Generalization. Suppose we have a game in which a player choosing k of n numbers purchases r tickets, selecting numbers at random. For $i = 1$ to r, let A_i be the event "ith ticket matches j out of n numbers." Let the approximate probability $P(A_1 \cup A_2 \cup \cdots \cup A_r) \approx \sum_{i=1}^{r} P(A_i)$ be expressed as $s : 1$, and let the exact probability for $P(A_1 \cup A_2 \cup \cdots \cup A_r)$ using inclusion/exclusion be expressed as $t : 1$, where s and t are rounded to the nearest integer. Then $t - s = 0$ or $t - s = 1$.

The Chain Rule for Matrix Exponential Functions

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This short note serves as an extension of Liu's note [**4**]. The problem is to determine the extent to which the chain rule for scalar exponential functions (i.e., $(\exp(f(t)))' = \exp(f(t))f'(t))$ extends to the context of matrix exponential functions.

If A is an $n \times n$ matrix, it is well known ([**2**], [**3**]) that the series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

(I denoting the $n \times n$ identity matrix) converges to an $n \times n$ matrix denoted by $\exp(A)$. One can then prove (see [**3**]) that

$$\exp(tA)' = A\exp(tA) = \exp(tA)A. \tag{1}$$

(All derivatives will be with respect to a real parameter t.) The question is whether the chain rule (1) extends to more general matrix exponential functions than just $\exp(tA)$. That is, if $B = B(t)$ is an $n \times n$ matrix of differentiable functions, is it true that

$$\exp(B)' = B'\exp(B) = \exp(B)B'?$$

Equation (1) says that the answer is 'yes' if B has the form $B = tA$, where A is a matrix of constants.

In general the answer is 'no.' Liu provided a counter-example in [**4**]. A more conceptual explanation is that matrix exponential manipulations do not work as in the scalar case unless the matrices involved commute. Such is the situation with the chain rule problem here.

Exercise 1. For any fixed value of θ, set

$$A = \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & -\cos\theta \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}.$$

Show that $A^3 = -A$, and that, for any value of t, $\exp(tA) = I + (\sin t)A + (1 - \cos t)A^2$.

Exercise 2. If A_1 and A_2 are $n \times n$ matrices, then $(A_1 + A_2)^2 = A_1^2 + 2A_1A_2 + A_2^2$ if and only if A_1 and A_2 commute.

Exercise 3. If A_1 and A_2 are $n \times n$ matrices that commute, then $\exp(A_1 + A_2) = \exp(A_1)\exp(A_2) = \exp(A_2)\exp(A_1)$.

Exercise 4. Show that the converse of Exercise 3 does not hold, as follows. Let

$$A_1 = 2\pi \begin{pmatrix} 0 & 0 & \sqrt{3}/2 \\ 0 & 0 & -1/2 \\ -\sqrt{3}/2 & 1/2 & 0 \end{pmatrix}, \qquad A_2 = 2\pi \begin{pmatrix} 0 & 0 & -\sqrt{3}/2 \\ 0 & 0 & -1/2 \\ \sqrt{3}/2 & 1/2 & 0 \end{pmatrix}.$$

Show that A_1 and A_2 do not commute, and use Exercise 1 to show that $\exp(A_1) = \exp(A_2) = \exp(A_1 + A_2) = I$.

I am grateful to Steve Mackey for the example in Exercise 4, and I refer the reader to [**1**, §1.2, §1.4] for background details.

In general, under suitable hypotheses, $\exp(A_1)\exp(A_2) = \exp(Z)$, for some matrix Z. The matrix Z can be expressed as a series $Z = A_1 + A_2 + \cdots$, where the additional terms involve iterated brackets (commutators) of A_1 and A_2, i.e., iterated expressions of the form $[A_1, A_2] = A_1A_2 - A_2A_1$. See [**1**, § 1.6, § 1.7] for more details.

Exercise 5. If $B_1 = B_1(t)$ and $B_2 = B_2(t)$ are $n \times n$ matrices of differentiable functions, then $(B_1B_2)' = B_1'B_2 + B_1B_2'$. In particular, $(B^2)' = B'B + BB'$.

Theorem. *If $B = B(t)$ is an $n \times n$ matrix of differentiable functions, then*

$$\exp(B)' = B'\exp(B) = \exp(B)B'$$

if and only if B and B' commute.

Proof. The 'if' part is an exercise that makes use of Exercise 5. The 'only if' part is also an exercise: observe that $\exp(B)B = B\exp(B)$, and differentiate both sides. Then notice that $\exp(B)$ is invertible, because its inverse is given by $\exp(-B)$ (using Exercise 3) . ■

Liu's counter-example in [**4**] has

$$B = \begin{pmatrix} t^2/2 & t \\ 0 & 0 \end{pmatrix}, \qquad B' = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}.$$

These matrices do not commute.

For an application to linear differential equations, suppose that $A = A(t)$ is an $n \times n$ matrix of integrable functions, and set

$$B = B(t) = \int_{t_0}^{t} A(s)\,ds. \tag{2}$$

Then $B' = A$.

Exercise 6. In the notation of (2), if A and B commute, show that

$$Y(t) = \exp\big(B(t)\big)Y_0$$

solves the initial value problem: $Y' = AY$, $Y(t_0) = Y_0$. In particular, show that this situation holds if A is a constant matrix, in which case $B = (t - t_0)A$.

Exercise 7. Suppose that $t_0 = 0$ and that

$$A(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Compute $B(t) = \int_0^t A(s)\,ds$, and show that A and B commute.

Exercise 8. Suppose A is the coefficient matrix of the companion equation $Y' = AY$ associated with the nth order differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0.$$

That is,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{pmatrix}.$$

Compute $B(t) = \int_0^t A(s)\,ds$, and show that A and B commute if and only if all the coefficient functions $p_i(t)$, $i = 1, 2, \ldots, n$, are constants.

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Extending Theon's Ladder to Any Square Root

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Introduction. Little is known of the life of Theon of Smyrna (circa 140 AD). At this time in the history of mathematics, there was a tendency to de-emphasize demonstrative and deductive methods in favor of practical mathematics. An excellent example of this is known as Theon's ladder, which describes a remarkably simple way to calculate rational approximations to $\sqrt{2}$. (See [**2**], [**3**], and [**5**].)

1	1
2	3
5	7
12	17
29	41
⋮	⋮