MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

24th Annual

AMC 10 A

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This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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The problems and solutions for this AMC 10 A were prepared by the
MAA AMC 10/12 Editorial Board under the direction of
Carl Yerger and Gary Gordon, co-Editors-in-Chief.

Problem 1:
What is the value of

$$3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}\ ?$$

(A) $\frac{31}{10}$ **(B)** $\frac{49}{15}$ **(C)** $\frac{33}{10}$ **(D)** $\frac{109}{33}$ **(E)** $\frac{15}{4}$

Answer (D): Working from the bottom up shows that

$$3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}=3+\frac{1}{3+\frac{1}{\frac{10}{3}}}=3+\frac{1}{3+\frac{3}{10}}$$
$$=3+\frac{1}{\frac{33}{10}}=3+\frac{10}{33}=\frac{109}{33}.$$

Note: Analogous to the golden ratio, the bronze ratio is the positive solution to the equation $x^2=3x+1$, which equals $\frac{3+\sqrt{3^2+4}}{2}$ and has the continued fraction expansion $3+\frac{1}{3+\frac{1}{3+\cdots}}$. The approximation evaluated in this problem is accurate to three decimal places (3.303). Assuming the continued fraction expansion has a limiting value x_0, it can be seen that $x_0=3+\dfrac{1}{x_0}$, which is equivalent to $x_0^2=3x_0+1$.

The bronze ratio, $\frac{3+\sqrt{3^2+4}}{2}$, is analogous to the golden ratio, $\frac{1+\sqrt{1^2+4}}{2}$. The golden ratio is associated with

$$1+\frac{1}{1+\frac{1}{1+\cdots}}.$$

Similarly, the silver ratio is $\frac{2+\sqrt{2^2+4}}{2}$, associated with

$$2+\frac{1}{2+\frac{1}{2+\cdots}}.$$

These metallic ratios are also related to generalizations of the Fibonacci sequence, arise as lengths of diagonals in regular polygons, and have many other geometric and algebraic interpretations as well.

Problem 2:
Mike cycled 15 laps in 57 minutes. Assume he cycled at a constant speed throughout. Approximately how many laps did he complete in the first 27 minutes?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

Answer (B): Let L denote the number of laps completed in the first 27 minutes. During the first 27 minutes Mike rode at a speed of $\frac{L}{27}$ laps per minute. On the entire ride, Mike's speed was $\frac{15}{57}$ laps per minute. Therefore $\frac{L}{27}=\frac{15}{57}$, so

$$L=\frac{15\cdot 27}{57}=\frac{5\cdot 27}{19}=\frac{135}{19}=7\frac{2}{19}\approx 7.$$

Problem 3:

The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (E): Let the numbers be a, b, and c. Then $a + b + c = 96$, $a = 6c$, and $c = b - 40$. From the first two equations it follows that $b + 7c = 96$. From the third equation it follows that $b + 7(b - 40) = 96$, which gives $8b = 376$. Hence $b = 47$, $c = 7$, $a = 42$, and $|a - b| = 5$.

Problem 4:

In some countries, automobile fuel efficiency is measured in liters per 100 kilometers while other countries use miles per gallon. Suppose that 1 kilometer equals m miles, and 1 gallon equals ℓ liters. Which of the following gives the fuel efficiency in liters per 100 kilometers for a car that gets x miles per gallon?

(A) $\frac{x}{100\ell m}$ **(B)** $\frac{x\ell m}{100}$ **(C)** $\frac{\ell m}{100x}$ **(D)** $\frac{100}{x\ell m}$ **(E)** $\frac{100\ell m}{x}$

Answer (E): Because the car can travel x miles per gallon, it uses $\frac{1}{x}$ gallons per mile. This can be converted to liters per 100 kilometers by multiplying by fractions equivalent to 1, treating units as factors and canceling common factors:

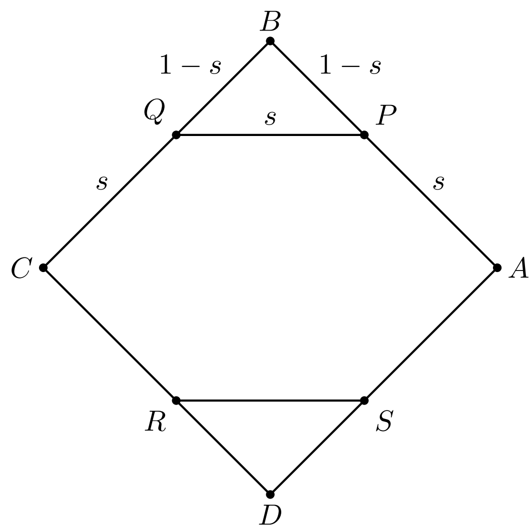
$$\frac{1}{x}\ \frac{\text{gallons}}{\text{miles}} = \frac{1}{x} \cdot \frac{\text{gallons}}{\text{miles}} \cdot \frac{m \text{ miles}}{1 \text{ kilometers}} \cdot \frac{\ell \text{ liters}}{1 \text{ gallons}} \cdot \frac{100}{100} = \frac{100\ell m}{x}\ \frac{\text{liters}}{100 \text{ kilometers}}.$$

Problem 5:
Square $ABCD$ has side length 1. Points P, Q, R, and S each lie on a side of $ABCD$ so that $APQCRS$ is an equilateral convex hexagon with side length s. What is s?

(A) $\frac{\sqrt{2}}{3}$ **(B)** $\frac{1}{2}$ **(C)** $2-\sqrt{2}$ **(D)** $1-\frac{\sqrt{2}}{4}$ **(E)** $\frac{2}{3}$

Answer (C): Without loss of generality, assume that P and Q lie on $\overline{AB}$ and $\overline{BC}$, respectively. Then $AP = PQ = QC = s$ and $PB = BQ = 1-s$, so $PQ = \sqrt{2}(1-s)$. Because $APQCRS$ is equilateral, it follows that $s = \sqrt{2}(1-s)$, from which

$$s = \frac{\sqrt{2}}{1+\sqrt{2}} = 2-\sqrt{2}.$$



Problem 6:
Which expression is equal to $\left| a - 2 - \sqrt{(a-1)^2} \right|$ for $a < 0$?

(A) $3-2a$ **(B)** $1-a$ **(C)** 1 **(D)** $a+1$ **(E)** 3

Answer (A): Because a is negative, $a-1<0$, so $\sqrt{(a-1)^2} = |a-1| = 1-a$. Therefore distributing gives $a-2-(1-a) = 2a-3$. Because a is negative, $2a-3<0$, so $|2a-3| = 3-2a$.

Problem 7:

The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

Answer (B): The positive integer n must be a multiple of 15 and a divisor of 180. There are 6 such integers, as shown in the following table.

n	$\operatorname{lcm}(n,18)$	$\gcd(n,45)$
15	90	15
30	90	15
45	90	45
60	180	15
90	90	45
180	180	45

The only choice that satisfies the conditions is $n = 60$, and the requested sum of digits is 6.

OR

Because $18 = 2 \cdot 3^2$ and $180 = 2^2 \cdot 3^2 \cdot 5$, it follows from $\operatorname{lcm}(n, 18) = 180$ that n is divisible by 2^2 but not 2^3, that n is divisible by 5 but not 5^2, and that n can have no prime factors other than 2, 3, or 5. Because $45 = 3^2 \cdot 5$ and $15 = 3 \cdot 5$, it follows from $\gcd(n, 45) = 15$ that n is divisible by 3 but not 3^2. Therefore $n = 2^2 \cdot 3 \cdot 5 = 60$, and the sum of its digits is 6.

Problem 8:

A data set consists of 6 (not distinct) positive integers: 1, 7, 5, 2, 5, and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all possible values of X?

(A) 10 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 40

Answer (D): First note that the mean of the elements of the data set excluding X is 4. The given condition will therefore be met if $X = 4$, and if $X \neq 4$, then the mean will not be X. Because X is a positive integer, the mean of the full data set must be greater than or equal to $21 \div 6 = 3.5$. The only other possibilities for the mean are therefore 5 and 7. The mean will be 5 if and only if $1+7+5+2+5+X = 6 \cdot 5$, which implies that $X = 30 - 20 = 10$. The mean will be 7 if and only if $1+7+5+2+5+X = 6 \cdot 7$, which implies that $X = 42 - 20 = 22$. The sum of the possible values of X is $4 + 10 + 22 = 36$.

Problem 9:
A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color—red, orange, yellow, blue, or green—so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?

(A) 120 **(B)** 270 **(C)** 360 **(D)** 540 **(E)** 720

Answer (D): There are 5 possible colors for the central rectangle in the lower row. There then are 4 possible colors for the rectangle at the left end of the lower row, then 3 choices for the rectangle at the upper left, then 3 for the rectangle at the upper right, and 3 for the rectangle at the lower right. In all there are $5 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = 540$ possible colorings.

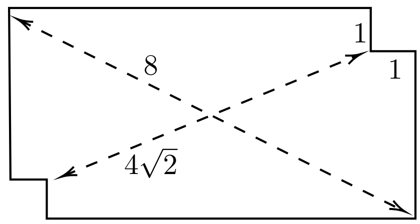
OR

There are $5! = 120$ ways to paint the regions using all 5 colors. If two regions are painted the same color, then those two regions must be either the upper left and lower right regions, or the upper right and lower left regions, or the lower left and lower right regions. Thus there are $3 \cdot (5 \cdot 4 \cdot 3 \cdot 2) = 360$ ways to paint the regions using four colors. The only way to paint the regions using only three colors is to paint the upper left and lower right regions the same color and the upper right and lower left regions the same color. This can be done in $5 \cdot 4 \cdot 3 = 60$ ways. The total is $120 + 360 + 60 = 540$.

Note: This problem can be viewed in terms of chromatic polynomials in graph theory, a concept introduced by George David Birkhoff in 1912.

Problem 10:

Daniel finds a rectangular index card and measures its diagonal to be 8 centimeters. Daniel then cuts out equal squares of side 1 cm at two opposite corners of the index card and measures the distance between the two closest vertices of these squares to be $4\sqrt{2}$ centimeters, as shown below. What is the area of the original index card?



(A) 14 **(B)** $10\sqrt{2}$ **(C)** 16 **(D)** $12\sqrt{2}$ **(E)** 18

Answer (E): Let x and y be the side lengths of the index card. Then the corners of the two cut squares are opposite vertices of a rectangle with sides $x-2$ and $y-2$. The Pythagorean Theorem applied twice yields

$$x^2+y^2=8^2=64 \qquad \text{and} \qquad (x-2)^2+(y-2)^2=\left(4\sqrt{2}\right)^2=32.$$

Expanding the second equation and substituting the first one gives

$$64-4(x+y)+8=32,$$

so $x+y=10$. Finally, the area of the rectangle is xy, which equals

$$\frac{1}{2}\left((x+y)^2-\left(x^2+y^2\right)\right)=\frac{1}{2}(100-64)=18.$$

Note: The original index card uniquely has dimensions $5\pm\sqrt{7}$ cm.

Problem 11:

Ted mistakenly wrote $2^m\cdot\sqrt{\frac{1}{4096}}$ as $2\cdot\sqrt[m]{\frac{1}{4096}}$. What is the sum of all real numbers m for which these two expressions have the same value?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (C): Notice that $4096=2^{12}$, so the first expression is

$$2^m\cdot\left(2^{-12}\right)^{\frac{1}{2}}=2^{m-6},$$

and the second expression is

$$2\cdot\left(2^{-12}\right)^{\frac{1}{m}}=2^{1-\frac{12}{m}}.$$

These two expressions have the same value if and only if $m-6=1-\frac{12}{m}$. Multiplying both sides of this equation by m, rearranging terms, and factoring gives $(m-3)(m-4)=0$. The solutions of this equation are $m=3$ and $m=4$. The requested sum of possible values for m is $3+4=7$.

Problem 12:

On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

(A) 7 **(B)** 12 **(C)** 21 **(D)** 27 **(E)** 31

Answer (A): Let T denote the number of truth-tellers, A_{o} the number of alternators who tell the truth on odd-numbered questions, A_{e} the number of alternators who tell the truth on even-numbered questions, and L the number of liars. These four variables must satisfy the following equations.

Total:	$31 = T + L + A_{\mathrm{e}} + A_{\mathrm{o}}$
Question 1:	$22 = T + L + A_{\mathrm{e}}$
Question 2:	$15 = L + A_{\mathrm{e}}$
Question 3:	$9 = A_{\mathrm{e}}$

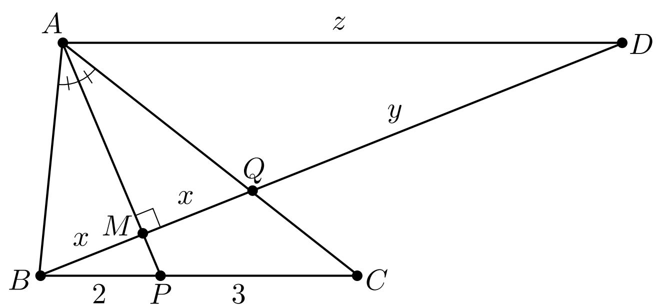
Subtract the third equation from the second to find that $T = 7$. Because they received candy only in response to the first question, they received 7 pieces of candy.

Note: It may be further deduced from the system of equations that $A_{\mathrm{e}} = 9$, $L = 6$, and $A_{\mathrm{o}} = 9$.

Problem 13:
Let $\triangle ABC$ be a scalene triangle. Point P lies on $\overline{BC}$ so that $\overline{AP}$ bisects $\angle BAC$. The line through B perpendicular to $\overline{AP}$ intersects the line through A parallel to $\overline{BC}$ at point D. Suppose $BP = 2$ and $PC = 3$. What is AD?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Answer (C): Let Q be the intersection point of $\overline{AC}$ and $\overline{BD}$, as shown.



Because $\overline{BQ} \perp \overline{AP}$ and $\angle BAP = \angle QAP$, it follows that $\triangle ABQ$ is isosceles with $AB = AQ$. Then by the Angle Bisector Theorem,

$$\frac{3}{2} = \frac{PC}{PB} = \frac{AC}{AB} = \frac{AC}{AQ} = 1 + \frac{CQ}{AQ} = 1 + \frac{BC}{AD} = 1 + \frac{5}{AD}.$$

Solving this equation yields $AD = 10$.

OR

Let M be the intersection point of $\overline{AP}$ and $\overline{BQ}$. As above, $\triangle ABQ$ is isosceles with $AB = AQ$, so $BM = MQ$. Let $BM = MQ = x$, $QD = y$, and $AD = z$. Because $\triangle AMD$ and $\triangle PMB$ are similar, it follows that

$$\frac{z}{2} = \frac{x+y}{x} = 1 + \frac{y}{x}.$$

Further, $\triangle AQD$ and $\triangle CQB$ are also similar, so $\frac{z}{5} = \frac{y}{2x}$. It follows that

$$\frac{z}{2} - 1 = \frac{y}{x} = \frac{2z}{5},$$

which yields $z = 10$.

Problem 14:
How many ways are there to split the integers 1 through 14 into 7 pairs so that in each pair the greater number is at least 2 times the lesser number?

(A) 108 **(B)** 120 **(C)** 126 **(D)** 132 **(E)** 144

Answer (E): If the lesser number in some pair is at least 8, then the greater number has to be at least $8 \cdot 2 = 16$, which is impossible. This means that the lesser number in each pair must be between 1 and 7, inclusive. Because there are exactly 7 pairs in the collection, it follows that the lesser numbers must be exactly the numbers 1 through 7.

The 7 must be paired with the 14. There are then 2 choices for the partner of 6—being paired with either 12 or 13. Now there are 3 choices for the partner of 5, because exactly 2 of the numbers in $\{10, 11, 12, 13, 14\}$ have already been picked. Continuing in this fashion, there are 4 possible partners for the 4, then 3 possible partners for the 3, then 2 possible partners for the 2, and finally 1 possible partner for the 1. Multiplying these numbers of choices together gives a final answer of $1 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 144$.

Problem 15:

Quadrilateral $ABCD$ with side lengths $AB = 7$, $BC = 24$, $CD = 20$, and $DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where a, b, and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?

(A) 260 **(B)** 855 **(C)** 1235 **(D)** 1565 **(E)** 1997

Answer (D): Observe that

$$\sqrt{7^2 + 24^2} = \sqrt{20^2 + 15^2} = 25.$$

If $AC < 25$, then $\angle ABC$ and $\angle ADC$ are both acute, so $ABCD$ cannot be cyclic. Analogously, if $AC > 25$, then $\angle ABC$ and $\angle ADC$ are both obtuse, and again $ABCD$ cannot be cyclic. Therefore $\triangle ABC$ and $\triangle CDA$ are both right triangles with hypotenuse 25.

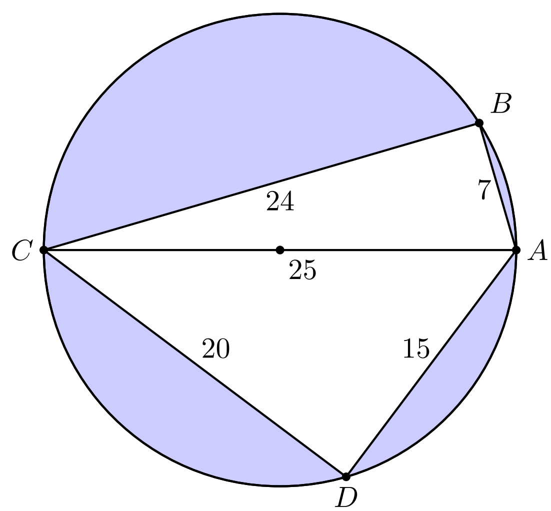
The area of $ABCD$ is $\frac{1}{2}(7 \cdot 24 + 15 \cdot 20) = 234$. Because $\angle ABC$ and $\angle ADC$ are right angles, $\overline{AC}$ is the diameter of the circumcircle, so the circumcircle has radius $\frac{25}{2}$ and its area is

$$\pi\left(\frac{25}{2}\right)^2 = \frac{625\pi}{4}.$$

Hence the required area is

$$\frac{625\pi}{4} - 234 = \frac{625\pi - 936}{4},$$

and the requested sum is $625 + 936 + 4 = 1565$.



Problem 16:

The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

Answer (D): Let $f(x) = 10x^3 - 39x^2 + 29x - 6$ and let a, b, and c be the roots of $f(x)$. Hence $f(x) = 10(x-a)(x-b)(x-c)$. Then the volume of the new box is

$$(2+a)(2+b)(2+c) = -(-2-a)(-2-b)(-2-c) = -\frac{f(-2)}{10} = -\frac{1}{10}(-80-156-58-6) = 30.$$

OR

Dividing by 10 gives the polynomial

$$x^3 - \frac{39}{10}x^2 + \frac{29}{10}x - \frac{3}{5},$$

whose roots are the same. If a, b, and c are the roots, then Vieta's Formulas give

$$abc = \frac{3}{5}, \qquad ab + bc + ac = \frac{29}{10}, \qquad \text{and} \qquad a + b + c = \frac{39}{10}.$$

The volume of the new box is

$$(a+2)(b+2)(c+2) = abc + 2(ab+bc+ac) + 4(a+b+c) + 8.$$

Substituting gives

$$\frac{3}{5} + 2 \cdot \frac{29}{10} + 4 \cdot \frac{39}{10} + 8 = 30.$$

Note: The solutions presented here did not require finding the roots. In fact, the roots of $f(x)$ are 3, $\frac{1}{2}$, and $\frac{2}{5}$. The dimensions of the new box are 5, $\frac{5}{2}$, and $\frac{12}{5}$, which gives a volume of $5 \cdot \frac{5}{2} \cdot \frac{12}{5} = 30$.

Problem 17:

How many three-digit positive integers $\underline{a}\,\underline{b}\,\underline{c}$ are there whose nonzero digits a, b, and c satisfy

$$0.\overline{\underline{a}\,\underline{b}\,\underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})\,?$$

(The bar indicates digit repetition; thus $0.\overline{\underline{a}\,\underline{b}\,\underline{c}}$ is the infinite repeating decimal $0.\underline{a}\,\underline{b}\,\underline{c}\,\underline{a}\,\underline{b}\,\underline{c}\,\ldots$.)

(A) 9 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 14

Answer (D): The given equation means

$$\frac{100a + 10b + c}{999} = \frac{1}{3}\left(\frac{a}{9} + \frac{b}{9} + \frac{c}{9}\right),$$

which simplifies to $7a = 3b + 4c$. Therefore $3b \equiv -4c \equiv 3c \pmod 7$, so $b \equiv c \pmod 7$. Given that the variables are nonzero digits, the possibilities for (b, c) are $(1, 1)$, $(2, 2)$, $(3, 3)$, $\ldots$, $(9, 9)$, $(1, 8)$, $(8, 1)$, $(2, 9)$, and $(9, 2)$. In each case the value of a is uniquely determined, and the 13 positive integers are 111, 222, 333, $\ldots$, 999, 518, 481, 629, and 592.

Problem 18:

Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \ldots, T_n$ returns the point $(1, 0)$ back to itself?

(A) 359 **(B)** 360 **(C)** 719 **(D)** 720 **(E)** 721

Answer (A): Denote by O the origin of the coordinate plane. Let P_k be the point obtained after the first k transformations, where $P_0 = (1, 0)$. Observe that, for each integer i and for every point P, the transformation $T_i(P)$ preserves the distance from P to the origin. Thus $OP_k = 1$ for every k. This means that the point P_k may be uniquely characterized by the counterclockwise angle θ_k between the x-axis and the ray $\overrightarrow{OP_k}$. (Here θ_k is measured in degrees.)

The transformation T_j rotates the plane j degrees counterclockwise around the origin and then reflects the plane across the y-axis. This means that $\theta_{k+1} = 180 - (\theta_k + k + 1)$ and

$$\begin{aligned}\theta_{k+2} &= 180 - (\theta_{k+1} + (k+2)) \\ &= 180 - \big(180 - (\theta_k + k + 1) + (k+2)\big) \\ &= \theta_k - 1.\end{aligned}$$

Note that $\theta_0 = 0$ and $\theta_1 = 179$. It follows that $\theta_{2m} = -m$ and $\theta_{2m+1} = 179 - m$ for every positive integer m. Therefore $(1, 0)$ first returns to its starting position when $m = 179$ and $n = 2 \cdot 179 + 1 = 359$.

Problem 19:

Let L_n denote the least common multiple of the numbers $1, 2, 3, \ldots, n$, and let h be the unique positive integer such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{17} = \frac{h}{L_{17}}.$$

What is the remainder when h is divided by 17?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Answer (C): Note that

$$1 + \frac{1}{2} + \cdots + \frac{1}{16} = \frac{m}{L_{16}}$$

for some integer m, so the given sum is

$$\frac{m}{L_{16}} + \frac{1}{17} = \frac{h}{L_{17}}.$$

Now $L_{17} = 17L_{16}$, so it follows that $h = 17m + L_{16}$. Therefore $h \equiv L_{16} \pmod{17}$. Note that $L_{16} = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. These factors modulo 17 satisfy $2^4 \equiv -1$, $5 \cdot 7 \equiv 1$, $9 \cdot 11 \equiv -3$, and $13 \equiv -4$, so the product is congruent to $(-1)(1)(-3)(-4) = -12 \equiv 5 \pmod{17}$.

Problem 20:

A four-term sequence is formed by adding each term of a four-term arithmetic sequence of positive integers to the corresponding term of a four-term geometric sequence of positive integers. The first three terms of the resulting four-term sequence are 57, 60, and 91. What is the fourth term of this sequence?

(A) 190 **(B)** 194 **(C)** 198 **(D)** 202 **(E)** 206

Answer (E): Let the terms of the arithmetic sequence be a_0, a_1, a_2, a_3, and let the terms of the geometric sequence be g_0, g_1, g_2, g_3. There must be constants b, d, c, and r such that $a_n = b + d \cdot n$ and $g_n = c \cdot r^n$ for $n = 0, 1, 2, 3$. Let the terms of the sum sequence be $x_n = a_n + g_n$. It is given that $x_0 = 57$, $x_1 = 60$, and $x_2 = 91$. With the goal of eliminating two of the constants, consider $x_2 - 2x_1 + x_0$. Then

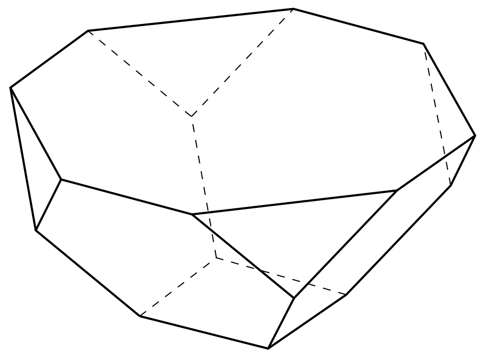
$$28 = 91 - 2 \cdot 60 + 57 = (b + 2d + cr^2) - 2(b + d + cr) + (b + c) = c(r-1)^2.$$

Because g_1 and g_2 are integers, r must be a rational number, say $\frac{p}{q}$ for some relatively prime positive integers p and q. Then the equation above becomes $c(p-q)^2 = 28q^2$. Because $\gcd(p-q, q) = \gcd(p, q) = 1$, either $(p-q)^2 = 4$ or $(p-q)^2 = 1$.

If $(p-q)^2 = 1$, then $c = 28q^2$. Because $c < 57$, it must be that $q = 1$ and $c = 28$. Then $p - q = 1$ (because $p \neq 0$), so $p = 2$ and $r = 2$. It follows that $b = x_0 - c = 57 - 28 = 29$, $d = a_1 - b = (60 - 2 \cdot 28) - 29 = -25$, and $a_2 = 29 + 2 \cdot (-25) < 0$, violating the conditions of the problem.

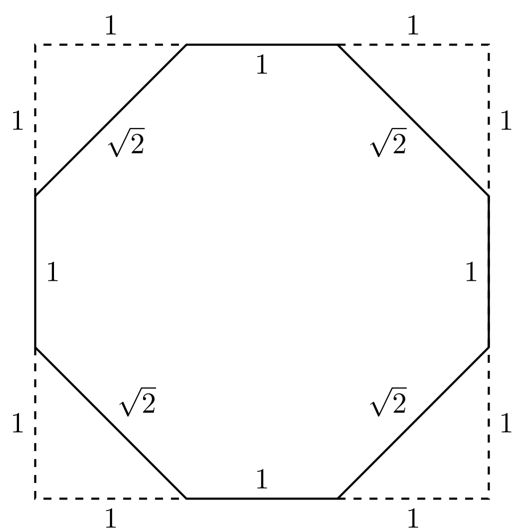
Therefore $(p-q)^2 = 4$ and $c = 7q^2$. Because $p - q$ is even, q must be odd and the only choice that makes $c < 57$ is $q = 1$. Then $c = 7$, $p = 3$, and $r = 3$. It follows that the geometric sequence is $7, 21, 63, 189$. Therefore $a_0 = 57 - 7 = 50$ and $a_1 = 60 - 21 = 39$, giving $d = -11$. Therefore $a_3 = 50 + 3(-11) = 17$ and $x_3 = 17 + 189 = 206$.

Problem 21:
A bowl is formed by attaching four regular hexagons of side 1 to a square of side 1. The edges of adjacent hexagons coincide, as shown in the figure. What is the area of the octagon obtained by joining the top eight vertices of the four hexagons, situated on the rim of the bowl?



(A) 6 **(B)** 7 **(C)** $5+2\sqrt{2}$ **(D)** 8 **(E)** 9

Answer (B): Adjacent hexagons with side length 1 are folded up so that a square with side length 1—the base of the bowl—exactly fits in the gap between them. By symmetry, a square of side length 1 must also fit in the other gap formed by those two hexagons. It follows that the distance between the closest vertices on adjacent hexagons on the top of the bowl are $\sqrt{2}$ units apart—the diagonal of a unit square. Therefore the required octagon is as shown below.



Its area is the area of the 3×3 square minus the areas of the four isosceles right triangular corners, namely

$$3 \cdot 3 - 4 \cdot \frac{1}{2} \cdot 1 \cdot 1 = 7.$$

Problem 22:

Suppose that 13 cards numbered $1, 2, 3, \ldots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass.

7	**11**	**8**	**6**	**4**	**5**	**9**	**12**	**1**	**13**	**10**	**2**	**3**

For how many of the 13! possible orderings of the cards will the 13 cards be picked up in exactly two passes?

(A) 4082 **(B)** 4095 **(C)** 4096 **(D)** 8178 **(E)** 8191

Answer (D): Choose a subset S of $\{1, 2, 3, \ldots, 13\}$, and place the cards numbered $1, 2, 3, \ldots, |S|$ in increasing order in the spots determined by S. Then place the remaining cards in increasing order in the remaining (complementary) spots. The resulting arrangement of cards will be picked up in at most two passes, and any other arrangement will require more than two passes. There are 2^{13} choices for the subset S, but any subset of the form $S = \{1, 2, 3, \ldots, k\}$ will result in the cards being picked up in one pass. There are 14 such "initial block" subsets ($\emptyset, \{1\}, \{1, 2\}, \ldots, \{1, 2, 3, \ldots, 13\}$) that result in picking up the cards in one pass. This gives $2^{13} - 14 = 8178$ possible orderings requiring exactly two passes.

OR

For each integer $1 \le m \le 13$, let A_m be the number of ways to arrange the 13 cards so that cards 1 through m are selected in the first pass with 1 or more cards left for a second pass. Observe that there are $\binom{13}{m}$ locations at which these cards can appear in the sequence, and within these locations the cards must appear in increasing order. The remaining $13 - m$ cards will occupy the remaining slots; because these cards will be selected in the second pass, they must appear in increasing order. As long as the first m cards do not occupy the first m spots in the sequence, two passes will be required to pick up all the cards. Thus $A_m = \binom{13}{m} - 1$. The total number of ways over all values of m is

$$A_1 + \cdots + A_{13} = \binom{13}{1} + \cdots + \binom{13}{13} - 13 = \left(2^{13} - \binom{13}{0}\right) - 13 = 2^{13} - 14 = 8178.$$

OR

For $i = 1, 2, 3, \ldots, 13$ let p_i denote the position of the number i in the list. Thus in the provided example, $p_1 = 9$, $p_2 = 12$, $p_3 = 13$, and so on. If $p_j < p_{j+1} < \cdots < p_k$, then the cards numbered $j, j+1, \ldots, k$ are picked up on the same pass. However, if $p_k > p_{k+1}$, then that pass is complete, and a new pass is required to pick up the card numbered $k + 1$. The sequence of numbers p_i is said to have a *drop* at k when $p_k > p_{k+1}$. The total number of passes required to pick up all 13 cards is one more than the number of drops in the sequence of numbers p_i.

Let D_n denote the number of permutations of the numbers $1, 2, 3, \ldots, n$ that have exactly one drop. If π_1 is a permutation of the numbers $1, 2, 3, \ldots, n-1$ with exactly one drop, then a permutation π_2 of the numbers $1, 2, 3, \ldots, n$ with exactly one drop can be created either by inserting n at the position of the drop, or by placing n at the righthand end of the sequence that defines π_1. Thus a permutation of the type π_1 generates two permutations of the type π_2. Now let π_3 be the permutation $1, 2, 3, \ldots, n-1$ with no drop. A permutation π_4 on $1, 2, 3, \ldots, n$ with exactly one drop can be created by inserting n either before 1, or between 1 and 2, or ..., or between $n-2$ and $n-1$, which is a total of $n-1$ places. Thus

$$D_n = 2D_{n-1} + n - 1.$$

Because $D_1 = 0$ and $D_2 = 1$, it follows by induction that $D_n = 2^n - n - 1$, and hence the answer is $D_{13} = 8178$.

Note: Let $\left\langle {n \atop k} \right\rangle$ denote the number of permutations of $1, 2, 3, \ldots, n$ that can be picked up in exactly $k+1$ passes. By generalizing the reasoning found above, it can be shown that these numbers are generated by the Pascal-like recurrence

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle.$$

These are the *Eulerian numbers.* Like the binomial coefficients, they have a symmetry:

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle.$$

Because these numbers count permutations in various classes,

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = n!.$$

However, unlike binomial coefficients, for which there exists a closed-form formula in terms of factorials, there seems to be no simple closed-form formula for the Eulerian numbers, although

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n.$$

The special case $k = 1$ is what was derived.

Problem 23:

Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1$, $PB = 2$, $PC = 3$, and $PD = 4$. What is $\frac{BC}{AD}$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

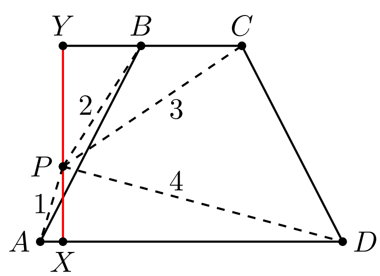
Answer (B): Let the trapezoid have vertices $A(-a,0)$, $B(-b,c)$, $C(b,c)$, and $D(a,0)$, where $a > 0$, $b > 0$, and $c > 0$. Let P have coordinates (p,q) for real numbers p and q. The given conditions imply

$$\begin{aligned}(p+a)^2 + q^2 &= 1,\\ (p+b)^2 + (c-q)^2 &= 4,\\ (p-b)^2 + (c-q)^2 &= 9, \quad \text{and}\\ (p-a)^2 + q^2 &= 16.\end{aligned}$$

Subtracting the fourth equation from the first gives $pa = \frac{-15}{4}$, and subtracting the third equation from the second gives $pb = -\frac{5}{4}$. Hence $\frac{BC}{AD} = \frac{b}{a} = \frac{1}{3}$.

OR

Let X and Y be feet of the perpendiculars from P to (parallel) lines AD and BC respectively. (In the diagram below, point Y lies to the left of segment $\overline{BC}$ and point X lies on segment $\overline{AD}$. The solution generalizes to other configurations without issue through the use of directed lengths.)



By the Pythagorean Theorem $AP^2 - AX^2 = XP^2 = DP^2 - DX^2$. It follows that $DX^2 - AX^2 = DP^2 - AP^2 = 4^2 - 1^2 = 15$, so

$$AD \cdot (DX - AX) = (DX + AX)(DX - AX) = DX^2 - AX^2 = DP^2 - AP^2 = 15.$$

Likewise, $BP^2 - BY^2 = YP^2 = CP^2 - CY^2$, so $CY^2 - BY^2 = CP^2 - BP^2 = 3^2 - 2^2 = 5$ and

$$BC \cdot (CY + BY) = (CY - BY)(CY + BY) = CY^2 - BY^2 = 5.$$

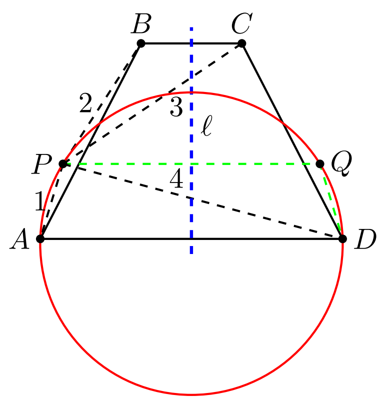
By symmetry, $AX + BY = DX - CY$, so $CY + BY = DX - AX$. Therefore

$$\frac{BC}{AD} = \frac{BC \cdot (CY + BY)}{AD \cdot (DX - AX)} = \frac{1}{3}.$$

OR

Let ℓ denote the common perpendicular bisector of $\overline{AD}$ and $\overline{BC}$, and let Q denote the reflection of P across ℓ. By symmetry, $AP = DQ = 1$, $AQ = DP = 4$, $BP = CQ = 2$, and $BQ = CP = 3$. Furthermore, $APQD$ is an isosceles trapezoid and is therefore a cyclic quadrilateral. By Ptolemy's Theorem $AD \cdot PQ + AP \cdot DQ = AQ \cdot PD$, so $AD \cdot PQ = 4^2 - 1^2 = 15$. Likewise, $BC \cdot PQ = 3^2 - 2^2 = 5$. Therefore

$$\frac{BC}{AD} = \frac{BC \cdot PQ}{AD \cdot PQ} = \frac{1}{3}.$$



Note: Such trapezoids do in fact exist. For example, if $a = 2$ and $b = \frac{2}{3}$ in the first solution, then $(p, q) = \left(-\frac{15}{8}, \frac{3\sqrt{7}}{8}\right)$ and $c = \frac{1}{24}\left(9\sqrt{7} + \sqrt{1463}\right)$. A degenerate example has vertices at $A(0, 0)$, $B(1, 0)$, $C(2, 0)$, and $D(3, 0)$, with P at $(-1, 0)$.

Problem 24:

How many strings of length 5 formed from the digits $0, 1, 2, 3, 4$ are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies this condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

(A) 500 **(B)** 625 **(C)** 1089 **(D)** 1199 **(E)** 1296

Answer (E): From the definition, any permutation of an acceptable string (a string satisfying the conditions of the problem) is acceptable. The following lists show the acceptable strings with the digits listed in nondecreasing order, based on the pattern of repeated digits. For example, pattern $(3, 1, 1)$ means that one digit occurs 3 times and each of two other digits appears once.

- Pattern (5): 00000 [1 string]
- Pattern $(4, 1)$: 00001, 00002, 00003, 00004, 01111 [5 strings]
- Pattern $(3, 2)$: 00011, 00022, 00033, 00111, 00222 [5 strings]
- Pattern $(3, 1, 1)$: 00012, 00013, 00014, 00023, 00024, 00034, 01112, 01113, 01114, 01222 [10 strings]
- Pattern $(2, 2, 1)$: 00112, 00113, 00114, 00122, 00133, 00223, 00224, 00233, 01122, 01133 [10 strings]
- Pattern $(2, 1, 1, 1)$: 00123, 00124, 00134, 00234, 01123, 01124, 01134, 01223, 01224, 01233 [10 strings]
- Pattern $(1, 1, 1, 1, 1)$: 01234 [1 string]

The number of different permutations of numbers in each list can be computed with multinomial coefficients.

$$\binom{5}{5} = \frac{5!}{5!} = 1$$
$$\binom{5}{4,1} = \frac{5!}{4!\,1!} = 5$$
$$\binom{5}{3,2} = \frac{5!}{3!\,2!} = 10$$
$$\binom{5}{3,1,1} = \frac{5!}{3!\,1!\,1!} = 20$$
$$\binom{5}{2,2,1} = \frac{5!}{2!\,2!\,1!} = 30$$
$$\binom{5}{2,1,1,1} = \frac{5!}{2!\,1!\,1!\,1!} = 60$$
$$\binom{5}{1,1,1,1,1} = \frac{5!}{1!\,1!\,1!\,1!\,1!} = 120$$

Finally, the number of strings with a given pattern that meet the conditions is obtained by multiplying the number of instances of that pattern by the number of different permutations for that pattern. Thus there are

$$1 \cdot 1 + 5 \cdot 5 + 5 \cdot 10 + 10 \cdot 20 + 10 \cdot 30 + 10 \cdot 60 + 1 \cdot 120 = 1296$$

strings in all.

OR

Replace 5 with a general value of b, and think of the problem as asking for the number of base-b nonnegative integers, with leading zeros allowed, that meet the given condition—that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j. Temporarily extend the numbers to base $b+1$, and consider all $(b+1)^b$ b-digit numbers. These numbers can be divided into $\frac{(b+1)^b}{b+1}$ teams of size $b+1$ such that every set of b-digit numbers of the form

$$\{\underline{a_1} \ldots \underline{a_b}, \underline{a_1 + 1} \ldots \underline{a_b + 1}, \ldots, \underline{a_1 + b} \ldots \underline{a_b + b}\}$$

forms a team, where addition of digits is done modulo $(b+1)$. Imagine testing whether each team member meets the stated condition as follows. Starting with the first digit and continuing through the bth digit, place the digits on a $(b+1)$-hour clock face so that digit i goes on hour i if hour i is empty. Otherwise digit i goes on the next empty hour clockwise after i. All b digits fit, and one hour will be left empty. The teams are constructed so that the empty hour for each team member is one greater, modulo $b+1$, than that of its predecessor, so exactly one team member will leave hour b empty. The team member that leaves hour b empty meets the stated condition because in order not to use hour b it must have at least i digits less than i for each $i < b$, and it must not contain digit b. Team members that fill hour b do so because for some i they fail to meet the stated condition. They fail to have i digits less than i, that is, they have $b - i + 1$ digits greater than or equal to i. These digits fill hours i through b.

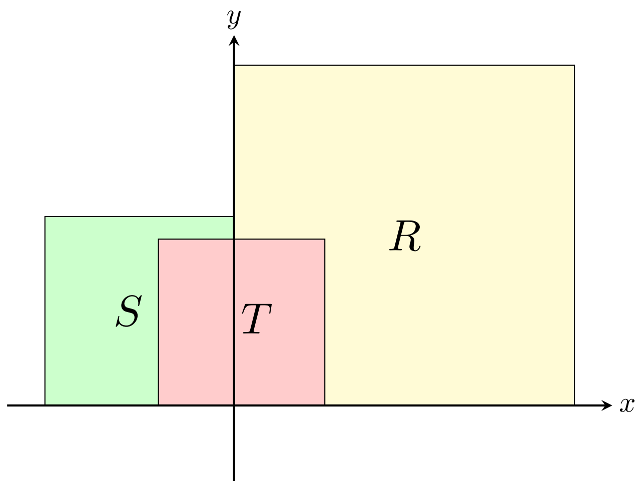
Each team has exactly one number that meets the stated condition, so the number of such numbers is

$$\frac{(b+1)^b}{b+1} = (b+1)^{b-1}.$$

For $b = 5$ the number is $6^4 = 1296$.

Note: This problem is related to the subject of parking functions. See sequence A000272 in the On-Line Encyclopedia of Integer Sequences. The idea of the second solution is attributed to Henry O. Pollak.

Problem 25:
Let R, S, and T be squares that have vertices at lattice points (i.e., points whose coordinates are both integers) in the coordinate plane, together with their interiors. The bottom edge of each square is on the x-axis. The left edge of R and the right edge of S are on the y-axis, and R contains $\frac{9}{4}$ as many lattice points as does S. The top two vertices of T are in $R \cup S$, and T contains $\frac{1}{4}$ of the lattice points contained in $R \cup S$. See the figure (not drawn to scale).



The fraction of lattice points in S that are in $S \cap T$ is 27 times the fraction of lattice points in R that are in $R \cap T$. What is the minimum possible value of the edge length of R plus the edge length of S plus the edge length of T?

(A) 336 **(B)** 337 **(C)** 338 **(D)** 339 **(E)** 340

Answer (B): Let $a-1$, $b-1$, and $c-1$ be the respective edge lengths of R, S, and T, and let A, B, and C be the respective sets of lattice points in R, S, and T. Then there is an integer k such that $a = 3k$ and $b = 2k$, and $|A| = a^2 = 9k^2$, $|B| = b^2 = 4k^2$, and $|C| = c^2$. Because $A \cap B$ contains b lattice points on the y-axis, $|A \cup B| = a^2 + b^2 - b = k(13k - 2)$. Thus $4c^2 = k(13k - 2)$, so k is even. Furthermore, if k is a multiple of 4, then the greatest power of 2 that divides the right side is odd, so $k \equiv 2 \pmod 4$.

Note that

$$\frac{|B \cap C|}{4k^2} = \frac{|B \cap C|}{|B|} = 27 \cdot \frac{|A \cap C|}{|A|} = 27 \cdot \frac{|A \cap C|}{9k^2},$$

implying that $|B \cap C| = 12 \cdot |A \cap C|$. If the lower left vertex of T is $(-j, 0)$, then the lower right vertex is $(c - 1 - j, 0)$, and

$$12 = \frac{|B \cap C|}{|A \cap C|} = \frac{j+1}{c-j}.$$

Therefore $j+1 = 12(c-j)$, from which $c \equiv -1 \pmod{13}$, and $k \equiv -2 \pmod{13}$. Hence $k = 52n-2$ for some positive integer n. The least possible value of k is 50, giving $(a, b, c) = (150, 100, 90)$. The squares in this case have side lengths 149, 99, and 89, and the requested sum is $149+99+89 = 337$. (Thus it turns out that R contains 22500 lattice points, S contains 10000 lattice points, T contains 8100 lattice points, 630 lattice points of T are contained in R, and 7560 lattice points of T are contained in S.)

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