the three-year period 1968–70, is the twentieth award of the Chauvenet Prize since its institution by the MAA in 1925. For the list of the names of previous winners, see this MONTHLY, 71(1964), p. 589, 72 (1965), pp. 2–3, 74 (1967), p. 3, 75 (1968), pp. 3–4, 77(1970), pp. 117–118, and 78 (1971), pp. 112–113.

Professor Trèves was born on April 23, 1930, in Brussels, Belgium. He received the first and second Baccalaureate degrees in Paris in 1949 and 1950, his *licence en science* and his Ph. D. at the Sorbonne in 1953 and 1958. From 1958 to 1961, he was an assistant professor at the University of California, Berkeley, from 1961 to 1964 an associate professor at Yeshiva University, and from 1964 to 1970 a professor at Purdue University. Since 1970, he has been a professor at Rutgers University.

Professor Trèves was an Alfred P. Sloan Fellow in 1960–62 and 1962–64. From June to November 1961 he was under the auspices of the Organization of American States at the *Instituto de Matematica Pura e Aplicada in Rio de Janeiro, Brazil*; in September 1965, he was a Visiting Professor at the Tata Institute of Fundamental Research in Bombay, India, and from 1965 to 1967, and again from May to June, 1970, he was a Visiting Professor at the Sorbonne in Paris.

Professor Trèves' significant contributions to various branches of analysis, but, in particular, to partial differential equations and functional analysis, are contained in his sixty publications.

In accepting the Award, Professor Trèves stressed that he was very much honored and thankful for having been awarded the 1972 Cauvenet Prize. He added that, because of the apparent increasing technicality of mathematical research, it is becoming ever more difficult to exchange information between mathematicians working in different fields — or even in the same field. He felt this to be a worrisome situation, which makes expository talks and articles more necessary than ever.

CONJECTURES AND COUNTEREXAMPLES IN METRIZATION THEORY

L. A. STEEN, St. Olaf College

Prologue. The search for necessary and sufficient conditions for the metrizability of topological spaces is one of the oldest and most productive problems of point set topology. Alexandroff and Urysohn **[4]** provided one solution as early as 1923 by imposing special conditions on a sequence of open coverings. Nearly ten years later R. L. Moore chose to begin his classic text on the Foundations of Point Set Theory **[41]** with an axiom structure which was a slight variation of the Alexandroff and Ury-

Lynn Steen received his M. I. T. Ph. D. in 1965 under Kenneth Hoffman and has been at St. Olaf College except for sabbatical leave in 1970–71 at the Inst. Mittag-Leffler, Sweden. His research is centered on topology, and he is the author with J. A. Seebach, Jr. of *Counterexamples in Topology* (HRW, 1970). *Editor.*

sohn metrizability conditions. After Jones [28], we now call any space which satisfies Axiom 0 and parts 1, 2, 3 of Axiom 1 of [41] a Moore space. Each metric space is a Moore space, but not conversely, so the search for a metrization theorem became that of determining precisely which Moore spaces are metrizable. The most famous conjecture was that each normal Moore space is metrizable.

It would probably be no exaggeration to say that for the last 30 years, the normal Moore space conjecture dominated the search for a significant metrization theorem and in the process played a major role in the development of point set topology. The conjecture itself was first stated in 1937 by Jones [28] who showed that if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal Moore space is metrizable. The next major result came nearly twenty years later when Bing [10] and Nagami [44] showed that every paracompact Moore space is metrizable. But Jones' result together with more recent ones of Heath [26] and Bing [8] indicated a close relationship between the normal Moore space conjecture and the continuum hypothesis which was shown by Cohen [18] in 1963 to be independent of the axioms of set theory. Quite recently Tall and Silver [54] used a Cohen model to show that the normal Moore space conjecture itself could not be proved from the present axioms of set theory.

Thus as metrization research shifts from topology to logic, we survey in this paper the chief topological milestones of the last half century. We shall not present proofs that are available in the literature, but shall concentrate instead on gathering together the most significant definitions, theorems, conjectures and counterexamples. The latter will be grouped together at the end of the paper and referenced throughout the text whenever appropriate. We begin at the beginning.

Basic Definitions. We shall assume throughout this paper that all topological spaces are Hausdorff. Most often we shall be concerned only with regular spaces, though this assumption will not go unwritten. **Regular** spaces are those which admit a separation of a point from a closed set by disjoint open neighborhoods. A space X is **normal** if each pair of disjoint closed sets can be separated by disjoint open neighborhoods, and **completely normal** if the same can be done for separated sets. A space is completely normal if and only if it is **hereditarily normal** [21], that is, if and only if every subspace is normal.

A subset of a topological space which can be written as the countable union of closed sets is called an F_σ**-set**; the complement of an F_σ-set can be written as a countable intersection of open sets, and is called a G_δ**-set** (or an **inner limiting** set). A space in which every closed set is G_δ (or equivalently, every open set is F_σ) will be called a G_δ**-space**; a normal space which is also a G_δ-space is called (by Cech [15]) **perfectly normal.** Every metric space is perfectly normal and every perfectly normal space is completely normal [33], so we have the following implications:

Metrizable $\Rightarrow$ perfectly normal $\Rightarrow$ completely normal $\Rightarrow$ normal $\Rightarrow$ regular.

Examples 5, 2, 10, and 6 show that none of these implications is reversible.

If a topological space has a countable dense subset it is called **separable**, if it has a countable basis it is **perfectly separable** (or **second countable**), and if it has a countable local basis at each point it is *first countable*. A space in which every subspace is separable is called **hereditarily separable**. If every open covering of X has a countable subcovering, X is called **Lindelöf** (or, by Russian mathematicians, **finally compact** [3]); clearly each perfectly separable space is both Lindelöf and hereditarily separable.

Since in a metric space the (open) balls of radius $1/n$ form a countable local basis at each point, every metric space is first countable. Metric spaces need not be second countable, but in metric spaces the properties of separable, hereditarily separable, second countable and Lindelöf coincide. Urysohn [**60**] proved in 1925 that every normal second countable space is metrizable, and, in response to a question proposed by Urysohn, Tychonoff [**59**] showed a year later that every regular second countable space is metrizable.

Developments. A collection of sets $F = \{U_\alpha\}$ is said to **cover** a space X if each point of X belongs to some U_α; if each U_α is open, the cover F is called an **open covering** of X. A cover $\{V_\beta\}$ of a space X is a **refinement** of a cover $\{U_\alpha\}$ if for each V_β there is a U_α such that $U_\alpha \subset V_\beta$. If $S \subset X$, the **star** of S with respect to a cover $F = \{U_\alpha\}$ is the union of all sets in F which intersect S; the star of S is denoted by $F^*(S)$, and the star of the singleton $\{x\}$ is usually denoted simply by $F^*(x)$.

A **development** for a topological space X is a countable family $\mathscr{F}$ of open coverings F_i such that if C is a closed subset of X and $p \in X - C$, there is a covering $F \in \mathscr{F}$ such that no element of F which contains p intersects C (i.e., such that $F^*(p) \cap C = \varnothing$). A space with a development is called **developable**. If $\mathscr{F} = \{F_i\}$ is a development where $F_i \subset F_{i+1}$ for all i, the family $\mathscr{F}$ is called a **nested development**, and if F_{i+1} is a refinement of F_i, $\mathscr{F}$ is called a **refined development**. Clearly each nested development is a refined development; Vickery [**61**] showed that every developable space has a nested development. Axiom 0 and parts 1, 2, and 3 of Axiom 1 of Moore [**41**] require precisely that a space be regular with a nested development $\{F_i\}$; such spaces are called Moore spaces (after Jones [**28**]), and are characterized by the fact that for each $p \in X$, $\{F_i^*(p)\}$ is a countable local basis. Vickery's theorem can be restated as follows: a topological space is a Moore space if and only if it is regular and developable.

Each metric space is a Moore space since the sequence of open coverings by metric balls of radius $1/n$ is a development; examples 6, 9, 14, and 15 show that Moore spaces need not be metrizable.

Semimetric Spaces. A **semimetric** for a Hausdorff space X is a symmetric function $d : X \times X \to R^+$ such that $d(x,y) = 0$ if and only if $x = y$, and if $x \in X$ and $E \subset X$, $\inf \{d(x,y) \mid y \in E\} = 0$ if and only if $x \in \bar{E}$, the closure of E; a Hausdorff space which admits a semimetric is called a **semimetric space**. If we did not require d to be symmetric, to assert the existence of a function with the remaining properties would

be equivalent to saying that the space X was first countable [**13**]. Thus a semimetric space may be thought of as a symmetric first countable space. In fact, some Russian mathematicians call these spaces **symmetrizable**.

Now every developable space has a natural semimetric: if $\{F_n\}$ is a nested development for X (with $X \in F_1$), we define $d(x,y) = \inf \{1/n \mid x,y \in U \in F_n\}$. Then d is a semimetric, but clearly not a metric since d is not continuous. (A semimetric space is metrizable if and only if it has a continuous semimetric [**13**].) Semimetric spaces share with metric spaces the property that every closed set is a G_δ [**35**], hence such spaces are G_δ-spaces. We use Figure 1 to summarize the implications for regular spaces; counterexamples to the converse implications are listed below each implication arrow.

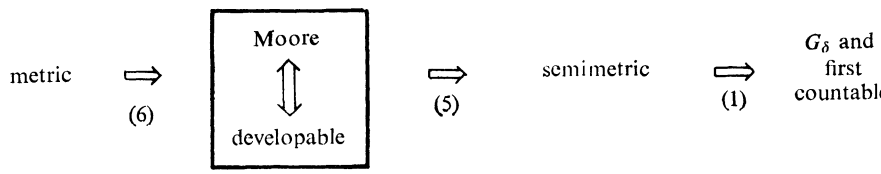

FIG. 1

Every known example of a Moore space which is not metrizable is also not normal; the normal Moore space conjecture asserts that it will always be thus. Jones [**28**] in 1937 mounted the first major attack on this conjecture, and succeed only in proving several weaker theorems: every normal Moore space is completely normal, and every separable normal Moore space is metrizable provided $2^{\aleph_1} > 2^{\aleph_0}$ — a fact implied by (but not equivalent to) the continuum hypothesis. Both of Jones' results have recently been strengthened: McAuley [**36**] observed in 1954 that a simple modification of Jones' proof will show that every normal semimetric space is completely normal, while in 1964 Heath [**25**] showed that a necessary and sufficient condition for the metrizability of a separable Moore space is that every uncountable subset M of the real line contains a subset which is not F_σ (in M). This condition is (perhaps not strictly [**25**]) weaker than that used by Jones, namely $2^{\aleph_0} < 2^{\aleph_1}$.

Jones actually showed that if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal space has the property that every uncountable subset has a limit point; Heath [**26**] called spaces with this property **$\aleph_1$-compact** and proved the converse to Jones' theorem: if every separable normal space is $\aleph_1$-compact, then $2^{\aleph_0} < 2^{\aleph_1}$.

Paracompactness. The most significant general approximation to the normal Moore space conjecture is the Bing-Nagami theorem that every paracompact Moore space is metrizable. To develop the concept of paracompactness and all its variations, we must first discuss the naming of various covers.

A cover is **point finite** if each point belongs to only finitely many sets in F, **locally finite** if each point has some neighborhood which intersects only finitely many members of F, and **star finite** if each set in F intersects only a finite number of other sets in F. A cover $V = \{V_\beta\}$ of X is a **star refinement** (or a **point star refinement**, or a Δ **refinement**) of a cover $\{U_\alpha\}$ if for each $x \in X$ there is some U_α such that $V^*(x) \subset U_\alpha$ (where $V^*(x)$ is the star of x with respect to $V = \{V_\beta\}$).

A Hausdorff space is called **fully normal** if every open cover has an open star refinement, **strongly paracompact** (or **star paracompact**) if every open cover has an open star finite refinement, **paracompact** if every open cover has an open locally finite refinement, and **metacompact** (or **pointwise paracompact**, or **weakly paracompact**) if every open cover has an open point finite refinement.

Fully normal spaces were first defined by Tukey [**58**] in 1940, while paracompact spaces were introduced by Dieudonné [**19**] in 1944. Tukey showed that every metrizable space is fully normal, while Dieudonné showed that every paracompact space is normal. The key link between these definitions was provided by Stone [**53**] in 1948 who showed that every metric space is paracompact by proving that every fully normal space is paracompact, and conversely. Although a regular semimetric space need not be paracompact (Example 6), Ceder [**16**] showed that each regular hereditarily separable semimetric space is paracompact. Smirnov [**48**] showed that a paracompact space which fails to be metrizable must fail for local reasons: every locally metrizable paracompact space is metrizable.

Also in 1948 Morita [**43**] introduced the concept (but not the name) of strongly paracompact spaces; he showed that each regular Lindelöf space is strongly paracompact while every strongly paracompact space is *a fortiori* paracompact. Kaplan [**32**] and Alexandroff [**1**] showed that each separable metric space is strongly paracompact, and that a nonseparable metric space need not be strongly paracompact (Example 11). We summarize in Figure 2 these results together with the counterexamples to the converse implications.

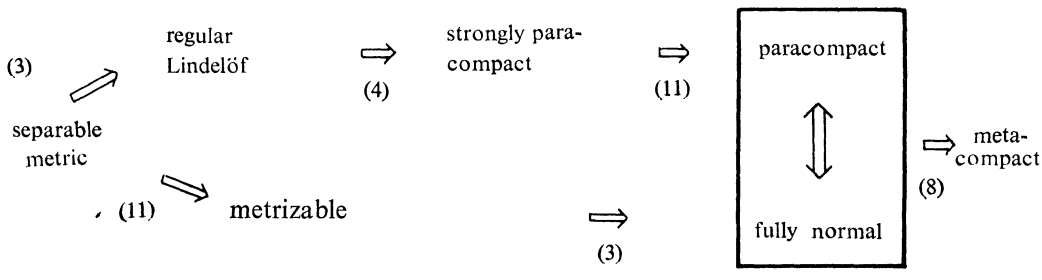

FIG. 2

A most important variation of paracompact spaces is that of **countably paracompact** spaces, those for which every countable open covering has a locally finite

open refinement. Morita [43] showed in 1948 that every metacompact normal space is countably paracompact, (see also Michael [40]) while in 1951 Dowker [20] proved that every perfectly normal space is countably paracompact. Dowker conjectured that every normal space is countably paracompact, and showed this conjecture equivalent to the conjecture that the product of a normal space with the closed unit interval I is normal by showing that X is countably paracompact and normal if and only if $X \times I$ is normal. Countably paracompact normal spaces are sometimes called **binormal**; they have been characterized in many ways by Mansfield [34] and Dowker [20]. Clearly every fully normal (i.e., paracompact) space is binormal, and every binormal space is normal.

Screenable Spaces. A collection $\mathscr{B}$ of sets is called **conservative** (or **closure preserving**) if for every subcollection $\mathscr{A} \subset \mathscr{B}$, the union of the closure of the members of $\mathscr{A}$ is closed. A conservative collection is **discrete** if the closures are pairwise disjoint. Equivalently a collection $\mathscr{B}$ of subsets of X is discrete if every point in X has a neighborhood which intersects at most one of the sets in $\mathscr{B}$.

Now a topological space is called (by Bing [10]) **screenable** if for each open covering F there is a sequence F_n of collections of pairwise disjoint open sets such that $\cup F_n$ is a refinement of F. The space is called **strongly screenable** if the F_n may be chosen to be discrete. A **perfectly screenable** space is one with a **σ-discrete base** — that is, a base which is the countable union of discrete families. A formally weaker condition is that of a **σ-locally finite base** — one which is the countable union of locally finite families. It follows directly from the definitions that every perfectly screenable space is strongly screenable, and *a fortiori*, screenable.

Stone [53] showed in 1948 that every metric space has a σ-discrete (and thus σ-locally finite) base. Shortly thereafter, Nagata [45] and Smirnov [50] showed that every regular space with a σ-locally finite base is metrizable, while Bing [10] showed that each perfectly screenable regular space is metrizable. A few years after Bing's work appeared, Nagami [44] showed that in regular spaces paracompactness is equivalent to strong screenability and that in binormal (i.e., countably paracompact and normal) spaces, screenable implies strongly screenable. Every strongly screenable developable space must be perfectly screenable since the discrete refinements of the development will form a σ-discrete base [10]. Thus every paracompact Moore space is metrizable, for by Nagami's theorem such spaces are strongly screenable and developable. Heath [25] showed that every screenable G_δ-space (thus every screenable developable space) is metacompact.

We summarize in Figure 3 the major implications for regular spaces (which are really the only ones of interest *vis-à-vis* metrizability). The relevant counterexamples are classified by the Venn diagram in Figure 4.

Collectionwise Normal Spaces. A (Hausdorff) topological space is called **collectionwise normal** if every discrete collection of sets (or, equivalently, closed sets) can be

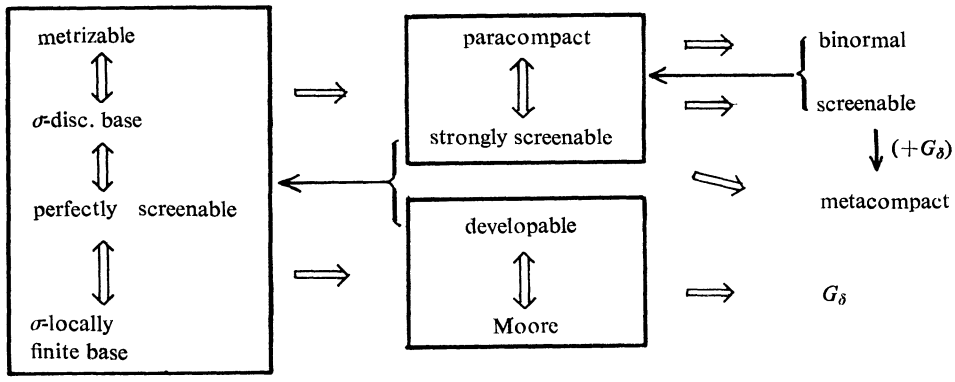

FIG. 3

covered by a pairwise disjoint collection of open sets, each of which covers just one of the original sets. If we weaken this property by requiring it of only countable discrete collections, we call the space **countably collectionwise normal.** On the other hand, we may strengthen collectionwise normal by requiring every **almost discrete** collection of sets (that is, a collection which is discrete with respect to its union)

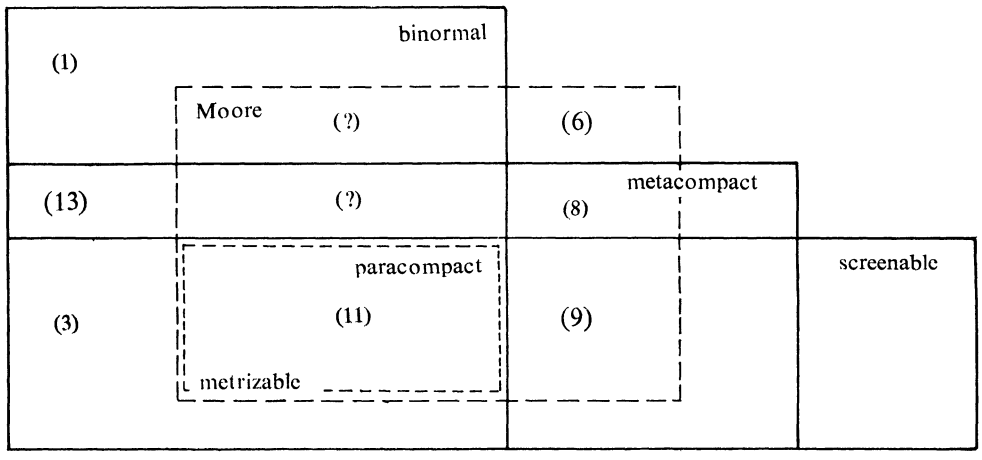

FIG. 4

to have a covering by pairwise disjoint open sets: such spaces are called **completely collectionwise normal.** A space is completely collectionwise normal if and only if it is hereditarily collectionwise normal [35], so each completely collectionwise normal space must be completely normal (i.e., hereditarily normal). Every metric space is completely collectionwise normal, so we summarize the implications in Figure 5. Examples 10 and 12 show that normal spaces need not be collectionwise normal, and that collectionwise normal spaces need not be completely collectionwise normal.

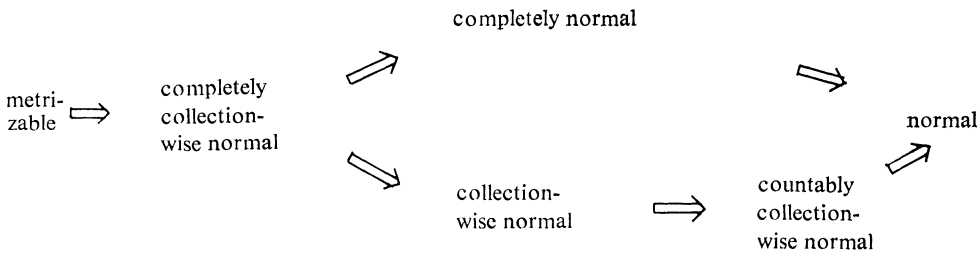

FIG. 5

Bing [**10**] showed that every fully normal (i.e., paracompact) space is collectionwise normal; Nagami [**44**] showed that every metacompact collectionwise normal space is strongly screenable. Nagami and Michael [**38**] showed that the converse holds for regular spaces. So for regular spaces, the concepts of fully normal, paracompact and strongly screenable coincide. Since each strongly screenable developable space is perfectly screenable and each regular perfectly screenable space is metrizable, we conclude again that every paracompact Moore space is metrizable. In fact, Bing [**10**] gave two slightly stronger results: every screenable, normal Moore space is metrizable (since every screenable normal developable space is strongly screenable) and every collectionwise normal Moore space is metrizable (since every such space is screenable). Thus to prove every normal Moore space metrizable, it would suffice to prove it collectionwise normal. In 1964 Bing [**8**] showed that every normal Moore space is countably collectionwise normal.

Several conditional converses of the basic implications have been established. Michael [**40**] showed that every collectionwise normal metacompact space is paracompact, while McAuley [**35**] showed that every collectionwise normal semimetric space is paracompact, and that every paracompact semimetric space is completely collectionwise normal.

In 1960 Alexandroff [**2**] developed a slightly different type of metrization theorem by defining the concept of a uniform base: a basis for X is a **uniform base** if for each $x \in X$ and each neighborhood U of x, only a finite number of the basis sets which contain x intersect $X-U$. Equivalently, a base $\mathscr{B}$ for X is uniform if for each $x \in X$ any infinite subset of $\{U \in \mathscr{B} \mid x \in U\}$ is a (local) basis at x. Since for each integer n the open covering of a metric space by balls of radius $1/n$ has a locally finite subcovering, each metric space has a uniform base, and each space with a uniform base is metacompact. Alexandroff showed that a collectionwise normal space with a uniform base is metrizable, and similarly that a paracompact space with a uniform base is metrizable. Heath [**25**] proved that a regular space has a uniform base if and only if it is metacompact and developable, from which both of Alexandroff's theorems follow.

Arhangel'skii [**5**] strengthened the definition of a uniform base by substituting for the point x an arbitrary compact set K: he called $\mathscr{B}$ a **strongly uniform base** if for

any compact subset $K \subset X$ and any neighborhood U of K, only a finite number of the basis sets intersect both K and $X - U$. Arhangel'skii showed [7] that a space is metrizable if and only if it has a strongly uniform base. Finally, a space is said to have a **point countable base** if it has a basis $\mathscr{B}$ such that no point is contained in more than countably many sets of $\mathscr{B}$. Each uniform base is point countable, and Heath [24] has shown that every semimetric space with a point countable base is developable. We summarize the preceding implications in Figure 6; the reader is invited to draw the corresponding Venn diagram.

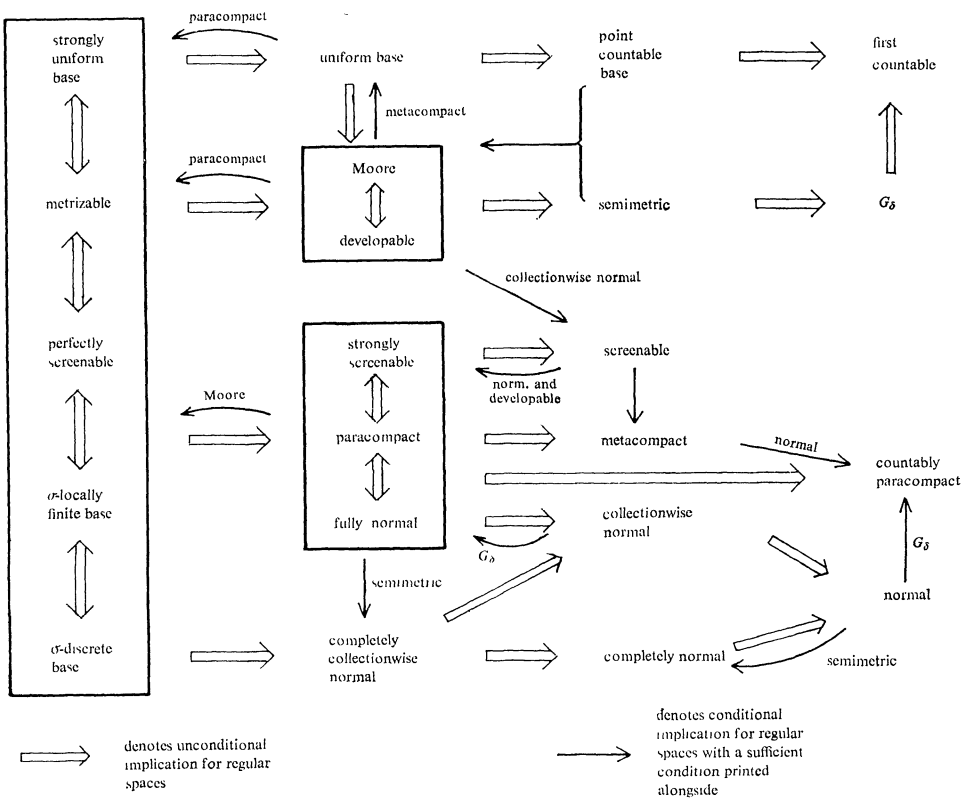

FIG. 6

Conjectures. The literature on the normal Moore space conjecture abounds in conditional theorems which assert that if some hypothesis is true, then some particular theorem is true. A famous example cited previously is Jones' theorem that if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal Moore space is metrizable. These theorems deal with implications among statements whose truth or falsehood is either not yet known, or which are in some cases (e.g., the continuum hypothesis) independent of the axioms of set theory.

We shall denote by CH the continuum hypothesis $2^{\aleph_0} = \aleph_1$; Gödel [22] and Cohen

[**18**] proved this hypothesis consistent with and independent of the Zermelo-Fraenkel (or Gödel-Bernays) axioms of set theory (hereafter referred to simply as "set theory"). We shall denote by ***WCH*** Jones' hypothesis that $2^{\aleph_0} < 2^{\aleph_1}$, since it is a weak version of *CH*: if $2^{\aleph_0} = \aleph_1$, then $2^{\aleph_0} = \aleph_1 < 2^{\aleph_1}$ by Cantor's theorem. Clearly the consistency of *CH* implies the consistency of *WCH*. The negation of *WCH*, namely $2^{\aleph_0} = 2^{\aleph_1}$ is called the Luzin Hypothesis (***LH***); Bukovsky [**14**] showed that *LH* is consistent with set theory. Thus *WCH*, the negation of *LH*, is independent of set theory.

Since every separable metric space has $2^{\aleph_0}$ Borel subsets *WCH* implies that every separable uncountable metric space has a subset which is not a Borel set; we shall call this ***BH***, for Borel hypothesis. Heath [**25**] used a special case of *BH* to strengthen Jones' theorem: we shall denote by ***HH*** the statement that every uncountable subspace M of the real line contains a subset which is not F_σ in M. Since every F_σ-set is a Borel set, *BH* implies *HH*; Heath showed that *HH* is equivalent to Jones' conjecture ***JC*** that every separable normal Moore space is metrizable. The consistency of the continuum hypothesis implies that of *JC*, while the independence of *JC* was proved by Tall and Silver [**54**] in 1970.

Heath also showed that Jones' conjecture follows from the hypothesis ***MMSC*** that every normal metacompact Moore space is metrizable; clearly *MMSC* is weaker than the normal Moore space conjecture ***MSC***. *MMSC* is equivalent to Alexandroff's conjecture ***AC*** that every normal space with a uniform base is metrizable [**3**]. Traylor [**57**] suggested the conjecture (***TC***) that every normal Moore space is metacompact. Since McAuley [**35**] showed that a separable normal metacompact Moore space is metrizable, Traylor's conjecture implies Jones' conjecture.

Several common conjectures center on semimetric spaces, a generalization of Moore spaces. Brown [**13**] suggested that every normal semimetric space is collectionwise normal, while Heath [**23**] appeared to strengthen this conjecture by suggesting that every normal semimetric space is paracompact. Actually since every semimetric collectionwise normal space is paracompact [**35**], these conjectures are equivalent; we shall denote them by ***NSP***. McAuley [**37**] proposed the weaker conjecture *SNSP* that every separable normal semimetric space is paracompact. The Bing-Nagami result that every paracompact Moore space is metrizable shows that *NSP* implies the Moore space conjecture *MSC*, and similarly, *SNSP* implies Jones' separable Moore space conjecture *JC*.

In [**10**] Bing showed that *MSC* is equivalent to the conjecture that every normal Moore space is collectionwise normal; in [**8**], he considered the weaker conjecture ***BC*** that every normal Moore space is collectionwise normal with respect to a discrete collection of points. (He termed a counterexample to *BC* one of *type D*.) Bing showed that *BC* is equivalent to the following set theoretic conjecture: If X is a set and if Y denotes the product $X \times X$ less the diagonal $\Delta = \{(x,x) \in X \times X)$, we call a subset $W \subset Y$ a **skew subset** if the projections $\pi_x(W)$ and $\pi_y(W)$ are disjoint. Bing's alternative to *BC* is the conjecture ***F*** that if $f: Y \to Z^+$ is a function from Y to the nonnegative integers with the property that for each skew subset $W \subset Y$ there is a function

$F_W : W \to Z^+$ which dominates f in the sense that max $[F_W(x), F_W(y)] > f(x,y)$ for all $(x,y) \in W$, then there is a function $F: X \to Z^+$ which dominates f in this sense for all $(x,y) \in Y$.

Bing also showed that *BC* implies *JC* by showing that any nonmetrizable separable normal Moore space would necessarily be a counterexample of type *D*. We summarize the relationships among these conjectures in Figure 7. Since all of the conjectures in this figure imply *JC*, none of them can be proved from the axioms of set theory. But the consistency of these various hypotheses (except of course for *CH* and its consequences) remains an open question.

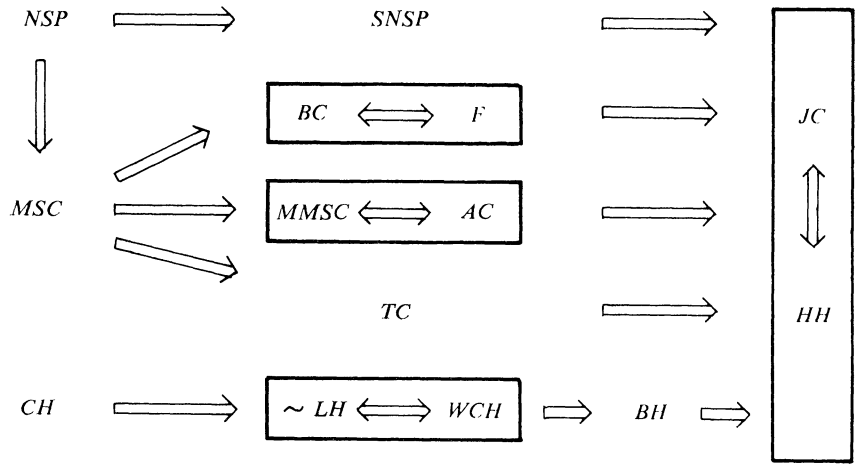

FIG. 7

We have already mentioned Dowker's conjecture ***DC*** that every normal space is countable paracompact; he showed this equivalent to the conjecture ***NP*** that the product of every normal space with the unit interval is normal [**18**]. Nagami [**44**] showed that a screenable normal countably paracompact space is paracompact and conjectured ***NC*** that every screenable normal space is paracompact. Clearly *DC* implies *NC*.

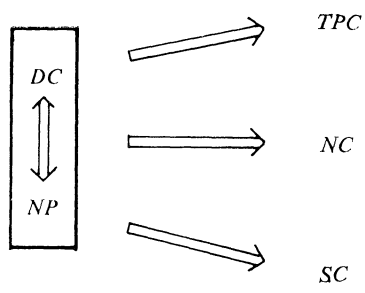

FIG. 8

Tamano [55] discusses a wide variety of theorems concerning the product invariance of normality and paracompactness and enunciates the following conjecture ***TPC***: If Y is metrizable and $X \times Y$ is normal then $X \times Y$ is paracompact. Tamano and Morita [42] have shown that to conclude that $X \times Y$ is paracompact it is sufficient to prove $X \times Y$ countably paracompact . Thus Dowker's conjecture implies Tamano's.

Souslin [50] asked whether a linearly ordered space must be separable whenever it satisfies the countable chain condition (that every disjoint collection of open sets is at most countable). We shall call this conjecture ***SC***; a counterexample (if it exists) is known as a **Souslin space**. A thorough discussion of this conjecture and related topics is provided by M. E. Rudin [47] who earlier showed [46] that if a Souslin space exists, then so must a counterexample to Dowker's conjecture. In other words, Dowker's conjecture implies Souslin's conjecture. Tennenbaum, Solovay, and Jech showed that Souslin's conjecture is consistent with [49] and independent of ([27, 56]) the axioms of set theory. Thus Dowker's conjecture cannot be proved from the present axioms of set theory. (*Added in proof:* In fact, it is false. Just recently M. E. Rudin constructed a counterexample to Dowker's conjecture.)

Epilogue. The concepts and examples discussed in this paper represent not so much the frontier as the established settlements of metrization research. Several recent papers by Ceder [16], Borges [11], [12], Michael [39], and Worrell and Wicke [62] contain such refinements as M_i-spaces, stratifiable spaces, $\aleph_0$-spaces, and θ bases. In each of these new areas there are significant and difficult conjectures similar to those enumerated above; the interested reader can pursue these issues in the papers cited in the bibliography, together with those listed in the excellent bibliographies of [3] and [6].

Since a metric is a map to the positive reals, it should not be surprising to find that the existence of certain esoteric metrics is intimately related to the existence of certain subsets of the real line. Example 7 provides a very specific instance of this relationship in that potential counterexamples to both Jones' and Dowker's conjectures depend on the existence of certain special subsets of the real line, while the independence theorems of Tall, Silver, Tennenbaum, Solovay, and Jech show that many topological problems depend on fundamentally undecidable problems of set theory. Thus many of the unresolved metrization conjectures may come to be viewed as one measure of the incompleteness of our present axiomatic view of metric spaces.

Examples

1. *Open Ordinal Space*. Let X be the set of all ordinal numbers strictly less than the first uncountable ordinal Ω; X carries the interval (or order) topology. Then X is completely collectionwise normal [51] but not fully normal [10].

2. *Closed Ordinal Space*. Let X be the set of all ordinal numbers less than or equal to the first uncountable ordinal Ω. X is compact in the interval topology, but not G_δ

since the closed set $\{\Omega\}$ is not a G_δ set. Thus X is neither perfectly normal nor semimetrizable. But of course it is strongly paracompact.

3. *Lower Limit Topology*. Let X be the real line with the topology generated by the sets of the form $[a,b) = \{x \in X \mid a \leqq x < b\}$. Bing [**10**] cites this space as an example of a regular, separable, strongly screenable (and therefore paracompact) space which is neither perfectly screenable nor developable.

4. *Stratified Plane*. If R is the real line with the Euclidean topology and S is the real line with the discrete topology, then $X = R \times S$ is a nonseparable strongly paracompact metric space.

5. *Bow-Tie Space*. Let X be the Euclidean plane with real axis L. If $d: X \times X \to R^+$ is the Euclidean metric on X, we define a semimetric δ as follows: $\delta(p,q) = d(p,q)$ if $p,q \in X - L$; $\delta(p,q) = d(p,q) + \alpha(p,q)$ if p or $q \in L$, where $\alpha\ (p,q)$ is the radian measure of the acute angle between L and the line connecting p to q. The topology on X is generated by the semimetric balls of small radius; a neighborhood ball of a point $p \in L$ looks like a bow-tie (Figure 9) or a butterfly, so this space is often called

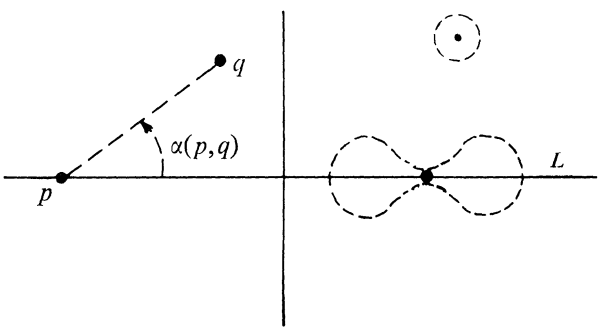

FIG. 9

the bow-tie or butterfly space. McAuley [**36**] introduced this space as an example of a regular semimetric space which is not developable. He showed furthermore that it is paracompact (thus completely collectionwise normal) and hereditarily separable.

6. *Tangent Disc Topology*. Let $P = \{(x, y) \mid x, y \in R, y > 0\}$ be the open upper half-plane with the Euclidean topology τ and let L denote the real axis. We generate a topology on $X = P \cup L$ by adding to τ all sets of the form $\{x\} \cup D$, where $x \in L$ and D is an open disc in P which is tangent to L at the point x (Figure 10). This important example was apparently introduced by both Niemytzki (see [**6**]) and Moore (see [**29**]) as a regular developable space which is not metrizable (since the uncountable closed subset L is discrete and thus not separable in the induced topology). The development which makes X a Moore space is the collection of

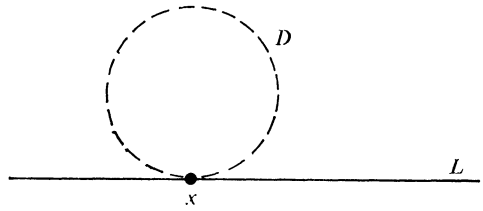

FIG. 10

open balls of radius $1/n$ (including the tangent discs $\{x\} \cup D$ if D has radius $1/n$). X is clearly not normal, and neither countably paracompact nor metacompact [52].

A common variation (see [30]) of the tangent disc topology is formed by replacing the tangent disc neighborhoods by sets of the form $\{x\} \cup T$ for each $x \in L$, where T is an inverted isosceles triangle in P with vertex at the point x and base parallel to L, such that the radian measure of the vertex angle equals the length

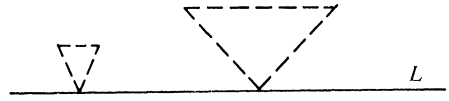

FIG. 11

of its adjacent sides (Figure 11). McAuley [36] discusses a different variation which is formed from the bow-tie space by rotating each of the bow-tie neighborhoods 90° (Figure 12). Bing [9] introduced a physical model which he called flow space by assuming that water is flowing from left to right across the unit square at the rate of $(1-x)$ feet per second. Flow space is the closed unit square, and a neighborhood $N_p(t)$ of a point p is the set of all points in X which a swimmer could reach in less than t seconds (Figure 13).

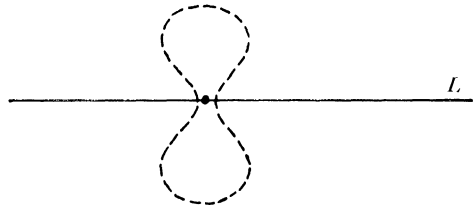

FIG. 12

7. *Tangent Disc Subspaces.* If S is a subset of the real line L, and $Y = P \cup L$ is the tangent disc space, we let X be the subspace $P \cup S$ with the topology induced from Y. The space X is second countable if and only if S is countable, so, since X is regular, X is metrizable if and only if S is countable. X will always be a Moore space since it has the same development as Y, and similarly it will always be separable since the rational lattice points in P are dense in X. Jones [28] showed that every subset of cardinality c of a separable normal space has a limit point; since S cannot

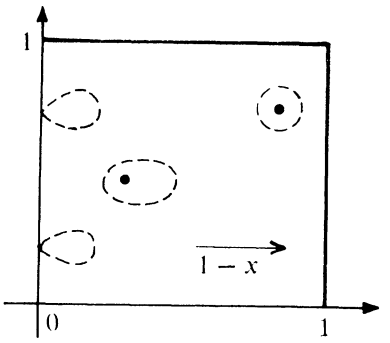

FIG. 13

have any limit points, X cannot be normal when S has cardinality c. Bing [**10**] showed that X is normal if every subset of S is a G_δ-set in (the relative topology of) S; but every uncountable G_δ-subset of the Euclidean real line has cardinality c (by Mazurkiewicz' theorem [**33**, p. 441]). Thus X would be a normal nonmetrizable Moore space if S were uncountable but of cardinality less than c with the additional property that every subset of S is G_δ in S. Such an S could contain only countable G_δ-subsets of the real line. Clearly the existence of a set with these properties cannot be proved within ordinary set theory since it would constitute a counterexample to the continuum hypothesis. However, Jones [**39**] constructed a set S of cardinality $\aleph_1$ such that every countable subset of S is G_δ in S.

Younglove [**63**] studied this example as a possible counterexample to Dowker's conjecture that every countably paracompact space is normal and proved that if S is a G_δ-set, then X is countably paracompact if and only if S is countable. Thus X could be a counterexample to Dowker's conjecture only if S was not a G_δ-subset of the real line L.

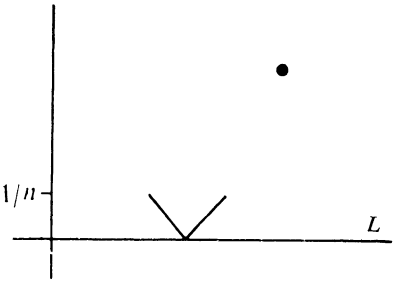

FIG. 14

8. *Tangent V Topology*. If X is the upper half plane including the real axis L, we let each point of $X - L$ be open and take as a neighborhood basis of points $x \in L$ a "V" with vertex at x, sides of slopes ± 1 and height $1/n$ (Figure 14). Heath [**25**]

showed that X is a metacompact Moore space which is not screenable. Clearly X is neither normal nor separable.

9. *Picket Fence Topology*. If X is the upper half plane including the real axis L, we let each point of $X - L$ be open, and take as a neighborhood basis of rational points $x \in L$ the vertical line segments of height $1/n$ with lower end point at x. The neighborhood basis of irrational points $x \in L$ consists of line segments of slope 1 and height $1/n$ with their base at the point x (Figure 15). Heath [**25**] introduced this as a simple example of a screenable Moore space which is not normal.

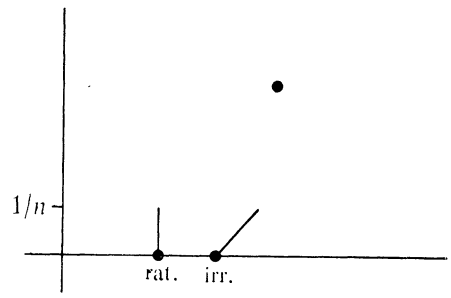

FIG. 15

10. I^I. Let $X = I^I$ be the uncountable Cartesian product of the closed unit interval $I = [0,1]$ with the Tychonoff topology; that is, X is the set of all functions from I to I with the topology of pointwise convergence. Since X is compact and Hausdorff, it is normal; but it is not completely normal [**52**] since it contains a subspace homeomorphic to Z^I, the uncountable product of the positive integers, which Stone [**53**] showed was not normal. Thus X is strongly paracompact and collectionwise normal but neither perfectly normal nor developable.

11. *Hedgehog*. If K is a cardinal number, a hedgehog X of spininess K is formed from the union of K disjoint copies of the unit interval $[0,1]$ by identifying the zero points of each interval. A metric for X can be defined by $d(x, y) = |x - y|$ if x and y belong to the same segment (or spine), and $d(x, y) = x + y$ otherwise. Alexandroff [**3**] cites a hedgehog of uncountable spininess as an example of a metric space which is not strongly paracompact.

12. *Bing's Power Space*. If S is some uncountable set with power set P, let $X = \prod_{\lambda \in P}\{0,1\}_\lambda$, where $\{0,1\}_\lambda$ is a copy of the two point discrete space. (If we let 2 denote the two point discrete space, we have $X = 2^{2^S}$.) Since the elements of X are collections of subsets of S, each ultrafilter on S is a point in X; let M denote the subset of X consisting of all principal ultrafilters of S. Then if x_s is the point in X whose λ-th coordinate $(x_s)_\lambda$ equals 1 if and only if $s \in \lambda$, we have $M = \{x_s \in X \mid s \in S\}$. If X has the

Tychonoff topology τ, $X - M$ is dense in X. Bing [**10**] generated a new topology on X by adding to τ all points of $X - M$ as open sets; we shall denote the topology thus generated by σ. M inherits from (X,σ) the discrete topology; furthermore, any two disjoint closed subsets of M are contained in disjoint open subsets of X [**52**]. It follows that X is normal but not perfectly normal [**10**], metacompact [**52**] or collectionwise normal (since M is an uncountable discrete collection of points without disjoint open neighborhoods for all of its points).

13. *Michael's Power Subspace.* If $X = 2^{2^S}$ is Bing's Power Space, we let Y be the subspace $M \cup L$, where M is the subset of all principal ultrafilters of S and L is the collection of all finite families in $X - M$. Michael [**40**] selected this subspace as an example of a normal metacompact space which is not collectionwise normal.

14. *Cantor Tree.* Let C denote the Cantor set in the unit interval $[0,1]$; the midpoints of the components of $[0,1] - C$ are 1/2, 1/6, 5/6, 1/18, 5/18, etc. Let D be the tree (or dendron) in the lower half plane whose vertices are (1/2,–1), (1/6,–1/2), (5/6,–1/2), (1/18,–1/4), (5/18,–1/4), etc.

Then the space X is defined as $D \cup C$ (Figure 16), where D inherits the Euclidean topology from the plane, while a basis neighborhood of a point $c \in C$ is a path Γ in the tree D whose upper limit is the point c, together with open segments at each branch point of Γ sufficiently short to avoid including any other branch point. Jones [**31**] cites this example of Moore as the first example of a nonmetrizable Moore space. The fact that X is nonmetrizable follows from the observation that it is separable but not perfectly separable. Jones [**31**] shows that X is not normal.

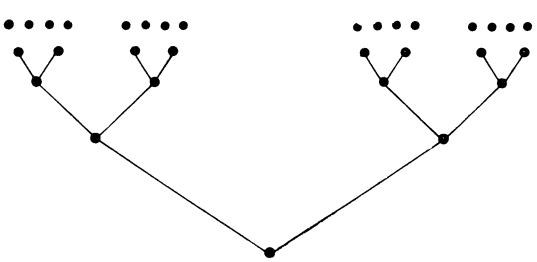

FIG. 16

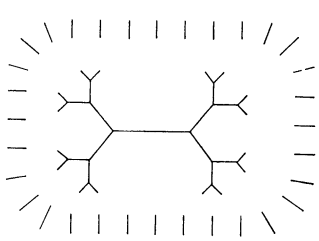

FIG. 17

15. *Moore's Road Space.* Let two roads start at the origin of the plane and proceed in opposite directions for one mile each. Let each then branch into two roads which continue for one mile each before each of these now branches into two roads. Continue this in such a way that none of the new roads ever intersect, and so that all roads proceed indefinitely far from the origin. This process generates c roads; at the "end" of each we adjoin a straight ray of infinite length. This collection of roads is the space X (Figure 17), and we generate a topology from a basis of open discs. This

"automobile road" space was introduced by Moore as a graphic variation of the Cantor tree (Example 14); it has the same properties [**31**].

1 = *example has property*

0 = *example does not have property*

Example	Hausdorff	Regular	Normal	Comp. Normal	Perf. Normal	Coll. Normal	Count. Col. Nor.	Comp. Col. Nor.	Metrizable	Separable	2nd Count.	1st Count.	Hered. Separ.	Lindelöf	Developable	Moore	Semimetric	G_δ	Fully Normal	Str. Paracompact	Paracompact	Metacompact	Count. Paracmpt.	Binormal	Screenable	Str. Screenable	Perf. Screenable	σ-loc. Fin. Base	Uniform Base	Pt. Count. Base
1 Open Ord. Space	1	1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	
2 Closed Ord. Sp.	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	
3 Lower Limit	1	1	1	1	1	1	1		0	1	0	1		1	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	
4 Stratified Plane	1	1	1	1	1	1	1	1	1	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5 Bow-Tie	1	1	1	1	1	1	1	1	0	1	0	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	
6 Tangent Disc	1	1	0	0	0	0	0	0	0	1	0	1	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	
8 Tangent V	1	1	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	0	0	0	1		0	0	0	0	0	1	1
9 Picket Fence	1	1	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	0	0	0	1		0	1	0	0	0	1	1
10 I^I	1	1	1	0	0	1	1	0	0	1	0	0		1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	
11 Hedgehog	1	1	1	1	1	1	1	1	1	0	0	1	0	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
12 Power Space	1	1	1		0	0		0	0		0			0	0	0	0	0	0	0	0	0				0	0	0	0	
13 Power Subspace	1	1	1		0	0		0	0		0					0	0	0	0	0	0	1	1	1	0	0	0	0		
14 Cantor Tree	1	1	0	0	0	0	0	0	0	1	0	1		0	1	1	1	1	0	0	0				0	0	0	0		
15 Moore's Road Space	1	1	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	0	0	0				0	0	0	0		

FIG. 18

References

1. P. Alexandroff, General duality theorems for non-closed sets of n-dimensional space, Mat. Sbornik, 21 (63) (1947) 161–231.

2. ———, On the metrization of topological spaces, Bull. Acad. Polon. Sci., 8 (1960) 135–140.

3. ———, Some results in the theory of topological spaces, obtained within the last twenty-five years, Russian Math. Surveys, 15 (1960) 23–83.

4. P. Alexandroff and P. Urysohn, Une condition nécessaire et suffisante pour qu'une classe (L) soit une classe (B), Comptes Rendus, 177 (1923) 1274–1276.

5. A. Arhangel'skii, Bicompact sets and the topology of spaces, Soviet Math. Dokl., 4 (1963), 561–564.

6. ———, Mappings and spaces, Russian Math. Surveys, 21 (1966) 115–162.

7. ———, On the metrization of topological spaces, Bull. Acad. Polon. Sci., 8(1960) 589–595.

8. R. H. Bing, A translation of the normal Moore space conjecture, Proc. Amer. Math. Soc., 16 (1965) 612–619.

9. ———, Challenging conjectures, this MONTHLY, 50th Anniv. Issue, 74 (1967) 56–64.

10. ———, Metrization of topological spaces, Canad. J. Math., 3 (1951) 175–186.

11. C. R. Borges, On the metrizability of topological spaces, Canad. J. Math., 20 (1968) 795–804.

12. ———, Stratifiable spaces, Pacific J. Math., 17 (1966) 1–16.

13. M. Brown, Semi-metric spaces, Summer Institute on Set Theoretic Topology, Madison, A.M.S., (1955) 64–66.

14. L. Bukovsky, Borel subsets of metric separable spaces, General Topology and its Relations to Modern Analysis and Algebra, 1966 Prague Symp., Academic Press, 1967, 83–86.

15. E. Čech, Sur la dimension des espaces parfaitement normaux, Bull. Intern. de l'Acad. de Bohême (Prague), 33 (1932) 38–55.

16. J. G. Ceder, Some generalizations of metric spaces, Pacific J. Math., 11 (1961) 105–125.

17. E. W. Chittenden, On the metrization problem and related problems in the theory of abstract sets, Bull. Amer. Math. Soc., 33 (1927) 13–34.

18. P. J. Cohen, The independence of the continuum hypothesis, I, II, Proc. Nat. Acad. Sci. U.S.A., 50 (1963) 1143–1148; 51 (1964) 105–110.

19. J. Dieudonné, Une généralisation des espaces compacts, J. Math. Pures Appl., 23 (1944) 65–76.

20. C. H. Dowker, On countably paracompact spaces, Canad. J. Math., 3 (1951) 219–224.

21. S. Gaal, Point Set Topology, Academic Press, New York, 1964.

22. K. Gödel, The Consistency of the Continuum Hypothesis, Princeton, 1940.

23. R. W. Heath, On certain first countable spaces, Topology Seminar, Wisconsin (1965); Princeton (1966) 103–113.

24. ———, On spaces with point-countable bases, Bull. Acad. Polon. Sci., 13 (1965) 393–395.

25. ———, Screenability, pointwise paracompactness, and metrization of Moore spaces, Canad. J. Math., 16(1964) 763–770.

26. ———, Separability and $\aleph_1$ compactness, Colloq. Math., 12 (1964) 11–14.

27. T. Jech, Non-provability of Souslin's Hypothesis, Comm. Math. Univer. Carolinae, 8 (1967) 291–305.

28. F. B. Jones, Concerning normal and completely normal spaces, Bull. Amer. Math. Soc., 43 (1937) 671–677.

29. ———, Metrization, this MONTHLY, 73 (1966) 571–576.

30. ———, Moore spaces and uniform spaces, Proc. Amer. Math. Soc., 9 (1958) 483–486.

31. ———, Remarks on the normal Moore space metrization problem, Topology Seminar, Wisconsin (1965); Princeton (1966) 115–119.

32. S. Kaplan, Homology properties of arbitrary subsets of Euclidean spaces, Trans. Amer. Math. Soc., 62 (1947) 248–271.

33. C. Kuratowski, Topologie I, Monografie Matematyczne, Vol. 20, Warsaw, 1958.

34. M. J. Mansfield, On countably paracompact normal spaces, Canad. J. Math., 9 (1957) 443–449.

35. L. F. McAuley, A note on complete collectionwise normality and paracompactness, Proc. Amer. Math. Soc., 9 (1958) 796–799.

36. ———, A relation between perfect separability, completeness, and normality in semi-metric spaces, Pacific J. Math., 6 (1956) 315–326.

37. ———, Paracompactness and an example due to F. B. Jones, Proc. Amer. Math. Soc., 7 (1956) 1155–1156.

38. E. A. Michael, A note on paracompact spaces, Proc. Amer. Math. Soc., 4 (1953) 831–838.

39. ———, $\aleph_0$ spaces, J. Math. Mech., 15 (1966) 983–1002.

40. ———, Point finite and locally finite coverings, Canad. J. Math., 7 (1955) 275–279.

41. R. L. Moore, Foundations of point set theory, Amer. Math. Soc. Coll. Publ. 13, New York, 1932.

42. K. Morita, Products of normal spaces with metric spaces, Math. Ann., 154 (1964) 365–382.

43. ———, Star-finite coverings and the star-finite property, Math. Japon., 1 (1948) 60–68.

44. K. Nagami, Paracompactness and strong screenability, Nagoya Math. J., 8(1955) 83–88.

45. J. Nagata, On a necessary and sufficient condition of metrizability, J. Inst. Polytech., Osaka City Univ., 1 (1950) 93–100.

46. M. E. Rudin, Countable paracompactness and Souslin's problem, Canad. J. Math., 7 (1955) 543–547.

47. ———, Souslin's conjecture, this MONTHLY, 76 (1969) 1113–1119.

48. YU. M. Smirnov, On metrization of topological spaces, Uspehi Mat. Nauk., 6 (1951) 100–111 (A.M.S. Transl. No. 91).

49. R. Solovay and S. Tennenbaum, Iterated Cohen extensions and Souslin's problem, Fund. Math.

50. M. Souslin, Problème 3, Fund. Math., 1 (1920) 223.

51. L. Steen, A direct proof that a linearly ordered space is hereditarily collectionwise normal, Proc. Amer. Math. Soc., 24 (1970) 727–728.

52. L. Steen and J. A. Seebach, Counterexamples in Topology, Holt, Rinehart and Winston, New York, 1970.

53. A. H. Stone, Paracompactness and product spaces, Bull. Amer. Math. Soc., 54 (1948) 977–982.

54. F. D. Tall, New results on the normal Moore space problem, Proc. of Washington State Univ. Conf. on General Topology, 1970, 120–126.

55. H. Tamano, Normality and product spaces, General Topology and its Relations to Modern Analysis and Algebra, 1966 Prague Symp., Academic Press, 1967, 349–352.

56. S. Tennenbaum, Souslin's Problem, Proc. Nat. Acad. Sci. U. S. A., 59 (1968) 60–63.

57. D. R. Traylor, On normality, pointwise paracompactness and the metrization question, Topology Conference, Arizona State Univ., (1967) 286–292.

58. J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940.

59. A. Tychonoff, Über einen Metrisationssatz von P. Urysohn, Math Ann., 95 (1926) 139–142.

60. P. Urysohn, Zum Metrisationsproblem, Math. Ann., 94 (1925) 309–315.

61. C. W. Vickery, Axioms for Moore spaces and metric spaces, Bull. Amer. Math. Soc., 46 (1940) 560–564.

62. J. M. Worrell, Jr., and H. H. Wicke, Characterizations of developable topological spaces, Canad. J. Math., 17 (1965) 820–830.

63. J. N. Younglove, Two conjectures in point set theory, *Topology Seminar*, Wisconsin (1965), Princeton (1966), 121–123.