The Adventures of π-Man: Measuring the Universe

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Measuring the Earth with a Pocket Watch. Our indefatigable hero π-man memorized all the theorems in the 13 volumes of Euclid's *Elements*, and when he was done, declared himself a geometer. Imagine his dismay when he found out that geometry (γεομετρψ) *really* means "measuring the Earth."

Undaunted, π-man undertook to measure the circumference of the Earth. Unfortunately, he had no scientific instruments to help him in this labor, only an old pocket watch.

Adventure #1.

π-man journeyed to the south and found himself on the equator at dawn of the first day of spring. He climbed a tree to watch the sunrise. Just as π-man saw the sun peep over the horizon, he fell out of the tree, landing on the ground two seconds later, according to his watch.

Recovering his breath while lying prone on the ground, 32 seconds later π-man saw the sun rise again.

"Aha!" our champion exclaimed. "The circumference of the earth is 25,000 miles."

Adventure #2. The next night π-man flew back home to Denver. At midnight he set his watch up on its edge, with the face to the west and the minute hand pointing at the northern horizon. He waited as the minute hand rose higher and higher until finally it was pointing at the North Star. This took 6 minutes, 40 seconds.

Illustration courtesy of Greg Nemec

Then, 32 seconds later π-man saw the sun rise again. "Aha!" our champion exclaimed. "The circumference of the earth is 25,000 miles."

The next night he drove to Colorado Springs, 69 miles due south, and repeated the experiment. This time it took 6 minutes, 30 seconds for the minute hand to reach the altitude of the North Star.

"Just as I thought," said π-man."25,000 miles."

It's simple, Simon. In adventure #1 it took two seconds for π-man to fall from the top of the tree to the ground. Knowing that the acceleration due to gravity is 32 feet per second squared, π-man used the resulting formula $h(t) = -16t^2 + h_0$ to compute the height of the tree, 64 feet.

Next, π-man considered the fact that he saw the sun rise twice, first from a height of 64 feet, then a total of 34 seconds later while lying on the ground. As shown in Figure 1, the angle θ through which the Earth rotated between the two sunrises is determined by the ratio $\theta/360° =$ time /24 hours. Thus $\theta = (34 \text{ seconds}/86400 \text{ seconds}) \times 360° \approx 0.14°$.

Again from the picture, $\sec\theta = (R + 64)/R = 1 + 64/R$, where R is the radius of the earth. Thus $R = 64/(\sec\theta - 1) \approx 2.1\times10^7$ feet, or about 4000 miles, and the circumference is $C = 2\pi R \approx 25{,}000$ miles.

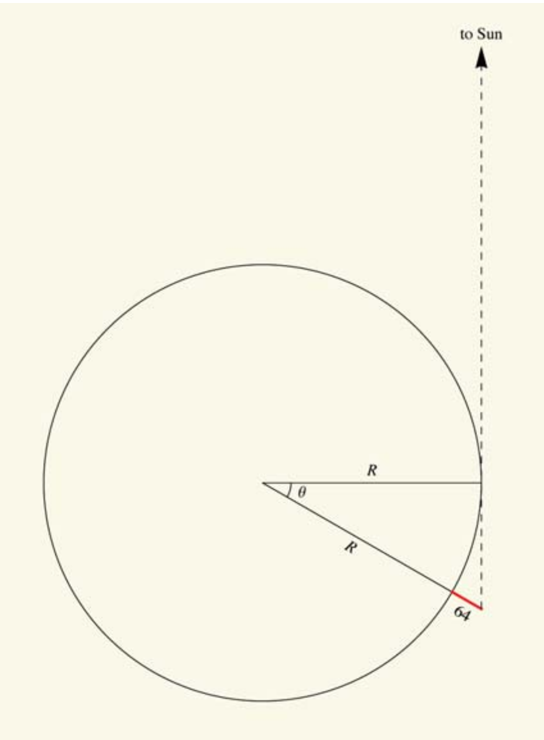

Figure 1. Measuring the Earth with a pocket watch.

In adventure #2, π-man is using his pocket watch as an *astrolabe*—a device for measuring the altitude of stars. As shown in Figure 2, the altitude θ of the North Star from any place in the Northern Hemisphere is the same as the observer's *latitude*, θ'. π-man determined the latitude of Denver by the ratio

latitude (degrees North)/360° = 6.667 minutes/60 minutes,

and similarly for Colorado Springs. This gives a latitude of 40° for Denver and 39° for Colorado Springs, for a difference of 1 degree.

Now π-man followed the lead of Eratosthenes, who brilliantly applied similar reasoning 2200 years ago. Since the distance of 69 miles between the two cities is 1/360th of the distance around the entire earth, we conclude that the circumference is $C = 69 \times 360 \approx 25{,}000$ miles.

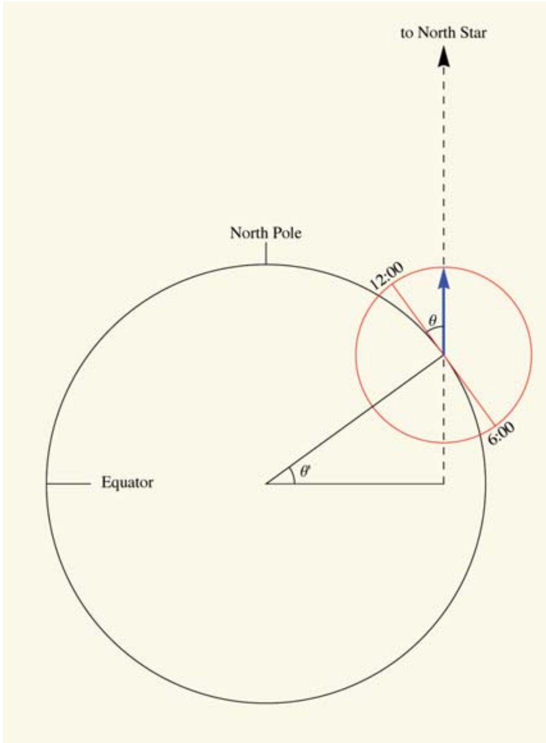

Figure 2. Measuring latitude.

Adventure #3: Measuring Time with a Yardstick. π-man met a variety of interesting folk while going to the fair. Once there, he took a ride on the merry-go-round. He was curious to know how fast the merry-go-round was spinning. Alas, as he took out his pocket watch to time a revolution, he dropped it and it broke.

Undeterred, π-man whipped out his meter-stick and measured the radius of the merry-go-round from the center to the rim. The radius was exactly five meters.

Then he measured the circumference by walking slowly around the rim. Now, according to π-man's geometry book, the circumference ought to be

$C_e = 2\pi r = 10\pi \approx 31.41592653589793\underline{23846}$ meters.

But by actual measurement the circumference turned out to be

$C_o = 31.41592653589793\underline{31502}$ meters.

Yes! The merry-go-round is spinning at the rate of 4 revolutions per minute.

π-man explains. Here is a "paradox" from the early days of Einstein's Special Theory of Relativity. Consider a spinning disc (the merry-go-round in π-man's adventure). As measured by an observer outside the disc, the relation between the circumference and the radius of the disc, whether the disc is spinning or not, follows the familiar Euclidean model $C = 2\pi r$.

But if the same disc is measured by someone riding along with it on the perimeter, C turns out to be greater than $2\pi r$. This is because of the *Lorentz-Fitzgerald contraction*, observed even before Einstein's time in experiments designed to measure the speed of light. The exact formula for the circumference is

(1) $C = 2\pi r/\sqrt{1 - v^2/c^2} \approx 2\pi r(1 + v^2/(2c^2)) = 2\pi r + \pi r v^2/c^2$

where v is the tangential velocity of the rim of the disc and where $c \approx 3 \times 10^8$ m/sec is the speed of light.

In π-man's measurements, $r = 5$ meters and the difference between the observed value of C and the expected Euclidean value $2\pi r$ was 7.656×10^{-16} meters. From equation (1) this gives a tangential velocity of $v = 2.0944$ m/sec and an angular velocity of $\omega = v/r = .4189$ radians per second, or about 4 revolutions per minute.

Adventure #4: Weighing the Earth with a Yardstick. Having previously measured the circumference of the Earth with nothing but a pocket watch, π-man's next task was to determine its mass. Fortunately, π-man had all the tools he needed—his trusty meter-stick and a tunnel through the center of the Earth. Using these resources, π-man measured the diameter of the Earth.

Knowing both the radius and the circumference of the Earth, π-man again tested the formula $C = 2\pi r$. But this time he found that the radius was 1.48 millimeters too long!

"Very interesting," said π-man. "The mass of the Earth is 6,000,000,000,000,000,000,000 metric tons."

π-man explains. According to Einstein's General Theory of Relativity, the presence of mass distorts the geometry of space. A formula due to Karl Schwarzschild quantifies this distortion in the case of a large spherical body like the Earth. According to the Schwarzschild solution of the Einstein field equations, if a great circle is drawn around a spherical gravitating mass of uniform density, the observed radius r_o of the circle is *longer* than the expected Euclidean value $r_e = C/2\pi$. The exact formula is

(2) $r_o = r_e\sqrt{c^2 r_e / 2GM}\,\arcsin\left(\sqrt{2GM / c^2 r_e}\right)$

where M is the mass of the sphere, c is the speed of light and $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is the universal gravitational constant.

π-man also remembered his calculus. For very small values of the argument, the first two terms of the MacLauren series for the arcsine function give an excellent approximation: $\arcsin(t) \approx t + t^3/6$.

For $t = \sqrt{2GM / c^2 r_e}$, G is tiny and c^2 is huge. In fact, $2GM$ is always much less than $c^2 r_e$ for ordinary material objects. If $2GM = c^2 r_e$, the object is, by definition, a black hole. Thus from (2) we have

(3) $r_o \approx r_e\sqrt{c^2 r_e / 2GM}\left(\sqrt{2GM / c^2 r_e} + \left(\sqrt{2GM / c^2 r_e}\right)^3 / 6\right)$

$= r_e + GM / 3c^2$

Solving for M, we obtain $M = 3c^2(r_o - r_e)/G$. For $(r_o - r_e) = .00148$ meter and for c and G as above, the mass of the Earth is $M = 6 \times 10^{24}$ kilograms.

π-man's Last Bow: Measuring the Universe with Nothing at All. π-man broke his pocket watch while riding on the carousel. π-man lost his meter-stick while measuring the sun. He gave away his straightedge and his pair of shiny compasses. π-man the geometer was done.

But in the repose of retirement, π-man turned his attention to contemplating the size, shape, and ultimate fate of the universe.

π-man looked through the Hubble telescope and found that there were ten thousand galaxy clusters at a distance of one billion light years. Then he cranked up the power and counted 40,100 at a distance of two billion light years.

"Ah," sighed π-man "The curvature of space is –0.00000000005 per light year. The universe is infinite and will expand forever, leaving us to freeze to death in the dark."

π-man explains. One way to distinguish between Euclidean and non-Euclidean geometry is by measuring the surface area of spheres. For a sphere of radius r in three-dimensional Euclidean space we have the surface area formula $A = 4\pi r^2$. But in a three-dimensional space with constant positive or negative scalar curvature k, the formula must be modified accordingly. For positively curved space,

(4) $A = (4\pi / k^2) \sin^2(kr)$.

And for negatively curved space,

(5) $A = (4\pi / k^2) \sinh^2(kr)$,

where sinh is the hyperbolic sine function:

$$\sinh(t) = \frac{e^t \quad e^{-t}}{2}.$$

In formula (4) the surface area is always *less* than $4\pi r^2$, and in formula (5) the surface area is always *greater* than $4\pi r^2$. This gives us a method of determining whether the universe is Euclidean or non-Euclidean.

According to the standard hot big bang model in cosmology, if space is positively curved then the universe is finite and the expansion of the universe will slow down and eventually reverse. If the curvature of space is negative, then the universe is infinite and its expansion will continue unimpeded forever. If the universe is Euclidean (curvature = 0), then the universe will also continue to expand, but at an ever-slowing rate. (This model was drastically complicated, however, by the discovery in 1998 of "dark energy," which is believed to accelerate the expansion of the universe and thus to mimic the effect of negative curvature.)

The experiment described in π-man's adventure is one way in which cosmologists go about measuring the global curvature of the universe. So far, all such measurements are compatible with the Euclidean model. The curvature of the universe is either zero or so close to zero that we have not yet been able to measure it.

But π-man did!

Assuming a uniform distribution of matter in the universe (and after compensating for the evolution of the universe and its contents over the last two billion years!) we may use galaxy counts as a surrogate for the surface area of large imaginary spheres centered at the earth. In the Euclidean model, the number of galaxy clusters at distance r should grow like the square of the distance: $A = 4\pi r^2$. Since in π-man's experiment he saw *more* than four times as many galaxy clusters when he doubled the radius from one billion light-years to two, we must have $A > 4\pi r^2$. Therefore we are in the situation of negative curvature: $A(r) = (4\pi/k^2)\sinh^2(kr)$, with $A(1) \sim 10$ and $A(2) \sim 40.1$. (r is measured in billions of light years and A in thousands of galaxy clusters times a constant factor depending only on the average density of matter in the universe).

Forming the ratio of our two area-surrogates, we have

$40.1/10 = A(2)/A(1) = \sinh^2(2k)/\sinh^2(k)$, or

$\sqrt{4.01} = \sinh(2k)/\sinh(k) = 2\sinh(k)\cosh(k)/\sinh(k) = 2\cosh(k)$,

and the curvature of the universe is $k = \cosh^{-1}(1/2\sqrt{4.01}) = -0.05$ (billion light-years)$^{-1}$.

Suggestions for further reading

Kitty Ferguson, *Measuring the Universe, Our Heroic Quest to Chart the Horizons of Space and Time*, Walker and Company, 1999.

Stephen Hawking and George F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973.

Leornard Mlodinow, *Euclid's Window: The Story of Geometry from Parallel Lines to Hyperspace*, Free Press, 2002.

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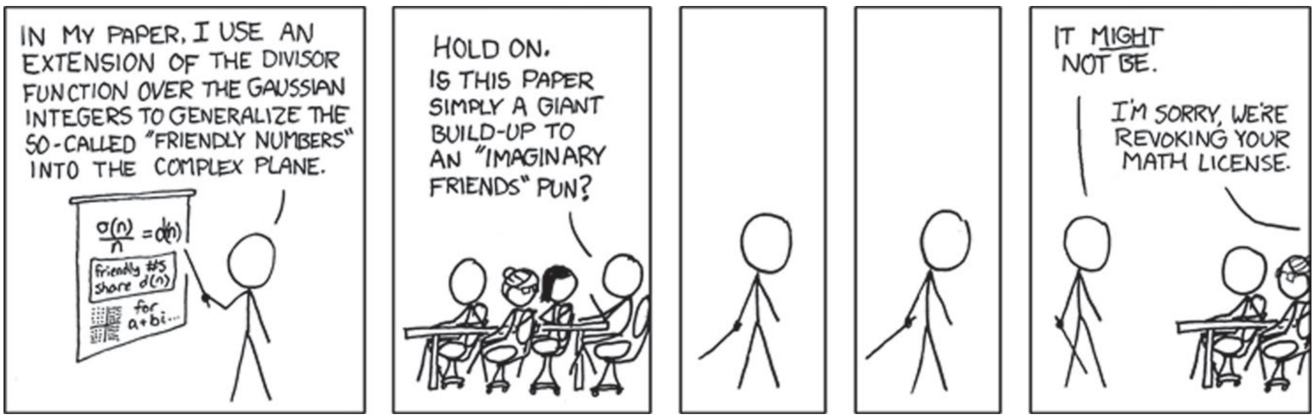
Math Paper



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