Geometry Playground Activity
Projective Geometry

If we use the half-sphere model, projective geometry looks quite similar to spherical geometry. When we use the plane model, projective geometry looks quite similar to Euclidean geometry. How is it different from each of these? How and why would mathematicians use projective geometry? Let's begin by starting Geometry Playground and choosing the projective geometry tab.

1. Using the **construct points** and **construct segments** tools, construct a triangle. Measure the angles of your triangle using the **measure angles** tool. Do you notice anything strange? Have you discovered at least one way that projective geometry is different from Euclidean geometry? Using the **transform** tool (in the manipulate menu), move one or more of the points of your triangle around. What is the largest sum of angle measures you can find? The smallest?

2. Clear the geometry using the **New** tool (in the file menu). Then using the **construct points** and **segments** tools, construct a segment. Measure its length using the **measure distance** tool. What is the largest distance you can create? Note that no units appear on the distance. What should the units be, or does it matter?

3. Without clearing your previous construction, use the **construct circles** tool to construct a circle with one of the segment's endpoints as its center and the other as a point on the circle. Measure the area of your circle. Again, no units appear. What should the units be, or do they not matter? Is $A = \pi r^2$? Can you estimate the area of the entire projective space? Explain your answer.

4. Clear the geometry using the **New** tool. Using the **construct points**, **segments**, and **circles** tools, make a small stick figure. Using the **transform** tool, move your stick figure's right arm up. You now have a right-handed figure. Again using the **transform** tool, move the entire space around. What happens if you move the figure over the boundary of the model? (Keep in mind that the boundary is only an *apparent* boundary to someone living outside of the geometry. To someone living in the geometry, there is no boundary, so they cannot "move over it.") Have you discovered at least one way that projective geometry is different from spherical geometry?

5. Clear the geometry. Let's try to imagine projective space with a disc removed. To do this, first create a circle, and consider its interior as the disc that we are removing. Increase the size of the disc until it is almost, but not quite, as large as possible. What remains? (It is easier to see if you position the center of the circle near the "edge" of the half-sphere model.) How are the "ends" of the remaining piece glued together? (It might help to move the point on the circle over the "edge" of the model.)

6. Why do mathematicians care about projective geometry? Let's investigate **Euclidean** geometry to find out. Construct a line in Euclidean geometry. In the menu, choose **display → model → projective** to see the Euclidean plane, including your line, radially

projected onto a half sphere. Switch back to **Projective** geometry. Make sure you are in the **half-sphere** model, and create a line. Switching back and forth between the two geometries, can you make the two lines you have constructed to appear the same? What might be an advantage to considering a "Euclidean line" as a "Projective line"?