AN ELEMENTARY DERIVATION OF THE CAUCHY, HÖLDER, AND MINKOWSKI INEQUALITIES FROM YOUNG'S INEQUALITY

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1. Introduction. Three of the most famous "classical inequalities" are those of Cauchy, Hölder, and Minkowski. These inequalities are "pulled out of the hat" so frequently in mathematical proofs that an early acquaintance with them would be useful for most students.

We shall deduce these three inequalities from an inequality involving integrals due to W. H. Young. In section 2 we present an elementary geometric proof of Young's inequality. In section 3 we obtain an important special case of Young's inequality from which we deduce the Hölder and Minkowski inequalities for finite sequences in sections 4 and 5. Finally in section 6 we use the inequality in section 3 to prove Hölder's and Minkowski's inequalities for Riemann integrals of continuous functions.

In our presentation Cauchy's inequality appears simply as a special case of Hölder's inequality. Historically, Cauchy's inequality was published in 1821, whereas Hölder's generalization did not appear until 1889. Minkowski's inequality appeared in 1896, while Young's inequality, which we use as a point of departure, was not published until 1912. See the References for details.

This approach seems desirable for an elementary discussion for several reasons:

First: All three of these famous inequalities are achieved easily and quickly. Moreover, the geometrically obvious necessary and sufficient condition for equality to hold in the case of Young's inequality leads directly to necessary and sufficient conditions for equality in the other cases.

Second: The Hölder and Minkowski inequalities are demonstrated for real exponents. The usual extra limiting process needed to proceed from rational to real exponents is avoided because of the elementary calculus assumed in applying Young's inequality.

Third: There is an appealing novelty in deducing inequalities for sequences from one containing integrals. This is the reverse of the usual procedure in intermediate courses in analysis, where sequences are studied first and properties of integrals deduced later.

Note 1. In his two-volume work, *Trigonometric Series*, Zygmund begins the section on inequalities by assuming Young's inequality as geometrically obvious. He then deduces from it, in a most elegant and condensed fashion, the above inequalities together with many others for finite and infinite sequences of complex numbers and for Lebesgue integrals.

Note 2. In the book entitled *Inequalities*, by Hardy, Littlewood, and Polya, there are 404 theorems each containing one or more inequalities which are frequently used by mathematicians. The inequalities of Young, Hölder, Cauchy, and Minkowski appear as Theorems 156, 11, 7, and 25 respectively. Obviously this approach is considerably different from that of Zygmund. We are indebted to this comprehensive work for much of our bibliography.

Note 3. *Analytic Inequalities*, by Kazarinoff gives a most readable introduction to inequalities for the undergraduate student. We recommend this 90-page book most highly. The Hölder and Minkowski inequalities are deduced on pages 67–74.

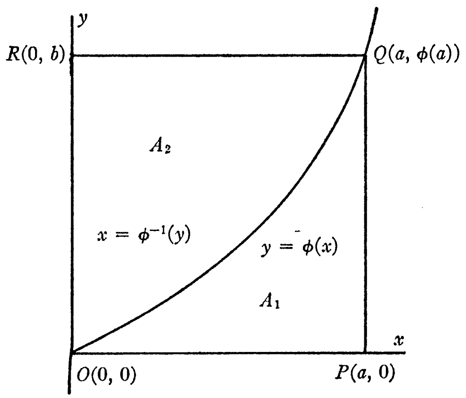

FIG. 1. $b=\phi(a)$, $ab=A_1+A_2$.

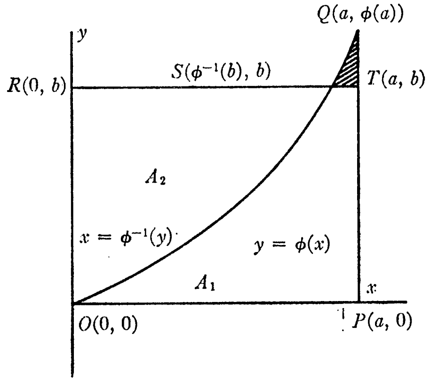

FIG. 2. $b<\phi(a)$, $ab<A_1+A_2$.

2. **Young's inequality.** Let $y=\phi(x)$ be a continuous, strictly increasing function for $x\geqq 0$, and let $\phi(0)=0$ and $\phi(a)=b$, where a and b are any positive real numbers. (See Figure 1.) Solving this equation for x in terms of y we obtain $x=\phi^{-1}(y)$. We call ϕ^{-1} the function inverse to ϕ, and it too is a continuous, strictly increasing function. Note that $\phi^{-1}(0)=0$ and $\phi^{-1}(b)=a$, and that the equations $y=\phi(x)$ and $x=\phi^{-1}(y)$ have the same graph.

From elementary calculus we know that the areas A_1 and A_2 in Figure 1 are given by

$$A_1 = \int_0^a y\,dx = \int_0^a \phi(x)dx \tag{2.1}$$

and

$$A_2 = \int_0^b x\,dy = \int_0^b \phi^{-1}(y)dy = \int_0^b \phi^{-1}(x)dx. \tag{2.2}$$

Since ab, the area of the rectangle $OPQR$ in Figure 1, is equal to the sum of the areas A_1 and A_2, it follows from (2.1) and (2.2) that

$$ab = \int_0^a \phi(x)dx + \int_0^b \phi^{-1}(x)dx. \tag{2.3}$$

Suppose now that b is not equal to $\phi(a)$. (See Figure 2 for the case $\phi(a)>b$ and Figure 3 for the case $\phi(a)<b$.) In each figure, ab is the area of rectangle $OPTR$. But this area is smaller than the sum of areas A_1 and A_2 by the amount STQ which is shaded in the figures.

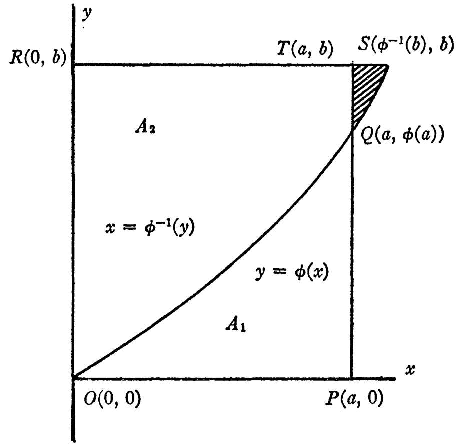

FIG. 3. $b>\phi(a)$, $ab<A_1+A_2$.

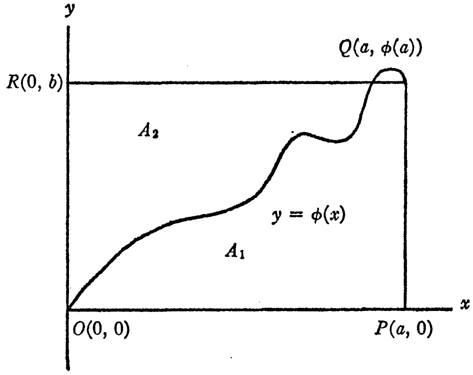

FIG. 4. $b=\phi(a)$, $ab\leqq A_1+A_2$.

Combining these three cases we obtain *Young's inequality* which states:

Let $\phi(x)$ and $\phi^{-1}(x)$ be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each other. Then for a, $b\geqq 0$ we have

$$ab \leqq \int_0^a \phi(x)dx + \int_0^b \phi^{-1}(x)dx. \tag{2.4}$$

From our previous discussion it is clear that equality holds if and only if

$$b = \phi(a). \tag{2.5}$$

Note 4. Had $\phi(x)$ not been assumed to be strictly increasing, but merely continuous, as suggested in Figure 4, we could not use the simple formula (2.2) for area A_2. Fortunately, in our applications of Young's inequality, the functions $\phi(x)$ will vanish at the origin and be continuous and strictly increasing for $x\geqq 0$.

3. Applications of Young's inequality. In this section we shall obtain inequalities from Young's inequality by choosing particular functions $\phi(x)$.

Example 1. Let $\phi(x)=x$. Then $\phi^{-1}(x)=x$ and (2.4) yields

$$ab \leqq \int_0^a xdx + \int_0^b xdx = \frac{a^2}{2} + \frac{b^2}{2} \cdot$$

This is the well-known inequality

$$2ab \leqq a^2 + b^2. \tag{3.1}$$

The condition $b=\phi(a)$ which is necessary and sufficient for equality in (3.1) is in this case

$$b = a. \tag{3.2}$$

Note 5. The usual way of proving (3.1) and (3.2) is simply to use the fact that $(a-b)^2\geqq 0$.

We now get the inequality essential to our purposes by choosing

$$\phi(x) = x^\alpha \qquad (\alpha > 0).$$

In this case the inverse function is $\phi^{-1}(x) = x^{1/\alpha}$, and (2.4) yields

$$ab \leqq \int_0^a x^\alpha dx + \int_0^b x^{1/\alpha} dx = \frac{a^{\alpha+1}}{\alpha+1} + \frac{b^{(1/\alpha)+1}}{(1/\alpha)+1}.$$

If in this last inequality we let $r = \alpha + 1$ and $r' = (1/\alpha) + 1$, then we may write

$$ab \leqq \frac{1}{r} a^r + \frac{1}{r'} b^{r'}. \tag{3.3}$$

Since $[1/\{\alpha+1\}] + [1/\{(1/\alpha)+1\}] = 1$ we see that r and r' are related by $(1/r) + (1/r') = 1$ or equivalently

$$r' = \frac{r}{r-1}. \tag{3.4}$$

Thus the inequality (3.3) holds under the conditions $a > 0$, $b > 0$, $r > 1$, and $(1/r) + (1/r') = 1$.

The condition for equality in (3.3) is derived again from (2.5). In this case the condition $b = \phi(a)$ becomes

$$b = a^\alpha = a^{r-1}. \tag{3.5}$$

Note 6. An even more roundabout way of obtaining (3.1) and (3.2) would be to let $r = r' = 2$ in (3.3) and (3.5).

Note 7. It is by allowing α to be any positive real exponent in the preceding integration that we are able to prove Hölder's inequality for real rather than rational exponents only in the next section. Note that r and r' may be any real numbers greater than unity and related by the formula in (3.4).

Note 8. Inequality (3.3) appears as Theorem 61 in Hardy, Littlewood, and Polya but is not there credited to any individual.

Note 9. If $a = 0$ and (or) $b = 0$ the reader can verify that (3.3) still holds. This fact will enable us to prove, in sections 4, 5, and 6, the Hölder and Minkowski inequalities in cases in which some terms of the sequences or some values of the functions involved are zero.

One could choose values for r in (3.3) such as 3, π, 5, sec 31°, etc., and obtain all kinds of nonobvious but also noninteresting inequalities. However, we shall use (3.3) in the next section to deduce Hölder's inequality which is also nonobvious but is not noninteresting.

4. Hölder's inequality. In order to keep printing and visual complexities at a minimum, we shall prove the three classical inequalities for finite sequences of positive real numbers only. The two sequences involved will be denoted by $\{a_i\} \equiv \{a_1, a_2, \cdots, a_n\}$, and $\{b_i\} \equiv \{b_1, b_2, \cdots, b_n\}$.

Note 10. We could prove our three inequalities for moduli of complex numbers by inserting appropriate absolute value signs throughout the following proofs. But this would clutter up the pages considerably. The following proofs are also valid for infinite sequences provided all sums involved are finite.

Let

$$(4.1)\qquad \begin{aligned} S &= (a_1^r + a_2^r + \cdots + a_n^r)^{1/r} \equiv (\sum a_i^r)^{1/r}, \\ T &= (b_1^{r'} + b_2^{r'} + \cdots + b_n^{r'})^{1/r'} \equiv (\sum b_i^{r'})^{1/r'}. \end{aligned}$$

Replace a and b in (3.3) by a_i/S and b_i/T respectively. Then for $i=1, 2, \cdots, n$ we get the n inequalities

$$(4.2)\qquad \frac{a_i}{S}\frac{b_i}{T} \leqq \frac{1}{r}\left(\frac{a_i}{S}\right)^r + \frac{1}{r'}\left(\frac{b_i}{T}\right)^{r'} \qquad (i = 1, 2, \cdots, n).$$

Adding up the right and left hand sides of the n inequalities (4.2) and, using (4.1) and (3.4), we get

$$(4.3)\qquad \begin{aligned} \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{ST} &\leqq \frac{1}{r}\left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{S^r}\right) \\ &\quad + \frac{1}{r'}\left(\frac{b_1^{r'} + b_2^{r'} + \cdots + b_n^{r'}}{T^{r'}}\right) \\ &= \frac{1}{r}(1) + \frac{1}{r'}(1) = 1. \end{aligned}$$

Finally multiplying both extremes of (4.3) by ST, we have $a_1b_1+a_2b_2+ \cdots +a_nb_n \leqq ST$ which is Hölder's inequality. Using the $\sum$ notation and (4.1), we write *Hölder's inequality* in the usual form,

$$(4.4)\qquad \sum a_ib_i \leqq (\sum a_i^r)^{1/r}(\sum b_i^{r'})^{1/r'},$$

where r and r' are real numbers greater than unity and $(1/r)+(1/r')=1$.

We shall have equality in Hölder's inequality if and only if we have equality in each of the n inequalities (4.2). This happens if and only if (3.5) is satisfied n times, that is if

$$(4.5)\qquad \frac{b_i}{T} = \left(\frac{a_i}{S}\right)^{r-1} \qquad (i = 1, 2, \cdots, n).$$

Since S and T are independent of i, we see that (4.5) implies there is a constant k $(k=T/S^{r-1})$ such that

$$(4.6)\qquad b_i = ka_i^{r-1} \qquad (i = 1, 2, \cdots, n).$$

In this case, we say that the sequences $\{a_i^{r-1}\}$ and $\{b_i\}$ are proportional. Conversely, if $\{a_i^{r-1}\}$ and $\{b_i\}$ are proportional, so that there is a constant k for which (4.6) holds, one can show that $k=T/S^{r-1}$ and thus that (4.6) implies

(4.5). This proves that equality holds in Hölder's inequality if and only if (4.6) is satisfied.

If we now raise both sides of (4.6) to the r' power, we have

$$b_i^{r'} = k^{r'} a_i^{(r-1)r'} = k^{r'} a_i^{r} \qquad (i = 1, 2, \cdots, n). \tag{4.7}$$

Since $k^{r'}$ is also a constant, we may conclude from (4.7) that Hölder's inequality becomes an equality if and only if the sequences $\{a_i^r\}$ and $\{b_i^{r'}\}$ are proportional.

If we now let $r=r'=2$ in (4.4) and (4.6), we have *Cauchy's inequality*,

$$\sum a_i b_i \leqq \left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2} \tag{4.8}$$

with equality holding if and only if

$$b_i = k a_i. \tag{4.9}$$

Note 11. So far we have proved Cauchy's inequality for positive a_i and b_i. It is easy to see that the inequality will still hold if some or all of the a_i and b_i are allowed to be negative; for the right hand side will be unaffected by a change in sign of the a_i or b_i, and the left side will certainly not increase numerically. Thus Cauchy's inequality holds when a_i and b_i are any real numbers. See Note 9 for the case in which a_i or b_i assume the value zero.

5. Minkowski's inequality. For any real number $r>1$ we may write

$$\begin{aligned} \sum (a_i + b_i)^r &= \sum (a_i + b_i)(a_i + b_i)^{r-1} \\ &= \sum a_i(a_i + b_i)^{r-1} + \sum b_i(a_i + b_i)^{r-1}. \end{aligned} \tag{5.1}$$

Application of Hölder's inequality (4.4) to each of the two sums on the right hand side of (5.1) yields

$$\begin{aligned} \sum (a_i + b_i)^r \leqq {} & \left[\sum a_i^r\right]^{1/r}\left[\sum (a_i + b_i)^{(r-1)r'}\right]^{1/r'} \\ & + \left[\sum b_i^r\right]^{1/r}\left[\sum (a_i + b_i)^{(r-1)r'}\right]^{1/r'}. \end{aligned}$$

Factoring the right hand side and using (3.4) we have

$$\sum (a_i + b_i)^r \leqq \left[\sum (a_i + b_i)^r\right]^{1/r'}\left[\left(\sum a_i^r\right)^{1/r} + \left(\sum b_i^r\right)^{1/r}\right]. \tag{5.2}$$

Dividing both sides of (5.2) by the first factor on the right hand side gives us

$$\left[\sum (a_i + b_i)^r\right]^{1-(1/r')} \leqq \left(\sum a_i^r\right)^{1/r} + \left(\sum b_i^r\right)^{1/r}.$$

Noting that $1-(1/r')=1/r$, we write this as

$$\left[\sum (a_i + b_i)^r\right]^{1/r} \leqq \left(\sum a_i^r\right)^{1/r} + \left(\sum b_i^r\right)^{1/r} \qquad (r \geqq 1), \tag{5.3}$$

which is *Minkowski's inequality*.

Note 12. If $r=1$, (5.3) obviously becomes an equality. If $r<1$, the inequality is reversed. See Theorem 25 of Hardy, Littlewood, and Polya.

In order to find necessary and sufficient conditions for equality in Minkowski's inequality we note that Hölder's inequality was applied to each of the two sums on the right hand side of (5.1). To ensure equality in each of these applications of Hölder's inequality we must satisfy two sets of conditions of the type (4.6), namely:

$$(5.4)\qquad \begin{aligned} (a_i + b_i)^{r-1} &= k_1^{r-1} a_i^{r-1} \\ (a_i + b_i)^{r-1} &= k_2^{r-1} b_i^{r-1} \qquad (i = 1, 2, \cdots, n) \end{aligned}$$

where the role of k in (4.6) is played by k_1^{r-1} and k_2^{r-1} respectively. Extracting the $(r-1)$st roots in (5.4) gives us the equivalent conditions

$$(5.5)\qquad \begin{aligned} a_i + b_i &= k_1 a_i \qquad b_i = (k_1 - 1)a_i \\ &\text{or} \\ a_i + b_i &= k_2 b_i \qquad b_i = \{1/(k_2 - 1)\} a_i. \end{aligned}$$

We can satisfy the 2 sets of equations in (5.5) simultaneously if and only if $k_1 - 1 = \{1/(k_2 - 1)\}$. In this case we may write (5.5) as one set of n equations

$$(5.6)\qquad b_i = k a_i \qquad (i = 1, 2, \cdots, n),$$

where

$$(5.7)\qquad k = k_1 - 1 = \frac{1}{k_2 - 1}\cdot$$

Since (5.4) implies (5.6) we now know that (5.6) is necessary for equality in Minkowski's inequality. Question: Does (5.6) imply (5.5) and (5.4)? Answer: Yes, for if we are given two proportional sequences $\{a_i\}$ and $\{b_i\}$ and hence a value for k such that (5.6) is satisfied, we may use (5.7) to compute $k_1 = k+1$ and $k_2 = (1/k)+1$ and be sure that (5.5) and hence (5.4) are satisfied. Thus (5.6) is also sufficient to ensure equality in (5.3).

Note 13. For $r = 2$ we can give elementary geometric interpretations for the Hölder (Cauchy) and Minkowski inequalities. Let $n = 3$, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be vectors having the components $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ respectively. We know from elementary vector analysis that the lengths of vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $(\boldsymbol{A}+\boldsymbol{B})$ are given respectively by

$$|\boldsymbol{A}| = \sqrt{\left\{\sum_1^3 a_i^2\right\}}, \qquad |\boldsymbol{B}| = \sqrt{\left\{\sum_1^3 b_i^2\right\}}, \text{ and}$$

$$|\boldsymbol{A}+\boldsymbol{B}| = \sqrt{\left\{\sum_1^3 (a_i + b_i)^2\right\}}.$$

Thus the Minkowski inequality,

$$\sqrt{\left\{\sum_1^3 (a_i + b_i)^2\right\}} \leqq \sqrt{\left\{\sum_1^3 a_i^2\right\}} + \sqrt{\left\{\sum_1^3 b_i^2\right\}},$$

asserts the obvious geometric fact that

$$(5.8)\qquad |A+B| \leqq |A| + |B|.$$

This is the so called *triangle inequality.*

We also know from elementary vector analysis that the scalar or "dot" product of two vectors A and B, defined as $|A|\ |B| \cos\theta$, is equal to $\sum_1^3 a_ib_i$. (Here θ is the angle between the vectors A and B.) Since $\cos\theta \leqq 1$ we know that

$$(5.9)\qquad |A|\ |B| \cos\theta \leqq |A|\ |B|.$$

This is equivalent to Cauchy's inequality,

$$\sum_1^3 a_ib_i \leqq \sqrt{\left\{\sum_1^3 a_i^2\right\}}\ \sqrt{\left\{\sum_1^3 b_i^2\right\}}.$$

It is geometrically obvious that we shall have equality in (5.8) and (5.9) if and only if the vectors A and B have the same direction. We know that two vectors have the same direction if and only if their components are proportional, which in this case means that

$$b_i = ka_i \qquad (i = 1, 2, 3).$$

We recognize this last equation as (4.9) and (5.6). This geometric argument has thus led us to our necessary and sufficient condition for equality to hold in both the Cauchy and the Minkowski inequalities for the case when $r=2$ and $n=3$.

6. Integral inequalities. We shall now state and prove the Cauchy, Hölder, and Minkowski inequalities for Riemann integrals of continuous functions.

If $f(x)$ and $g(x)$ are continuous, nonnegative functions on the closed interval $c \leqq x \leqq d$, then the following inequalities are true.

Cauchy-Schwarz:

$$(6.1)\qquad \left[\int_c^d f(x)g(x)dx\right]^2 \leqq \left[\int_c^d f^2(x)dx\right]\left[\int_c^d g^2(x)dx\right].$$

Equality holds if and only if

$$(6.2)\qquad g(x) \equiv kf(x).$$

Hölder:

$$(6.3)\qquad \int_c^d f(x)g(x)dx \leqq \left[\int_c^d f^r(x)dx\right]^{1/r}\left[\int_c^d g^{r'}(x)dx\right]^{1/r'},$$

where $(1/r)+(1/r')=1$ and r and r' are real numbers greater than 1. Equality holds if and only if

$$(6.4)\qquad g(x) \equiv k[f(x)]^{r-1}.$$

Minkowski:

$$(6.5)\qquad \left[\int_c^d \{f(x)+g(x)\}^r dx\right]^{1/r} \leqq \left[\int_c^d f^r(x)dx\right]^{1/r} + \left[\int_c^d g^r(x)dx\right]^{1/r},$$

where r is any real number greater than or equal to 1. The necessary and sufficient condition for equality is (6.2).

Note 14. Cauchy's inequality for integrals is called the Cauchy-Schwarz inequality and is again the special case of Hölder's inequality for which $r=r'=2$. It was first proved by Cauchy for finite sums, by Buniakowski for classical integrals, and by Schwarz for Lebesgue integrals.

Note 15. Since $f(x)$ and $g(x)$ are continuous nonnegative functions so are $f^r(x)$ and $g^{r'}(x)$; hence all the above Riemann integrals exist. The proofs that follow are valid for Lebesgue integrals provided all integrals exist. Equality will hold for Lebesgue integrals if the conditions for equality stated in (6.2) and (6.4) hold for almost all x (instead of for all x) on $c \leqq x \leqq d$.

Our proof of (6.3) is strictly analogous to the proof of Hölder's inequality in section 4. Assume that neither $f(x)$ nor $g(x)$ is identically zero on $c \leqq x \leqq d$, and let

$$S = \left(\int_c^d f^r(x)dx\right)^{1/r} \quad \text{and} \quad T = \left(\int_c^d g^{r'}(x)dx\right)^{1/r'}. \tag{6.6}$$

Since $S \neq 0$ and $T \neq 0$, we may choose $a=f(x)/S$ and $b=g(x)/T$ in (3.3) obtaining

$$\frac{f(x)}{S}\,\frac{g(x)}{T} \leqq \frac{1}{r}\,\frac{f^r(x)}{S^r} + \frac{1}{r'}\,\frac{g^{r'}(x)}{T^{r'}} \qquad (c \leqq x \leqq d). \tag{6.7}$$

Since S and T are definite integrals, they are constants; hence we may factor them out from under the integral signs when we integrate both sides of (6.7). This yields

$$\begin{aligned} \frac{1}{ST}\int_c^d f(x)g(x)dx &\leqq \frac{1}{r}\left[\frac{\int_c^d f^r(x)dx}{S^r}\right] + \frac{1}{r'}\left[\frac{\int_c^d g^{r'}(x)dx}{T^{r'}}\right] \\ &= \frac{1}{r}[1] + \frac{1}{r'}[1] = 1. \end{aligned} \tag{6.8}$$

Multiplying (6.8) by ST yields $\int_c^d f(x)g(x)dx \leqq ST$ which, in view of (6.6), is Hölder's inequality (6.3).

Question: Are we sure that integrating both sides of an inequality like (6.7) really preserves the inequality yielding the inequality (6.8)? *Answer*: Yes, for if we regard the left side of (6.7) as a continuous function $\phi(x)$ and the right hand side of (6.7) as a continuous function $\psi(x)$, we may call upon the following theorem from elementary calculus:

Let $\phi(x)$ and $\psi(x)$ be continuous functions satisfying the inequality $\phi(x) \leqq \psi(x)$ for all x on $c \leqq x \leqq d$. Then we have

$$\int_c^d \phi(x)dx \leqq \int_c^d \psi(x)dx. \tag{6.9}$$

Moreover, equality holds in (6.9) *if and only if*

$$\phi(x) = \psi(x) \qquad \text{for all } x \text{ on } c \leqq x \leqq d. \tag{6.10}$$

This theorem is obvious if interpreted in terms of areas.

From (6.9) and (6.10) we now know that our integrals are equal when and only when their integrands are equal for all x on $c \leqq x \leqq d$. Thus we have equality in Hölder's inequality if and only if equality holds in (6.7) for all x on $c \leqq x \leqq d$. It then follows from (3.5) that

$$\frac{g(x)}{T} = \left[\frac{f(x)}{S}\right]^{r-1} \qquad \text{for all } x \text{ on } c \leqq x \leqq d \tag{6.11}$$

is a necessary and sufficient condition for equality in (6.3). Since S and T are constants, (6.11) is equivalent to (6.4).

To prove Minkowski's inequality for integrals, the reader must go through the same procedures with integrals as we did with sums in section 5. Substituting $k\,f(x)$ for $g(x)$ in both sides of (6.5) and showing that each side reduces to $(1+k)\left[\int_c^d f^r(x)dx\right]^{1/r}$ is the easy way to show that $g(x) = k\,f(x)$ is sufficient to ensure equality. An argument analogous to that leading to (5.6) would show that $g(x) \equiv k\,f(x)$ is also necessary for equality in Minkowski's inequality.

Note 16. If we allow $f(x)$ and $g(x)$ to be discontinuous, $\phi(x)$ and $\psi(x)$ in (6.9) may also be discontinuous. In this case the integrals in (6.9) will remain equal even though $\phi(x) \neq \psi(x)$ at a denumerable set of values of x on $c \leqq x \leqq d$. Thus we are assured of equality in (6.1), (6.3), (6.5) if the condition stated for equality in each case holds for all but a denumerable set of values of x on $c \leqq x \leqq d$.

Note 17. We could have proved each of the three inequalities for integrals from the corresponding inequality for sequences, by approximating the integrals by step functions and using the corresponding inequality for sequences on the step functions. To write out such proofs in detail is much more lengthy and cumbersome than our method.

Note 18. A very condensed derivation of the Hölder, Cauchy-Schwarz, and Minkowski inequalities from Young's inequality is presented on pages 20–22 of Volume 1, *Metric and Normed Spaces*, of the two-volume work, *Elements of the Theory of Functions and Functional Analysis*, by Kolmogorov and Fomin. These inequalities are there crucial in proving that the distance, $\rho(x, y)$, between points x and y of the authors' numerous examples of metric spaces satisfies the *triangle inequality*:

$$\rho(x, y) + \rho(y, z) \geqq \rho(x, z).$$

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A REMARK CONCERNING THE DEFINITION OF A FIELD

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A convenient and well-known method of defining a field utilizes the notion of an abelian group. Thus, by a field we mean any algebraic system, say $(S, +, \cdot, 0, 1)$, where $+$ and $\cdot$ are binary operators on S, $0 \in S$ and $1 \in S$, which possesses the following properties.

(i) $(S, +, 0)$ is an abelian group.

(ii) $(S-\{0\}, \cdot', 1)$ is an abelian group, where $\cdot'$ is $\cdot$ restricted to $S-\{0\}$.

(iii) The following propositions are true about $(S, +, \cdot, 0, 1)$:

(a) $0 \neq 1$

(b) $\forall x \forall y \forall z[x\cdot(y+z) = x\cdot y + x\cdot z]$

(c) $\forall x \forall y \forall z[(y+z)\cdot x = y\cdot x + z\cdot x]$.

At first sight, it might appear that (c) is unnecessary. The purpose of this note is to demonstrate that (c) is *not* a consequence of the other postulates.

Consider the algebraic system $(\{0, 1\}, +, \cdot, 0, 1)$, where $+$ and $\cdot$ are defined by the following tables:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	1
1	0	1

Clearly, $(\{0, 1\}, +, 0)$ is an abelian group, and $(\{1\}, 1)$, is an abelian group. Furthermore, the left-hand distributive law holds, since $\forall x \forall y[x\cdot y = y]$ holds in the given algebraic system. However, the right-hand distributive law fails, since $(1+1)\cdot 1 = 1$ while $1\cdot 1 + 1\cdot 1 = 0$. It follows that (c) is not a consequence of the remaining field postulates.

To clarify the intended meaning of postulates (i), (ii), and (iii), we spell out these statements in terms of propositions about the algebraic system $(S, +, \cdot, 0, 1)$.