the mathematics concepts, principles, and skills, as well as reasoning about mathematics based on representations and models. This knowledge looks deeply at student understanding and explanations as well as methods used by students as they solve problems [2, 43]. The use of personal response systems in this discourse reveals the level of mathematical understanding of students and provides a method for instructors to review student analysis of plausible reasoning. Personal response systems help to launch the conversation about misconceptions, misrepresentations, and what can be learned about reasoning of our students. Not only is this conversation directed at undergraduate learning, it also creates an environment where students learn about questioning techniques for use in their future classrooms. The teaching moves of a faculty member may appear more transparent and provide models of how they may learn from their own students in the future.

12

Using Clickers in Professional Development Workshops

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12.1 Introduction

Much of the existing literature on the use of an electronic personal response system is in the context of increasing participation in a lecture setting. In this paper, we give several examples of using an electronic response system, or *clickers*, during a workshop for the professional development of in-service teachers. Clickers were used in three distinct settings during the workshop: a) to analyze existing mathematical content knowledge, b) to promote awareness of national mathematics testing data, and c) to conclude a mathematical experiment. In each case, the principal goal of the use of this technology was the promotion of productive mathematical discourse. Participant comments and self-evaluation of clicker use indicate that benefits similar to those found in the lecture setting were also realized in this workshop setting.

12.2 Background

District teams of teachers in grades 3 to 9 including special education teachers participated in an eight-day summer workshop to enhance their mathematical content knowledge. With the disparate mathematical backgrounds of the 48 teachers, ranging from teachers with one undergraduate mathematics course to those with an undergraduate mathematics degree, it was anticipated that strategies would be needed to promote productive discourse between the workshop leaders and the teachers and among the teachers themselves. For this reason, the use of clickers in the workshop was anonymous, that is, teacher identity was not associated with a clicker identification number. Springer and Dick [78] describe a discourse move in a mathematics classroom as “a deliberate action taken by a teacher to encourage, facilitate, participate in, or influence the discourse.” Each of the settings in which clickers were used was deliberate and indeed, with a primary goal of promoting discourse.

While there were situations when same-grade teachers were grouped together for planning and curriculum discussions, the district-team grouping was maintained for most of the workshop, and in all three settings in which clickers were utilized. The following quote from a workshop participant reveals that the purpose for grouping by district teams was realized.

Many times I say things like, “Oh, the fourth grade teachers will really be impressed that you know this.” But I need to think beyond that. I need to think to fifth grade, middle school, high school, and even college. If I instill a love and passion for math in third grade, my students will have a better chance of having love and passion for math in their years to come! That’s powerful!

12.3 Content Knowledge

Algebraic thinking and mathematical processes were weak areas for students in nearly all of the participating districts as determined by a statewide examination. As a prelude to building content knowledge, clickers were used as a scripted discourse move [48] to assess the teachers existing knowledge base. Teachers voted to choose the correct property represented in an equation such as the one illustrated (Figure 12.1). With the wide range of mathematical content knowledge among the participants, we predicted that few teachers would have volunteered answers if the questions had not been posed with the clicker technology. This is evidenced by the fact that just 52% of the teachers correctly answered this question as the commutative property. The following teacher reflection on using clickers with these questions summarizes many of the comments made by other teachers and extends this workshop setting to an elementary classroom:

... with a classroom of more knowledgeable math teachers (just like some students have more math knowledge) I would have hung back until others shared and I was SURE I had the right answer.

All the advantages of classroom voting as detailed in Cline [16] were present in this setting: a) each teacher needed to ponder the question as an individual, b) the workshop leaders and the participants both received immediate feedback concerning the concept, c) the results of the voting launched productive discourse, and d) the participants found the activity entertaining.

When asked specifically, “Would you have volunteered to answer the operations-properties questions if they had been posed in an open-fashion without clickers?” nine of the 28 teachers responding said yes, though five of those were qualified with statements such as “but I would have been a bit more apprehensive” or “Yes, but not as freely. . .” Nineteen of the teachers gave a “no” reply, with two emphatic reactions, “NO WAY” and “NOT ME!”

12.4 Awareness

An essential component of the professional development of teachers involves a considerable discussion of assessment. The content choices for the workshop were based on student performance on a state-wide examination. Teachers are always eager to learn of new sources of open-response questions, performance tasks and other assessments to complement their curriculum. In this second setting for clicker use, teachers were first introduced to the National Assessment of Educational Progress (NAEP) website of the U.S. Department of Education, Institute of Educational Sciences, National Center for Educational Statistics. Following an exploration of released items, sample student responses, and grading rubrics, the teachers voted on what percentage of eighth grade students nationwide answered a particular question (Figure 12.2, [81], 2005-8M3 No.:10, U.S. Department of Education) correctly. Referencing the possible results in Figure 12.3, the number of teachers choosing the responses a, b, c (correct), d, and e were 5, 13, 11, 4 and 2 respectively.

As anticipated, this led to a discussion of what types of schools participate in this exam, and whether this data represents typical students for teachers in this program. The participating teachers are employed in rural districts and few were aware of the NAEP exam. This discussion was followed by an electronic vote about where the largest achievement gap occurred. With many teachers voting for the city/urban/rural (7 of 31) or the eligible/not for free/reduced lunch category (13 of 31), the results brought about an awareness concerning their preconceptions and achievement

$$(3 + 4) + 2 = 2 + (3 + 4)$$

illustrates which property?

- a. Commutative
- b. Associative
- c. Reflexive
- d. Transitive
- e. Distributive

Figure 12.1.

Linear connections

■ In the equation $y = 4x$, if the value of x is increased by 2, what is the effect on the value of y ?

- a. It is 8 more than the original amount
- b. It is 6 more than the original amount
- c. It is 2 more than the original amount
- d. It is 16 times the original amount
- e. It is 8 times the original amount

Figure 12.2.

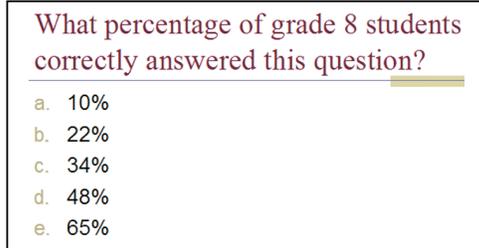


Figure 12.3.

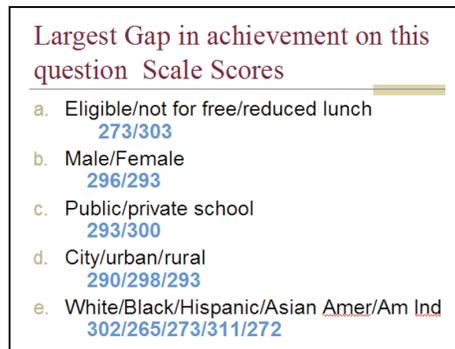


Figure 12.4.

gaps. Figure 12.4 shows the gap categories with the corresponding average total scores for the respective groups indicated below the category. The setting for this clicker use is an example of a provisional discourse move, for if the discussion had not migrated to one of participation on the NAEP exam, the workshop leaders would have initiated that discussion before the clicker question on achievement gaps. As a result of this engagement, the teachers now have a resource for assessment, and the ability to make valid statements based on research about achievement gaps.

12.5 Mathematical Experiment

The use of clickers to conclude a physical experiment is quite different from the use of clickers typically reported in mathematics courses in a lecture setting. Groups of teachers were given a unique vase, a unit of volume measure, a centimeter ruler and graph paper. Each group recorded the height of water in their vase with each additional unit of water added to the vase, and then graphed the height of the water as a function of the volume of water in the vase. Mathematically, this is quite interesting, since most expressions relating volume and height (such as $V = \pi r^2 h$, $V = \frac{1}{3} \pi r^2 h$) are written with volume as a function of height. Yet, to physically graph a relationship, it is far more natural to add units of volume and measure the resulting height rather than add water to a specific height and then measure the corresponding volume. After all graphs were completed, each graph was labeled with a letter, A–K and the vases were randomly numbered I–XI. There was no apparent connection between graph labels and vase numbers. The graphs were arranged on a wall and the vases lined up on tables in no particular order and without the unit of measure. The experiment concluded with participants viewing all the graphs and vases, attempting to match the graphs to the vases (Figure 12.5).

The use of clickers to determine the correct matches was purely an improvisational discourse move [48] with the decision to use clickers made within minutes of the matching. With only choices a–e on the clickers, the vases were grouped a) I, II or III b) IV or V, c) VI or VII, d) VIII or IX and e) X or XI. The popular choice for graph A was then confirmed as correct or not by the group that constructed graph A. For every graph, the majority vote did indeed identify the correct group of vases. Following this vote, a secondary vote was taken based on the vases remaining in the group. Again, the group who had developed the graph confirmed the match and numerous participants volunteered to explain why the match was correct. We believe that once the correct match was determined, confidence in their reasoning prompted the teachers to offer explanations, as is seen in the following teacher comment on the vase problem and clicker usage:

... the clickers *added* interest because I was able to compare myself in a “secret” way with others. Maybe it fed my ego! Maybe it built my confidence!



Figure 12.5.

With eleven vases, the voting was time-consuming, yet still engaging for the last few graphs. Participants noted in a whole group discussion following the voting, that in a smaller setting, such as one the teachers would see in their own classrooms, fewer vases would be used and the time for voting would not be an issue. The whole class discussion generated ideas for using the vase problem at all grade levels. In third grade, the focus could be on measurement, while in ninth grade, curve fitting of data was a suggested use of this experiment. Many of the middle grades teachers thought an excellent extension would be to have students draw a vase to match a given graph.

12.6 Teacher Participant Reactions

The participating teachers were sent an email survey concerning the use of the clickers during the summer workshop. Twenty eight teachers responded. When asked “Did the use of clickers in an anonymous fashion contribute to your willingness to use the clickers?” 25 teachers said yes, one teacher had no preference and two replied no. One of the no replies included the statement “I believe they would increase the willingness of young people incredibly.” The following teacher quote includes many of the sentiments expressed by others.

Because it didn't single someone out as being wrong or right. It felt safe, and helped me to see I wasn't the only one struggling with the question. Sometimes we think we are the only ones not getting it, and then we get stressed, and that stress just interferes with thinking.

Another teacher extended his response to his own students, “I think many of my students are not willing to respond because of risk of ridicule for being wrong, right, or the teacher's pet.” The importance of anonymous use of the clickers was referred to repeatedly in survey responses.

When asked whether seeing the results of their fellow teachers was of interest or not, one teacher felt it was not necessary, one felt it was somewhat important, and one deemed the results were only important on the vase problem. The remaining 25 teachers felt that seeing the results was important.

When asked whether the use of clickers promoted discussion with other teachers seated nearby, all but one teacher responded positively. The teacher quote that follows confirms that the conversation was indeed productive for some teachers.

I found the conversation enlightening—confirming my response, making me feel more sure that I was correct, or giving me a different direction to think that I had not come up with on my own. I feel the conversation was VERY beneficial to learning, and kept me on task and thinking about the problem even after I voted.

One survey question considered whether these teachers, at elementary, middle and high school level would use clickers themselves. Every teacher indicated that if made available, they would use clickers.

Absolutely yes. Every other teacher at my grade level would want to borrow them, too!

The use of an electronic voting system in the professional development of teachers was well received. The clickers promoted engagement in the problem and prompted discussion among teachers with significantly different mathematical background. The teacher comment below summarizes many of the comments made by the participants.

Clickers are interactive, technological, they are fun, the pressure is off but the learning is on!

12.7 Summary

While the literature on the use of personal response systems in mathematics has been limited to the undergraduate lecture, the benefits extend to the professional development of in-service teachers in a workshop setting. In addition, our observations show that these benefits may extend outside of the lecture and into the lab, when clickers are used in conjunction with a physical experiment. Furthermore, the literature on discourse moves applies to the teacher-directed choices to utilize response system technology.

Acknowledgement The workshop described in this paper was funded through a Title IIB State of Wisconsin, Department of Public Instruction, Math-Science Partnership Grant, *Creating Mathematics Excellence*, WI070906. Anne Wallisch, Principal Investigator.

13

Using ConcepTests in College Algebra

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I have been using ConcepTests since the fall of 2000 in a variety of mathematics classes. I use ConcepTests to engage students, provoke discussion, and help lead the class through challenging material. In this paper, I will discuss how I use these questions in my college algebra class. I will start with background on my course and the mechanics of using ConcepTests, move to different uses of ConcepTests with examples, and then finally raise some challenges for ConcepTests in college algebra.

13.1 Background

At Centenary, college algebra is one of three mathematics courses offered primarily for “core” — that is, students take the course to satisfy the mathematics requirement for the college. While a few students will take the course in preparation for precalculus, the vast majority takes the course as their last college mathematics course. Accordingly, our version of the course focuses less on symbolic manipulation and more on modeling and data analysis. For at least the last seven years, the department has used the Kime/Clark text [46], which is close in spirit to the Harvard calculus series. The text emphasizes different modes of thinking about functions (graphical, tabular, etc..) and places the study of linear and exponential functions in a modeling context. In addition, the course covers some basic descriptive statistics along with a section on regression.

Class sizes tend to be around 30 students with most of the students at sophomore level or higher. There are typically a large number of majors in the arts (music, communication, etc) and the social sciences.

I almost always use the two cycle format of ConcepTests — that is, I present a question and hold one vote after about thirty seconds of silent consideration, then encourage class discussion in small groups (often aided by my roaming about), and finally take a second vote which ideally leads to some debate and discussion by the class as a whole. The small group discussion between votes usually lasts anywhere from half a minute to several minutes. When I can tell the discussion is flagging, I call for an immediate second vote. In short, I see the primary function of ConcepTests as provoking student discussion for the purpose of clarifying understanding.

For that reason, I seldom give ConcepTests where the answers can be easily computed (e.g., solving an equation algebraically or checking a graph with a graphing calculator). While those types of tests can be useful for students, I prefer to simply give problems in class to check student skills. I also do not grade students on their answers to ConcepTests; I do not want the anxiety surrounding grades to interfere with the discussion.

While the questions are shown to the class using PowerPoint slides, I do not use clickers to collect answers and instead have relied on index cards given to the students. Besides being cheap, this approach allows students to vote

for several correct answers. In class, after the thirty seconds of silent consideration, I ask all students to hold up their cards. Some students do need prompting at this point. I can quickly look over the class and see a basic distribution of the answers. When I ask students to hold up cards again for the second vote, I can see where students have changed their votes. (For example, I may note that a group of students who voted across the board in the first vote have all converged to one answer in the second vote — that may be a good group to call upon during discussion.) However, with index cards, I cannot easily collect data on student answers, and students have less anonymity than they would with clickers. The lack of anonymity sometimes causes students to look around and vote with the majority rather than vote their own opinion.

13.2 Examples

In looking over my ConcepTests, I primarily use the tests in one of four ways: assessing different modes of expression, testing valid inferences based on rules, assessing the interpretation of mathematics, and motivating new material.

13.2.1 Different Modes of Expression

In my class, I most often use ConcepTests to help students navigate the different modes in which mathematics can be expressed (e.g., symbolically, graphically, numerically). For these tests, I will present some material using one mode of thinking and then introduce the ConcepTest to challenge the students to think in different modes. I will illustrate this using three ConcepTests from the section on linear functions.

Before the ConcepTest below, the class would have covered the slope-intercept form of the line and plotted some lines to explore the effects of m and b on the graph of the line. In words, we would have described these effects (e.g., “if m is positive, the line is increasing”). I would then present this ConcepTest (which is similar to homework questions in the text):

All four linear functions below are graphed in Figure 13.1. Which equation is the marked line?

- A) $y = 4 + x$
- B) $y = 4 - 2x$
- C) $y = 2 - 2x$
- D) $y = 2 - 4x$

Correct answer: B)

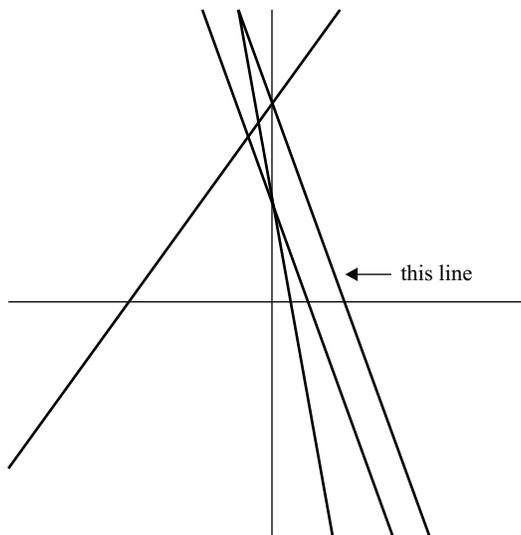


Figure 13.1. Four lines

Note that I do not place a scale on the axes. I do not want students to provide a calculator-based answer, but an answer based on the interpretation of the slope and intercept. A nice feature of this test is that students usually end up identifying all four lines in the discussion phase of the ConcepTest. I can then use the class discussion after the second vote to hear different students provide the reasons for different functions.

I use the next ConcepTest after we have discussed the idea that a linear function is one with a constant rate of change. By this point, we have also used models to motivate linear functions, so my students have seen the idea that equal changes in the input produce corresponding equal changes in output.

(Multiple answers possible)

Which of the tables shows a linear function?

- A) $A(x)$
- B) $B(x)$
- C) $C(x)$
- D) None of the above

x	$A(x)$	x	$B(x)$	x	$C(x)$
1	-1	1	1	1	3
2	-4	2	4	2	5
3	-7	3	9	4	9
4	-10	4	16	8	17

Correct answers: A) and C)

Most students have no problem answering A) — they see the equal changes in input producing equal changes in output. However, the other two choices present some interesting discussion. Most students reject B) as an answer, but the reasons for rejection are usually two-fold. Some students will recognize B) as the square function and reject it because it does not follow the $mx + b$ format. Others will calculate the average rate of change. I find it useful to highlight both responses so students can think about the issue symbolically and numerically.

Many students have initial problems with C). They associate the nonlinear pattern in the input with a nonlinear function. During the discussion phase, I will remind students to check the average rate of change. Almost always, we have a class argument about C) with the result being that students gain a wider understanding of how a linear function can be represented.

As one last example of using ConcepTests to help students think in different modes, let me give an example covering direct proportionality. Before I have given this test, we have discussed the form of directly proportional relationships ($y = mx$), the graphs (lines passing through the origin), and a test for directly proportional (output divided by input must be constant).

(Multiple answers possible)

- Which of these functions do not show a directly proportional relationship?

Correct answers: A), B), and C)

- Which of these functions are not linear?

Correct answers: B) and C)

x	$A(x)$	x	$B(x)$	x	$C(x)$
1	-1	1	-3	1	-1
2	-4	2	-6	2	4
3	-7	3	-9	3	-9
4	-10	4	-10	4	16

I find this test valuable for several reasons. First, students approach the first question using a variety of tools. For example, some students will “count back” to find the value when $x = 0$. Others will apply the “output divided by input” test but not check all the data on function $B(x)$. I can then have a productive discussion after the votes on the methods that work. Second, at this point, students typically start confusing linear functions and directly proportional relationships. The second test helps students clarify the difference and help them understand the idea that the latter is a subset of the former.

13.2.2 Valid Inferences

A second important way I use ConcepTests is to test whether students can make valid inferences based on the rules we have introduced in class. I often use this kind of test when we are discussing symbolic simplification.

The next two ConcepTests are examples of how I assess student inferences using the laws of logarithms:

If you know that $\log a$ is about .301, which of the following statements is true?

- A) $\log(100a)$ is about 2.301
- B) $\log(a^2)$ is about .090601
- C) Both A) and B) are true
- D) Neither A) nor B) are true

Correct answer: A)

Suppose you know that $\ln(a)$ equals .25. What can you say about $\ln(ae^4)$?

- A) It equals 4.25.
- B) It equals $.25 + e^4$
- C) It equals .254.
- D) None of the above.

Correct answer: A)

The first test would be introduced shortly after discussion of the laws of common logarithms; the second test would be used after the reintroduction of those laws with natural logarithms. In both cases, I am using problems that cannot be simply evaluated on a calculator. (Note that students at this point are usually not fluent in solving logarithmic equations, so very few of them solve for a .)

I want the students to evaluate reasoning, so in both problems, I raise the possibility of incorrect reasoning. In particular, I often use the question structure of Both A) and B) and Neither A) nor B) to give the students practice on following mathematical arguments. After the votes on both questions, I take the class through the different options and ask for reasoning on each. These types of questions are easy to generate for all types of symbolic simplification.

13.2.3 Interpreting Mathematical Data

A third way I use ConcepTests in my class is to assess how my students interpret mathematical data. Given that this will be most students' only math class, I want to ensure that students can critique the use of mathematical language and images.

For example, during the section on modeling with linear functions and after the introduction of the average rate of change, I will use the following test:

Suppose the average rate of change of jobs in Caddo Parish with respect to time is 10,000 jobs/year for the period from 1995 to 2005. Which statement is accurate?

- A) Each year, the number of jobs increased by 10,000.
- B) The number of jobs never decreased.
- C) Both A) and B)
- D) Neither A) nor B)

Correct answer: D)

Here I want students to appreciate the difference between the constant slope of a linear function and the much more chaotic behavior of a real-life function. I find the class discussion to be particularly helpful here — the students often do a better job of presenting counterexamples to A) and B) than I do! This test also reinforces the idea that the choice of endpoints is crucial in linear modeling.

Another example of interpretation is the following ConcepTest, with the graphs shown in Figure 13.2 A)–C):

Which graph has the line with the largest slope?

Correct answer: All graphs show a line with the same slope.

While this is a trick question (and thus not the best choice for a ConcepTest), it is a ConcepTest students handle quickly. By the time of the second vote, almost all the students have seen that all the slopes are equal. Here the benefit of the test is the discussion — the test provides a starting example for discussion of bias in graphs and the fact that any graph involves choices of scale.

13.2.4 Motivation of New Material

Finally, I use ConcepTests to motivate new material. Since the very structure of ConcepTests provokes students' engagement, I find it useful to use a modeling example in a ConcepTest to start discussion.

For example, after the class has discussed linear modeling, I will use the next question to help introduce solving systems of linear equations:

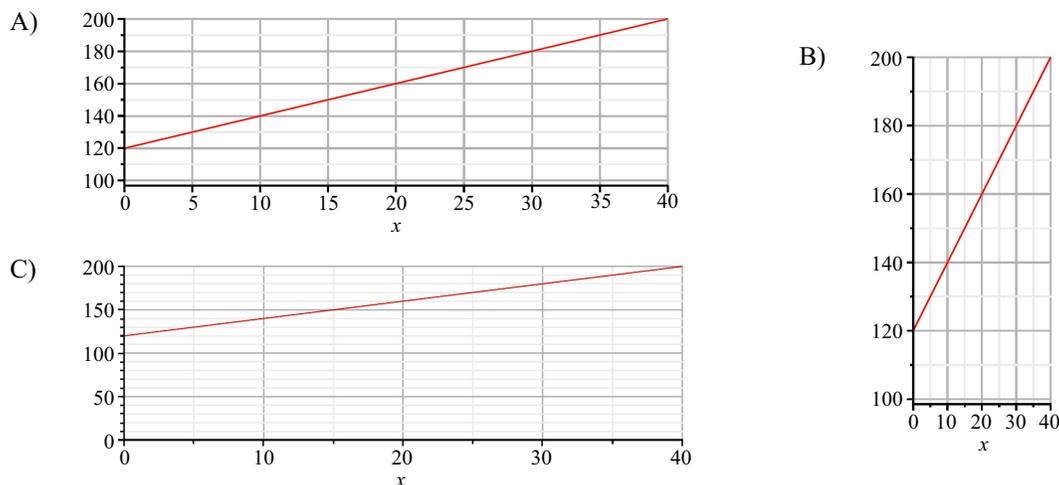


Figure 13.2.

You have been offered three different jobs: job R has a base salary of \$55,000 with annual raises of \$2000, job S has a base salary of \$42,000 with annual raises of \$3200, and job T has a base salary of \$48,000 with annual raises of \$2000. Assuming you will be working at this job for many years, how should you rank the jobs according to the income you will receive?

- A) R, S, T
- B) S, R, T
- C) R, T, S
- D) None of the above

Correct answer: B)

While the students can answer the question using their knowledge of slope and intercept, the test helps them think about the implications. Moreover, since some students will admit being uncomfortable with the vagueness of “many years,” we can move to a discussion to exactly when each job will be better than another and introduce solving systems of linear equations. Indeed, the question gives us a great opportunity to move from interpretation to graphical thinking to symbolic manipulation.

Another example of how I use ConcepTests is in the introduction of logarithms. After discussion of exponential functions, our text has a section on logarithmic time scales. The idea is to help students see logarithms as tools for accurately placing events on these scales. The goal is to connect logarithms with the exponents on powers of 10. I use the following question near the end of this section:

On your logarithmic scale of time, where does 500 years ago fit?

- A) Closer to the 10^2 mark than to the 10^3 mark
- B) Closer to the 10^3 mark than to the 10^2 mark
- C) Exactly halfway between the 10^2 and the 10^3 mark

Correct answer: B)

Not surprisingly, C) is the most popular answer and sometimes wins even after the second vote. The students who do best on the question are those who try and evaluate $10^{2.5}$ on their calculator. While the question is frustrating for some students due to the numerical content, it does model the type of thinking they need to successfully work with logarithms.

13.3 Conclusions and Discussion

As in my other mathematics classes, I have seen greater engagement and discussion in college algebra as a result of using ConcepTests. Students are more excited in class, and I am able to let students reach conclusions through the voting process.

I must mention one more challenge. While in my calculus classes, I often give one to two ConcepTests per class, I typically give fewer tests in my college algebra class, usually one to two per week. I have already mentioned one reason for the lower frequency (my penchant for giving problems in class), but I think another reason is the perceived lack of concepts in a college algebra class. Even with a strong reform setting, I find it difficult to move away from the idea that college algebra is meant to prepare students for symbolic manipulation in higher level (and more “conceptual”) courses. As I have put more emphasis on where students have trouble with the material, I have found it easier to develop tests, but I believe more effort is needed to get a strong understanding of the concepts of college algebra.

14

An Example of Multi-Purpose Use of Clickers in College Algebra

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14.1 Introduction

College algebra is a course with a great many and well-documented teaching and learning problems, as well as a great variety of proposed solutions. (See for example [35, 38, 75] and their bibliographies.) In what is likely the most popular model for the course, students listen to the professor lecture and watch the recitation instructor work exercises. Students, especially underclassmen, often poorly synthesize the information that they see and hear during class. The corresponding failure to transfer knowledge and concepts is not easy for students to recognize – and even when they do, there are often few viable activities available for them to bridge the gap. The bulk of the learning experience becomes memorization of a few techniques which are judged most likely to appear on the exam. My journey toward addressing some of these course issues has been strongly influenced by my involvement with the GoodQuestions for Calculus project [58]. The goals of the course design presented below are

- to increase student attendance and participation,
- to provide students and instructors with instantaneous feedback regarding students' basic skills,
- to address misconceptions by challenging intuition via conceptual questions,
- to improve reasoning, communications skills and classroom experience by including a peer-learning component to the conceptual questions,
- to provide external activities which both prepare students before the classroom experience and assist them with knowledge synthesis afterward.

To make explicit one final goal, it is crucial for the course design to intrude minimally on the instructor's current time and energy allotment for the course.

14.2 How This Class Works

At the University of Louisville, college algebra is taught chiefly in large lectures. Up to 150 students are enrolled in each large lecture section of the course. The lecture portion of the course meets for two hours and thirty minutes each week. Each student is also enrolled in a recitation section corresponding to their lecture section. The recitation meeting

lasts for 50 minutes each week, and is conducted by a graduate teaching assistant in sessions of at most twenty-five students. In a typical year, around 2,000 students will enroll in the course altogether. The activities described below in the sample lesson plan were conducted during one of the large lecture meetings of the course. The following is an excerpt from the course syllabus:

This class consists of several different kinds of activities in a recurring sequence.

1. Before each lecture when new material is to be covered there is a reading assignment from the course text, and a corresponding pre-class warm-up assignment on-line. These questions are typically one or two of the most basic exercises at the end of the assigned reading section in the book. The last question on each assignment is created to be an opportunity for the student to submit a question concerning the reading assignment. Half of the score for this component of the course is based simply on participation.
2. At the beginning of lecture, the class will respond to the questions from the warm-up assignment (as necessary) and also answer a few other exercises from the section using the clicker. After students respond to each exercise, we will work out the answer together as a class. To encourage pre-class preparation and in-class participation, bonus credit on exams is awarded based on a combination of participation and successful answers to the clicker questions.
3. After a brief lecture answering questions asked by students about the reading assignment and discussing the ideas, the class will respond to a sequence of concept questions, leading up to the discussion question for that class. The main discussion question is first answered by the students individually. After the students' first reactions are polled, time is allotted for discussion between pairs of students regarding why certain answers should be correct and others incorrect. Next, the students' answers are polled again. Finally, the instructor leads a discussion of which answer is most correct, and why.
4. After working on exercises in small groups during recitation, students will complete an electronic homework assignment covering this material. This homework is automatically graded by the computer system, and can both explain the solutions to the exercises to the student in a step by step fashion, and generate new problems for the students as often as needed. If a student is persistent and completes the assignments before the deadline, he or she should always be able to obtain a perfect homework score.

14.3 Sample Lesson Plan

The following sequence of paragraphs represent an example lecture period from one of my classes, titled *Linear Relationships: Slope and Average Rate of Change*. The italicized commentary includes the motivation for using each activity. The large lecture meetings take place two times each week in a lecture hall with three large projection screens, and last one hour and fifteen minutes. All of the items listed below appear during the lecture on the large screens via data projection of pdf slides, and are annotated during the lecture through the use of a wireless bluetooth tablet. Students have access to the annotated files for reference after the lecture is completed.

14.3.1 Reading question one

To check if a function is linear, we could just check that whenever the x -variable changes by 3 the y -variable always changes by the same amount (not necessarily 3), regardless of which x -value we started with.

- A) True B) False

14.3.2 Reading question two

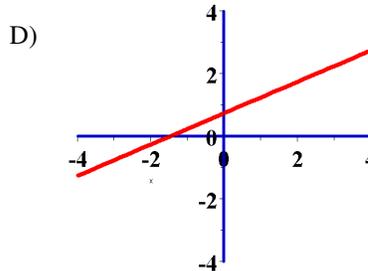
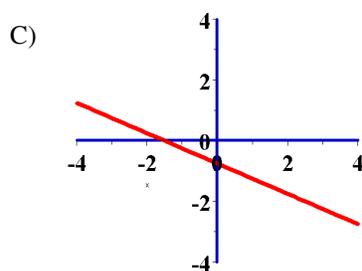
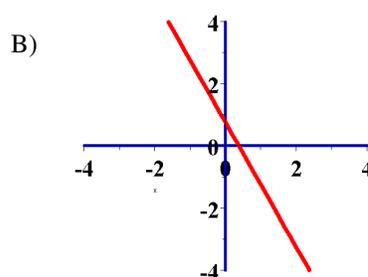
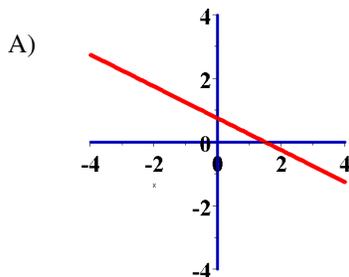
What is the solution of $mx + b = 0$?

- A) The y -intercept
B) The slope
C) The x -intercept
D) There is no graphical representation of this solution.

This brief review of the reading questions helps us to remember what we read. When many students have struggled to answer the questions correctly, the lecture plan can be adjusted immediately to allow more time for explaining basic information. Typically the reading questions are more computational in nature than the ones used for this session.

14.3.3 Class Exercise one

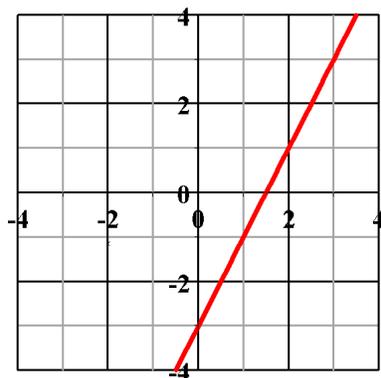
Which is the graph of $y = -\frac{1}{2}x + \frac{3}{4}$?



Answer: A.

14.3.4 Class Exercise two

What is the equation for the line



- A) $y = mx + b$
- B) $y = 2x - 3$
- C) $y = 2x + 3$
- D) $y = -2x + 3$

Answer: B.

Each of these questions is first answered by the students by clicker, and then explained by soliciting suggestions for proceeding from individual students. By the end of these explanations, most of the main ideas from this section of the material have been addressed.

14.3.5 Mini Lecture answering student questions

- Can you change the x -value of a linear function without changing the y -value and still have it be a linear function?
- Do the graphs of non-linear equations always form curved lines?
- Why aren't vertical lines functions?
- What exactly classifies an Identity function?
- How do you find the slope of the line containing the given points $(z + q, z)$ and $(z - q, z)$?

These questions are taken directly from the students on the previous evening's reading assignment. This selection (out of the 150 or so questions that might have been received) represents a combination of the most frequently asked questions and the questions that are most "on-point" with the main ideas of the section. Not all of them are answered, but just seeing what questions are asked helps students to improve their thinking while reading or preparing future material. This section also gives the students a greater sense of ownership since their questions help determine what we talk about in class.

14.3.6 First Discussion lead-in question

Suppose you have a line with slope 3. If the y -value changes by 6, how much does the x -value change?

- A) 3
- B) 2
- C) 1
- D) -2

Why doesn't it matter what the initial y -value is?

Answer: B.

14.3.7 Second Discussion lead-in question

You have been traveling for a while at a constant speed of 50 miles per hour. If you need to travel 150 miles further than you have traveled so far, how long should it take, if we continue to travel at the same speed?

- A) 2 hours
- B) 3 hours
- C) 4 hours
- D) It depends on how far you have gone so far.

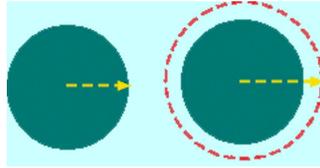
How is this question the same as the previous one?

Answer: B.

These questions are very much the same from a computational standpoint, since both are linear relationships and in each the variable which is typically taken to be the dependent variable (y in $y = mx + b$ or D in $D = rt$) is changed, determining a change in what is usually taken to be the independent variable (x , or t). This is precisely the same computation that is used to solve the following question, since the relationship between circumference and radius is typically written as $C = 2\pi r$.

14.3.8 Discussion Question

Imagine that there is a rope around the equator of the earth. Add a 60 foot long segment of rope to it. The new rope is held in a circular shape around the equator so that it is the same distance off the ground everywhere.



Then the tallest of the following that can walk beneath the rope without touching it is:

- A) An amoeba
- B) An Ant
- C) A Dog
- D) Me (the student)

Answer: D.

The typical first response to this question is that a 60 foot increase in the circumference should not make any appreciable difference in the radius. The students gradually come to understand both that the linear relationship indicates that the change in the radius must be the same regardless of the initial value of the circumference (so that the result would be the same if the earth were the size of a tennis ball) and that the height of a person is also not appreciable with respect to the size of the radius of the earth. This process illustrates the power of understanding and recognizing linear relationships between physical quantities in our world.

It should be noted that leading an effective full class discussion is a learned skill. In this model, students are more encouraged to respond than would be typical in a large lecture course because they already know that a certain percentage of the other students have agreed with their answer selection. Rarely is it necessary to call on students who do not volunteer their response, but at times I have found it helpful to call on students whose ideas I have heard and know to be profitable for the discussion while circulating about the class during the peer-discussion time. It is however crucial for the instructor to both honor the value of every student response and to completely refrain from injecting his or her own perspective on the problem until the students have exhausted their own resources. This keeps students participating, since you clearly respect their thinking, and avoids short-circuiting the process, which occurs when students believe that only the instructor's perspective is of value. My personal technique for accomplishing these goals is to repeat each student response to the entire class, e.g., "This is what I hear you saying: ... Is that what you were trying to express?" It helps to keep in mind that each incorrect response that a student volunteers in a large lecture setting usually represents an problem for many other students as well. In fact, these incorrect responses are especially valuable because they help to highlight common misconceptions in a much more effective way than a perfectly correct and complete response would.

14.4 Rationale and Effectiveness

14.4.1 Rational for each of the course components

By consistently reading before class, checking their comprehension by answering questions and by asking questions about the points in the text which they find confusing, students will improve their ability to comprehend technical material. The process of answering computational exercises using the clicker provides feedback to the students regarding how well they understood the reading assignment and how well they have learned how to work the exercises during the lecture. The development of communication, language and reasoning skills through the peer-discussion of the discussion style clicker questions will help students to become more confident when using mathematics throughout the college curriculum. The group work which occurs during the recitation hour represents another opportunity for peer-learning and for feedback regarding manipulative skills, as well as an opportunity to interact directly with the recitation instructor. All homework exercises are graded automatically by the computer to give the student an immediate response.

14.4.2 Discussion of effectiveness of course design

The results listed here are a comparison with the previous iteration of the course taught by the instructor, in standard lecture format, without using clickers.

Improved Attendance: Average attendance in lecture is 83% (up 13%).

Improved Performance: The rate at which students withdrew or earned a final grade of D or F (WDF rate) for the course is down to 30%. This represents a 25% drop in the WDF rate.

Engaged Students: The computational exercises fit naturally with the clicker functionality to transform the student from a listener to a doer during class. Combining the clicker with painstakingly designed discussion questions changed the students from a group listening to a leader to a group of people who think, make decisions, and communicate with each other to understand ideas. Although there might be other ways to accomplish this kind of transformation of the large-lecture classroom, I have been hard-pressed to find a method as convenient and cost-effective as this one.

Helped Students to Help Themselves: In the Fall 2006 semester, I taught 2 sections of College Algebra to a total of 300 students. When asked whether or not they found the clicker questions helpful, 83% indicated that they had. When asked how helpful these questions were, 95% indicated that the questions were a little helpful, somewhat helpful, or very helpful. Following are some of their comments:

- “I felt that the interactive nature of the clicker exercises helped me to follow along with the material being covered and gave me hands-on practice in working the problems while the material was still fresh.”
- “It’s nice to be able to answer questions without having to worry about points (although the extra credit is nice!). It is also helpful to see how other people answered. If I missed a question that most of the class got correct I know I probably missed something in the reading or in lecture. If a lot of people missed it I knew you would spend more time on it.”
- “I was skeptical about the clicker when the class started, but I am completely sold on it. It was a huge help and kept me engaged in the class. Usually I feel lost in such a big class. I thought it was great.”
- “Very helpful, allow you to put what you are learning to the test. A lot of times I think I know how to do something, but the clicker exercises allow you to actually try a problem and be sure.”

14.5 Instructor Impact

Since a significant amount of the information distribution portion of this course takes place before class, the instructor preparation activities are somewhat different than in a typical course. The instructor needs to:

- choose Pre-class reading assignment questions, check the overall success rates before class and collect a sample of student questions from the reading.
- choose additional exercises which will guide the progression of the class toward answering the student questions, and prepare multiple choice answers to these exercises (as is typically required by the clicker system).
- create a sequence of concept style questions leading up to the discussion question of the day.

Typically, the first two of these activities are not very time consuming, and in fact simply replace the usual time spent creating lecture notes and choosing example problems to work. The third activity is rather more involved. Combining one’s knowledge of students’ prior experience with the material with one’s own depth of understanding of mathematical concepts to create questions which challenge students’ intuition in a formative but not overwhelming fashion is quite a delicate task. (See [59, 80] for more on this process). However, this is exactly the sort of creative task which contains the power to rescue mathematics instructors with specialized training in mathematics from the wearying routine of preparing and recycling non-interactive lectures.

In the absence of existing banks of well-designed and tested discussion questions, this method of teaching would be a good choice for only those few instructors who can afford to spend significantly more than the usual amount of time preparing for and teaching the course. On the other hand, creating one or two good discussion questions to go along with a previously developed selection of questions can be a manageable and rewarding task for most instructors. Banks of discussion or “GoodQuestion” style questions have been in development for some time now [59, 80], and

my attempt to continue this development into the college algebra curriculum is available on the MathQuest website at mathquest.carroll.edu/libraries/GoodQuestionsForCA.pdf. It is my hope that this collection of questions will offer readers a starting point for exploration of this teaching style and will encourage instructors to begin writing discussion style questions of their own to share with the community.

14.6 Conclusion

To elevate learning inside the classroom, more of the information distribution must occur outside the classroom. This means that students must do something to prepare for class, whether it is reading, answering questions, asking questions, or watching videos. I spent several years trying to figure out how to get students to participate in these sorts of activities. What I discovered was that what happens in the classroom itself dictates whether or not students participate in pre-class preparation. It is absolutely necessary to participate in pre-class preparation to take advantage of the learning that takes place in an attractive classroom atmosphere. Learning is the great motivator—it feels good to learn something, and when students experience that feeling in the classroom, they will want to come back, and they will want to be prepared so they can experience the feeling again. The use of the clicker is an integral part of this process since it is through the clicker questions that the students see most clearly how well they have learned.

Second, simply requiring students to buy clickers and tossing out clicker questions during class is not a guarantee of success. It takes time to create and revise concept-based questions that challenge students' intuitions in a profitable way, and it also takes some experience to know how to respond to the results of clicker questions. It is very difficult at times to know when to allow the student discussion to move forward at its own pace, and when to interject with some extra information or direction for consideration—only experience can teach us the judgment and patience that this requires. I would venture to say that even if you were given a complete set of perfectly ideal materials, it would still take a full semester's use of those materials to learn how to respond to student ideas in a way that will optimize the positive conceptual learning outcomes. On the other hand, it is also the case that a tentative and imperfect inclusion of clicker activities is likely to result in outcomes that are at least as good as those resulting from a lecture-heavy format. This principle removes the barriers to trying new teaching strategies of all kinds.

Finally, keep in mind that the biggest learning gains come from allowing the clicker questions to become a teaching strategy that transforms your classroom from an instructor-centered forum into a place where students are the central participants and must become a cooperative group of learners. This requires a more significant commitment to spend class time using clicker questions, to spend preparation time thinking about how the concepts in your course can be addressed by clicker questions, and to spend more than one semester experimenting and learning how clicker questions can be made to work best for you and your students. Once that commitment is made, however, the multiple and significant benefits will become abundantly clear.

15

ConcepTests—Classroom Voting: A Catalyst for an Interactive College Algebra Classroom

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This article will concentrate on the use of ConcepTests in a course on College Algebra and some reasons why I think their use is effective in enhancing student learning. I use them as an aid in promoting student discussion and learning of mathematical concepts rather than as a means of determining a student's grade. It is really for formative assessment — giving both the instructor and students a means of assessing how well the students are understanding a concept or procedure. The ConcepTests used in the article are found in a ConcepTest Supplement to an algebra textbook, Robinson, Lomen, et al. [70].

15.1 Method

Because of the variety of forms these questions may have, instructors can use them in a manner that fits comfortably with their teaching style. Four possible ways are:

1. As an introduction to a topic. This works especially well if the topic is closely related to a previous lesson, or is something with which most students have some familiarity.
2. After presentation of a specific topic. Here a ConcepTest may be used to see if the students have grasped the concept, or if the topic needs more discussion or examples.
3. As a review of material that has been thoroughly discussed.
4. As a means of checking homework. Here students work on them outside of class, either individually or in groups. Then in class, students can vote on their results and discuss the strategies used in obtaining their answers.

For the first three situations above, after a question is presented students are given a short time (one to four minutes, depending on the question) to think about the question and then vote for the answer they think is correct. Providing all of the students do not vote for the correct answer, students are then given a short period of time to discuss the ConcepTest with adjacent students and then are asked to vote again. I then call on various students to explain the reasoning they used to obtain their answer. Having students discover and hear a variety of reasoning methods used in the solution is a crucial aspect of using this method. For the last situation, only one vote is taken before discussion takes place. Most questions can be used in any of these four ways, provoking different discussions in each case.

15.2 Examples

The first set of examples show possibilities of using ConcepTests as part of the introduction to a topic.

Polynomials

The following question would be given after a discussion of the process of adding and multiplying polynomials.

- Given that $p(x)$ is a polynomial of degree 6, leading coefficient -4 , and constant term 7, and $q(x)$ is a polynomial of degree 5, leading coefficient 2, and constant term -3 , what is the degree of $p(x)q(x)$?
 (a) 5 (b) 6 (c)* 11 (d) 30 (e) None of these

While (c) is the correct answer, students often choose (b) as they confuse multiplication with addition. A few choose (d) as they remember that you multiply when dealing with exponents.

A companion question would be: What is the y -intercept of the graph of $p(x)q(x)$? You then can repeat the question for $p(x) + q(x)$.

Vertical and horizontal shifts

A second example is a ConcepTest which is given after introducing vertical and horizontal shifts of a function, emphasizing their graphical aspects.

- If $(6, 2)$ is a point on the graph of $y = f(x)$, then which of the following points must be on the graph of $y = f(x + 2)$?
 (a)* $(4, 2)$ (b) $(4, 4)$ (c) $(8, 2)$ (d) $(8, 4)$ (e) None of these

If students understand the graphical significance of horizontal translations, they realize that (b) and (d) cannot be correct. After discussing why choice (a) is correct, the question can be rephrased to state “which choice “CANNOT” be on this graph” and asked again. Because $(4, 2)$ is on the graph, the only point that can be excluded is $(4, 4)$, because f is a function (single-valued). This shows how it is easy to use a current question to review previous topics, or how the answer to one question can easily bring up others.

After this question is discussed, the following question is asked:

- If $(6, 2)$ is a point on the graph of $y = f(x)$, then which of the following points must be on the graph of $y = f(x + 2) + 1$?
 (a) $(4, 1)$ (b)* $(4, 3)$ (c) $(8, 1)$ (d) $(8, 3)$ (e) $(8, 4)$

The point of this question is to combine a horizontal translation with a vertical one, so the answer will be (b). Most students answer this question correctly on their first vote. How many more questions like this would be asked depends on how the class responded to these questions.

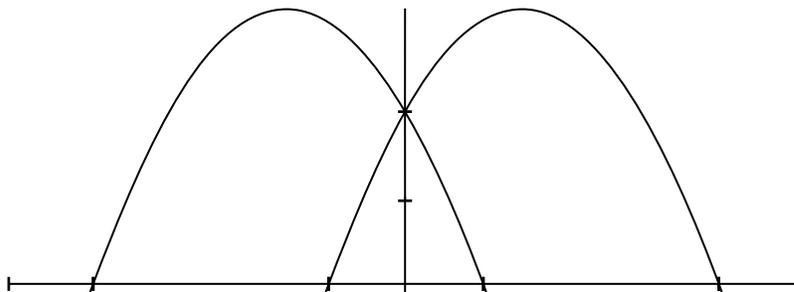
Scaling and translations

Later on in the course after discussing scaling inputs and outputs, the following could be asked.

- How might the graph of $y = 2x + 4$ be obtained from the graph of $y = x + 2$?
 (a)* Scaling the output by a factor of 2
 (b) Scaling the output by a factor of $1/2$
 (c) Scaling the input by a factor of 2 and shifting result up 2 units
 (d)* Scaling the input by a factor of $\frac{1}{2}$ and shifting the result up 2 units
 (e) Scaling the input by a factor of $\frac{1}{2}$

In the initial vote, the majority of the students will choose a correct answer of (a). Then after discussion with peers, a few groups will come up with the other correct answer, namely (d). Because I am especially concerned with developing students’ reasoning abilities, for a question like this one I will have students discuss among themselves and formulate reasons why they decided to eliminate choices (b), (c), and (e). For this question I am expecting graphical reasons. The next question is in this same spirit.

5. How can the parabola on the left side of the figure below be obtained from the graph of the parabola on the right side?



- (a) Horizontal translation to the right
- (b)* Horizontal translation to the left
- (c) Reflection across the horizontal axis
- (d)* Reflection across the vertical axis
- (e) None of these

Students initially choose correct answer (b) more often than correct answer (d). After discussing this with peers, almost all students understand why these are the only two possible answers. Using questions with more than one possible answer encourages students to realize that developing their mathematical reasoning is just as important as finding a correct answer.

Rational Functions

The next example shows a ConcepTest which could be used in the introductory part of a lesson on rational functions. After a discussion on how to find zeros and asymptotes of rational functions and giving examples which had both, I would ask

6. Consider the rational function $y = (x^2 - 4)/(x^2 + 2)$. Adding 2 to the numerator of this rational function changes which of the following (more than one may apply)?
- (a)* The x -intercept
 - (b)* The y -intercept
 - (c) The horizontal asymptote
 - (d) The vertical asymptote
 - (e) None of these

For this question, I ask the class “Who thinks that choice (a) applies?” and so on for the remaining choices. After having students explain the reasons for their answers, I pose the same question for $y = (x^2 - 4)/(x^2 - 2)$, followed by the same question with $y = (3x^2 - 4)/(x^2 - 2)$.

The following class session (after they have completed their homework assignment on this section) I would give them two questions similar to those above and then ask:

7. Consider the rational function $y = (x^2 - q)/(x^2 - 2)$. Increasing the value of q changes which of the following?
- (a) The x -intercept
 - (b) The y -intercept
 - (c) The horizontal asymptote
 - (d) The vertical asymptote
 - (e) None of these

The point of this question is to emphasize that the answers depend upon whether q is positive or not. If q is always negative, only the y -intercept changes. (Many students think of parameters as having only positive values, usually positive integers. This is actually an advantage here as the special case $q = 2$ will often be discovered.) If q is positive and not 2, then increasing q changes both the x -intercepts and y -intercept.)

The previous questions illustrate their use as a part of the introduction of a topic. However, they could also be used later on to check to see if the concepts had been mastered, or if more time should be spent on the topic. This is a means for the instructor to obtain instant feedback on how students are doing and thus more accurately determine a proper pace of developing material.

Review Questions

The three questions that follow I have used in reviewing a topic:

8. For what values of c is the vertical asymptote of $y = \ln(2^x - c)$ at $x = 4$?

- (a) 1 (b) 2 (c) 15 (d)* 16 (e) None of these

The purpose of this question is to check understanding of domain and range for composite functions.

9. Consumer experts advise us never to pay the sticker price for a car. A rule of thumb is to pay the sticker price minus 20% of the sticker price, plus \$200. We have \$18,000 budgeted for a car. What sticker prices are within our budget? Which inequality best represents this problem?

- (a) Let x represent the sticker price, $x - 0.2 + 200 \leq 18,000$
 (b) Let x represent the sticker price, $x - 0.2x \leq 18,200$
 (c)* Let x represent the sticker price, $x - 0.2x \leq 17,800$
 (d) Let x represent the purchase price, $x - 20\% - 200 \leq 18,000$
 (e) Let x represent the purchase price, $18,000 \leq x - 0.2x + 200$

This question seeks to check mastery of setting up word problems and the use of inequalities, two areas of difficulty to many students.

10. A quadratic equation is given by $y = ax^2 + b$, $a \neq 0$ and $b \neq 0$. Which of the following are true? Doubling the value of a

- (a) shifts the graph of this equation up by a factor of 2
 (b) shifts the graph of this equation down by a factor of 2
 (c) changes the value of the x -intercept
 (d) changes the value of the y -intercept
 (e) none of the above

If a and b have the same sign the answer is (e), if they have opposite signs the answer is (c). Students can also discuss what happens if $a = 0$ or $b = 0$.

15.3 Implementation

Instructors at the University of Arizona have implemented “classroom voting” in a room which has a built in Personal Response System (similar to that used on “So You Want To Be A Millionaire”). In an ordinary classroom with each student using an electronic, hand-held “clicker,” votes can be instantly displayed as a histogram on the instructor’s computer. Students can also vote with no technology at all, by holding up colored index cards (A = red, B = blue, C = green, etc.), or by simply raising their hands so that the instructor will get a good understanding of the results just by glancing around the room. After the final vote, the instructor can then call on various students, asking them to explain what they voted for and *the reasoning they used to reach their conclusion*.

I have displayed questions using Power Point, an overhead projector, and writing them on the board, and have used clickers, colored cards and raising hands for voting outcomes. The method I feel is most effective for me in a class of about thirty-five students is using an overhead projector and having students raise their hands. Because I

determine how much time to spend on a topic by student response to the questions, it is much easier to manipulate several transparencies than manipulate questions using Power Point. Also, if some misunderstanding is discovered, I can quickly use a transparency I was planning to use later, or write a new question on the board which addresses this situation. Having students raise their hands to vote eliminates the time required to hand out and collect clickers before and after class. It also creates an informal setting and helps emphasize reasoning rather than answers. Since students need to use paper and pencil for some questions, not having clickers to deal with as well is also a plus.

15.4 Results

It is almost common sense to realize that students learn best when they are doing the mathematics for themselves, rather than passively following the instructor's work. If you call on individual students during a lecture to ask them a question, you can create a more interactive classroom, but that still leaves the remainder of the students passively observing. However, with classroom voting you can ask a question and receive a reply from every single student, requiring them all to grapple with the key issues which allows all the students to be involved with answering the question. The more deeply students are involved in the lesson, the more they will understand, and they more they will remember.

Chickering and Gamson [15] list the following Faculty Inventory for good teaching practices:

1. Encourage Student–Faculty Contact
2. Encourage Cooperation Among Students
3. Active Learning
4. Give Prompt Feedback
5. Emphasize Time on Task
6. Communicate High Expectations
7. Respect Diverse Talents and Ways of Learning

The very nature of classroom voting as described above clearly results in achieving items 1–5 above. As far as item 6 is concerned, I find that the inherent competitive nature in many students gives the class an unstated, but real goal to do as well as possible, even though their answers are not recorded for a grade. Allowing students to both work by themselves and with peers and then explaining different procedures for obtaining an answer addresses item 7.

I have found that since using ConcepTests and “classroom voting” very few students drop my class, and most students have perfect attendance. They say that they look forward to coming to class because there they can use their brains and discuss ideas and procedures instead of simply listening and taking notes. Student evaluations of my courses mention that voting makes class more interesting, the material more understandable, and they have less difficulty with their homework. If given a choice, they would choose a class that uses voting rather than one that does not.

15.5 Conclusion

Using ConcepTests and “classroom voting” in a mathematics class can be very effective for several reasons: Students learn from their peers, who understand the common mistakes that may no longer be obvious to the instructor. Further, students learn to “talk math” with each other on a regular basis, verbally expressing their own mathematical reasoning and learning to evaluate the reasoning of others. Every individual gives their own vote, so there is a real motivation for each person to figure out what's right, and to not just copy the votes of their peers. Also students often explain correct mathematical reasoning that would never occur to the instructor, but makes very good sense to a student who would not understand the instructor's explanation. The key element is to have students think and communicate Mathematics in class.

16

Using Clickers to Encourage Communication and Self-Reflection in Precalculus

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16.1 Introduction

While it is imperative that future teachers of mathematics are able to have a profound understanding of mathematical ideas [53], so they are able to communicate effectively in a K–12 classroom, it is equally important that all college students communicate mathematics in an articulate and proficient manner. Future business leaders or lab technicians may use spreadsheets to do much of the heavy lifting in regards to mathematical computation in the business world, yet must still be able to convey their mathematical understanding of those numeric outputs to others on their team or the clients that employ their services. A common theme among recent documents published by the Mathematical Association of America [54], the National Council of Teachers of Mathematics [62], and the American Mathematical Association of Two-Year Colleges [1], is to provide classroom situations where students have thought-provoking and interesting conversations about mathematical topics and concepts. We have found that clickers allow us to break down some of the invisible classroom walls set up by students' discomfort communicating in front of their peers. In place of these walls, we have found students who are responsive and willing to share their thoughts and feelings about their own personal understanding of a mathematical concept. We attribute this to the fact that each student has taken a personal ownership of an answer, because of the inherent forced participation on the part of the clicker.

Although mathematics is typically not thought of as a discussion course, through use of clickers we are able to generate more classroom discussion and have students explain their reasoning for selecting a specific solution. We try to choose questions that do not heavily emphasize computation or drill, as those questions will be covered through other classroom activities. Instead we choose questions for use with clickers which focus on conceptual understanding. Thus a student will not take on a single identity towards the mathematics they study, and will find a true balance in their mathematical understanding [75]. Our choice for clicker items are often questions where answers can be modified based on parameter changes within the model of the scenario presented. We often say to students that as a result of the discussion, they should either, A: convince someone else in your group that your answer is the correct one, or B: you will be convinced by someone else in the group that their answer is correct. We emphasize that reasoning behind their solution choices is more important than simply voting for the correct answer with no knowledge of why it is correct.

An important byproduct of having students engaged in rich discussion is that misconceptions students might hold

about mathematical topics are challenged. According to the National Research Council [63] there is evidence that suggests that metacognition develops gradually and is as dependent on knowledge as experience. It is difficult for students to engage in self-examination and reflection in areas where one does not have a full understanding. Thus students need to experience situations where their incorrect beliefs and misconceptions are challenged. It is often easy for a student to sit through a lecture and nod in total agreement with what the instructor says or directs them to do. By asking students to vote on a question, it gives them a personal investment in the problem. When a student's conception is challenged, it makes them more interested in what others have to say about the concept, to understand why a particular belief is incorrect. By engaging in rich discussion, students may reprocess thoughts about a topic which they may have mistakenly thought they had previously mastered.

16.2 The Course

During the spring semester of 2008 we began using a personal response system in a precalculus course. The course was taught in a manner where conceptual understanding was emphasized as equal to teaching procedural skills [75]. Due to the timing of the university ordering the clickers, they were not available to our students at the beginning of the semester. Once they became available, we used them on average one to two times per week, typically at the beginning of the hour. We normally gave the students four to six clicker items during that day, which could typically take up 50 to 60 percent of the allotted classroom time during that class period.

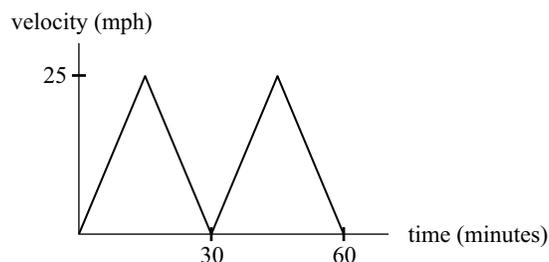
As mentioned earlier, the types of questions you use with your personal response system are very indicative of the type of conversation you might hope to elicit from the class. During this semester of using the personal response system our textbook [21] was accompanied by a database of conceptest questions that were written to coincide with the text. We chose to use many of these items as our clicker questions that semester. Where necessary, we made modifications to the items or wrote additional questions that would help us focus on specific areas where we anticipated our students would have misconceptions. In the following section we will discuss three questions that were used in this class. We detail how we used the questions in our class, as well as the types of reasoning and communication that would be expressed by our students when engaging in discussion and self-reflection about the given topic.

16.3 Clicker Questions

The first example clicker question [21] is one that was used near the beginning of the semester to assist students in using terminology associated with functions, as well as to assist in the interpretation of graphs.

Clicker Question #1

The figure below shows the velocity of a cyclist traveling due east from home.



Part a) Since $v(15) > v(30)$, the cyclist is further from home at $t = 15$ than at $t = 30$. Answer a-true or b-false.

Part b) There is a t -value where $v(t) = 10$. Answer a-true or b-false.

The correct answer to part a is false. $v(15) > v(30)$ implies that the biker is traveling at a greater velocity at time $t = 15$ minutes, compared to time $t = 30$ minutes. The correct answer to part b is true. There are multiple locations on the graph where $v(t) = 10$, as you can draw a horizontal line at $y = 10$, and the line would intersect the graph at four different t -values on the domain $[0, 60]$.

In this series of questions associated with the given graph, the main goal is to assist students with their understanding of functional notation. In part a, students will discuss the graphical interpretation of the two notations being

compared. They must also be aware of the fact that the function represents velocity, and not distance. It often takes some discussion among the large group to convince some of the students in the course why their thinking is incorrect.

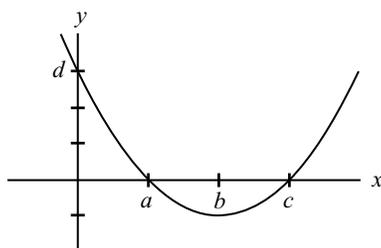
In part b, students will struggle with the understanding of the meaning of function. Students will often struggle with the interpretation of how this question differs from asking what does $v(10)$ equal. These are great questions for students to repeatedly think about and discuss with their peers. It gives them a chance to internalize through class discussion, what features of the graph are being expressed by the equation. In example questions like this, it gives the instructor an ability to open up the question to further interpretation. You might ask a student to discuss the practical meaning is of the equation. Thus they must link back to the context associated with the functional notation.

The second example clicker question [21] is one that was used in the middle of the semester when discussing quadratic models and their graphs. The function being graphed is based strictly on parameters, and does not make use of actual numbers in the equation choices. This forces students to think about underlying concepts associated with the model, rather than being able to fall back on simply substituting in numbers, or making use of their graphing calculators to assist them in answering. They must also think about the meaning of different forms for writing quadratic models, and how the different parameter values affect the function's graph.

Clicker Question #2

Which of the following are possible equations for the function whose graph is given in the figure?

- a) $y = (x - b)^2 + d$
- b) $y = (x - b)^2 - d$
- c) $y = (x - a)(x - c)$
- d) $y = ax^2 + bx + c$
- e) None of the above



The correct choice is c, since this equation correctly represents the zeros of the graph. As the students discuss the choices associated with this question, it gives them an opportunity to think about different forms associated with quadratic models.

Solution a will often lure many students towards it, because they will see $+d$ as being associated with the y -intercept. Students can be asked to describe what happens to the equation if $x = 0$. They will quickly find that y does not simply equal d , when $x = 0$.

Solution b is one that is chosen by those students thinking about the vertex form of a quadratic equation. A clever student might try to argue that this model does fit the graph, going under the assumption that the person drawing the graph has done a poor job of scaling the negative y -axis. The student might believe the vertical component of the vertex is d units below the x -axis. Although this is not the intended solution, a student giving that argument is using some creative reasoning to fit the situation, and brings forth the notion of how important it is to scale and label axes appropriately when drawing graphs.

Solution c, the correct choice, does allow for further explanation besides stating it is correct. We would want the students voting for this choice to further the discussion about the zeros of the function. We would want to hear these students explain how the graph would be modified if the function choice were $y = (x + a)(x + c)$. We would also want them to be able to discuss why the graph is concave up, and how the expansion of the mathematical expression ties to this fact.

For many students that might have chosen d as the solution, there is the opportunity to discuss the fact that just because the value of c is three tick marks to the right of the origin, does not necessarily mean that it is scaled the same as d , which is three tick marks above the origin. Since actual numbers are not used in this question, it allows for freedom to return to the discussion of the effect scaling and labeling of the axes can have on the graph.

Although solution e is not the intended choice, there is valuable discussion that can come from a student choosing this option. A student might disregard choice c on the basis that when you expand the expression, the constant term ends up being ac rather than d . Although there is the possibility that $ac = d$, the fact that these issues are not simply cut and dry to the student, nor to the problem, can lead to rich mathematical discussion.

The third example clicker question [21] is one that was used near the end of the semester when discussing trigonometric functions. The question asks the students when two trigonometric functions are equal to one another. What the question does not provide is a specified domain. The specified domain plays an important role with trigonometric equations due to their periodic nature. This is done intentionally so we are able to gain a sense of how students are able to handle a situation which could present them with a different number of potential solutions, based on the modification of the domain.

Clicker Question #3

For how many angles of x , does $\cos x = \sin x$?

- a) 1
- b) 2
- c) 3
- d) 4
- e) Not enough information is provided

Because no domain is specified, the best answer would probably be that there are an infinite number of angles for which these functions are equal. Of the answers provided, the best answer is e. However, the purpose of the question is to provoke thought and dialogue, and by using the question as structured we have been able to gain some valuable discussion when asking students why they may have voted for one of the other four solutions. Based on homework problems students may have done, they may become locked in to thinking that all trigonometric questions focus on the domain $[0, 2\pi]$. At the same time, a student that might have answered that there is only one angle that makes this equation true, may have their eyes opened to the fact that not everything is restricted to simply the first quadrant. This allows for some good discussion of thinking about the trigonometric situation strictly in terms of right triangles, or if students should expand their thinking into a unit circle convention. It is interesting to find that some students thinking about the unit circle convention might believe that if an answer is possible, it should be an even number of solutions, since cosine and sine are equal to one another at exactly two points on the unit circle. Thus it is important to facilitate conversation where students will come to a conclusion that the domain could be limited to simply half of a revolution around the unit circle.

This is a clicker question where each answer gets a share of the students' votes. It is important to point out to students that their answer could be correct, based on how we define the domain of the problem. This reinforces the importance of communication to the student, and how defining a problem for someone to solve is important. Without the correct parameters of the problem, it is easy to answer a different question than the one posing the question may have intended.

16.4 Purposes and Models

The clicker unit we currently use is the i-clicker. As students vote for an answer, the instructor base unit records the student votes. Students are able to change their answers as many times as they like until the instructor closes the voting period. The base unit has an LCD screen which informs the instructor of the percentage of students voting for each answer. We find this quite useful, as it allows us time to formulate a direction of conversation once the voting is complete. Once the instructor closes the voting session, we are aware of the class results, but not individual results. Individual results can be viewed in a spreadsheet format, but this is often cumbersome to do during class. This is best done outside of class. If we choose to, we can display a bar graph to the class which represents the number of students that chose each answer. We do this occasionally, but not all the time. Typically we ask for volunteers to defend an answer. If we are able to move around the room and listen to student conversations, then we are able to select groups of students to initiate the conversation depending on what we hear them discussing.

Our goal for using the questions encompasses three different purposes, with no particular ordering to them. One purpose was to introduce a topic students might have studied previously in a mathematics course to see what they remembered about the topic. Clicker Question #1 was used for that purpose since it contains a contextual situation that students could relate to and use their real-life knowledge to assist them in reasoning and answering. Another purpose

was to ask questions which coincided with material that had been discussed during the past week of class sessions. We used Clicker Question #2 for that purpose. Quadratic models were discussed earlier in the week using definitive parameters. The clicker question asked the students to think about the models and graphs in a more generalized manner using parameters instead of numbers. A third purpose was to elicit a reaction from students about problems which are dependent on gaining more understanding before a clear solution can be chosen for the problem. Clicker Question #3 was used for that purpose. Many students become cognizant that they need a specified domain for the equation before a specific answer can be given.

A positive aspect of using clickers is the fact that they give students a sense of anonymity, yet at the same time allowing them to participate in the class activity. We try to move one step beyond that. As students feel safer participating in the conversation, we often begin asking them to defend their answer to the entire class. We have found anecdotally that by using the clickers to stimulate conversation, it often carries over into classroom sessions that do not make use of the clickers. Students become excited to share their thinking and reasoning behind what they believe to be correct about the concept at hand. This could be attributed to the newness factor of using them, but we thought it was encouraging that students were excited to engage in discussing mathematics. It is important though, that the classroom environment is set up in a way which encourages this, and makes students feel safe about sharing their thoughts.

We tend to use a variety of different models for discussion during the voting session. The choice of models is sometimes dependent on the results of the initial vote. One model we use with students is to ask them to vote singularly, without any discussion with their peers. Once everyone has had a chance to vote, we show the students the results of the voting. An ideal situation is when multiple answers get a share of the vote. Depending on the distribution of the vote, the instructor has the flexibility of choosing how they want to engage the class into conversation. If a majority of students have chosen the wrong answer, we find it advantageous to hear from a student who voted for the incorrect answer. This is not always possible, as students are asked to share by volunteering, and sometimes they are not comfortable being the first volunteer. Typically if there isn't a choice that receives a majority of the vote, students are more likely to share their thoughts and reasoning. By leading with students that have flawed reasoning, it gives those with the correct solution an understanding of where their classmates may have an incorrect understanding.

We have found some of the most useful discussions to occur when almost no one votes for the correct answer. This doesn't happen often, but the discussions become very powerful and meaningful for the entire class. In situations like this though, students will quickly learn that they should not always go with the majority rules attitude. It helps to strengthen the declaration that students should not believe that the majority is always correct when discussing future questions.

A second model we often use is to ask students to first answer the question by themselves. We then show the voting results to the class and ask them to discuss the question with their pre-assigned group members. After that, they come back and revote on the question. After the revote, we show the students how the vote has changed. When using this model, it is nice to encourage those students that might have changed from an incorrect choice to a correct answer after conferring with classmates. Again it is useful to try and spend some time walking around the classroom to listen to the conversation going on in the groups. That way we gain an understanding of which groups of students it would be beneficial to ask to volunteer during the discussion time. If a member of a group that has changed their thinking is willing to share, this allows the entire class to hear about the metacognitive process that was employed, and how the misconception was challenged in the student's own line of thinking. At the same time, it gives the student a chance to verbalize a new understanding of the topic, thus confirming for the student if he/she understands what the others in the group stated about the correct solution.

A third model implemented in our class happens when we have the class vote split between two different answers during the initial vote. We show the class that the vote is split, and then ask for volunteers from each of the solution groups to give an explanation of why they chose their particular solution. After each solution has been defended, we ask students to do a revote, without discussing it with anyone else. Depending on the results of the revote, we gain a sense of whether or not those choosing an incorrect solution were able to re-evaluate the situation and come to a correct conclusion based on something that was shared by the correct respondent. If this does not happen, then we continue the discussion asking for each of the choices to once again be defended to see if anyone has now chosen to select a different answer. It is this group of students, those changing from an incorrect to correct answer, who are important for convincing their peers who stick to their misconception, why their misconception is incorrect.

Although beneficial, clicker questions can use a significant amount of class time. During this particular precalculus

course, we chose to use clickers approximately once a week. On the day we used clickers, we would typically ask four to six questions during the 50-minute class period. Depending on the depth in that day's questions, they could typically take 30 or more minutes to discuss and answer.

16.5 Summary

While much of the literature on the use of personal response systems in mathematics has been geared towards just-in-time teaching, we feel an exemplary use of clickers is to facilitate discussion amongst students, while challenging them to self-reflect on their mathematical understanding of topics. We have found that by using appropriate questions, structured in an appropriate manner, students are able to express their understanding and misconceptions to not only the instructor, but also to themselves. Through these types of situated cognitive learning experiences, students will have these discussions to fall back on and review in their minds the next time they are presented with similar situations.

17

Writing and Adapting Classroom Voting Questions: New Functions from Old

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17.1 Introduction

There are an increasing number of classroom voting question sets already available. This paper is a guide on how to utilize existing questions, how to write a few new questions and how to put them to use during a lesson. The example lesson “New Functions from Old” is from the third day of class of review for a Differential Calculus course. This lesson was used in two different sections of the course, each taught during a 50-minute period. Lesson goals: recognizing even and odd functions; identifying invertible functions; finding the equation for the inverse of an invertible function; composing functions, and graphing vertical and horizontal shifts and stretches, reflections about x - and y -axis and across the line $y = x$. The functions that have been studied so far are lines and exponentials but the students should also be familiar with quadratic, cubic and trigonometric functions through prerequisite courses.

17.2 Organization and preparation

For the following lesson example, I used the voting questions as a follow up for each portion of the traditional lecture instruction. The lesson plan was used on the same day for both a morning and an afternoon section of Differential Calculus. My notes on the lesson reflect this tendency to lecture then vote in small time increments, which keeps the class at a comfortable pace. If the voting questions are used at the end of class as a kind of catch all, it is often the case that time is an issue and it ends up that no voting questions are used instead of at least a few of them sprinkled throughout the lesson. The set of questions that I used for this course are from the bank of questions developed by Carroll College (mathquest.carroll.edu) and the resources accompanying the Harvard Calculus text. Course packets of the entire semester’s worth of questions are purchased by the students from the bookstore at the beginning of the semester.

Votes were collected either by a show of hands, labeled index cards, or by using electronic personal response units. The afternoon section had so few people in the class that the students eventually opted to just shout out their answer when the vote was called for instead of registering their votes electronically, as such I have less consistent data for that class. The drawback for this method is that it is easy for a student to wait to vote until they hear their neighbors’ response. However this did not seem to be a major issue since the discussion is what really counts, and it becomes immediately apparent who just guessed. For the electronic voting method, there is a set of “clickers” in each classroom along with the appropriate receiver and computer software. Each student has his or her own clicker assigned to them.

The question is “opened” and then students register their votes as they complete the question. Once all but a few of the students have registered their votes, I make a last call for votes and then “close” the vote. A pie chart of the distribution of votes is projected on a screen for the students to see. Seeing how everyone voted seems to be what students enjoy most about the electronic voting.

I collect information on how long it takes for all the students to vote on the question as well as the percentage of votes each answer receives. This is to help me further determine which of the questions are “good questions.” It is important to keep in mind that the voting is nothing more than a gimmick to get the students to focus, make a decision and then engage in classroom discussion. It is easy to get distracted and focus only on the voting part instead of the discussion part. To this end, I also jot down notes in the margin of my instructor’s edition of voting questions of how the discussion went, whether there were any unexpected issues or if the discussion was particularly rigorous and illuminating. These notes serve to remind me, or someone else who might use the questions in the future that this particular question was especially valuable or perhaps should be avoided.

In organizing the voting questions for use, I use the order that they are printed out in the course packet as a rough guide, but put them in the order that flows the best with my lesson plan. The students don’t seem to mind that the questions are presented out of order; in fact it seems like there is more focus if questions are out of order. I prefer to give the students a hard copy of the questions, rather than a power point presentation or overhead transparencies, so that students can take notes in the margins of the questions as well as use the questions for later review.

Prior to writing out my lesson I prepare by looking over the chapter in the textbook, viewing the online homework questions that are assigned and by previewing the available voting questions. Once I have a clear idea of the goals for the lesson as well as which of the available questions I will probably use, I go about the process of thinking about what other questions I could create to supplement the lesson. I view writing these voting questions as two slightly separate steps: writing the questions and writing the answers. A good question can be ruined by not having good, plausible distracters and vice versa. A rough guide to writing my own question/answer sets usually includes the following self questions:

- What specific lesson goal am I trying to serve?
- Is there notation that could be used that will further the course goals?
- Is there some extension of a current idea that will tie into a later concept?
- What is the correct answer? Unless the choice of “none of the above” is provided, the correct answer should be included as a possible choice.
- Is there more than one way to represent the correct answer, maybe a representation that would allow for students to make a common simplification in order to get to that final form?
- Is there a common computational error that students make? If so, what would be the answer if that mistake were made?
- Is there some “common knowledge” answer or something that students might partially remember from a past course that could be easily confused with the answer to the current problem?
- If all else fails, is there some way that complete nonsense can be phrased to sound really technical and therefore serve as a plausible distracter?

Once I have sketched out a few questions and played around with writing the answers, I select the few that I want to use and then I test them out in class.

17.3 Lesson: New Functions from Old

This lesson used 6 voting questions and was given in both a morning and an afternoon section of Differential Calculus.

For the first 5–10 minutes, we took care of any old business such as homework questions and a follow-up voting question from the previous class’s lesson:

1. Which of the following is an exponential function which has a y -intercept of 4 and goes through the point (2, 9)?
 - a) $f(x) = 4 \cdot 1.25^x$
 - b) $f(x) = 4 \cdot 1.5^x$

- c) $f(x) = 4 \cdot 2.25^x$
 d) $f(x) = 2 \cdot 1.25^x$
 e) $f(x) = 2 \cdot (\sqrt{9/2})^x$
 f) $f(x) = 2 \cdot 1.5^x$

The correct answer is (b), as indicated by the bolded percentage in the vote distribution. Vote distributions were, in order: 6/16/63/15/0/0 taking 3:25 minutes and 0/75/25/0/0/0 taking 2:17 minutes. Note that the morning class only had a small percentage of votes for the correct answer, whereas the majority voted incorrectly. The afternoon section had the majority vote for the correct answer but a significant number had still voted incorrectly. This question had great discussion and as a class we realized that another option should be included. $4 \cdot 2.4^x$ was a common mistake, since 2.5 is the slope of the line through the given points.

For the main part of the day's lesson we discussed how to modify a basic function and create an entire family of functions. For example take $y = x^2$ as a basic function, then the family of functions can be represented by $y = ax^2 + bx + c$, for a, b, c real values or in vertex form $y = a(x + h)^2 + k$, with a, h, k real values. The following function modifications were lectured on, by giving a brief definition and graphical example:

1. *Stretch/shrink*: $a \cdot f(x)$ or $f(a \cdot x)$, for varying values of a .
2. *Shift vertically*: $f(x) \pm k$, for varying values of k .
3. *Shift horizontally*: $f(x \pm h)$, for varying values of h .
4. *Reflect across the x-axis*: $-f(x)$.
5. *Reflect across the y-axis*: $f(-x)$.

In order to illustrate a combination of these function modifications and facilitate discussion, we used voting question #2:

2. A function is given in Figure 1.10 below. Which one of the other graphs could be a graph of $f(x + h)$?

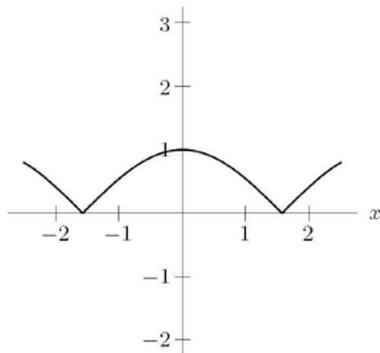
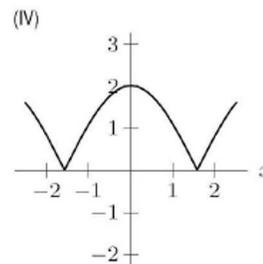
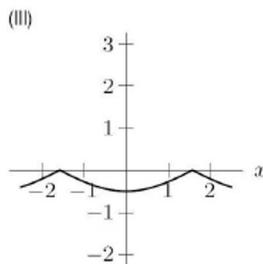
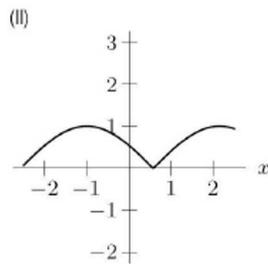
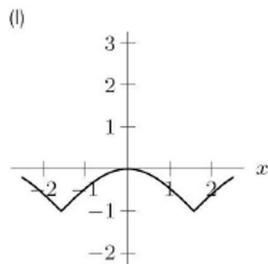


Figure 1.10



- a) I
 b) II
 c) III
 d) IV

Time: 1:20

Distribution: 14/78/8/0 and 25/50/17/8

The correct answer collected the majority of votes in both the morning and afternoon sections. The time needed to complete the voting process was identical for both sections even though the morning section has about three times the number of students in the class. Students discussed how they eliminated options (I) and (III) since those looked like x -axis reflections rather than horizontal shifts. Then another student argued that really (I) would have to be a vertical shift, not a reflection. Then the students debated between (II) and (IV), where (II) was finally the winner since it was noted that in order for the graph to cross the y -axis at 2, it would have had to undergo some sort of stretch. Had the discussion not gone as well as it did, the follow-up question was planned to be “what kind of modification happened to produce each given graph?”

The next question was one that I had written to show students how to write a formula given a certain list of function modifications.

3. Take $f(x)$ and in order:

Shift right h units

Reflect across the y -axis

Reflect across the x -axis

Shift up k units.

What is the formula for the new function?

a) $f(x + h) + k$

b) $f(x - h) + k$

c) $-f(-x - h) + k$

d) $-f(-x + h) + k$

Time: 1:30

Distribution: 0/0/83/17

This question produced great discussion! I only used this question with the afternoon section and wished that I had presented this question to the morning class too. The majority of the students voted for option c) which would be the correct answer if the first two modifications had switched order. This highlighted the importance of order of operations. After the discussion we went through the process step by step in order:

First “Shift h units right”: $f(x - h)$

Then “ y -axis reflect”: $f(-(x - h))$

Then “ x -axis reflect”: $-f(-(x - h))$

Finally “shift up k units”: $-f(-(x - h)) + k$

Simplifying gives the final answer: $-f(-x + h) + k$

When we got to the end, the students got that “light bulb” look on their faces that told me that this was a good thing to review in detail.

In the next segment of class we discussed even and odd functions. I gave a graphical description in terms of reflections as well as the definition using function notation: *A function $f(x)$ is even if $f(x) = f(-x)$. A function is odd if $f(x) = -f(-x)$ or similarly $f(-x) = -f(x)$.* We immediately used a voting question to discuss what functions are in each category:

4. True/False All functions are either odd or even.

Time: 31 seconds

Distribution: 83/16

Students made their decision very quickly on this question, and most of them held the common misconception that a function is either odd or even. I asked a student at random how they voted, and they happened to be one of the few students who voted correctly. The student gave the counterexample of e^x which is neither even nor odd. I followed up with a sketch of $(x + 3)^2$, asking if it was an even function. Again, most students thought that it was since it is in the x^2 family. Once both the y -axis and then x - and y -axis reflections were sketched, students were able to see that just because the function was in the “even powered family” there was something special about the basic function that the

whole family did not necessarily share. We used the power functions x^{2n} and x^{2n+1} to show that the names “even” and “odd” are helpful reminders of examples of each type of function.

The next portion of the lesson was on invertibility. We discussed the horizontal line test, how to solve for an inverse function, using inverse function notation, and how to graph an inverse by reflecting over the line $y = x$. I immediately used a voting question to give students a chance to put these ideas into action:

5. Which of the following functions IS invertible:

- a) $f(x) = -x^2 + 7$
- b) $g(x) = e^{3x/2}$
- c) $h(x) = \cos(x)$
- d) $k(x) = |x|$

Time: 2:00, distribution: 9/91/0/0 and Time: 1:51, distribution 4/56/33/7. Note that the time needed to collect votes from a large class and a small class only differs by a few seconds.

The discussion ran along the lines of “the *graph* passes the HLT” verses “the *formula* passes the HLT.” The follow-up questions were to 1) compute the inverse of the invertible function and 2) graph the function, its inverse and the line $y = x$ on the same axis. Note that the composition with natural logarithm to find the inverse of $e^{3x/2}$ is not something that has been directly discussed in this course, but would be something that the students had seen in a prerequisite course. We did not talk about inverse cosine and how to restrict the domain appropriately, but this was brought up in a later lesson.

The final idea presented was function composition. We discussed alternate notations and how $f(g(x))$ reads “ f of g of x ” which helps us remember that we are inputting an entire function, $g(x)$ in for the independent variable in the other function, $f(x)$. I decided to go about this in a way that lends itself to utilizing the chain rule once we get to differentiation:

6. Write $h(x) = e^{3x/2}$ as a composition of functions, $f(g(x))$. In order $f(x)$, $g(x)$:

- a) $e^x, \frac{3x}{2}$
- b) $\frac{3x}{2}, e^x$
- c) $\frac{x}{2}, e^{3x}$
- d) $e, \frac{3x}{2}$

Time: 1:50

Distribution: 56/7/37/0

While the majority were able to reach the correct answer, those that did make a mistake made the error in the order of “inside” and “outside” functions. The follow up question on this was to have the students determine the following: 1) given the functions f and g for each part a–d, what is $f(g(x))$? and 2) Given $h(x) = e^{3x/2}$, what is the composition needed (if possible) for the given f and g in each part a–d? Note that both (b) and (c) are $g(f(x))$.

At this point, we had used up 49 minutes of our 50 minute class. I assigned them voting question #7 to look at on their own. At the beginning of the next class period I would choose a random student to defend his or her answer and reasoning. Since this question was prepared by students outside of class time, distribution data was not collected.

7. $P = f(t) = 3 + 4t$, find $f^{-1}(P)$.

- a) $f^{-1}(P) = 3 + 4P$
- b) $f^{-1}(P) = \frac{P-3}{4}$
- c) $f^{-1}(P) = \frac{P-4}{3}$
- d) $f^{-1}(P) = 4(P + 3)$
- e) $f^{-1}(P) = \frac{P+3}{4}$

17.4 Lesson Reflection

I find that immediately after the lesson is the best time to determine what voting questions should be cut or modified and what questions are “good questions.” One way to modify a voting question is to recognize that there was a “common mistake” option that was not included in the given answers. I determine this by asking the class if there was an answer that was not included in the options that they came up with in their problem solving. Interestingly enough, students like to share their mistakes with the class. While things are still fresh in my mind I like to jot down a few notes on ideas for voting questions to fill in any gaps. In my notebook for voting questions I’ve got sticky notes that say things like “we need a question here that is about...” so that even if I can’t come up with the good question for that idea right then, I have a reminder when I teach the class the next time. It seems a lot easier to write a good question when the material is fresh and there is access to the misconceptions or mistakes that have been brought up in class. It is possible to write some questions way in advance, but I find that difficult to do out of context. All of these notes, voting times, voting pattern distributions, and modifications are to build a set of “good questions.” The criteria that I use to determine a “good voting question” is based on the following:

- Did it illustrate the lesson goal it was written for?
- Does the question illuminate a common misconception?
- Is the length of time needed to vote on the question justified by the amount/level of information that the students receive?
- Was there interesting/illuminating discussion?
- What was the distribution of votes? Was it fairly uniform or was there one question that got more votes than others? If so, was it the right answer or a common misconception?
- Did the students like the question? Did they feel they were being “tricked” by the question/answers?
- Were correct grammar and punctuation used in the writing of the question?
- Did some unexpected positive or negative aspect come up with this question?

For this particular lesson on *New Functions from Old*, my reflections included the following:

Note on function modification *vertical shift*:

Insert question: “Given the graph of x^2 , with point $(2, 4)$ marked, what are the new coordinates of the function shifted down m units?”

- a) $(2, 4)$
- b) $(2, -m)$
- c) $(2 - m, 4)$
- d) $(2, 4 - m)$
- e) None of these

The idea here is that instead of doing an example on the board, a voting question can be used so that the students are the ones thinking about the problem at hand, not just the teacher. I drew an example on the board and went through this exact same process, but I was the one feeding the thought process to the students, instead of allowing them to think through on their own what “shift down m units” means in terms of a single point. I had a similar note for *horizontal shift*.

My next note was on one of the supplemental questions that I had carefully prepared and planned on doing during class but changed my mind during the lesson:

Let $f(x) = x + 1$ and $g(x) = 3x^2 - 2x$. What is $g(f(x))$?

- a) $3x^2 - 2x + 1$
- b) $(3x^2 - 2)(x + 1)$
- c) $3x^2 + 4x + 1$
- d) $3(x_1)^2 - 2x$

I decided against doing this question and only did #6. I was concerned about the time commitment to have the students complete the necessary algebraic simplifications to arrive at the correct answer. Perhaps a different version with less involved algebra would have been better. I also could have given this question as a second take home question.

17.5 Conclusion

As in any course, with any teaching method, trying to have a ground covering course that is not also in danger of being “a mile wide and an inch deep” seems unattainable at times. As can be seen in this single lesson plan, some of the planned voting questions could not be used due to time constraints. So the important lesson here is to choose the few best questions to use so that the most can be gleaned from each moment spent on the voting and discussion. This is why I collect the time/distribution data as well as make any notes about how the discussion went; I want to have the set of distilled questions that are the best of the best. My recommendation is to just jump in and try writing a question or two. In the beginning, there will be “mistakes” where the question just flops: the time spent is not balanced by the amount of understanding or the students are just confused by what is being asked, or the question isn’t phrased correctly to lead them to the level of understanding desired. But then again there are the occasions that the new voting question works better than could have been imagined.

I find that the creative process involved in writing these questions is enjoyable and also deepens my own understanding of the subject and how students think about the material. The time I spend reflecting on the day’s selection of voting questions help smooth out the voting/discussion process and builds a more powerful question set. As a fairly new teacher, I feel that the reflection time really helps me focus and become a better teacher with a deeper understanding of students’ misconceptions. As I become more comfortable with the method, the good questions come easier and quicker, which means that my voting question prep time for a 50 minute lesson is usually not more than 10–20 minutes. This is, of course, partly due to having the bulk of the questions already written and organized by someone else; “stealing” is always easier than creating something from scratch! If however, a set of questions is not available, it is still possible to write a question or two for use in the classroom and slowly build your own voting question bank. The hardest and most important step is to just get started. Happy Voting!

18

Enhancing Student Participation and Attitudes in a Large-Lecture Calculus Course

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18.1 Introduction

Good communication is essential in every relationship. This includes the relationship that I cultivate with my students as individuals and with my class as a whole. A personal response system, “clickers,” is a tool that I have employed to aide communication.

While there are a variety of options for engaging students in the classroom, I specifically chose this technology because my students are part of the Millennial generation and I believed clickers would be a means to engage them using a digital form of communication. They’ve grown up with technology playing a role in nearly every facet of their lives, with this likely to increase as time passes. A May 2008 report from In-Stat predicts “a steady growth rate culminating in the number of US millennia’s subscribing to mobile social networking reaching nearly 30 million by the year 2012 [46].” Many attributes of the clicker emulate my students’ current social networks. At the simplest level, my students become members of a group by enrolling in the course. However, the ability to efficiently ask the entire class for input on a topic, followed with the means to immediately share the overall results with them, actually allows them to communicate with each other, as well as with me.

18.2 Background

I teach at a regional university with an enrollment a little over 10,000 which offers 77 different majors, and graduate programs in 22 different fields. It is a medium-sized campus of a major university that views itself as an alternative to both large research universities and small liberal arts colleges. The Department of Mathematics and Statistics consists of 26 faculty members ranging from instructor to full professor and 25 graduate students pursuing a Master of Science in Applied and Computational Mathematics degree. Often the introductory courses are taught by the instructors, who all have earned a master’s degree in mathematics.

The majority of calculus courses are taught in a large lecture format in which 150+ students meet three times a week with the instructor and twice a week in discussion groups of approximately 25 with a graduate student. Clickers are not used by all instructors; those who use them only use them in the large lecture meeting. All of the following discussion is based on experiences I have had in a large lecture calculus course. The population wIthin the class is typically over 85% freshmen students.

18.3 Behind the scenes

The preparation done behind the scenes is critical if one is to have any success using the clicker. This includes (but is not limited to) learning to use the system, making decisions about how the grades will be integrated with the current grade-keeping method, pedagogical decisions on learning outcomes, and preparing questions. Problems will undoubtedly arise and I've found it important to not only have the resources to help the student solve the problem, but to also make a concerted effort to remain upbeat while determining the solution.

I will not include a full description of the clickers here, because the general system has been described extensively elsewhere [12, 40, 42]. However, I want to mention the importance of clearly explaining the system to the class at the beginning of the semester. I find they are interested in how it works. In turn, their understanding helps them know how to use the clickers correctly. Some general rules that I follow are: keep the format of the question slides the same, start the countdown to end the question only after half the class has responded and encourage the students to talk to each other while solving the problem.

Since this system is new to many students, I suggest making use of a numbered response grid. In the system that I use each number corresponds to a specific student and when the student answers a question the background of their grid square turns light blue. This only indicates to the student that an answer was received, but gives no indication of the accuracy of the answer. In large classes it is best to use a rotating grid, displaying 2530 numbers at a time. Distributing clicker numbers with such a large class could be arduous. I have chosen to use a web-based grade posting program and enter their clicker number as their first assignment. Naturally this number is not used in the computation of the student's overall grade.

In the grade breakdown for the semester the points acquired from clicker questions come under the heading of "Participation." Participation points cannot be made up due to absence. Questions are not asked every day, but typically there are thirty questions asked during a semester with only one per day. However, at the end of the semester, I drop the points from the three lowest clicker questions to help mitigate the volume of complaints and excuses (some being valid reasons) for missing class. The portion of the points allocated for attendance and those given for the accuracy of the solution is up to the instructor. For example, in my classes each question is worth two points with one point granted for participating and the second point for determining the correct solution. Overall, participation is worth roughly eight percent of the course grade.

Finally, I find many benefits in using a Tablet PC for lecture delivery. A Tablet PC is a laptop equipped with touchscreen technology, which allows the user to operate the computer with a stylus or a fingertip, instead of a keyboard or mouse. The biggest benefit is the seamless switch between delivery and inquiry whenever I want to pose a clicker question. My lectures are handwritten during class using Microsoft Office OneNote and the clicker questions are prepared ahead of time using a vendor program imbedded within Microsoft Office Power Point. I prefer to use a program like OneNote for teaching because the students are able to see the mathematics as it is constructed and a Tablet PC allows me to do this. Also, I find I prefer to face my students when I am teaching. Additionally, although I choose not to use the feature, many available programs on the Tablet PC typically used for writing with a stylus can easily convert files to a format which can be made available to students online.

18.4 In front of the class

I usually ask questions at the beginning or at the end of the hour. The questions are typically a review of the material learned in the previous few meetings. I have found that this has the effect of requiring the students to focus on calculus. An additional benefit is students arrive to class on time. While most of my questions are prepared ahead of time, it is important to know how to write questions on the fly because students have requested that I ask a clicker question to help settle a classroom discussion. Occasionally I ask questions that are non-content based to get a feel for the students' thoughts on recent individual performance or for input regarding scheduling a review session prior to an exam. By doing this, I can easily get a pulse on what is going on with my students and in turn they recognize that I care about their thoughts and opinions beyond their ability to answer a mathematical question correctly. Finally, I post the question and solution online the following day so students can review the concepts at their leisure. The rapid turn around in feedback is a crucial component of the overall usefulness of clickers as a learning tool. Commonly called "The Value of Feedback," rapid assessment is the first step in combating students' illusions of knowing. It involves confronting them on a regular basis with evidence of their knowing or lack of it [79].

In order for students to see value in the system beyond an attendance-taking mechanism, it is critical to have really good questions. The main criteria I use to determine if a question is good is to first consider whether the concept is one that students are challenged by. Then I assess whether the question is written in a way that will determine if a student is at least one level beyond just knowing how to do a specific calculation and actually has a good understanding of when and why to do said calculation. For example, instead of asking “Find the value of c , that is guaranteed to exist by the Mean Value Theorem for $f(x) = x^2 - 4x$ on the interval $[0, 2]$,” one could ask:

A car is initially traveling 30 miles per hour and after a half hour it is traveling 65 miles per hour with an average speed during the half hour of 45 miles per hour. What is the speed that is guaranteed to exist by the Mean Value Theorem ?

- a) 30
- b) 45
- c) 50
- d) 55
- e) 65

The correct choice is 45 miles per hour. It is more important to me that students understand the concept of the Mean Value Theorem rather than just being able to “plug and chug” their way through a memorized and regurgitated formula.

I strive to write questions that illuminate common misconceptions. It is important to note that the software for the clickers I use only allows for multiple choice questions, which poses some constraint on question design, but can also be a great learning tool. In the answer choices, I often include an answer that can only be reached by falling prey to the misconception. While this sounds like a trap or trick, getting it wrong in class with very minimal point consequences helps prevent students from making the same error on the subsequent assignments or exams. For example, when we first start to learn the power rule for derivatives, I ask:

Find the first derivative of $f(x) = \pi^2$.

- a) 0
- b) 2
- c) 2π
- d) $2x$

The correct answer is 0, but students will often choose 2π . This provides me with an opportunity to stress the difference between differentiating a constant and differentiating a variable. I think the beauty of this problem is the simplicity of it. I’m interested in reminding them of a very simple concept that is often initially misunderstood and I intentionally chose a question that did not contain anything more than computing the derivative of a constant which is written as a constant raised to a power.

Also, when we cover the method of partial fraction decomposition as a technique of integration, I am careful to include an integral where the integrand has a higher degree in the numerator and thus requires long division. Many reach an answer without using long division, but hopefully for the last time. At the very least, this gives me a chance to remind the entire class that this is a common error.

Another example of a question I ask in which I include multiple distracters that play to common errors is:

If the rate, R (grams/sec) of a chemical reaction is dependent on the temperature, T (in degrees Celsius), such that $R(T) = 5(-0.03T^3 + 10T + 10)$ determine the temperature that produces the greatest reaction rate.

- a) 10.5
- b) 18.7
- c) 101
- d) 401

The correct answer is 10.5. The other answers are reached by misunderstanding or miscalculation. For example, 18.7 is the zero (or root) of $R(T)$ and 401 is the maximal rate achieved with a temperature of 10.5 degrees Celsius.

In order to generate interesting discussions, I occasionally like to pose a question that has multiple correct answers. For example, during our discussion of numerical integration, we cover the following methods: Left, Right and Midpoint Rules, Trapezoidal Rule, and Simpson's Rule. I ask the following:

Which of the rules provides the best approximation of the integral of the absolute value function from negative two to two using four subintervals?

- a) Right Hand Rule
- b) Midpoint Rule
- c) Trapezoidal Rule
- d) Simpson's Rule

Many, many students automatically choose Simpson's Rule because, in general, it is an improvement over the others. However, Simpson's Rule uses parabolas and, due to the linearity of this particular function, all the other methods approximate the integral exactly. The question provides students with the opportunity to really think about how these approximation methods work geometrically.

I strive to write questions that really focus on one specific concept and pose the question in a manner that digs deep to see that they truly understand that concept. For example, when discussing the use of a tangent line to approximate a function a typical question that I ask is:

If a tangent line is used to approximate $1/4.1$, will the approximation be an over or under estimate?

- a) Overestimate
- b) Underestimate
- c) Neither
- d) Huh?!

In this question the function is $f(x) = 1/x$ and the ideal point of tangency is 4. The correct answer is underestimate, but students will often choose overestimate because 4.1 is larger than the ideal point of tangency at 4. I've found that it helps to show a simple graph of the function and the tangent line on the answer slide. This is a visual means of driving home the importance of whether the tangent line lies above or below the curve. You can even take this opportunity to discuss concavity, if you wish to explore a little further.

I find that my ability to test the depth of their understanding of a concept increases if I make an effort to minimize my use of actual functions and present more general cases. When we are discussing concavity a question that I often ask is:

If f is positive, increasing and $g = 1/f$, then g is ...

- a) Increasing
- b) Decreasing
- c) Neither
- d) Umm ... what?!

The correct answer is decreasing. This can be determined using the Quotient Rule and in addition to checking their ability to use the rules of differentiation, the question probes a little deeper to assess their understanding the relationship between derivatives and words like positive, increasing, etc.

It is equally important for me to prepare ample discussion as a follow-up to the question. The amount of time I spend on this discussion portion depends on results. I always explain the question and the reason that I included it. If the majority of the class has answered correctly, after a three minute or so explanation I congratulate them and move on. On the other hand, if the majority of the class did not answer correctly, I still congratulate them and explain that now we all have a great opportunity to learn. I follow with a more in-depth explanation.

Often during the discussion portion of a clicker question, I invite the class to participate by explaining how they arrived at the correct choice or how they went astray. Voluntarily participating in a large lecture class room would

typically be very intimidating for most students, but I start setting the climate in the classroom to encourage discussion on the very first day of classes by organizing an icebreaker activity taking place in their discussion group that requires each of them to talk to another student or students for a few minutes and then present their findings to the larger group of twenty-five. Speaking in front of a large group is not something that some students will ever be comfortable with. To mitigate any fear they may have, I make it clear to them at the beginning of the semester that they can always “pass” on answering a question with no fear of repercussion. Occasionally, I find that there are some students who want to answer everything including questions directed to other students. I’ve handled these cases by sending the student an email letting them know that while I am impressed with their grasp of the material, I would appreciate them pausing for a moment or two before raising their hands. I explain that I do not want the other students to become either complacent in knowing that someone else will answer or become discouraged and in either case not even try to solve the question.

Finally, I have used the clickers to conduct a post exam survey to initiate discussion about the class results of an exam. I ask my students to answer the question:

How do you feel your performance was on the Exam?

- a) Better than I expected.
- b) About what I expected.
- c) Worse than I expected.
- d) Way worse than I could have imagined.

I follow this with questions about whether they believe their score accurately reflects their level of understanding of the material and an inquiry into how they prepared for the exam. Each question is designed to encourage the students to reflect on their contributions to learning. Furthermore, the discussion following each question provides students with the opportunity to comment on the exam. It’s not unusual for a student to express that they really never understood a topic and were hoping not to see it on the exam or that they expected a particular topic that did not appear.

18.5 Results

I have used the clickers to poll five of my classes, asking them what they think of learning with clickers. Nearly 750 students have been polled. The poll I have used is, in general, to determine the student’s opinion of all the instructor technologies I use in my courses. The topics range from the class webpage to the few lectures I give using PowerPoint to an overall view of technology. Overall, my classes have been very positive about clickers. When asked the question, “What do you think of clickers as a learning tool for this class?” on average, 67 percent of the class responded that they either like or strongly like and 22 percent responded that they dislike or strongly dislike the system with the remainder of the class undecided. In addition, when asked, “Did the clickers help you with learning in this class?” on average, 59 percent of the class responded that the clickers helped them a little or very much and 6 percent responded that clickers hindered their learning a little or a lot. At this time, I have made no attempt to measure learning gains, but I plan to construct a study in the future.

Student attitudes about clickers have changed since I started using clickers in 2004. Initially there were very few faculty members using them, so students often complained about the cost. They were also much more critical of any problems with the system and were quite vocal with doubts about the usefulness of clickers. The support system provided by our Information Technology Department has increased the number of faculty using the clickers, aided in ample training, and in turn, has drastically reduced the students’ negativity.

I have found some downsides to using clickers. In particular, I have noticed an increase in behavioral issues. I attribute this to students who are really not interested in attending class and are now compelled to attend by the points associated with the clicker questions. The disruption is typically talking, using cell phones for text messaging, or using laptop computers. I have even encountered situations where students have given their clicker to another student to “click in” for them. This problem is easily remedied at the outset of the semester by having a teaching assistant count the number of students physically in the class and discussing that the discrepancy between that and the number of clicker responses is as much an incidence of cheating as looking at another person’s exam.

18.6 Conclusion

In conclusion, I have found clickers to be an efficient and effective means of engaging my large lecture calculus course. My students are favorable about clicker use and seem to be benefiting. I firmly believe that the support provided by both the Information Technology Department and fellow faculty members has played a large role in the success that I have experienced.

19

Good Questions for Mathematics Education: An Example from Multivariable Calculus¹

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Mathematics *education*? Aren't I missing an "a" in the middle of that word? No, I mean education, a process by which good questions educate, or

- draw out latent understanding of how the world works,
- lead students to build a deep foundation of fundamental understanding,
- engage students in bringing order to, or making sense of, complex phenomena from basic given facts.

In this article I present some examples of educating questions that were developed and tested via clicker technology through the Good Questions Project². I will summarize some of the student feedback, and the research on teaching and learning that provides a framework for understanding why "educating" questions may be effective.

The Good Questions project grew out of an effort to introduce young instructors to the benefits of teaching by asking. It was inspired by Cornell mathematician David Henderson's rich knowledge and experience of teaching geometry [40] by posing questions that lead students to construct and refine mathematical concepts based on their own experiences. We were also inspired by the success of Harvard physicist Eric Mazur [55] in using ConceptTests and peer instruction to teach physics. We wanted to develop questions to stimulate thoughtful discussions of key concepts in calculus, and put them in the hands of instructors in their formative years as TAs and postdoctoral fellows. We wanted to know if discussing the questions would lead to better student understanding and better performance in a traditional freshman calculus course. With support from Cornell and the National Science Foundation we wrote a collection of questions and tested them using clicker technology in the Fall of 2003. A paper describing the project and some of our results in more detail was originally published in PRIMUS [58] and is included in this collection. The data from that project suggested that regular use of questions that engaged students in deeper discussions, might be related to stronger overall performance as measured by a comprehensive final examination. That result encouraged me to look at the "deep" questions and to see if there was some research that would illuminate why some questions seem to be more effective at helping students learn than others.

¹This paper is based on a paper of a similar title that was presented at the invited paper session "Inquiry Based Learning" at Mathfest, August 10, 2006, Knoxville Tennessee.

²Support for the Good Questions project was provided by the National Science Foundation's Course, Curriculum, and Laboratory Improvement Program under grant DUE - 0231154.

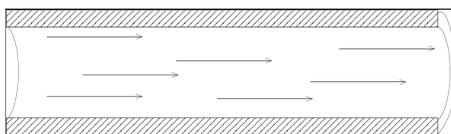
Research on learners and learning, and teachers and teaching summarized in *How People Learn* [8] identifies three key findings on teaching and learning that are derived from a solid research base and that have strong implications for how we teach mathematics. I've copied them below. I use some specific examples to illustrate how these findings are reflected in what I have come to call educating questions.

Key findings:

- 1. Students come to the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that are taught, or they may learn them for the purposes of a test but revert to their preconceptions outside the classroom.**
- 2. To develop competence in an area of inquiry, students must: (a) have deep foundation of factual knowledge, (b) understand facts and ideas in the context of a conceptual framework, and (c) organize knowledge in ways that facilitate retrieval and application.**
- 3. A metacognitive approach to instruction can help students learn to take control of their own learning by defining learning goals and monitoring their progress in achieving them.**

A clear implication of the first finding is that teachers should draw upon and engage students' prior or latent understanding. How can clicker questions be used? Let us consider an example. The concept of flux in vector calculus is essential to building models of fluid flow and for the statement of the divergence theorem. What do students know about flux from their everyday life, that could help them develop a deeper understanding of the definition we use in calculus? Here is a simple question I have found particularly useful in engaging what students know and introducing them to the mathematical concept of flux.

Suppose the velocity of the particles of water flowing down a pipe is 3 meters per second, and the cross sectional area of the pipe is 5 square meters.



How much water flows out of the pipe per second?

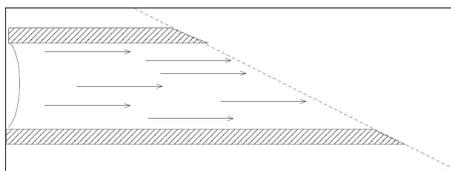
[Answer: 15 cubic meters per second]

Most students (as well as the proverbial man on the street) are very quick to respond that water is flowing out of the pipe at the rate of 15 cubic meters per second. Of course they are correct. This is a simple multiplication problem, area of the base times the height of a column of water. This is what I would call the first stage of an educating question, engaging what students already know and what they understand about how the world works.

The second stage of an educating question engages students in building a solid foundation of factual knowledge by eliciting misperceptions, partial understanding and uncertainty, and by employing precise mathematical thinking to illuminate and resolve the conflict.

Here is the second part of the question:

Now suppose our pipe is cut on the diagonal.



At what rate is water flowing out of the pipe?

- 15 cubic meters per second,*
- More than 15 cubic meters per second.*
- Less than 15 cubic meters per second.*
- It is impossible to determine the rate.*

[Answer: a)]

When I ask this question with anonymous polling I find that 20% of students choose a), 70% choose b), essentially no one chooses c), and 10% choose d). What does this reveal about students' understanding of flux? Students who answer 2) see that the area of the opening is greater than 5 square meters. Since the water is flowing down the pipe at 3 meters per second, they reason that the volume of water crossing that opening in one second is greater than 15 cubic meters. At this point I change the phrasing of the question.

Suppose you put a bucket under a garden hose, does the rate at which the bucket fills up depend on whether or not the end of the hose is cut on the diagonal?

[Answer: no]

Almost all students say no, cutting the end of the hose on the diagonal does not affect the rate at which the bucket fills up, and hence does not affect the rate at which water flows out of the hose. How does the mathematical definition of flux accommodate an elementary understanding of flux, as the area of the surface times the velocity of the fluid crossing the surface, and at the same time resolve the apparently conflicting numerical answers? By examining the question of how much water crosses the surface of the opening in one second, students see that the that question is equivalent to the problem of computing the volume of a cylinder given the area of its base and its slant height. Students see that they can compute the volume that flows out in one second in two ways. One way involves finding the component of the slant height that is perpendicular to the base; the other involves finding the component of the area that is perpendicular to the slant height. Either way, the dot product of two vectors—one representing the speed and direction of the fluid and the other the area and orientation of the cut face—does the job.

To recap so far, and to connect this example to the key findings in *How People Learn* [8], the first phase of educating draws out what students know and understand about the world. The second phase of educating is the “leading” phase, it engages students in examining and refining their understanding in a way that resolves conflicts and misconceptions and fosters a deep foundation of factual knowledge that is part of conceptual framework of formal mathematical understanding.

The third phase is the reflective and interpretive phase, the metacognitive phase, in which students monitor their understanding of the concept and extend or transfer it to other settings. A good question to foster metacognition about flux is:

Assume that it is raining and the droplets all fall straight down with the same speed. You want to minimize how wet you get.

- a) It is better to walk through the rain.*
- b) It is better to run through the rain.*
- c) It doesn't matter if you walk or run, you will get just as wet either way.*

[Answer: b)]

This question is quite a bit of fun to discuss with students. They readily see that flux is an applicable concept. I usually suggest that students simplify the problem by computing how much water would pass through a horizontal planar region (representing head and shoulder area), and a vertical planar region (representing the frontal area). Students discuss their ideas about whether they think you get wetter when you run into the rain or whether it just seems that way because you are accumulating water over a shorter period of time. The answer to the walk versus run question requires computing the flux across the horizontal plane and the flux across the vertical plane. In setting up the computations it becomes clear that the amount of water that crosses the horizontal plane during the trip depends on the length of time you are out in the rain. The amount of water that crosses the vertical plane during the trip is independent of the length of time the trip takes, and depends only on the length of the trip. In the end students see what common sense tells them: It is better to run.

This third phase of educating consists of reflecting on how one knows what one knows, and fostering the ability to transfer or use that knowledge to deduce results in new situations. This is part of the metacognitive process of learning.

A major goal of teaching mathematics is to prepare students to use and adapt what they have learned to solving new problems in diverse settings. The ability of students to transfer what they have learned requires that students have a solid understanding of basic concepts and have opportunities to apply them to interesting and engaging problems. Transfer of classroom math knowledge to everyday environments is an important indicator of how well students can

use what they are learning. Clicker questions, especially those that educe, that draw out, that lead, and that promote sense making, can be a powerful pedagogical tool.

20

Integrating Classroom Voting Into Your Lectures: Some Thoughts and Examples from a Differential Equations Course

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20.1 Introduction

One of the largest barriers to adopting a new teaching technique is the startup costs in time invested. In this article, I share the ways in which I prepare for a class that uses voting questions as an integral part of the class. I believe it is possible to prepare for class in roughly the same amount of time it would take to prepare a standard lecture for a class that you are teaching for the first time.

One of the largest obstacles to overcome in using voting questions in the classroom is preparing a lecture which uses them and works well in the classroom. Effectively using voting requires a serious time investment in the classroom, so it is not possible to take a traditional lecture and simply add voting questions at pertinent points. We must rethink our methods for preparing lectures. The best strategy for me was to start with the voting questions that I wanted to use and then to fill in any additional points.

Let me begin by giving my experience with clickers — clickers add technology to voting, but everything I say seems just as applicable without the technology component. I am a new faculty member and just completed my first year of full-time teaching in the Spring of 2008. I discovered clickers halfway through the Fall semester in 2007 as a way to spice up a linear algebra class that met twice a week for 1 hour and 40 minutes. I used clickers primarily as concept checks and to provide a forum for students to be active and work examples. They were essentially an add-on to the course to promote student activity. In the Spring semester, I made clickers an integral component to both a Calculus II and an Ordinary Differential Equations course from day 1.

When I chose to use clickers in the classroom, I resolved that I would not sacrifice any content that I would normally cover. Instead, I wanted to keep the same content and add a deeper conceptual understanding by forcing the students to actively engage and discuss the underpinnings of what was happening in class. This meant, for me, that I used my voting questions in two distinct ways: to break the ground on new ideas and open a discussion of them as well as to guide students through involved examples.

20.2 Preparing for lecture

At the beginning of the Spring semester, my lecture preparation took a very long time. I wrote a full traditional lecture and then cut back examples for voting questions and previewed concepts with more conceptual voting questions. This

method of lecture preparation was insupportable. It took far too long to prepare a normal lecture and then to figure out what the right clicker questions were to go with it and how to integrate them. Even worse, by doing things this way, I almost guaranteed that I would not have enough time in class to do everything I planned. It also led to a distinct lack of fluidity in class.

As the semester progressed, I evolved a much different strategy. I began with the clicker questions. This is the strategy that I will discuss here. If your goal is to prepare an effective lecture without sacrificing too much in preparation time, it is very helpful to have access to a large database of clicker questions, such as the ones maintained by Carroll College¹, Cornell², University of Oklahoma³, and others. These resources include databases for precalculus, calculus, linear algebra, differential equations, and statistics.

To illustrate my strategy, I will show how I developed two lectures for my differential equations course. The course met for one hour and fifteen minutes twice a week and had an enrollment of 26 students. I usually used the one-vote method; however, for particularly conceptual problems that divided the class during discussion, I would spontaneously ask for a second vote to cement the discussion. The first lecture I will discuss was what I would consider a “normal” day. The second lecture was an experiment in which I taught the entire class period based on the discussion generated around voting questions.

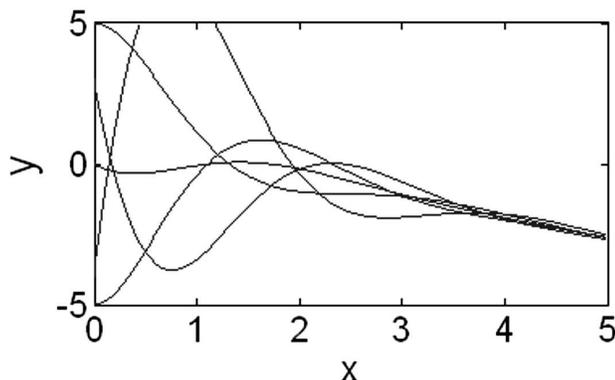
20.2.1 A normal lecture example

The class was on the forced harmonic oscillator differential equation. This class immediately followed one in which we had classified the behavior of all types of unforced harmonic oscillators.

I began my class preparation by deciding what ideas I wanted to communicate: 1) the strategy for solving the forced harmonic oscillator (and how it was similar to solving first-order non-homogeneous differential equations), 2) how to pick a suitable guess and then solve for the particular solution, 3) what parts of a solution are constants that can be adjusted for initial conditions and what parts should be solved for, and 4) how forcing affects things qualitatively. Then I went to the Carroll College Project Math QUEST questions for “Second Order Differential Equations: Forcing.”

I would begin the lecture with the following question:

1. The functions plotted below are solutions to which of the following differential equations?⁴



- (a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 3 - 3x$
- (b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 3e^{2x}$
- (c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = \sin \frac{2\pi}{9}x$
- (d) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3x - 4$
- (e) None of the above

¹<http://mathquest.carroll.edu/>

²<http://www.math.cornell.edu/~GoodQuestions/>

³<http://www.ou.edu/statsclickers/>

⁴Solution to 1: (a)

This question focused on the effect of forcing in the differential equation and introduced forcing as opposed to the unforced equations we had studied. It also made it immediately clear that we were dealing with something different and new. I decided to give very little preparation for this question and had the intention of circling back to this question at the end of class to emphasize geometrically what forcing can look like.

After choosing the first question, I chose a series of questions to highlight the process of producing the general solution of a forced harmonic oscillator:

2. The general solution to $f'' + 7f' + 12f = 0$ is $f(t) = C_1e^{-3t} + C_2e^{-4t}$. What should we conjecture as a particular solution to $f'' + 7f' + 12f = 5e^{-2t}$?⁵
 - (a) $f(t) = Ce^{-4t}$
 - (b) $f(t) = Ce^{-3t}$
 - (c) $f(t) = Ce^{-2t}$
 - (d) $f(t) = C \cos 2t$
 - (e) None of the above

3. The general solution to $f'' + 7f' + 12f = 0$ is $f(t) = C_1e^{-3t} + C_2e^{-4t}$. What is a particular solution to $f'' + 7f' + 12f = 5e^{-6t}$?⁶
 - (a) $f(t) = \frac{5}{6}e^{-6t}$
 - (b) $f(t) = \frac{5}{31}e^{-6t}$
 - (c) $f(t) = \frac{5}{20}e^{-6t}$
 - (d) $f(t) = e^{-3t}$
 - (e) None of the above

4. If we conjecture the function $f(x) = C_1 \sin 2x + C_2 \cos 2x + C_3$ as a solution to the differential equation $y'' + 4y = 8$, which of the constants is determined by the differential equation?⁷
 - (a) C_1
 - (b) C_2
 - (c) C_3
 - (d) None of them are determined.

Question 2 recalled the idea of first solving the unforced situation then guessing a particular solution to the nonhomogeneous case. Question 3 takes the next step and emphasizes that we can determine the coefficient of our guess. Finally, Question 4 helps to sort the pieces out and forces the students to distinguish which coefficients are part of a general solution (and thus would be determined once we have an initial value) and which coefficients are part of a particular solution and should be nailed down immediately.

After choosing these four questions, I already had a solid framework for the lecture. I decided that I should add a bit more to flesh things out. I added a short bit of lecture between Questions 1 and 2 to recall how we solve nonhomogeneous first-order differential equations and then outline the technique for forced harmonic oscillators. Questions 2 through 4 and the discussion following after would then see me through the mechanics. After this, I intended to spend the rest of the class time working one example in detail so that they had a full example in their notes. Then I would discuss how we decide what to guess. The final five minutes of class would be spent going back over Question 1 now that we know analytically what solutions should look like. This helped to connect the analytic work with the geometric intuition.

This lecture was fairly typical of my classes. I began with a framework and added clickers. Then I fleshed out any additional details that I did not think would be covered sufficiently in the discussion following the clickers. I found that this led to a much more “organic” lecture and really enabled me to turn lecture into a conversation with students where they were working directly with the material.

⁵Solution to 2: (c)

⁶Solution to 3: (a)

⁷Solution to 4: (c)

20.2.2 How the class went

The class where I gave this lecture went fairly well. In retrospect, I probably gave too little information leading up to the first question. Students really struggled with this question at first. Only 19% got this question right on the first try. However, I also think it put them into the correct frame of mind for the rest of the lecture, which went very smoothly.

Students very quickly recalled the pertinent strategy for the first-order non-homogeneous case, and we quickly came up with the method to give the general solution to a forced harmonic oscillator. At this point, we did the next 3 clicker questions. On each of these three questions, over 90% of the class chose the correct answer. I knew at this point that the computational skills that the students needed from the class had been successfully communicated. I was able to quickly work a full example from the beginning with student input to guide me.

Finally, I decided to focus on the “guess” that comprises forming a particular solution. By this point in the course, students were fairly familiar with this technique, so I talked a good bit without writing. A student jumped in to ask for a particular example involving a polynomial forcing function (I had mentioned you would need more than one undetermined coefficient), and we worked an example on the spot. This naturally brought us back to the question that opened the lecture, and we were able to see much more the component parts of a solution and understand the picture that was given. In particular, we connected with our analytic work to talk about why the oscillations were dying out and why things were starting to look linear as time progressed.

20.2.3 A lecture entirely by clickers

At the extreme end, I did decide to give one lecture entirely by clickers. It was a lecture where we explored the consequences of analytic work on solutions of homogeneous systems of differential equations. We were interested in the qualitative consequences of that work.

We were dealing with autonomous linear systems of differential equations (specific examples focused on 2×2 matrices) when the eigenvalues are distinct, real, and non-zero. We needed to address the cases where both eigenvalues were positive, both were negative, and one was positive and one negative. We also needed to recall how the eigenvalues and eigenvectors played a role in the general solution and determine what the straight-line solutions should do. Preparing for this class took longer than the previous one since I decided to write several questions of my own (however, I now have these questions for future times I teach the course).

The first three questions of the lecture were designed to recall the general solution to this type of system — particularly, the role that eigenvalues and eigenvectors play in the solution — and to determine what direction solutions move along lines determined by the eigenvectors. Incidentally, this case also broached the ground on having one positive and one negative eigenvalue.

1. You have a linear, homogeneous, system of differential equations with constant coefficients whose coefficient matrix A has the eigensystem: eigenvalues -5 and -2 and eigenvectors $\langle -1, 2 \rangle$ and $\langle -4, 5 \rangle$, respectively.

Then a general solution to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ is given by:⁸

$$(a) Y = \begin{bmatrix} -k_1 e^{-5t} + 2k_2 e^{-2t} \\ -4k_1 e^{-5t} + 5k_2 e^{-2t} \end{bmatrix}$$

$$(b) Y = \begin{bmatrix} -k_1 e^{-2t} - 4k_2 e^{-5t} \\ 2k_1 e^{-2t} + 5k_2 e^{-5t} \end{bmatrix}$$

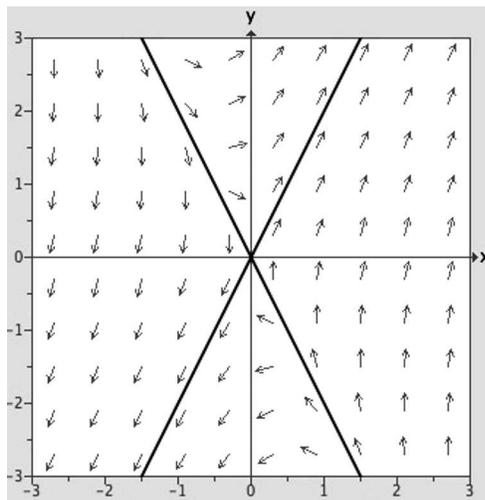
$$(c) Y = \begin{bmatrix} -k_1 e^{-5t} - 4k_2 e^{-2t} \\ 2k_1 e^{-5t} + 5k_2 e^{-2t} \end{bmatrix}$$

$$(d) Y = \begin{bmatrix} -k_1 e^{-2t} + 2k_2 e^{-5t} \\ -4k_1 e^{-2t} + 5k_2 e^{-5t} \end{bmatrix}$$

2. The differential equation $\frac{d\vec{Y}}{dt} = A\vec{Y}$ has two straight line solutions corresponding to eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

⁸Solution to 1: (c)

and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ that are shown on the direction field below. We denote the associated eigenvalues by λ_1 and λ_2 .⁹



We can deduce that λ_1 is

- (a) positive real
 - (b) negative real
 - (c) zero
 - (d) complex
 - (e) There is not enough information
3. We consider the same system as in the previous question. We can deduce that λ_2 is¹⁰
- (a) positive real
 - (b) negative real
 - (c) zero
 - (d) complex
 - (e) There is not enough information

The first real challenge would come in Question 4:

4. We continue to consider the same system. Suppose we have a solution $\vec{Y}(t)$ to this system of differential equations which satisfies initial condition $\vec{Y}(t) = (x_0, y_0)$ where the point (x_0, y_0) is not on the line through the point $(1, -2)$. Which statement best describes the behavior of the solution as $t \rightarrow \infty$?¹¹
- (a) The solution tends towards the origin.
 - (b) The solution moves away from the origin and asymptotically approaches the line through $\langle 1, 2 \rangle$.
 - (c) The solution moves away from the origin and asymptotically approaches the line through $\langle 1, -2 \rangle$.
 - (d) The solution spirals and returns to the point (x_0, y_0) .
 - (e) There is not enough information.

I anticipated that discussion after this question would completely resolve this case. With the groundwork on straight line solutions laid, I went directly to systems where both eigenvalues were negative. The first question was designed to get students to realize that solutions would be drawn into the origin in this case:

⁹Solution to 2: (a)

¹⁰Solution to 3: (b)

¹¹Solution to 4: (b)

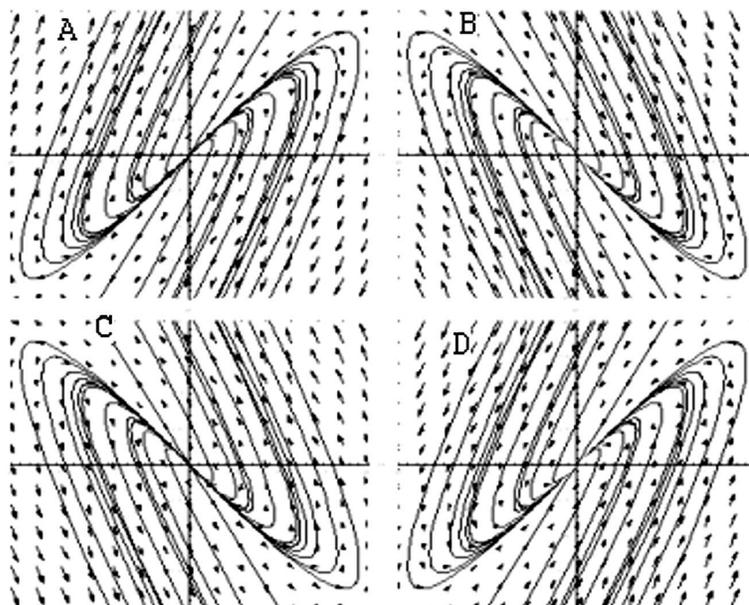
5. Suppose you have a linear, homogeneous system of differential equations with constant coefficients whose coefficient matrix has the following eigensystem: eigenvalues -4 and -1 and eigenvectors $\langle 1, 1 \rangle$ and $\langle -2, 1 \rangle$ respectively. The function $\vec{Y}(t)$ is a solution to this system of differential equations which satisfies initial value $\vec{Y}(0) = (-15, 20)$. Which statement best describes the behavior of the solution as $t \rightarrow \infty$?¹²
- The solution tends towards the origin.
 - The solution moves away from the origin and asymptotically approaches the line through $\langle 1, 1 \rangle$.
 - The solution moves away from the origin and asymptotically approaches the line through $\langle -2, 1 \rangle$.
 - The solution spirals and returns to the point $(-15, 20)$.
 - There is not enough information.

The next question, Question 6, asked the more difficult question of how the solution was drawn to the origin:

6. Suppose we have the same system as in the previous question in addition to a solution $\vec{Y}(t)$ which satisfies $\vec{Y}(0) = (x_0, y_0)$ where the point (x_0, y_0) is not on the line through the point $(1, 1)$. How can we best describe the manner in which the solution $\vec{Y}(t)$ approaches the origin?¹³
- The solution will approach the origin in the same manner as the line which goes through the point $(1, 1)$.
 - The solution will approach the origin in the same manner as the line which goes through the point $(-2, 1)$.
 - The solution will directly approach the origin in a straight line from the point (x_0, y_0) .
 - The answer can vary greatly depending on what the point (x_0, y_0) is.
 - The solution doesn't approach the origin.

I anticipated that this question would be tough going, but that enough students would get it for a fruitful discussion. I also planned to revote on this question after the discussion to give the entire class a chance to affirm the correct solution. I concluded the work on negative eigenvalues with a question that forced them to put together their ideas geometrically.

7. Which phase portrait below corresponds to the linear, homogeneous, system of differential equations with constant coefficients whose coefficient matrix has the following eigensystem: eigenvalues -5 and -2 and eigenvectors $\langle -1, 2 \rangle$ and $\langle -4, 5 \rangle$, respectively?¹⁴



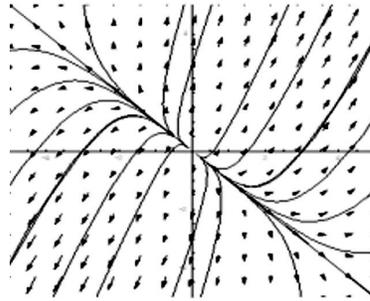
¹²Solution to 5: (a)

¹³Solution to 6: (b)

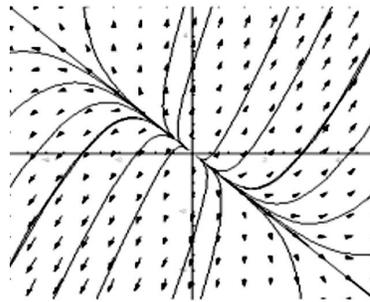
¹⁴Solution to 7: (b)

Finally, we turned to the case where both eigenvalues were positive. Instead of keeping the picture for the end, I used the first question to present a phase portrait and have the students articulate why both eigenvalues must be positive. This broke the ground for a followup question about the dominant eigenvector, which asks for a more precise analysis of the solutions.

8. Using the phase portrait below for the system $Y' = AY$, we can deduce that the eigenvalues are:¹⁵



- (a) of mixed sign
 (b) both negative
 (c) both positive
 (d) Not enough information is given
9. Using the phase portrait below for the system $Y' = AY$, we can deduce that the dominant eigenvector is:¹⁶



- (a) $\langle -1, 1 \rangle$
 (b) $\langle 1, 3 \rangle$
 (c) $\langle 1, -2 \rangle$
 (d) There is no dominant eigenvector because there is no vector that is being approached by all of the solution curves.
 (e) Not enough information is given

At this point, I planned to pause for a short recap on the board of everything we had just discussed. I would summarize all of the main points for the three cases and then introduce notation labeling the equilibrium points as saddles, sinks, and sources respectively. Finally I would close the lecture by presenting a system and asking the students to classify the equilibrium point.

10. Classify the equilibrium point at the origin for the system¹⁷

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{Y}.$$

- (a) Sink
 (b) Source
 (c) Saddle
 (d) None of the above

¹⁵Solution to 8: (c)

¹⁶Solution to 9: (b)

¹⁷Solution to 10: (c)

20.2.4 How the class went

This class went extremely well. On Question 1, 95% of the students voted for the correct answer, suggesting that they recalled the previous lecture and were ready for the day's discussion. I went directly to Question 2, where 68% voted for the correct question. I consider this to be quite good since I had not at all discussed the geometry before the question. The class discussion correctly highlighted the features students should look for, and 90% got the next question (which is the same idea) correct.

Question 4 was the first real test that required significant analysis. On this question, 65% chose the correct answer, and 35% decided that there was not enough information. Again, the class discussion was on point and resolved the issue directly.

Question 5 went smoothly, and then we got to Question 6. My suspicion that this would be the most challenging question in the set was correct, and only one student voted for the correct answer. For the discussion, I polled several students and had them articulate exactly why they voted for what they did. I saved the student that I suspected had the correct solution for after several people had given their thoughts. The student with the correct answer gave an excellent explanation, which I asked him to repeat to help him improve his precision of language. Then, I told the students to find someone who voted for something different than they had and try to convince them of an answer. I reopened the voting. This time 58% voted for the correct solution. I felt this was still a bit low, so I used the opportunity to summarize the correct behavior to reinforce the point. I should also note that it was a good sign that so few got the correct answer on the second vote for a different reason. This suggested that students were really focusing on the content and not trying to tailor their responses to signals they picked up from me.

The other questions went smoothly until Question 9. Here, only 10% had the correct solution. I had not defined the term “dominant eigenvector” and had left the students to figure out the term based upon the context. This seemed to be the main hangup, so the discussion clarified the issue.

Finally I did a quick summary and gave the notation for the types of equilibrium points and pictures we had seen. I wrapped up in the final minute with the last question asking for a classification of an equilibrium point. Here 89% of the students got the correct answer, suggesting that they were still able to compute eigenvalues and connect them to the summary just presented.

Incidentally, on the next midterm, I asked the following question:

6. [20 points] Describe the different kind of phase plane pictures that can occur for constant coefficient linear systems of the form

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

when the eigenvalues of the system are *real*, nonzero, and *distinct*. Include pictures of typical types of systems and discuss the the eigenvalues corresponding to each type. In each case, you should classify the equilibrium point at the origin.

Student success on this problem was everything that I could have hoped for.

I should also make a note on the time spent in class. A colleague was observing my class that day, and the first 10 minutes of class were devoted to a survey. This class was a bit shorter than normal since the survey and other administrative details ate up a good chunk of time. That said, we definitely covered the questions comfortably, and I was never rushed. Also, the colleague (who had not had experience with voting in the classroom) was very positive about everything he observed in the class.

20.2.5 Reflection upon the two classes

While these two classes illustrate different approaches to using voting questions in the classroom, given the chance I would teach both classes again in the same manner. The first class is the usual model that I use to build a lecture with voting. This style is very effective at keeping the students active and at having the students contend with serious

mathematical ideas but still allows the instructor to maintain control of the classroom and pace. In fact, if covering a set amount of content is important for a particular day, a class prepared on the first model can easily be modified on the fly by dropping voting questions that you don't have time for. The students will still have the question that you dropped and can work it on their own later, but you maintain enough flexibility to finish as much material as you need.

I would tend to use the second model for classes that are focused primarily on consequences of previous work. In the class described, we were focused on consequences of the general analytic solution to the differential equations we were studying. The students already had all of the tools and just needed the time and prompting to connect to geometric and qualitative behavior of the solutions. This is an ideal situation for voting and discussion. The focus is less on covering a lot of ground and more on putting the ideas together.

20.3 Conclusion

If you are new to voting questions and considering adding them to your repertoire, I would summarize my thoughts as follows. First, decide what role you want the voting questions to play in your class. Do you want them to be mainly computational and serve to keep the students alert? Do you want them to play a more conceptual role and challenge the students on the concepts? Do you want them to come before you lecture about a subject? Do you want a blend of different types of questions? Do you want a portion of the class learning to come through post-voting discussion?

Once you have decided how you want to use voting questions, build your lecture by starting with the questions that you will use (assuming you have access to a thorough database). Then add any additional content that won't be addressed by the voting questions and subsequent discussion.

I think the advantages to doing things this way are numerous. By really focusing on the voting questions, you leave yourself the flexibility to really go where the student discussion leads. This will lead to a more spontaneous, energetic class that is focused on where the students are and where they need to be. You will also have a lot of fun!

Finally, do not be afraid to experiment and give your students challenging questions. Often the hardest conceptual questions lead to the most fruitful discussions and progress. Your students might surprise you quite often as well!

21

Classroom Voting Questions that Provoke Teachable Moments in Differential Equations

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Classroom voting has become recognized as a powerful teaching technique in many courses including the calculus sequence (see e.g., [16, 59]). To build on this success, we have developed a library of over 300 classroom voting questions for differential equations with the primary goal of creating useful student discussions that will help teach this material (questions available at <http://mathquest.carroll.edu>).

After writing this collection, the next step was to test the questions by using them in our courses and collecting data on the voting results. The authors teach at a variety of different institutions, allowing us to see how well this teaching method worked with different groups of students. Cline, Zullo, and Parker teach at Carroll College in Montana, a small private liberal arts college with a traditional residential student body, and Harris teaches at a similar institution in Kentucky. Stewart teaches at Hood College in Maryland, a small private liberal arts college with a mix of residential and commuter students. Storm teaches at Adelphi, a university in New York with a strong regional draw that is primarily a commuting campus. George teaches at a public high school in Montana.

We found that classroom voting can be a very powerful way of revealing student misconceptions, so that they can be directly addressed. Rather than trying to anticipate the common difficulties that students have and designing a lesson to prevent students from making these errors, instead we found it more useful to ask questions which provoked the errors, helping students to confront these issues directly. As such, often the most useful discussions are produced by questions where significant percentages of students voted for incorrect answers. Sometimes we have large numbers of students voting for several different options, which results in very rich discussions as the students defend a variety of plausible perspectives. Alternatively, we might see a strong majority voting for one particular incorrect answer. In such cases there may not be much peer-to-peer discussion before a vote, but afterwards, when the class discovers that this majority is incorrect, they are usually surprised and intrigued to learn how they were mistaken.

In order to identify the questions which provoke the most fruitful discussions, we have been recording voting statistics, and incorporating these into the teacher's edition of our question collection. Each instructor can use these past results to help select the most effective questions for a given class period. Of course, the way that a particular class votes on a specific question can be affected by a great many factors. For instance, the results depend heavily on

the context in which the question is asked: at the very beginning of the lesson to introduce a topic, in the middle of the lesson to apply an idea for the first time, or after the lesson as practice or review. Thus we see that many questions provoke very different responses from different classes. However, there are some questions which tend to produce fairly consistent results, posing a significant challenge to different classes, taught by different instructors at different institutions. Such questions generally focus on an issue of particular difficulty, and thus warrant further examination and attention in the classroom.

To identify the most consistently challenging questions, we reviewed our database of over 300 voting questions and considered only those where we had statistics from three or more classes, and further, where on average less than 50% of students voted correctly. The questions below met these criteria, and upon review it appears that all of these questions help students confront important challenges in this subject matter. They span a wide variety of subjects, and often reveal fundamental issues that students struggle with.

21.1 Units

Units can be a powerful mathematical tool to determine the meaning of different quantities in a differential equation or a function, and we expected that our students would be using this tool on a regular basis. Two questions considered in this paper concern units. The first was asked near the beginning of the course, when students were just beginning their study of first order linear equations:

The evolution of the temperature of a hot cup of coffee cooling off in a room is described by $\frac{dT}{dt} = -0.01T + 0.6$ where T is in $^{\circ}\text{F}$ and t is in hours. What are the units of the numbers -0.01 and 0.6 ?

- (a) -0.01°F , and 0.6°F
- (b) -0.01 per hour, and 0.6°F per hour
- (c) -0.01°F per hour, and 0.6°F
- (d) neither number has units

	a)	b)	c)	d)
Class 1	0%	10%	85%	5%
Class 2	0%	48%	52%	0%
Class 3	8%	0%	92%	0%
Class 4	0%	33%	67%	0%

These results reveal considerable confusion about the most basic ideas concerning how units function in any equation, such as the principle that quantities which are equal or are summed must have the same units, and the idea that when we take the product of two quantities, the result has the product of their units.

The other question is from the lesson on second order differential equations, concerning how units operate in a trigonometric function:

A differential equation is solved by the function $y(t) = 3 \sin 2t$ where y is in meters and t is in seconds.

What units do the numbers 3 and 2 have?

- (a) 3 is in meters, 2 is in seconds
- (b) 3 is in meters, 2 is in per second
- (c) 3 is in meters per second, 2 has no units
- (d) 3 is in meters per second, 2 is in seconds

	a)	b)	c)	d)
Class 1	0%	37%	0%	63%
Class 2	10%	41%	45%	3%
Class 3	18%	9%	36%	36%

This question is more complex, because the students must understand that the input and the output of the sine function are dimensionless and therefore that the coefficient 2 must cancel out the units of seconds, and that the units of y must come from the coefficient 3.

21.2 Euler's Method

This procedure for approximating a solution for any differential equation is appealing and powerful. However, the relative lack of computational complexity may lead students to miss important issues:

We have used Euler's method to approximate a differential equation with the difference equation $z_{n+1} = 1.2z_n$. We know that the function $z(0) = 3$. What is the approximate value of $z(2)$?

- (a) $z(2) \approx 3.6$
- (b) $z(2) \approx 4.32$
- (c) $z(2) \approx 5.184$
- (d) Not enough information is given.

	a)	b)	c)	d)
Class 1	0%	77%	18%	6%
Class 2	0%	60%	0%	40%
Class 3	31%	46%	8%	15%

Without knowing the step size that was used in deriving this difference equation, we don't know how many steps to take and so we can only conclude that not enough information is given. The most popular answer is (b), where the students have assumed that the step size is equal to one. This helps emphasize the difference between the independent variable of our function, which can take on the value of any real number, and the index of our difference equation, which must be an integer.

21.3 Equilibria

Understanding what equilibria are and how they organize the solution space of a differential equation is very important. This question was asked when introducing the concept of equilibria, with uniformly devastating results:

The differential equation $\frac{dy}{dt} = (t - 3)(y - 2)$ has equilibrium values of

- a) $y = 2$ only
- b) $t = 3$ only
- c) $y = 2$ and $t = 3$
- d) No equilibrium values

	a)	b)	c)	d)
Class 1	0%	11%	89%	0%
Class 2	15%	8%	77%	0%
Class 3	4%	0%	96%	0%
Class 4	0%	0%	100%	0%
Class 5	0%	0%	100%	0%

This question required less than two minutes for each vote, yet was a profound surprise for the students. They understood that the derivative would be zero at an equilibrium value, but in the discussion after the vote did they get the idea that an equilibrium value is a value of the function for which the derivative equals zero over all values of the independent variable, so that if we choose it as an initial condition, then the function will remain at this value permanently. This question also helps to reemphasize that solutions to differential equations are functions. Using ideas like continuity in order to determine what types of behaviors are possible is a rich area, requiring students to grapple with some key conceptual issues, as in the question below:

We know that a given differential equation is in the form $y'(t) = f(y)$ where f is a continuous function of y . Suppose that $f(5) = 2$ and $f(-1) = -6$.

- a) y must have an equilibrium value between $y = 5$ and $y = -1$.
 b) y must have an equilibrium value between $y = 2$ and $y = -6$.
 c) This does not necessarily indicate that any equilibrium value exists.

	a)	b)	c)
Class 1	33%	10%	57%
Class 2	57%	43%	0%
Class 3	26%	15%	59%
Class 4	0%	0%	100%

There are two layers to this problem. First they must interpret the notation, and then they must apply the idea of continuity. Because f is continuous, this means that if f goes from positive to negative, then there must be a zero, an equilibrium value between them and so the correct answer is (a). Votes for (b) indicate a lack of understanding of the notation and its meaning since 2 and -6 are y' not y . (c) is a catch-all for students who didn't parse the notation or can't recall and apply the implications of continuity. Further the students have learned to be cautious about our use of language in mathematics. "Must" is a very strong word, and so they may be afraid that there are counterexamples that they haven't noticed, pushing them to vote for (c) as the less strongly phrased option.

The following question uses the same basic format of an autonomous differential equation and the idea of continuity to create a much more challenging scenario:

We know that a given differential equation is in the form $y'(t) = f(y)$ where f is a continuous function of y . Suppose that $f(6) = 0$, $f(14) = 0$, and $y(10) = 10$.

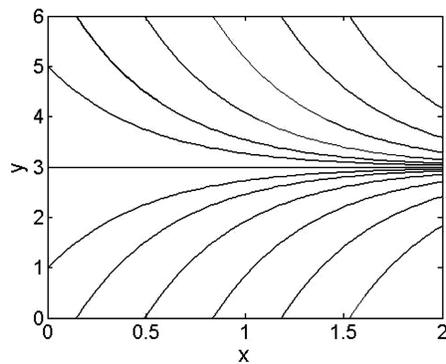
- a) This means that $y(0)$ must have been between 6 and 14.
 b) This means that $y(20) = 0$ is impossible.
 c) This means that $y(20) = 20$ is impossible.
 d) All of the above.
 e) None of the above.

	a)	b)	c)	d)	e)
Class 1	14%	5%	5%	0%	76%
Class 2	0%	0%	4%	0%	96%
Class 3	0%	0%	17%	0%	83%
Class 4	17%	22%	5%	11%	44%

This question focuses on the fundamental idea that no solution trajectory can cross an equilibrium, and so the equilibria help to organize all possible solutions. Here we have equilibria at 6 and 14, and we know that the function is between these at $t = 10$. This means that all past and future values of y are confined between these equilibria and so all three statements are true. The large numbers of students voting for (e) tell us that they really didn't get anywhere with this problem. Again, "impossible" and "must" are strong words, and so they avoid them by voting for (e). In classes 1 and 2, students took about two and a half minutes before registering their votes, and were fairly confident in their answer, so it was quite dramatic when they discovered that no one got it right, producing a powerful teachable moment. Even though none of the students voted correctly in three of these classes, we were able to guide very useful student discussions after the vote, by posing some simple leading questions, like "Does this system have any equilibria?" and "Is it possible for a solution to cross an equilibrium value?" Most of the students were genuinely surprised to discover that no one had voted correctly, and so they were intrigued by what they had missed.

Many questions in our database require students to interpret graphical information. Here we ask them how to determine possible values for parameters:

The figure below plots several functions which all solve the differential equation $y' = ay + b$. What could be the values of a and b ?



- a) $a = 1, b = 3$
- b) $a = 2, b = -6$
- c) $a = -1, b = -3$
- d) $a = -2, b = 6$
- e) $b = 3$ but a is not easy to tell.

	a)	b)	c)	d)	e)
Class 1	0%	24%	24%	48%	5%
Class 2	0%	24%	3%	0%	72%
Class 3	9%	9%	0%	27%	55%

The graph shows that $y = 3$ is a stable equilibrium, thus $a < 0$, causing the exponential part of the solutions to decay away, and allowing us to rule out (a) and (b). The equilibrium value of this equation is $-\frac{b}{a} = 3$, and so we can rule out (c) which would give an equilibrium value of -3 . The large numbers of students voting for (e) indicate confusion between the nonhomogenous term b , and the nonzero equilibrium that this produces.

Students are often unused to thinking about what types of values a parameter could have, and how these could affect the different categories of solutions produced by an equation, as we see in this question from later in the course, as a preparation to introduce the concept of a bifurcation:

How many equilibria does the differential equation $y' = y^2 + a$ have?

- a) Zero
- b) One
- c) Two
- d) Three
- e) Not enough information is given.

	a)	b)	c)	d)	e)
Class 1	0%	13%	8%	4%	75%
Class 2	0%	6%	67%	0%	28%
Class 3	7%	15%	37%	0%	41%

In this case if a is negative, then there are two equilibria, if a is zero then there is one equilibrium, and if a is positive, then no equilibrium exists. Dealing with parameters is often a major challenge for students, from calculus into higher courses. The substantial number of students voting for (c), two equilibria, indicates they have not thought carefully about what this requires them to assume about a .

21.4 Nonhomogeneous Differential Equations

It is possible that students may master the complex procedures for solving nonhomogeneous differential equations without developing a real understanding of why these procedures work, and where they come from. Unless the students can construct a deeper understanding, then even this superficial procedural knowledge will probably not be retained. Thus the following question serves as a powerful probe to determine how substantial the students' conceptual knowledge really is:

When we have $y' = 7y + 2x$ we should conjecture $y = C_0e^{7x} + C_1x + C_2$. Why do we add the C_2 ?

- Because the $7y$ becomes a constant 7 when we take the derivative and we need a term to cancel this out.
- Because when we take the derivative of C_1x we get a constant C_1 and we need a term to cancel this out.
- Because this will allow us to match different initial conditions.
- This does not affect the equation because it goes away when we take the derivative.

	a)	b)	c)	d)
Class 1	0%	21%	58%	21%
Class 2	0%	8%	62%	31%
Class 3	0%	13%	80%	7%
Class 4	5%	20%	35%	40%

The most popular answer is (c), which is concerning. The ability of a general solution to match different initial conditions comes from the solution of the associated homogeneous equation. Here C_2 is part of the particular solution, and thus has nothing to do with initial conditions. This indicates that in none of these classes did the students have a meaningful understanding of *why* the various terms in the general solution are necessary. In the following classes, where we add a sine or a cosine as a nonhomogeneous term to a first order equation, we can revisit this idea again, asking why we must use both a sine and a cosine to create a particular solution.

21.5 Uniqueness of Solutions

The issue of whether solution trajectories for a given set of initial conditions are unique can be a very abstract concept for students. However the following question uses this idea in a very tangible setting, in order to help them build a more concrete understanding of what this means (adapted from [68]):

Based upon observations, Kate developed the differential equation $\frac{dT}{dt} = -0.08(T - 72)$ to predict the temperature in her vanilla chai tea. In the equation, T represents the temperature of the chai in $^{\circ}\text{F}$ and t is time. Kate has a cup of chai whose initial temperature is 110°F and her friend Nate has a cup of chai whose initial temperature is 120°F . According to Kate's model, will there be a point in time when the two cups of chai have exactly the same temperature?

- Yes
- No
- Can't tell with the information given

	a)	b)	c)
Class 1	88%	12%	0%
Class 2	73%	27%	0%
Class 3	62%	28%	10%

Because the derivative given by the differential equation is continuous, this tells us that solutions are unique. Thus, although both cups of chai asymptotically approach the same equilibrium temperature, they never actually reach this temperature, and they never merge. This can be non-intuitive for students, and some may point out that after a sufficient length of time, the temperatures of the two cups will be so close that no thermometer could distinguish between them. This type of discussion can be very powerful, because it stimulates the students to think carefully about the relationship between our mathematical model, the actual state of reality, and what exactly we can measure.

21.6 Systems of Differential Equations

Using eigenvalues and eigenvectors to understand the behavior of systems of differential equations is another rich source of complexity and misconceptions. The following question asks students to qualitatively predict the behavior of a system based on the eigenvalues and eigenvectors:

The eigenvalues and eigenvectors for the coefficient matrix A in the linear homogeneous system $Y' = AY$ are $\lambda_1 = 4$ with $v_1 = \langle 1, 2 \rangle$ and $\lambda_2 = -3$ with $v_2 = \langle -2, 1 \rangle$. In the long term, phase trajectories:

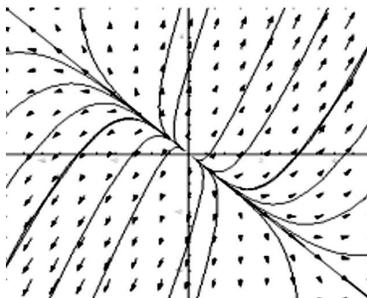
- become parallel to the vector $v_2 = \langle -2, 1 \rangle$.
- tend towards positive infinity.
- become parallel to the vector $v_1 = \langle 1, 2 \rangle$.
- tend towards 0.
- None of the above

	a)	b)	c)	d)	e)
Class 1	0%	52%	48%	0%	0%
Class 2	0%	29%	53%	18%	0%
Class 3	0%	43%	21%	14%	21%

Option (b) gets significant numbers of votes and is not implausible in this context. However phase trajectories will be split between positive and negative infinity, depending upon the initial conditions, so this answer is not generally correct. Instead, solutions will tend to become parallel to the dominant eigenvector.

Interpreting graphical information and determining what it means is a particular challenge for many students. In the following case we present students with the phase portrait and solution trajectories for a two dimensional system, asking them to relate these to the eigenvectors and eigenvalues of this system:

Using the phase portrait below for the system $Y' = AY$, we can deduce that the dominant eigenvector is:



- $\langle -1, 1 \rangle$
- $\langle 1, 3 \rangle$
- $\langle 1, -2 \rangle$
- There is no dominant eigenvector because there is no vector that is being approached by all of the solution curves.
- Not enough information is given.

	a)	b)	c)	d)	e)
Class 1	42%	25%	25%	0%	8%
Class 2	47%	12%	0%	29%	12%
Class 3	17%	61%	4%	9%	9%
Class 4	57%	10%	0%	33%	0%

In this diagram the $\langle -1, 1 \rangle$ eigenvector is certainly very eye-catching, and so substantial percentages of students vote for (a). However, upon closer reflection we see that trajectories are diverging away from this direction, and thus $\langle 1, 3 \rangle$ must be the dominant eigenvector.

21.7 Conclusions

It is important to note that the questions presented here are some of the most difficult and challenging ones in our collection of over 300 questions, and that unlike these, most of the questions we used caused a majority of students to vote correctly. However when we used the questions presented above, we found that often only a small minority of students would vote correctly, and sometimes none at all. An instructor might be concerned about using questions that provoke such results, and worry that unless a significant number of students vote correctly, it might be difficult to create a productive classroom discussion. However, our experience is that on the contrary these questions produced some of the best discussions. We found that students rose to the challenge, and were intrigued to figure out the flaws in their reasoning. If the necessary ideas were not quickly forthcoming in the post-vote class-wide discussion, it was not difficult for the instructor to pose a few leading questions and help the students resolve the issues. Note that in these classes, voting was done purely for student learning and the accuracy of votes did not contribute to student grades, which prevented student anxiety about voting incorrectly. We often emphasize throughout our courses that these are not quizzes and the purpose of voting is to help the students discover the ideas together.

What can we learn from this group of questions as a whole? These questions tend to be well focused. Rather than trying to deal with several issues at once, they usually zero in on one particular challenge, one particular misconception. The effective distracters are well thought out, and aimed at this one specific error, deliberately designed to provoke students into making this mistake in class, in an environment where the mistake can be corrected. None of these questions are computational in nature, asking students to work out a solution to a specific problem. Instead they focus on qualitative issues and fundamental definitions, asking why we do things in a particular way, or what sorts of behaviors a system might have. This focus on a specific, key conceptual issue appears to be a hallmark of very effective questions, and may be an important principle to use when writing new voting questions.

It might appear to be equally easy to write voting questions which are very easy or very hard. However, our experience is that once a question is successfully integrated into a lesson, most questions produce a substantial majority voting for the correct answer. Writing voting questions that are genuinely and regularly challenging for our students is a difficult task, and yet these tend to be the questions which create the richest discussions and appear to stimulate the greatest learning.

These conceptually challenging questions are worthwhile investments in class time. When selecting the voting questions for a particular class period, it is often useful to start by considering the available questions which have robustly produced the weakest results from other classes. The class discussions following the voting on these questions can be among the richest. Thus the questions we present here are real gems, and we plan to use them regularly when we teach this material.

Acknowledgements This paper is based on work supported by a National Science Foundation Grant (DUE 0536077 and 0836775). Any opinions, findings, and conclusions, or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

22

Teaching Linear Algebra with Classroom Voting: A Class Period on Linear Independence

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Over the past few years, Holly Zullo, Mark Parker and I have been working on the NSF funded Project MathQUEST (DUE-0536077) to develop and test a library of over 300 classroom voting questions for linear algebra. We have found that this teaching technique fits well with the course material. We were consistently impressed with the power of this teaching method to engage students and create a more active learning environment, while at the same time students reported that voting made mathematics class more enjoyable. Good classroom voting questions can bring up the important concepts, so that students engage in this material during class time, and leave class well prepared to work through the homework exercises.

To develop this collection of questions, we began by dividing up the topics for the course among the three of us. One person would consider a particular topic, thinking about key ideas, common misconceptions, and the sorts of leading questions that we could ask in order to help the students discover the fundamental concepts. Then we would meet together and the other two people would work through the questions before reading the teacher's commentary that accompanies each question. We would then discuss each question individually, make revisions, and propose new questions in order to adequately cover the topic. Our collection currently contains 16 classroom voting questions on the topic of linear independence and is available at:

mathquest.carroll.edu/libraries/LA.teacher.00.06.pdf.

In order to show how linear algebra can be taught with classroom voting, I discuss a 50-minute class period, from spring semester 2008, in which I taught the topic of linear independence using a lesson that integrated three of these classroom voting questions. The course is titled "MA 334 Differential Equations and Linear Algebra II," and it is a four credit course, required of all mathematics and engineering majors and minors, usually taken in the spring of a student's sophomore year. During this semester, 18 students were enrolled in the course, and all were present during this period.

In the previous class period we studied linear combinations of vectors in R^n . Before class I reviewed our collection of 16 questions, and selected five questions which I could potentially use. In the end I had sufficient class time to have the class vote on three of these questions.

I began the class by quickly defining linear independence, writing an informal definition up on the board: *A set of vectors is linearly independent if none of them can be written as a linear combination of the other vectors in the set, or equivalently, the only way to produce the zero vector from a linear combination of the vectors in the set is the trivial*

way. I decided not to give them any examples or provide any further discussion, instead asking them to work out the meaning of this definition with a voting question:

Which set of vectors is linearly independent?

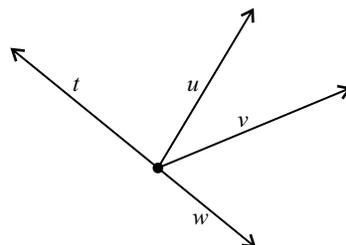
- (a) $(2, 3), (8, 12)$
- (b) $(1, 2, 3), (4, 5, 6), (7, 8, 9)$
- (c) $(-3, 1, 0), (4, 5, 2), (1, 6, 2)$
- (d) None of these sets are linearly independent.
- (e) Exactly two of these sets are linearly independent.
- (f) All of these sets are linearly independent.

The students worked hard for five minutes, discussing and debating this question until most of them had clicked in and so I called for the rest of the class to register their votes. In this course voting has no direct effect on grades. To ensure that all students vote, I simply perform a head count at the beginning of class, then hold each vote open until all students have clicked in. The results for this question were 25% for (a), 50% for (b), and 25% for (d). If we were using this question for assessment, these results would be a disaster, but in this context, they meant that we had a good starting point for discussion. Further, the post-vote discussion quickly revealed that although 25% of the class had correctly voted for (d), no one was immediately able to give a clear line of reasoning why this was correct.

Post-vote discussions are often most useful if the instructor is very Socratic, asking students to explain their reasoning, but refusing to confirm or deny whether the student is correct, and instead simply going on to the next person and asking for their thinking. Thus, I began by selecting a student and asking him to explain what he had voted for and why. The student had voted for (a), explaining that if you have only two vectors in a set, they must be linearly independent: You can't make a linear combination with only one vector, so you could never write one vector as a linear combination of one other vector. I selected another student who agreed with this reasoning. However a student volunteer explained that he had used the second definition of linear independence, and had made it work: If you multiply the first vector by negative four, and add this to the second vector, then you get the zero vector. At this point I refocused the discussion, asking how it was possible for this set to be linearly independent by one definition but linearly dependent by another definition if the two definitions were equivalent. No one really knew where to go with that, but one student admitted that we must have done something wrong. So, I asked this student to explain what a linear combination was: The student successfully explained that to get a linear combination you multiply each vector by a scalar and then you add them up. I followed this up, asking what you would get if you made a linear combination from a single vector. Now several students got the point, volunteering to explain that when you make a linear combination with a single vector, you can get any multiple of that vector, and so in this case the vector $(8, 12)$ can be written as a linear combination of the other vector in the set, $(2, 3)$, because it's a multiple of this vector.

Once this fundamental issue was settled, I picked another student and asked him to explain what he had voted for and why. This student had voted for (b), explaining that (c) could not be right because you can get the third vector by simply adding up the first two. However the student said that he could not see any way to write $(7, 8, 9)$ as a linear combination of $(1, 2, 3)$ and $(4, 5, 6)$. I replied by asking how you could tell whether it is really impossible to write one vector as a linear combination of two others, or whether you're just not seeing the right way to do this. The student couldn't identify any general method to resolve this issue, but at this point another student volunteered with a solution for this particular problem, pointing out that if you multiply $(4, 5, 6)$ by 2, and then subtract $(1, 2, 3)$, you get $(7, 8, 9)$, and so this means that this set is dependent. With all three sets of vectors established as dependent, we went on to another question:

Which subsets of the set of the vectors shown below are linearly dependent?



- (a) u, w
- (b) t, w
- (c) t, v
- (d) t, u, v
- (e) None of these sets are linearly dependent.
- (f) More than one of these sets is linearly dependent.

This question resulted in another protracted pre-vote discussion, lasting four minutes, before a strong majority of the class had voted, and I called time. The chart showed that 44% of the class had voted for (b), 6% for (c), and 50% for (f). The first student I called on was confused and had interpreted the definition of linear independence backwards. This may have been an issue because up until this point I had only discussed independence, however the question asked about dependence. So I defined both terms: A set of vectors is linearly independent if you cannot write one as a linear combination of the others. A set of vectors is linearly dependent if you can write one as a linear combination of the others. Once this issue was cleared up, the student could explain that set (b) was dependent, because t and w were multiples of each other. The next student I called on stated that set (d) was linearly dependent as well, and so I asked how he knew this. He explained that there must be some way to make a linear combination of these vectors add up to zero, but wasn't sure exactly how this could be done. I asked how you add vectors graphically. He explained the tip-to-tail technique, and then began proposing different possibilities, which I sketched on the board. He came to the idea that $u - v$ would be in the same direction as t , and so if we multiplied t by the right number, we could get these three to all cancel out. Once this issue was resolved, a student brought up the more general question, conjecturing that no set of three vectors in two dimensions could be linearly independent, and I confirmed his graphical intuition on this point.

Now I returned to the question that had come up earlier: Suppose you have a set of vectors. What's a general method for determining whether or not they are linearly independent? This was not obvious to the students, so I wrote out the problem on the board more formally: I want to find constants c_1, c_2, c_3, \dots so that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \dots = \vec{0}$. Now a student volunteered the idea that we want to put these vectors into a matrix and put it in reduced row echelon form. I followed up by asking how these vectors would go into this matrix: Would we put them in the rows or the columns? Another student explained that the vectors would be the columns of the matrix, and that we could then augment it with a column of zeros, to see if it was possible to get the zero vector as a linear combination of the others. This is often an issue my classes: I introduce the Gaussian elimination process with the task of using an augmented matrix to solve a system of nonhomogeneous equations, and the students conclude that if we put a matrix in reduced row echelon form, it must be an augmented matrix. So, to deal with this, I asked: If we start out with a column of zeros, and we perform row operations, could there ever be anything but zeros in this column? Phrased in this way, the students could correctly state that this column would always contain zeros, and so augmenting the matrix is unnecessary.

Now the students were ready for the next question:

Suppose you wish to determine whether a set of vectors is linearly independent. You form a matrix with those vectors as the columns, and you calculate its reduced row echelon form,

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

What do you decide?

- (a) These vectors are linearly independent.
- (b) These vectors are not linearly independent.

The class discussed this question for four minutes and thirty seconds, then 41% voted for (a) and 59% voted for (b). However the post-vote discussion quickly showed that although a majority had voted correctly, they could not provide a sound mathematical argument to back up their vote. One stated that we did not receive the identity matrix, and so the vectors were dependent, but when I asked "Why would an identity matrix tell us that these are independent?" he

had no response. So I asked: what set of equations does this matrix represent? The first student replied by saying that $c_1 = 1$, $c_2 = 1$, $c_3 = 2$, and since we got a solution, this means that there is a nontrivial way to make the zero vector, so they are independent. Now another student volunteered that this is not an augmented matrix, so we had four vectors to begin with, so we need four coefficients, not three. Rather than clarifying this myself, I let the students sort out the issue, which happened fairly readily because most students were quite invested in the process. I could then write out the equations on the board: $c_1 + c_4 = 0$, $c_2 + c_4 = 0$, and $c_3 + 2c_4 = 0$. Now that we had the equations right, I could ask: What does this mean? At this point the conclusion was fairly straight forward: We had a nontrivial solution to the problem, and so the vectors are not linearly independent.

After this discussion, time was quickly running out, and so I summarized the key points of the lesson and dismissed the class. Within this class period I called on all of the 18 students present by name at least once, and many students spoke up several times. In the days that followed, I found that this lesson had given the students sufficient preparation for the homework assignment, which included several exercises where students had to evaluate a set of vectors for linear independence, and find a nontrivial way to produce the zero vector from a linear combination of these if the set was dependent.

In the class period discussed above, I posed three voting questions, the discussion and resolution of which consumed almost the entire period. However I selected three questions which together addressed the important points in the lesson: In order to give good answers to these questions, the students had to demonstrate mastery over the teaching goals. This also illustrates the importance of the post-vote discussions. Merely because students voted correctly did not mean that they understood the issues. But the requirement of choosing one of the options forced them to really consider the question, and the knowledge that they could be called upon in the post-vote discussion pushed them to take the question seriously. As a result, they were invested in the question, they genuinely wanted to see how things would be resolved after the vote, and so they were firmly engaged in the discussion that followed.

In general, I have found that to be successful, classroom voting must be integral to the lesson, an absolutely essential part of the teaching process. The voting must be used in such a way that if a student were to tune out for the voting questions and discussion, this student would miss vital points and would probably be unable to do the homework. Conversely, if the voting is used as an optional add-on, so that anything taught through voting is also taught through traditional lecture, then the students quickly recognize this, and become impatient with the voting, unwilling to invest much effort in something which is not crucial to the learning process. The lesson described above thus demonstrates how classroom voting can be used to teach new material, through a process of question, discussion, resolution, and generalization.

Acknowledgements This paper is based on work supported by a National Science Foundation Grant (DUE 0536077 and 0836775). Any opinions, findings, and conclusions, or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

23

Lesson Planning with Classroom Voting: An Example from Linear Algebra

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23.1 Introduction

Planning a lesson with classroom voting requires many of the same strategies as planning a lesson to be delivered through a lecture. First and foremost, a good lesson plan begins with a clear set of goals, and those goals must be kept in mind as the lesson is developed. Of course, the material must be approached in a logical order. Finally, a bit more flexibility is needed in a classroom voting lesson plan than a traditional lesson plan, since any given discussion can run much longer or shorter than anticipated.

Not too long ago, teaching with classroom voting involved the time-consuming process of writing questions for each class period. Now several people have made their questions available as public libraries (see the links at mathquest.carroll.edu/resources.html), saving the newcomer vast amounts of time. However, working with a large library poses a different set of problems when planning a lesson, in that there are often so many excellent questions that there is not time to use all of them. This, then, leaves us in the familiar situation of having so much that we want to share with our students about a particular topic that we are forced to pick and choose. If we do not choose carefully, then we spend valuable class time on ideas that may be a lot of fun and interesting, but which are not central to our course goals.

To make sure I stay on track, when I plan a lesson with a vast question library I begin by explicitly writing down my goals for the lesson. (Yes, we should always do this as educators, but how many of us really do?) This step can help me immediately rule out some questions because they simply do not support my lesson goals. Next I study the questions and think carefully about the material to decide whether I need to introduce the topic with a brief lecture, or whether there is a question that will introduce the topic for me. Whenever possible, I like to ask questions that will lead the students to discover ideas on their own rather than tell them about those ideas through lecture. If I do have to introduce an idea with lecture, then I use the voting questions to deepen the students' understanding and to provide them with examples. As I select questions for my lecture, I make a note about the point of each question. This helps me to double-check that the question supports my lesson goals, and it helps me in class if I run short of time and need to skip some questions. All of this lesson-planning is done in conjunction with planning the homework assignment. As with any sort of lesson, I want to make sure that the topics presented in class are reinforced on the homework; students should have enough background to do the homework, yet the homework problems should push the students to deepen their thinking. I'll illustrate these ideas with an actual linear algebra lesson plan from Spring 2008, in which I used the Project MathQUEST linear algebra question library (mathquest.carroll.edu/la.html).

23.2 Planning the Lesson: Matrix Multiplication and Inverses

My full lesson plan for this material is shown in Figure 23.1. Students already had some exposure to matrix multiplication from a computer lab earlier in the week, and I assume they saw it in high school as well. Many students have seen matrix inverses in high school, but most don't have a great working knowledge of them. So my goals for the day were to be sure students know how matrix multiplication works, have them be comfortable writing a system of equations in $Ax = b$ format, and to introduce matrix inverses. In particular, I wanted them to know how to find the inverse of a matrix by augmenting the matrix with the identity matrix and row reducing, and to understand how the invertibility of the coefficient matrix relates to the number of solutions to a system of equations.

With these goals in mind, I studied our question bank. We had 13 questions related to matrix multiplication and ten questions on matrix inverses. Given an unlimited amount of time, I would love to have asked all of them; but with limited class time for each topic, I had to select only a few.

Goals:

- Be sure they know how matrix multiplication works.
- Write system of equations in $Ax = b$ format.
- Introduce matrix inverses.
 - Find with rref
 - Tie to solutions of systems

Matrix Multiplication

Voting #1 (how to multiply matrices — they should do by hand!)
 Voting #2 (more on how to multiply)
 Voting #3 (application — when do matrix operations make sense?)

Matrix Inverses

Define inverse of a matrix.
 Find inverse by solving $AB = I$.
 Can also use A^{-1} on calculator or Matlab.

Voting #4 (how a matrix can fail to have an inverse)
 Voting #5 (solutions to $AX = 0$)
 Voting #6 ($AB = AC$, does $B = C$?)
 Voting #7 (must always multiply on same side)
 Voting #8 (can solve by taking inverse of both sides)

Figure 23.1. Lesson Plan for Matrix Multiplication and Matrix Inverses

23.3 Matrix Multiplication

Since I expected the students to already know how to multiply two matrices, I opened the lesson by asking Question 1 — just to be sure they all had the idea. I emphasized that they should do this question by hand, not with their calculators.

Question 1. Calculate $\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$.

(a) $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -2 \\ 2 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix}$

- (d) None of the above
 (e) This matrix multiplication is impossible.

[Correct answer: b]

The results indicated that this question was, indeed, worthwhile: 4% voted for (a), 54% for (b), 14% for (c), 25% for (d), and 4% for (e). The students spent a little over two-and-a-half minutes working on this problem. The post-vote discussion was short; someone explained how to multiply matrices, and the class was ready for bigger challenges. Note that answer (c) comes from the misconception of multiplying the entries pair-wise, as matrix addition is performed, and answer (d) is the catch-all for computational errors.

My next question to the class was an attempt to get them to focus on the row-times-column idea of matrix multiplication.

Question 2. If

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 3 \\ -2 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

what is the (3, 2)-entry of AB ? (You should be able to determine this without computing the entire matrix product.)

- (a) 1
 (b) 3
 (c) 4
 (d) 8

[Correct answer: c]

This time 75% of the students correctly answered (c) after two-and-a-half minutes of work. The follow-up discussion centered on avoiding doing the whole multiplication and finding the quick way to do this problem.

With the mechanics of matrix multiplication firmly in hand, my final question on this topic was an application problem. We had not discussed how matrix multiplication might be used in applied settings; I hoped the students would figure that out by using their understanding of the multiplication process.

Question 3. You have a business that sells tables and chairs. You have brown tables and white tables, and corresponding chairs. Your May sales are 4 brown tables, 6 white tables, 20 brown chairs, and 24 white chairs, which is represented by the matrix $M = \begin{bmatrix} 4 & 6 \\ 20 & 24 \end{bmatrix}$, where the first row is tables, the second row is chairs, the first column is brown items, and the second column is white items. Your June sales are given by the analogous matrix J , where $J = \begin{bmatrix} 6 & 8 \\ 22 & 32 \end{bmatrix}$. Which of the following matrix operations would make sense in this scenario? Be prepared to explain what the result tells you.

- (a) $M + J$
 (b) $M - J$
 (c) $1.2J$
 (d) MJ
 (e) All of the above make sense.
 (f) More than one, but not all, of the above make sense.

[Correct answer: f]

The students again took two-and-a-half minutes to vote, with the following results: 36% for (a), 4% for (d), and 57% for (f). The students who voted for answer (a) generally voted before checking the other answers. A brief discussion clarified that the students who voted for answer (f) had correctly identified that only answer (d) would not make sense.

23.4 Matrix Inverses

With matrix multiplication behind us, it was time to start matrix inverses. I didn't think this was a topic I could just dive into without introduction, so I gave a brief lecture of about 10 minutes in which I defined the inverse of a matrix, demonstrated that students could find the inverse by augmenting their matrix with the identity matrix and row reducing, and quickly showed an easy way to find inverses on the calculator. Then we turned back to voting, beginning with Question 4.

Question 4. When we put a matrix A into reduced row echelon form, we get the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. This means that

- (a) Matrix A has no inverse.
- (b) The matrix we have found is the inverse of matrix A .
- (c) Matrix A has an inverse, but this isn't it.
- (d) This tells us nothing about whether A has an inverse.

[Correct answer: a]

Only 22% of the students correctly voted for answer (a), 7% picked (b), 26% (c), and 44% chose (d). Clearly the students did not get as much as I might have hoped out of my little talk about finding inverses by row reducing. This question provided a great opportunity to discuss that idea in more detail to understand why this result means that our matrix is not invertible.

The next question required students to connect several ideas.

Question 5. We find that for a square coefficient matrix A , the homogeneous matrix equation

$$AX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial solution,

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This means that

- (a) Matrix A has no inverse.
- (b) Matrix A has an inverse.
- (c) This tells us nothing about whether A has an inverse.

[Correct answer: b]

The students struggled with this question for nearly three minutes before registering their votes. 7% voted for (a), 15% correctly voted for (b), and 78% voted for (c). The ensuing discussion provided an excellent opportunity for students to actively engage their minds as they sorted through the implications of the zero-vector being the *only* solution to this system. This question was the class's first exposure to this sort of connection, and while they had difficulties with this question, the concepts here were revisited several times on homework, exams, and in class. Later questions on this idea in future class periods generated much larger percentages of students voting correctly.

Class time was nearly over at this point, so I had to pick just one of the three questions remaining in my lesson plan. This is where my notes about each question come in handy. If there is still a critical idea that has not been covered, I will certainly pick that question. Sometimes I'm in the lucky position where all of the remaining questions are "interesting side trips" or "fun depths to explore," and in that case I will usually glance through the voting statistics from previous years and pick a question that seems to be challenging. This time I really wanted to cover the next question in line, so I stayed with my original order.

Question 6. *True or False* If A , B , and C are square matrices and we know that $AB = AC$, this means that matrix B is equal to matrix C . [*Correct answer: False*]

One-third of the students voted True on this question, while two-thirds voted False. The post-vote discussion allowed us the opportunity to discuss how we solve this equation when A is invertible, as well as the implications of A not being invertible. When time allows, it is nice to ask students for specific matrices that demonstrate the fallacy of this statement.

The questions I did not get to were as follows:

Question 7. *True or False* Suppose that A , B , and C are square matrices, and $CA = B$, and A is invertible. This means that $C = A^{-1}B$. [*Correct answer: False*]

Question . We know that $(5A)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What is matrix A ?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix}$

(d) $\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$

(e) There is no matrix A which solves this equation.

[*Correct answer: c*]

23.5 Lesson Summary

This lesson plan probably could have been completed, except that I had started class by wrapping up some material from the previous day. When I don't get through all of the questions I had hoped to, I will frequently pick one to start the next class period with, thus providing an opportunity for review. This time, though, I decided I had met my lesson goals and the issues in the remaining two questions would be handled sufficiently in the homework assignment, so I began the next day with a new topic.

23.6 Conclusions

Any lesson, with any teaching method, should be carefully planned. If anything, though, that care needs to be elevated when planning a lesson that includes a significant amount of student discussion. The length and quality of discussions can vary greatly from class to class, so the lesson plan must be flexible enough to allow for these variations and still cover the necessary material. An added risk with classroom voting is spending too much time on questions that, as instructors, we find fun and interesting, but that really are not central to our learning objectives.

I recommend carefully writing out the goals for each lesson, before thinking about voting questions. When selecting questions for the lesson, whether from a question library or by writing your own, be sure that each question is in line with the lesson goals and that you know what you hope to teach with the question. Finally, select two or three "wish list" questions that you can include or leave out, depending on the pace of the actual class period. Then be prepared for all the fun and lively discussions that will propel your students through the concepts of mathematics.

Acknowledgements This paper is based on work supported by a National Science Foundation Grant (DUE-0536077). Any opinions, findings, and conclusions, or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

24

Using Clickers to Enhance Learning in Upper-Level Mathematics Courses

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24.1 Introduction

I began using clickers in the classroom in Fall 2005. Before using them, I was skeptical about their value. Indeed, my initial bias was very much against multiple choice questions and against what I perceived as the impersonal nature of the technology. Nonetheless, I was intrigued by (and helping to co-author) the collection of ConcepTests available as a supplement to *Calculus*, by Hughes-Hallett, et.al. I decided to experiment with using both the ConcepTests and the clickers in my calculus classes in 2005. From the very first semester, my students loved this addition to the course. Also, to my great surprise, the use of clickers and ConcepTests significantly *increased* the interactive conversations in the classroom. Student engagement with the material was noticeably higher and objective evaluations (such as grades on final exams) improved. I have been using the clickers ever since, and (at the urging of my students) have expanded their use into my upper level classes as well. This paper is about using the clickers in a 200-level Introduction to Proofs course and a 300-level Abstract Algebra course.

All the clicker questions used in the Introduction to Proofs course and the Abstract Algebra course (as well as a 300-level Graph Theory course) were written by me, and I am happy to share them. They are available off my page on the St. Lawrence University website myslu.stlawu.edu/~plock. A link to this page can also be found on the Math QUEST homepage.

24.2 Background

St. Lawrence University is a highly selective private liberal arts college of about 2,000 students located in Northern New York State. The number of students in classes in which I've used the clickers has ranged from 19 to 40. Students purchase the clickers individually and use the same clicker throughout the semester. They are expected to bring the clicker (which is about the size of a cell phone) to every class. The clicker software automatically keeps track of student responses and keeps an automatic gradebook. Each student's "clicker grade" is a factor in the final grade in the course but minimally: the clicker grade counts about 5% of the final grade. Furthermore, about half of a student's clicker grade is based only on participation (being there and thinking about the questions and entering an answer, even if it is wrong) while the other half is based on the number of correct answers. This balance of some credit for effort

and more credit for accuracy has worked well. I believe that having the grade count is important to having students care about getting the right answer and having the grade not count too much is important to student enjoyment of this aspect of the course. The focus on participation also meant that I could feel free to ask significantly more difficult and thought-provoking questions, and to ask questions about topics that were just being introduced.

Clickers are used by some faculty members as an important piece of student assessment, with questions being asked after students have had a chance to internalize the concepts. The clicker questions in this scenario replace to some extent the use of pencil and paper quizzes or problem sets. This is *not* how I use the clickers. My clicker questions are asked in real time, as topics are being introduced and discussed. The clicker questions in most cases are, in fact, the same questions I have always asked my class. The difference is that instead of some students thinking about the answer and one or two actually answering, now *all the students* have to think about the answer and engage the current topic enough to select an answer. The point of the clicker questions, for me, is to increase student engagement and participation in the class as we engage new material, rather than as a new form of assessment. My goal is to keep students focused and actively thinking throughout the class time.

24.3 Clickers in the Classroom

In my courses, clicker questions are a part of most classes, although not all. Whether or not clicker questions are asked in a given class depends more on whether I had time to make up the questions than on whether such questions worked for the topic. As I got better at making up clicker questions, I found they seemed to work for almost any topic. I also found that coming up with the questions became very natural. The biggest draw on my time is the time spent typing them into PowerPoint, not conceiving them. Once the clicker software is installed on a machine, converting the questions from PowerPoint slides to clicker questions is extremely easy and quick.

In those classes in which I ask clicker questions, there are generally between 5 and 10 questions asked in a 90-minute class, usually taking between 5 and 15 minutes but occasionally lasting significantly longer if the questions really spark conversation. The clicker questions are sometimes asked all together (sometimes at the beginning, sometimes at the end, and everything in between) and sometimes spread out throughout the class.

Clicker questions are displayed on a PowerPoint slide and students are given an appropriate amount of time (which varies widely depending on the difficulty of the question, from 10 seconds to 5 minutes or longer) to think about the answer. The timer is then started and the students have 30 seconds to “click in” with their answers. When time runs out, a histogram appears showing the distribution of answers. The students cannot tell who answered what — in this sense, the answers are anonymous — although an internal gradebook keeps track of all answers so that I have all the information. After the histogram appears, we talk as a class about the ideas in the question. We talk for a longer time if the answers were all over the place and we move right on (and cheer!) if the answers were 100% correct.

In my classes, each clicker question is asked only once, although in many cases I have a follow-up clone question. The first question gets the students engaged with an idea, while the second question (virtually identical to me but sometimes not at all to them) — asked after discussing the first — plants the concept more firmly in their minds and lets me know whether they now understand. Students answer the questions individually, rather than in groups, although I never explicitly rule out conversation. I know that others have had great success using the clicker technology in conjunction with pair or group conversations followed by repeating the question. I made the decisions I did (individual responses obtained only once per question) solely in the interest of time. Class time is very valuable and I wanted clickers to be a part of the class but not to dominate the class.

Clickers can be used in many effective ways. One option is to ask deep thought-provoking questions that generate significant classroom conversation. That option is discussed in depth elsewhere in this volume; in this article I will focus more on using clicker questions to keep students engaged *as a topic is introduced*. The questions described in this article, therefore, are more straightforward and are intended to serve as a vehicle to help all students stay engaged and thinking throughout the class. These questions are asked during the class in which the topic is introduced, and are designed to ensure that students understand the basics before we move on. Even these relatively straightforward questions dramatically increase classroom conversations. Many more questions, comments, and “what if” type questions are generated when every student in the class needs to process the ideas as we discuss them. These concepts are not easy and getting students to actively engage with them right from the beginning can be a huge aid in the learning process.

24.4 Using Clicker Questions in an Introduction to Proofs Course

Our Introduction to Proofs course (officially called *A Bridge to Higher Mathematics*) is a required sophomore-level course for our majors and is a prerequisite for our upper-level theory courses. The goal is to get students comfortable with some of the basic ideas in mathematics (basic set theory, one-to-one and onto functions, equivalence relations, basic ideas in cardinality) while helping them develop facility in writing mathematical proofs. In teaching the writing of proofs, I hope that the students will get comfortable with the basic mechanics of such things as induction proofs, proofs of if-then statements, set containment proofs, etc. On the other hand, I also want the students to appreciate the explorative and creative nature of mathematics and to feel able to develop their own style in writing proofs. For me, getting this balance right is one of the most difficult parts of teaching this course. Using clicker questions to help students understand the basic ideas and the rudiments of proof has allowed me to increase class time on the exploratory aspects of mathematics.

In the class in which set operations are introduced, I use many clicker questions such as:

Assume $A = (-\infty, 3)$ and $B = [2, 5]$. What is $A \cap B$?

- (1) $(-\infty, 2)$ (2) $(-\infty, 2]$ (3) $(-\infty, 5)$ (4) $(-\infty, 5]$ (5) $(2, 3)$
 (6) $[2, 3)$ (7) $(2, 3]$ (8) $[2, 3]$ (9) $(3, 5]$ (10) $[3, 5]$

[Answer: (6).]

This is an easy and quick question, and a natural one to ask in a course such as this. The advantage of using it as a clicker question is that before we continue on to do more with set operations, every student in the class has to focus and internalize the notation and meaning of intersection. It greatly reduces the chances that they will be completely tuning me out!

In an early class on proofs, I use these two back-to-back questions:

“If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.” What is the best first sentence of a formal direct proof?

- (1) Assume $A \cup C$. (2) Assume $A \subseteq B$. (3) Let $x \in B \cup C$.
 (4) Let $x \in A$. (5) Let $x \in A \subseteq B$. (6) Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$.

[Answer: (2).]

And then:

“If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.” What is the best second sentence of this proof?

- (1) Assume $A \cup C$. (2) Assume $A \subseteq B$. (3) Let $x \in A \cup C$.
 (4) Let $x \in A$. (5) Let $x \in B \cup C$. (6) Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$.

[Answer: (3).]

Every single student (in a class of 39) got the basics of implication proofs and set containment proofs very quickly. This has not been my previous experience in teaching this course. These slides also provoked additional conversation as the students became comfortable with set notation. We talked about, for example, which of the answer options in the two previous questions even made sense as a sentence.

From a class on composition of functions:

Assume $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions. What is the domain of $g \circ f$?

- (1) A (2) B (3) C (4) f (5) g
 (6) $B \circ A$ (7) $B \circ C$ (8) $f \circ g$ (9) $g \circ f$ (10) $A \cup B$

[Answer: (1).]

And from a class on one-to-one and onto functions:

Define $f : R \times R \rightarrow R$ by $f(x, y) = x \cdot y$. (f is the multiplication function). Is f one-to-one?

- (1) Yes (2) No (3) I have absolutely no idea.

[Answer: (2).]

These are pretty standard questions. The benefit is not that the questions are brilliant. The benefit is that the students, as soon as you have defined composition or a one-to-one function, are required to focus on what the concept really means. Getting the basic ideas down quickly makes the harder stuff a great deal easier.

The questions can also be more thought-provoking and designed to get the students to begin thinking about harder ideas. In each of the next two questions, students received credit for every answer (right or wrong) since we were just beginning our discussion of cardinality and the point was not to check understanding of a concept but rather to get them thinking in new ways. Both questions prompted lots of conversation and set the stage for our discussions on cardinality.

Does there exist a function from the set of positive integers to the set of all integers which is onto?

- (1) Yes (2) No

[Answer: (1).]

Does there exist a function from the set of positive rational numbers to the set of positive integers which is one-to-one?

- (1) Yes (2) No

[Answer: (1).]

I also asked clicker questions before these two that asked about functions in the other direction (i.e., Does there exist a function from the set of all integers to the set of positive integers which is onto? Does there exist a function from the set of positive integers to the set of positive rational numbers which is one-to-one?) These are easy questions for students who really understand the concepts, and are a great way to take the pulse of a class on how many are getting it. The first question (onto function from integers to positive integers) sparks a good conversation about what to do with zero.

The following question is one of only a few that I ask twice. (I love this question and it did not originate with me. I've been using it in different forms for so many years that I have long since lost track of where it came from. I send my thanks and apologies to the creative genius behind it.)

Here is a statement about the set of four cards shown below: "If a card has a vowel on one side, then it has an odd number on the other side."



If you want to turn over the minimum number of cards to determine if the statement is true, which cards do you have to turn over?

- (1) A (2) 6 (3) B (4) 5 (5) A and 5
 (6) A and B (7) A and 6 (8) B and 6 (9) None of them (10) All of them

[Answer: (7).]

I ask this question first very early in the course, before we have talked about implications and the converse and contrapositive. Only a few students get the correct answer, and many will argue with me about the answer. Apparently, the contrapositive and, indeed, the whole notion of implication, are not naturally intuitive. The question generates lively discussion. I ask a very similar question again near the end of the course. I'm happy to report that most of them get it right the second time around.

24.5 Using Clicker Questions in an Abstract Algebra Course

The role the clicker questions play in my Abstract Algebra Course is very similar to the role they play in my Introduction to Proofs course. The course is a Group Theory course, and as soon as we define a group and do a few examples in class, we have a series of clicker questions such as the following:

Is $(\mathbb{Z}^+, +)$ a group?

- (1) Yes (2) No

[Answer: (2).]

Why is $(\mathbb{Z}^+, +)$ not a group?

- (1) I have no idea. What the heck is a group?
 (2) $+$ is not a binary operation on this set.
 (3) $+$ is not associative.
 (4) There is no identity (and hence no inverses).
 (5) There are no inverses.
 (6) All of the above (except (1)).

[Answer: (4).]

Cosets seem to be a particularly hard concept for the students, so some quick clicker questions such as the following can help them internalize the concept.

If H is the subgroup $\langle 3 \rangle$ in Z_{12} , then $H^2 =$

- (1) 5 (2) 14 (3) $\{3, 6, 9, 0\} \cup \{2\}$
 (4) $\{5, 8, 11, 2\}$ (5) $\{5, 6, 9, 0\}$ (6) $\{3, 2, 5\}$

[Answer: (4).]

Clicker questions can require the students to think ahead about what makes sense *before* you write it on the board, as in the following question:

Let H be a normal subgroup of a group G . We can define $\phi : G \rightarrow G/H$ by:

- (1) $\phi(ab) = \phi(a)\phi(b)$ (2) $\phi(a) = Ha$ (3) $\phi(Ha) = a$
 (4) $\phi(ab) = Hab$ (5) $\phi(ab) = HaHb$

[Answer: (2).]

I often use clicker questions to get the students to think through the outline of a proof, as in the following question:

To use the First Isomorphism Theorem to show that $Q_8/\langle I \rangle \approx V$, we first:

- (1) Define a function $\phi : Q_8 \rightarrow \langle I \rangle$.
 (2) Define a function $\phi : Q_8 \rightarrow Q_8/\langle I \rangle$.
 (3) Define a function $\phi : Q_8/\langle I \rangle \rightarrow V$.
 (4) Define a function $\phi : Q_8 \rightarrow V$.
 (5) Show $Q_8/\langle I \rangle$ and V are both abelian.
 (6) Show $Q_8/\langle I \rangle$ and V are both cyclic.

[Answer: (4).]

Follow-up questions ask them to outline the entire proof of this result.

As another example, the entire proof that every cyclic group is abelian is done in class with clicker questions, with every sentence a separate clicker question (“Of the options given, the best next sentence in the proof would be...”).

24.6 Outcomes

All of the clicker questions I’ve included here are questions that could be asked in class anyway or asked on homework or exams. The key differences with clicker questions are twofold. First, all students must participate and second, feedback is immediate. Also, clicker questions are fun!

I list here several outcomes I’ve observed since I started using clicker questions.

1. There is more laughter in the classroom. This is an outcome I did not expect, and is an advantage that should not be underestimated. The students enjoy the whole process of using the clickers and in every class I have used them in, it has spawned class jokes and increased the sense of community in the classroom. (I find it ironic that the clickers I worried would be so impersonal have had exactly the opposite effect. I think this makes me old.)

I also have started throwing in extraneous just-for-fun questions that are designed so that students get credit for every answer. For example, in February 2007, I asked:

Who won the SuperBowl on Sunday?

- | | |
|-----------------|------------------|
| (1) Dallas | (2) Indianapolis |
| (3) New England | (4) Pittsburgh |
| (5) Seattle | (6) Tampa Bay |
| (7) Brazil | (8) Notre Dame |

The correct answer is (2), but, yes, even answering “Brazil” received credit.

I asked the following question before Harry Potter Book 7 arrived and it caused so much conversation that I had to shut down the talking to get on with the class. It has been fun both for me and the students to ask some random questions such as this.

Do you believe Severus Snape is good or evil?

- (1) Good (2) Evil (3) Who the heck is Severus Snape???

2. There is more focus on conceptual understanding. This is another outcome I did not expect. The very act of coming up with the clicker questions has made me more conscious of the concepts that I really want to get across. Also, I’ve learned that making sure definitions are understood quickly is critical. If I introduce a definition at the start of class and then spend the next hour and a half elaborating on it, my students would be *very* confused and behind by the end of class if they did not really understand the initial definition. Having the capability now of finding out quickly and easily if they “get” the initial definition has made classes more productive. It has also allowed us to focus more as a class on the underlying concepts. This is much better than having me repeat a definition multiple times thinking that somehow the students will hear it correctly one of the times!

3. The clickers allow all students in the class to give feedback. The students who are lost appreciate the clickers because they provide an anonymous (to the other students) and not embarrassing way for them to let me know that they are confused and that I need to spend more time on that topic. The quiet and smart students like the clickers because they now have a way to let me know that they understand, even when it might not be their style to wave their hand in the air to volunteer ideas. Indeed, the clickers are liked by loud students and quiet students, smart students and not-so-smart students, competitive students and not-at-all competitive students. I first used the clickers in the Intro to Proof class because the students who had used them in my calculus classes strongly encouraged me to do so. I then first used the clickers in Abstract Algebra because the students who had used them in my Intro to Proofs class or my calculus classes strongly encouraged me to do so. The students almost universally seem to really enjoy this aspect of the class. (I should add that I think if I were using clickers as an efficient way to give quizzes instead of as a way to take the pulse of the class, this student response to the clickers would probably be substantially different.)

4. There is increased awareness on my part of actual student comprehension. I can usually, but not always, predict how students will do on a clicker question. Sometimes I have been spectacularly wrong! Sometimes students will surprise me positively, doing very well on a question that I thought they would find difficult. More often, I will see that they are confused on a question that I thought would be very easy. When this happens, I am usually able to clear up the confusion relatively quickly. The class from there on out is much more productive than it would have been if I had not been aware that the confusion even existed. Both the students and I appreciate this very quick way to gauge understanding.

5. There is additional variety in classroom activities and in assessment. Variety during class time is good! Variety in assessment is good! I like this extra way to mix it up a bit.

6. By objective measures, my students' understanding and ability have improved. On problem sets and final exams that are relatively consistent with earlier ones, students have performed significantly, although not dramatically, better. The improvement has been substantial enough that I have had to consciously make my exams harder in these courses. In certain specific learning areas, the clicker questions have had a dramatic and obvious positive impact on learning. (These include internalizing the meaning of key definitions as well as increased proficiency in the basics of proving theorems.) In other areas, the impact is negligible. (These would include the need to try to create a counterexample before starting a proof to develop some intuition for why it might be true.) For me, the increased student engagement in class is at least as important as the increased scores on the final exams (and, of course, the two outcomes are certainly correlated.)

7. All the choices we make in education impact our time — both our preparation time and our time in class — and clickers are no different. Class time is very valuable and we have to think carefully about how to spend it. The 10 to 15 minutes spent in class on clicker questions means that much less time to do other things: less time to cover additional material, to present another example on the board, to have students do an example at their seats, to have students do an exploration in groups. I continue to do all these things in my classes (variety is good!) while also using the clickers. Using the clickers just means that sometimes I have to do *less* of some of these things. In some classes, particularly if I have a difficult group exploration planned, I will opt to have no clicker questions at all, or to have only a couple. It is true that deciding to use clickers means that you will have less time to do other things.

As for professor time, there are pluses and minuses to using clickers. I love the automated grading of the questions and really appreciate that benefit. On the other hand, making up the questions does take time. I found that it took less time than I expected (and it only takes a significant amount of time the first time you use them in a course) but the time needed to make up the questions still needs to be a consideration in the decision of whether or not to use this new technology. The concern about time to make up questions is somewhat alleviated by the fact that the number of textbooks offering clicker questions as supplements (such as the ConcepTests for *Calculus* by Hughes-Hallett, et.al.) is growing, as is the number of open-source materials (such as the ones I'm offering here for 200 and 300-level courses.)

8. By far the biggest outcome, and the biggest advantage, in using clickers is that they provide an effective way to keep all the students active, thinking, and engaged throughout class. This is a hard thing to accomplish and I'm happy to have all the help I can get! Much as I would like to believe that my brilliant lectures and engaging interactive class style always has all my students on the edge of their seats, reality teaches me otherwise. The fact that *every student in the class* has to think about these questions is a huge advantage. Anything that has them actively participating instead of me just talking is a bonus.

24.7 Conclusion

I have thoroughly enjoyed using clickers in my classrooms — so much so that I continue to expand the courses in which I use them. I found them effective in calculus and even more effective in my upper-level courses. I do not, however, claim that they are a magic bullet that will suddenly improve all classes. I suspect that, like most pedagogical ideas, they will work well for some people and less well for others. I think that, used effectively, the benefits are well worth the effort.

I believe that clickers will continue to expand in popularity, and are likely to become a part of all of our everyday lives. I have already used our University-owned set of clickers to help with a meeting, anonymously poll people on sensitive issues, take votes at a faculty meeting, etc. The more places we use them, the more we get requests to use them again.

Clickers have made my classes more fun, my students more engaged, and the learning outcomes of my students higher. That works for me!

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