REAL PROOFS OF COMPLEX THEOREMS (AND VICE VERSA)

LAWRENCE ZALCMAN

Introduction. It has become fashionable recently to argue that real and complex variables should be taught together as a unified curriculum in analysis. Now this is hardly a novel idea, as a quick perusal of Whittaker and Watson's *Course of Modern Analysis* or either Littlewood's or Titchmarsh's *Theory of Functions* (not to mention any number of *cours d'analyse* of the nineteenth or twentieth century) will indicate. And, while some persuasive arguments can be advanced in favor of this approach, it is by no means obvious that the advantages outweigh the disadvantages or, for that matter, that a unified treatment offers any substantial benefit to the student. What is obvious is that the two subjects do interact, and interact substantially, often in a surprising fashion. These points of tangency present an instructor the opportunity to pose (and answer) natural and important questions on basic material by applying real analysis to complex function theory, and vice versa. This article is devoted to several such applications.

My own experience in teaching suggests that the subject matter discussed below is particularly well-suited for presentation in a year-long first graduate course in complex analysis. While most of this material is (perhaps by definition) well known to the experts, it is not, unfortunately, a part of the common culture of professional mathematicians. In fact, several of the examples arose in response to questions from friends and colleagues. The mathematics involved is too pretty to be the private preserve of specialists. Publicizing it is the purpose of the present paper.

1. The Greening of Morera. One of the most useful theorems of basic complex analysis is the following result, first noted by Giacinto Morera.

MORERA'S THEOREM [37]. *Let $f(z)$ be a continuous function on the domain D. Suppose that*

$$\int_\gamma f(z)dz = 0 \tag{1}$$

for every rectifiable closed curve γ lying in D. Then f is holomorphic in D.

Morera's Theorem enables one to establish the analyticity of functions in situations where resort to the definition and the attendant calculation of difference quotients would lead to hopeless complications. Applications of this sort occur, for instance, in the proofs of the Schwarz Reflection Principle and other theorems on the extension of analytic functions. Nor is its usefulness limited to this circle of ideas; the important fact that the uniform limit of analytic functions is again analytic is an immediate consequence (observed already by Morera himself, as well as by Osgood [39], who had rediscovered Morera's theorem).

Perhaps surprisingly, the proofs of Morera's theorem found in complex analysis texts all follow a single pattern. The hypothesis on f insures the existence of a single-valued primitive F of f, defined by

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta. \tag{2}$$

Here z_0 is some fixed point in D and the integral is taken over any rectifiable curve joining z_0 to z. The function F is easily seen to be holomorphic in D, with $F'(z) = f(z)$; since the derivative of a holomorphic function is again holomorphic, we are done.

Several remarks are in order concerning the proof sketched above. First of all, the assumption that (1) holds for all rectifiable closed curves in D is much too strong. It is enough, for instance, to assume that (1) holds for all closed curves consisting of a finite number of straight line segments parallel to the coordinate axes; the integration in (2) is then effected over a (nonclosed) curve composed of such segments, and the proof proceeds much as before. Second, since analyticity is a local property, condition (1) need hold only for an arbitrary neighborhood of each point of D; that is, (1) need hold only for *small* curves. Finally, the proof requires the fact that the derivative of an analytic function is again analytic. While this is a trivial consequence of the Cauchy integral formula, it can be argued that that is an inappropriate tool for the problem at hand; on the other hand, a proof of this fact without complex integration is genuinely difficult and was, in fact, only discovered (after many years of effort) in 1961 [**44**], [**10**], [**46**].

There is an additional defect to the proof, and that is that *it does not generalize*. Thus, it was more than thirty years after Morera discovered his theorem that Torsten Carleman realized the result remains valid if (1) is assumed to hold only for all (small) *circles* in D. It is an extremely instructive exercise to try to prove Carleman's version of Morera's theorem by mimicking the proof given above. The argument fails because it cannot even be started: the very existence of a single-valued primitive is in doubt. This leads one to try a different (and more fruitful) approach, which avoids the use of primitives altogether.

Suppose for the moment that f is a smooth function, say continuously differentiable. Fix $z_0 \in D$ and suppose (1) holds for the circle $\Gamma_r(z_0)$ of radius r, centered at z_0. Then, by the complex form of Green's theorem

$$0 = \int_{\Gamma_r(z_0)} f(z)dz = 2i \iint_{\Delta_r(z_0)} \frac{\partial f}{\partial \bar{z}}\, dxdy,$$

where $\Delta_r(z_0)$ is the disc bounded by $\Gamma_r(z_0)$ and $\partial f/\partial \bar{z} = \frac{1}{2}\,(\partial f/\partial x + i\, \partial f/\partial y)$. (There's no cause for panic if the $\partial/\partial \bar{z}$ operator makes you uneasy or you are not familiar with the complex form of Green's theorem; just write $f(z) = u(z) + iv(z)$, $dz = dx + idy$, and apply the usual version of Green's theorem to the real and imaginary parts of the integral on the left.) Dividing by an appropriate factor, we have

$$\frac{1}{\pi r^2}\iint_{\Delta_r(z_0)}\frac{\partial f}{\partial \bar{z}}dxdy = 0;$$

i.e., the average of the continuous function $\partial f/\partial \bar{z}$ over the disc $\Delta_r(z_0)$ equals 0. Make $r \to 0$ to obtain $(\partial f/\partial \bar{z})(z_0) = 0$. Since this holds at each point $z_0 \in D$, $\partial f/\partial \bar{z} = 0$ identically in D. Writing this in real coordinates, we see that $u_x = v_y$, $u_y = -v_x$ in D; thus the Cauchy-Riemann equations are satisfied and f is analytic.

Notice that we did not need to assume that (1) holds for all circles in D or even all small circles; to pass to the limit it was enough to have, for each point of D, a sequence of circles shrinking to that point. Moreover, since f has been assumed to be continuously differentiable, it is sufficient to prove that $\partial f/\partial \bar{z}$ vanishes on a dense set. Finally, and most important, *the fact that our curves were circles was not used at all*! Squares, rectangles, pentagons, ovals could have been used just as well. To conclude that $(\partial f/\partial \bar{z})(z_0) = 0$, all we require is that (1) should hold for a sequence of simple closed curves γ that accumulate to z_0 (z_0 need not even lie inside or on the γ's) and that the curves involved allow application of Green's theorem. It is enough, for instance, to assume that the curves are piecewise continuously differentiable.

To summarize, we have shown that Green's theorem yields in a simple fashion a very general and particularly appealing version of Morera's theorem for C^1 functions. It may reasonably be asked at this point if the proof of Morera's theorem given above can be modified to work for functions which are assumed only to be continuous. That is the subject of the next section.

2. Smoothing. Let $\phi(z)$ be a real valued function defined on the entire complex plane which satisfies

(a) $\phi(z) \geqq 0$,

(b) $\iint \phi(z)dxdy = 1$,

(c) ϕ is continuously differentiable,

(d) $\phi(z) = 0$ for $|z| \geqq 1$.

It is trivial to construct such functions; we can even require ϕ to be infinitely differentiable and to depend only on $|z|$, but these properties will not be required in the sequel. Set, for $\varepsilon > 0$, $\phi_\varepsilon(z) = \varepsilon^{-2}\phi(z/\varepsilon)$. Then, clearly, ϕ_ε satisfies (a) through (c) above and ϕ_ε vanishes off $|z| < \varepsilon$. The family of functions $\{\phi_\varepsilon\}$ forms what is known in harmonic analysis as an approximate identity (a smooth approximation to the Dirac delta function); workers in the field of partial differential equations, where the smoothness properties of the ϕ_ε are emphasized, are accustomed to call similar functions (Friedrichs) mollifiers.

Suppose now that f is a continuous function on some domain D and set

$$f_\varepsilon(z) = \iint f(z-\zeta)\phi_\varepsilon(\zeta)d\xi d\eta \qquad \zeta = \xi + i\eta, \tag{3}$$

where the integral is extended over the whole complex plane. This integral exists and defines a continuous function for all points z whose distance from the boundary of D is greater than ε. Moreover, $f_\varepsilon(z)$ is continuously differentiable for such points. Indeed, changing variable in (3), we have

$$f_\varepsilon(z) = \iint \phi_\varepsilon(z-\zeta)f(\zeta)d\xi d\eta$$

and the x and y derivatives can be brought inside the integral since we have chosen ϕ_ε to be continuously differentiable. Finally, we note that for any compact subset K of D, $f_\varepsilon(z)$ converges uniformly to $f(z)$ on K as $\varepsilon \to 0$. This expresses the delta-function-like behavior of the family $\{\phi_\varepsilon\}$. Here is the simple proof. By (b),

$$f(z) - f_\varepsilon(z) = \iint_{|\zeta| \leqq \varepsilon} \{f(z) - f_\varepsilon(z-\zeta)\}\phi_\varepsilon(\zeta)d\xi d\eta,$$

whence by (a) and (b)

$$\begin{aligned} |f(z) - f_\varepsilon(z)| &\leqq \iint_{|\zeta| \leqq \varepsilon} |f(z) - f(z-\zeta)|\,\phi_\varepsilon(\zeta)d\xi d\eta \\ &\leqq \sup_{|\zeta| \leqq \varepsilon} |f(z) - f(z-\zeta)|. \end{aligned} \tag{4}$$

Since K is compact, f is uniformly continuous on K; so (4) shows that

$$\sup_{z \in K} |f(z) - f_\varepsilon(z)| \to 0 \qquad \text{as } \varepsilon \to 0.$$

The proof of Morera's theorem is now easily completed. Suppose, for instance, that f is continuous on D and that there exists a sequence of positive numbers $r_1 \geqq r_2 \geqq r_3 \geqq \cdots \to 0$ such that

$$\int_{\Gamma_{r_n}(z)} f(w)dw = 0 \tag{5}$$

for each $z \in D$ whenever the circle $\Gamma_{r_n}(z) = \{w : |w - z| = r_n\}$ lies in D. Fix a compact set $K \subset D$ and take $\varepsilon < \frac{1}{2}\operatorname{dist}(K, \partial D)$. Then for $r = r_n < \frac{1}{2}\operatorname{dist}(K, \partial D)$ and $z \in K$ we have

$$\begin{aligned} \int_{\Gamma_r(z)} f_\varepsilon(w)dw &= \int_{\Gamma_r(z)} \left\{ \iint f(w-\zeta)\phi_\varepsilon(\zeta)d\xi d\eta \right\} dw \\ &= \iint \left\{ \int_{\Gamma_r(z)} f(w-\zeta)dw \right\} \phi_\varepsilon(\zeta)d\xi d\eta \\ &= \iint \left\{ \int_{\Gamma_r(z-\zeta)} f(w)dw \right\} \phi_\varepsilon(\zeta)d\xi d\eta \\ &= 0. \end{aligned}$$

Since f_ε is continuously differentiable, it is analytic on the interior of K; and since f_ε converges to f uniformly on K, f must be analytic there. Finally, because K is arbitrary, f is analytic on all of D.

Again, there is nothing particularly sacred about circles: if $\{\gamma_n\}$ is a sequence of simple closed piecewise continuously differentiable curves which shrink to the origin and $\gamma_n(z)$ is the image of γ_n under the map $w \mapsto w + z$, we may replace (5) by

$$\int_{\gamma_n(z)} f(w)dw = 0 \tag{6}$$

and the rest of the argument remains unchanged. Similarly, it is enough to assume that (5) or (6) hold only for a dense set of $z \in D$, since the full condition then follows from the continuity of f.

The result can be extended even further. The requirement that f be continuous may be relaxed to the assumption that f is measurable and integrable with respect to Lebesgue area measure on compact subsets of D. Of course, the conclusion now reads that f agrees almost everywhere with a function analytic on D. For a complete treatment, together with an historical discussion, see [**62**]. The use of smoothing operators is a standard tool among workers in partial differential equations and approximation theory; for a systematic exposition of its use in this last subject, see [**52**].

The success of the smoothing technique in dealing with Morera's theorem suggests using it to prove Cauchy's theorem. This is a good idea, but one which, unfortunately, simply does not work. Here's the rub. Suppose $f(z)$ is analytic in the disc D. We know (by Green's theorem) that $\int_T f(z)dz = 0$ for every triangle T in D *if* f is continuously differentiable. Of course, in general, f is *not* known *a priori* to be continuously differentiable; but we may construct $f_\varepsilon(z)$, as in (3), which is. *However, it is not clear that* $f_\varepsilon(z)$ *is holomorphic.* The problem is that while $f'(z)$ is known to exist for each $z \in D$, and is easily proved to be measurable, it is *not* known to be integrable; we cannot, therefore, differentiate f inside the integral sign of (3). (A similar difficulty arises in the proof of Hartogs' theorem: *If a function of two complex variables* $g(z_1, z_2)$ *is analytic in each variable separately, then it is analytic as a function of the joint variables* z_1, z_2). The argument does work if f' is assumed to be area integrable, but this assumption is (of course) unnecessary, and it seems best to base the proof of Cauchy's theorem on Pringsheim's device [**45**] of subdividing triangles. This is the pattern followed in most modern texts.

3. In circles. All the versions of Morera's theorem discussed up to now have depended in an essential fashion on the fact that (1) holds for a certain class of contours containing arbitrarily small curves. The obvious question to ask is what happens if (1) holds for circles which do *not* shrink in radius. In this situation, it is natural to assume that the function in question is defined on the entire complex plane. A satisfying answer is provided by the following result, proved in 1970 [**62**].

THEOREM. *Let f be a continuous function on the complex plane and suppose that there exist numbers $r_1, r_2 > 0$ such that*

$$\int_{\Gamma} f(z)dz = 0 \tag{7}$$

for every circle having radius r_1 or r_2 (and arbitrary center). Then f is an entire function unless r_1/r_2 is a quotient of zeroes of the Bessel function $J_1(z)$.

The hypothesis on f may be relaxed to the assumption of local integrability, and (7) need hold only for 'almost all' circles. The restriction on the pair r_1, r_2 is, however, essential: in case it is not satisfied, f may fail to be holomorphic anywhere.

The proof is considerably more involved than (and of an altogether different character from) the sort of argument we have seen in the preceding sections; essential ingredients include the harmonic analysis of an appropriate space of distributions and the Delsarte-Schwartz theory of mean-periodic functions. See [**62**], where related results are discussed, for details. One can also show that if f is continuous on the plane and (7) holds for every square (of arbitrary center and orientation) having side of fixed length, then f is entire. Again, a reference is [**62**]. Further perspectives on results of this sort will be found in [**63**].

4. Reflections on reflection. According to the Schwarz Reflection Principle, if $f(z)$ is analytic in $\Delta = \{z: |z| < 1\}$ and continuously extendible to an open arc γ of $\Gamma = \{z: |z| = 1\}$, and if the values of f corresponding to points of γ lie on a circular, or, more generally, an analytic arc γ^*, then f may be extended by 'reflection' to a function analytic in a domain containing $\Delta \cup \gamma$. The usefulness of this technique can hardly he overestimated: it provides an essential tool in problems involving the extension of conformal mappings and plays a traditional role in the 'slick' proof [**49**, pp. 322–325] of Picard's little theorem. Another application yields what is surely the simplest proof that a nonzero function analytic in Δ cannot vanish identically on an arc of Γ.

The question thus naturally arises whether an analogous result holds if γ^* is no longer analytic but simply smooth, C^∞ say. A negative answer is immediate. Indeed, let Γ^* be an infinitely differentiable, nowhere analytic, simple closed Jordan curve and let f map Δ conformally onto the interior D of Γ^*. The univalent function f, extends to a homeomorphism of $\Delta \cup \Gamma$ onto $D \cup \Gamma^*$ and induces a one-one correspondence between the points of Γ and those of Γ^*. However, f cannot be continued analytically across any subarc of Γ, for then f would establish an analytic correspondence between a subarc γ of Γ and a subarc γ^* of Γ^*. Thus γ^* would be analytic, contrary to hypothesis. This example is really quite striking, providing, as it does, an example of a (univalent!) function analytic on Δ and of class C^∞ on $\Delta \cup \Gamma$ which cannot be extended analytically across any arc of Γ.

What is not generally realized is that the example can be worked backward to provide an example of an infinitely differentiable, yet nowhere analytic, Jordan curve. This approach avoids altogether reliance on the plausible (and true) but nonobvious facts concerning smoothness and univalence of the boundary function which we invoked so shamelessly above. The tools we need are two, the first of which is the following simple lemma.

LEMMA. *Let* $f(z) = z + a_2 z + a_3 z^3 + \cdots$ *be analytic in* Δ. *Suppose that* $\sum_{n=2}^{\infty} n \mid a_n \mid < 1$. *Then* f *is continuous on* $\Delta \cup \Gamma$ *and univalent there.*

Proof. Continuity is clear from the absolute convergence of the series. Let $z, \zeta \in \Delta \cup \Gamma$. Then

$$\frac{f(z)-f(\zeta)}{z-\zeta} = 1 + \sum_{n=2}^{\infty} a_n(z^{n-1} + z^{n-1}\zeta + \cdots + \zeta^{n-1}).$$

Thus

$$\left| \frac{f(z)-f(\zeta)}{z-\zeta} \right| \geqq 1 - \sum_{n=2}^{\infty} n \mid a_n \mid > 0,$$

so that f is univalent.

The second ingredient we need is the celebrated Hadamard gap theorem.

HADAMARD GAP THEOREM. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ *have* Δ *as its disc of convergence. If* $n_{k+1}/n_k \geqq q$ *for some* $q > 1$ *and all large* k, *then* f *has* Γ *as its natural boundary*; *that is,* f *cannot be continued analytically across any subarc of* Γ.

The beautiful proof of this theorem due to L. J. Mordell ([**36**], cf. [**54**, p. 223]) should be standard fare in graduate courses in complex analysis.

The construction of the required function is now almost trivial. We choose the sequences $\{a_k\}$ and $\{n_k\}$ to satisfy

(a) $a_0 = n_0 = 1$,
(b) $\sum_{k=1}^{\infty} n_k \mid a_k \mid < 1$,
(c) $(a_k)^{1/n_k} \to 1$,
(d) $n_{k+1}/n_k \geqq 2$,
(e) $\sum_{k=0}^{\infty} n_k^j \mid a_k \mid < \infty \qquad j = 0, 1, 2, \cdots$.

A simple concrete example is provided by the function

$$f(z) = z + \sum_{n=5}^{\infty} z^{2^n}/n!.$$

By the lemma, f establishes a homeomorphism between Γ and a simple closed Jordan curve Γ^*. Since f satisfies the hypothesis of Hadamard's gap theorem, f cannot be extended analytically across any arc of Γ. Hence, Γ^* must be nowhere analytic since otherwise the Schwarz principle would apply. Finally, by (e), the series for $f^{(j)}(z)$ converges absolutely on $\{z : \mid z \mid \leqq 1\}$ for each j; thus f is infinitely differentiable on $\Delta \cup \Gamma$, so that Γ^* is a C^∞ curve.

Interestingly enough, one can trace the basic ideas of this section back to before the turn of the century, (see Osgood [**38**]). In particular, the lemma, which is usually attributed to the American topologist J. W. Alexander [**67**], was known to Fredholm as early as 1897 ([**38**, p. 17]).

5. **Extensions.** The reflection principle enables one (in certain circumstances) to extend a holomorphic function across an analytic arc to a somewhat larger domain. As we have seen, it is in general impossible to relax the condition of analyticity; nevertheless, the much weaker hypothesis of rectifiability suffices in case a continuous extension analytic in an abutting domain is already known. The precise result may be stated (somewhat informally) as follows.

THEOREM. *Let D be a domain and let J be a simple rectifiable Jordan arc dividing D into disjoint domains D_1 and D_2. Suppose f_j $(j = 1,2)$ is analytic in D_j and continuous on $D_j \cup J$ and that $f_1 = f_2$ on J. Then the function f obtained by setting $f(z) = f_j(z)$ for $z \in D_j \cup J$ is analytic in D.*

The proof is a standard application of Morera's theorem, with due care exercised in dealing with the assumption that J is merely rectifiable.

The precise nature of the hypothesis of rectifiability on J in the above theorem is by no means clear, and the proof (which we leave to the reader) does little to explicate it. My experience has been that students — especially good ones — generally guess that the result remains true if rectifiability is dispensed with. This, however, is *not* the case, as the following example shows.

Let K be a compact set of positive Lebesgue measure and set

$$f(z) = \iint_K \frac{d\xi d\eta}{\zeta - z} \qquad \zeta = \xi + i\eta. \tag{8}$$

The function $f(z)$ is obviously analytic off K and satisfies $f(\infty) = 0$; moreover since $\lim_{z \to \infty} zf(z) = -\iint_K d\xi d\eta \neq 0$, f is nonconstant on the unbounded component of K. We claim f is actually continuous on the complex sphere. Indeed, formula (8) exhibits f explicitly as the convolution of the locally (area) integrable function $1/\zeta$ with the bounded measurable function of compact support $\chi_K(\zeta)$, the characteristic function of K. Such a convolution is well known (and easily proved) to be continuous (see, for instance, [**5**, p. 154]).

Suppose now that $K = J$, a simple closed Jordan curve. The existence of such curves having positive area was first proved by Osgood [**41**] in 1902. (This is one of the relatively few examples in mathematics that retains its original vigor unimpaired: students today — even those who know about Peano curves — are as baffled and surprised by this fact as mathematicians were 70 years ago. The construction is not too complicated for presentation in class, and the example itself instills a healthy respect for the Jordan curve theorem.) One can actually construct J to have the additional property that it has positive area everywhere, that is, if D is an open set and $D \cap J \neq \emptyset$ then $D \cap J$ has positive area. The function f defined by (8) with $J = K$ is continuous on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and analytic off J; thus, it is analytic in both components D_1, D_2 of $\hat{\mathbb{C}} \setminus J$. However, f is not analytic at any point of J. Indeed, suppose f analytic at $z_0 \in J$ and let D be a small open disc about z_0 lying in the domain of analyticity of f. Set $J_1 = D \cap J$. Then

$$f(z) = \int_{J \setminus J_1} \frac{d\xi d\eta}{\zeta - z} + \int_{J_1} \frac{d\xi d\eta}{\zeta - z} = g(z) + h(z)$$

for $z \notin J$, and $g(z)$ is clearly analytic in D. Thus $h(z)$ must be analytic in D as well. But h is obviously analytic off $\bar{D}$ and continuous on $\hat{\mathbb{C}}$. Thus, according to the theorem of the present section, h is analytic on all of $\hat{\mathbb{C}}$, hence a constant. But $J \cap D = J_1$ has positive area, so that $h(z)$ is nonconstant. We have reached the desired contradiction.

Thus f cannot be continued analytically across any arc of J. In particular, the restrictions f_1, f_2 of f to the components D_1, D_2 of $\hat{\mathbb{C}} \setminus J$ determine analytic functions *which are not analytic continuations of one another*; indeed, J forms a natural boundary for each of these functions.

Actually, the requirement that J have positive measure was used merely to insure the existence of nontrivial functions continuous on $\hat{\mathbb{C}}$ and analytic off J. The same result can be obtained (but with more work) if the set in question has positive Hausdorff $(1 + \varepsilon)$-measure for some $\varepsilon > 0$ [**61**]. Even this condition is not necessary; in fact, Denjoy [**11**] has constructed an arc which is the graph of a function and which has the required property.

6. Blowing up the boundary. Questions involving length and area arise in conformal mapping as well. A conformal map, being analytic, must map sets of zero area to sets of zero area; however, distortion at the boundary is an *a priori* possibility. Writing $\Delta \cup \Gamma = \{z : |z| \leqq 1\}$ as before, let us assume that the univalent function $f(z)$ maps Δ conformally onto the Jordan region D. According to the Osgood-Taylor-Carathéodory theorem, f extends to a homeomorphism of $\Delta \cup \Gamma$ onto $D \cup \partial D$. (Proofs of this important result, announced by Osgood [**65**] and proved independently by Osgood and Taylor [**66**, p. 294], and Carathéodory [**6**], [**7**] are available in [**9**, pp. 46–49] and [**24**, p. 129–134]. The reader will find a comparison of the treatments in these references particularly instructive in the matters of style of exposition and attention to detail.) In case ∂D is rectifiable, a theorem of the Riesz brothers [**47**] insures that f and f^{-1} preserve sets of zero length (= Hausdorff one-dimensional measure). When ∂D fails to be rectifiable, however, all hell breaks loose. In particular, a subset of ∂D having positive area may correspond to a subset of Γ having zero Lebesgue (linear) measure! For the construction, we need an important result from plane topology.

MOORE-KLINE EMBEDDING THEOREM [**35**]. *A necessary* and *sufficient condition* *that a compact set $K \subset \mathbb{C}$ should lie on a simple Jordan arc is that each closed connected subset of K should be either a point or a simple Jordan arc with the property that $K - \gamma$ does not accumulate at any point of γ, except (perhaps) the endpoints.*

Now let K be a Cantor set having positive area; K may be realized, for instance as the product of two linear Cantor sets, each of which has positive linear measure. Construct countably many disjoint simple Jordan arcs $J_n \subset \mathbb{C} \setminus K$ such that the sequence $\{J_n\}$ accumulates at each point of K and at no other points of $\hat{\mathbb{C}}$ and with the additional property that if $z_0 \in \mathbb{C} \setminus (K \cup \{J_n\}) = R$ and $z \in K$, then any arc from z_0 to z which lies, except for its final endpoint, in R must have infinite length. By the Moore-Kline embedding theorem, we may pass a simple closed Jordan curve J through $K \cup \{J_n\}$. Let f be a conformal map from Δ to D, the domain bounded by J. Then f extends to a homeomorphism from Γ to J. Let $S = f^{-1}(K)$. That S has zero linear measure follows at once from the following theorem, due to Lavrentiev.

THEOREM. *Let f be a conformal homeomorphism of $\Delta \cup \Gamma$ onto the Jordan domain $D \cup J$. If $S \subset J$ is not rectifiably accessible from D then $f^{-1}(S) \subset \Gamma$ has zero measure.*

Proof. Since D is a bounded domain, its area, given by the expression $\iint_\Delta |f'(z)|^2 dxdy$, is finite. Thus

$$\int_0^{2\pi} \int_0^1 |f'(re^{i\theta})| \, rdrd\theta$$

$$\leqq \left(\int_0^{2\pi} \int_0^1 rdrd\theta \right)^{1/2} \left(\int_0^{2\pi} \int_0^1 |f'(re^{i\theta})|^2 r \, drd\theta \right)^{1/2} < \infty.$$

It follows that $\int_0^1 |f'(re^{i\theta})| \, r \, dr < \infty$ for almost all θ or, what is the same, $l(\theta) = \int_0^1 |f'(re^{i\theta})| \, dr < \infty$ almost everywhere. But $l(\theta)$ is the length of the image of the radius from 0 to $e^{i\theta}$ under f. So almost every point of Γ corresponds to a rectifiably accessible point of J, and we are done.

Actually, much more is true. It follows from a result of Beurling [**3**] (cf. [**9**, p. 56]) that the set of points on J which are not rectifiably accessible from D must correspond to a set of logarithmic capacity 0 on the unit periphery. It would take us too far afield to enter into a detailed discussion of the capacity of plane sets here; for our purposes it is enough to know that sets of capacity zero are exceedingly small. For instance, such a set must have zero Hausdorff ε-measure for all $\varepsilon > 0$. The first person to show that a set of capacity zero on Γ could correspond under a conformal mapping to a set having positive area was Kikuji Matsumoto [**33**]. He actually proved (what is implicit in the above discussion) that for each totally disconnected compact subset K of the plane there exists a Jordan domain D with boundary $J \supset K$ such that K corresponds under conformal mapping to a set of capacity zero on Γ. The discussion here (in particular, the ingenious proof of the central result) is based on an idea of Walter Schneider [**51**].

The *compression* of the boundary of the unit disc presents greater difficulties. Lavrentiev, however, has shown that a set of positive measure on Γ may be mapped onto a set of zero length under a conformal mapping of Jordan domains [**30**]. A more recent construction is due to McMillan and Piranian [**32**].

7. Absolute convergence and uniform convergence. Conformal mapping techniques are also useful in constructing examples concerning the convergence of power series and Fourier series. Below we offer some simple but instructive examples.

The first example of a power series which converges uniformly but not absolutely on the closed unit disc was given by Fejér [**15**], cf. [**25**, vol. 1, p. 122]. The following geometric example, due to Gaier [**68**] and rediscovered by Piranian (see [**1**, pp. 289, 314]), is particularly appealing. Let D be the region of figure 1, a triangle from

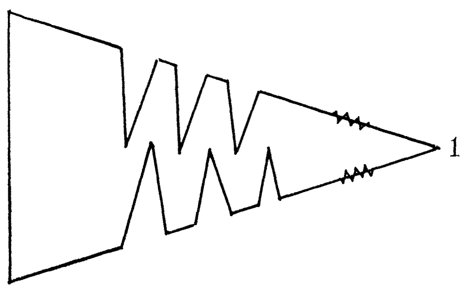

FIG. 1

which wedges have been removed in such a way that the vertex at $z = 1$ is not rectifiably accessible from the interior of D. Since D is a Jordan region, any conformal map of $\Delta = \{z: |z| < 1\}$ onto D extends to a homeomorphism of the closed regions. Suppose that $f(z) = \Sigma a_n z^n$ is such a homeomorphism satisfying $f(1) = 1$. Clearly,

$$\begin{aligned} \int_0^1 |f'(r)|\,dr &= \int_0^1 \Big|\sum_{n=1}^{\infty} n a_n r^{n-1}\Big|\,dr \\ &\leqq \int_0^1 \Big(\sum_{n=1}^{\infty} n|a_n| r^{n-1}\Big)dr = \sum_{n=1}^{\infty} |a_n|. \end{aligned} \tag{9}$$

Since the length of the image of $[0,1]$ under f is infinite and is given by the extreme left member of (9), the series for f is not absolutely convergent. That the series is uniformly convergent on the closed disc follows from a result due to Fejér.

FEJÉR'S TAUBERIAN THEOREM [**16**], [**55**, p. 357]. *Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ and suppose $\Sigma_{n=1}^{\infty} n|a_n|^2 < \infty$. If $\lim_{r\to 1-} f(re^{i\theta}) = f(e^{i\theta})$ exists, then the sum $\Sigma_{n=0}^{\infty} a_n e^{in\theta}$ exists and is equal to $f(e^{i\theta})$. Moreover, if $\lim_{r\to 1-} f(re^{i\theta}) = f(e^{i\theta})$ uniformly for $\theta_1 \leqq \theta \leqq \theta_2$, then $\Sigma_{n=0}^{\infty} a_n e^{in\theta} = f(e^{i\theta})$ uniformly for $\theta_1 \leqq \theta \leqq \theta_2$.*

This is a typical theorem of Tauberian type, the Tauberian condition being, of course, $\sum_{n=1}^{\infty} n\,|\,a_n\,|^2 < \infty$.

Proof. Set $s_N(e^{i\theta}) = \sum_{n=0}^{N} a_n e^{in\theta}$. Then

$$|f(re^{i\theta}) - s_N(e^{i\theta})| \leqq \sum_{n=1}^{N} |\,a_n\,|(1 - r^n) + \sum_{n=N+1}^{\infty} |\,a_n\,| r^n = S_1 + S_2.$$

Now $1 - r^n \leqq n(1 - r)$ (divide both sides by $1 - r$). Thus,

$$S_1 \leqq (1 - r) \sum_{n=1}^{N} n\,|\,a_n\,| \leqq (1 - r)\left(\sum_{n=1}^{N} n\right)^{1/2}\left(\sum_{n=1}^{N} n\,|\,a_n\,|^2\right)^{1/2} \leqq KN(1 - r),$$

where $K = (\sum_{n=1}^{\infty} n\,|\,a_n\,|^2)^{1/2}$. Here we have used the Cauchy-Schwarz inequality and the fact that $\sum_{n=1}^{N} = N(N+1)/2 \leqq N^2$. Applying Cauchy-Schwarz to S yields

$$S_2 = \sum_{n=N+1}^{\infty} \sqrt{n}\,|\,a_n\,|\,\frac{r^n}{\sqrt{n}} \leqq \left(\sum_{n=N+1}^{\infty} \frac{r^{2n}}{n}\right)^{1/2}\left(\sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/2}$$

$$\leqq \left(\frac{1}{N(1-r)} \sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/2}$$

since

$$\sum_{n=N+1}^{\infty} r^{2n}/n \leqq 1/N \sum_{n=0}^{\infty} r^n = 1/N(1 - r).$$

Having fixed N, we may, by the intermediate value theorem for continuous functions, choose $r = r_N$ such that $N(1 - r_N) = (\sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2)^{1/2}$. Clearly, as $N \to \infty$, $r_N \to 1$. Thus

$$|f(r_N e^{i\theta}) - s_N(e^{i\theta})| \leqq K\left(\sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/2} + \left(\sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/4},$$

and the right hand side tends to 0 as $N \to \infty$ since $\sum_{n=1}^{\infty} n\,|\,a_n\,|^2 < \infty$. Since $f(re^{i\theta}) \to f(e^{i\theta})$, $s_N(e^{i\theta}) \to f(e^{i\theta})$; hence $\sum_{n=0}^{\infty} a_n e^{in\theta} = f(e^{i\theta})$. Finally, all our calculations are uniform in θ, so if $f(re^{i\theta}) \to f(e^{i\theta})$ uniformly on some arc, then $\sum_{n=0}^{\infty} a_n e^{in\theta} = f(e^{i\theta})$ uniformly on that arc.

To apply Fejér's theorem to the situation at hand, simply note that if f maps Δ conformally onto the Jordan region D then (by the Osgood-Taylor-Carathéodory theorem) f extends continuously to $\Delta \cup \Gamma$ so that $f(re^{i\theta}) \to f(e^{i\theta})$ uniformly for $0 \leqq \theta \leqq 2\pi$. Since

$$\pi \sum_{n=1}^{\infty} n\,|\,a_n\,|^2 = \int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \, r \, dr = \text{area of } D < \infty,$$

the Taylor series for f converges uniformly on Γ.

We should observe that it is easy to modify the domain of figure 1 so that its boundary becomes analytic at every point except $z = 1$. The corresponding mapping function then extends (by Schwarz reflection) across $\Gamma\setminus\{1\}$ and yields a function univalent and analytic on a domain containing $(\Delta \cup \Gamma)\setminus\{1\}$ whose Taylor series converges uniformly but not absolutely on $\Delta \cup \Gamma$.

8. Fourier series. One of the loveliest applications of complex analysis to real variables occurs in the theory of Fourier series. The result in question is the so-called Pál-Bohr theorem, which may be stated as follows.

PÁL-BOHR THEOREM. *Let $f(e^{i\theta})$ be a continuous real-valued function on the unit circle Γ. There is a self-homeomorphism ϕ of Γ such that the Fourier series of $f \circ \phi$ converges uniformly.*

It is well known, of course, that the Fourier series of a continuous function may diverge on a dense subset of Γ [**28**, p. 58]; this gives the Pál-Bohr theorem added poignancy. On the other hand, a deep and famous result of Lennart Carleson [**64**] insures that the Fourier series of a continuous function converges *almost* everywhere in the sense of Lebesgue measure.

The Pál-Bohr theorem has an interesting history. It was first proved by Jules Pál in 1914 with the weaker conclusion that uniform convergence could be obtained on any *proper* closed subarc of Γ, however large. Bohr [**4**], in 1935, removed the restriction in Pál's theorem. Finally, in 1944, Salem [**50**] introduced a trick which yields the full strength of the result very quickly.

Proof of the Pál-Bohr Theorem. Regard f as a function on the interval $[-\pi, \pi]$ satisfying the periodicity condition $f(-\pi) = f(\pi)$. We rule out at the outset the trivial case in which f is identically constant. By adding, if necessary, a continuous periodic function of bounded variation, we may assume that $f(-\pi) = f(\pi) = f(x)$ for exactly one point $x \in (-\pi, \pi)$. (This is Salem's trick; see [**50**] for a complete verification.) Since the Fourier series of a continuous function of bounded variation converges uniformly, it is enough to prove the theorem under this additional assumption. Let g be a continuous periodic function on $[-\pi, \pi]$ which increases on $(-\pi, x)$ and decreases on (x, π). Then the image of $[-\pi, \pi]$ under the map $H(t) = g(t) + if(t)$ is a simple closed Jordan curve J in the plane.

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be a Riemann map of Δ onto the interior of J such that $F(-1) = H(-\pi)$. Then F extends to a homeomorphism of Γ onto J, and by the discussion following the proof of Fejér's theorem, the series $F(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ converges uniformly on Γ. The required homeomorphism of $[-\pi, \pi)$ is obtained by setting $\phi(t) = H^{-1}(F(e^{it}))$. Indeed, this is clearly a homeomorphism, and $f(\phi(t)) = f \circ H^{-1}(F(e^{it})) = \operatorname{Im} F(e^{it})$, which has a uniformly convergent Fourier series since $F(e^{it})$ does.

Perhaps surprisingly, the argument given above is (essentially) the *only* known proof of this theorem. Whether an analogous result holds for complex-valued functions remains an open question; of course, this is equivalent to the question of whether, given *two* real-valued continuous functions f, g on Γ, one can find a single homeomorphism ϕ for which $f \circ \phi$ and $g \circ \phi$ both have uniformly convergent Fourier series.

We learned of the Pál-Bohr theorem from the interesting survey article of Goffman and Waterman [**20**], and our treatment parallels the discussion given there. The decision to reproduce the proof in some detail was based on our feeling that this beautiful result deserves a wider public.

9. Harmonic conjugates. A somewhat different application of conformal mapping to problems involving Fourier series involves the construction of functions having certain prescribed bad boundary behavior. Thus, one may ask (and Prof. A. Devinatz did) for an explicit example of a function harmonic on Δ and continuous on $\Delta \cup \Gamma$ whose harmonic conjugate is discontinuous but bounded. Although the problem has been framed (for simplicity) in terms of harmonic functions, it is actually a pure real variable question concerning the lack of smoothness of a certain singular integral operator.

For the solution, consider the simply connected domain D, indicated in Figure 2, bounded by an (open) analytic curve J together with its asymptote the segment

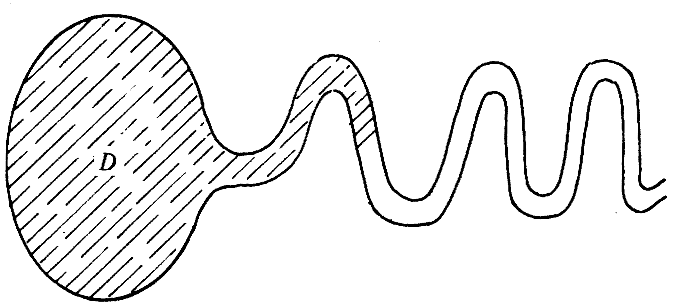

FIG. 2

$\{y: -1 \leqq y \leqq 1\}$ of the y-axis in the complex plane. Map Δ conformally onto D by the univalent function $f(z) = u(z) + iv(z)$. A standard result in conformal mapping [**55**, p. 353] insures that a single point of Γ, say 1, corresponds to the "bad" part of the boundary and that f establishes a homeomorphism between $\Gamma \setminus \{1\}$ and J. By the reflection principle, f actually extends analytically across $\Gamma \setminus \{1\}$. One proves that as $z \to 1$, $u(z) \to 0$; and it is now obvious that u is not only harmonic on Δ and harmonically extendible across $\Gamma \setminus \{1\}$ but also continuous on $\Delta \cup \Gamma$. On the other hand, the harmonic function v, which is clearly bounded, is *not* continuous at $z = 1$. The details of the proof will be easily supplied by anyone familiar with Carathéodory's important theory of prime ends [**7**], [**55**, pp. 352–355], [9]. An obvious modification yields a bounded continuous function whose conjugate is unbounded.

10. Tauberian theorems. Tauberian theorems, such as Fejér's, have an intrinsic interest quite independent of applications. Of these, the most celebrated is certainly that due to Littlewood, which states that if $\lim_{r\to 1-} \sum_{n=0}^{\infty} a_n r^n = L$ exists and $a_n = O(1/n)$, then $\sum_{n=0}^{\infty} a_n = L$. This result resisted considerable efforts at proof for several years before it was finally settled by Littlewood [31], whose argument required six pages of ingenious and delicate analysis. Much later, Karamata [27] introduced a new technique, based on approximation theory, resulting in an enormous simplification of the proof. Much less well-known is Wielandt's modification [59] of Karamata's proof, which yields a simple and transparent proof of the theorem in question. Below, we present Wielandt's proof of a strengthened version (due to Hardy and Littlewood [22]) of the Littlewood Tauberian theorem.

THEOREM. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* $|z| < 1$ *and suppose that the* a_n *are real and that* $na_n \leqq K$ *for some* $K > 0$. *If* $\lim_{r\to 1-} f(r) = L$ *exists as* $r \to 1-$, *then* $\sum_{n=0}^{\infty} a_n = L$.

The advantage of this result over the original Littlewood theorem lies, of course, in the fact that the order estimate on the coefficients is replaced by a *one-sided* bound.

Proof. Trivial normalizations allow us to assume that $L = 0$, $a_0 = 0$, $K = 1$. Consider the family $\mathscr{F}$ of real functions $\phi(x)$ on $(0,1)$ which satisfy

(a) $\sum_{n=1}^{\infty} a_n\phi(x^n)$ is convergent for $x \in (0,1)$,

(b) $\Phi(x) = \sum_{n=1}^{\infty} a_n\phi(x^n) \to 0$ as $x \to 1-$.

Clearly, if $\phi(x) \in \mathscr{F}$, $\phi(x^k) \in \mathscr{F}$ $(k = 1,2,\cdots)$ and $\mathscr{F}$ is closed under linear combinations. Since (by hypothesis) $x \in \mathscr{F}$, each polynomial vanishing at the origin belongs to $\mathscr{F}$. The proof depends on a simple lemma concerning the approximation of functions.

LEMMA. *Let* $\phi(x)$ *satisfy* (a). *Suppose that for each* $\varepsilon > 0$ *there exist polynomials* $p_1(x)$, $p_2(x)$ *such that* $p_i(0) = 0$, $p_i(1) = 1$ $(i = 1,2)$ *and*

$$p_1(x) \leqq \phi(x) \leqq p_2(x) \qquad \frac{p_2(x) - p_1(x)}{x(1-x)} = q(x) > 0,$$

where $\int_0^1 q(x)dx < \varepsilon$. *Then* $\phi(x)$ *satisfies* (b) *and hence belongs to* $\mathscr{F}$.

Proof of Lemma. Let $\Phi(x) = \sum_{n=1}^{\infty} a_n\phi(x^n)$, $q(x) = \sum_{k=0}^{r} b_k x^k$.
Then

$$\Phi(x) - \sum_{n=1}^{\infty} a_n p_1(x^n) = \sum_{n=1}^{\infty} a_n(\phi(x^n) - p_1(x^n)) \leqq \sum_{n=1}^{\infty} \frac{1}{n}(p_2(x^n) - p_1(x^n))$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}(1-x^n)x^n q(x^n) \leqq (1-x) \sum_{n=1}^{\infty} x^n q(x^n)$$

$$= (1-x) \sum_{k=0}^{r} b_k \sum_{n=1}^{\infty} x^{n(k+1)} = \sum_{k=0}^{r} \frac{b_k(1-x)x^{k+1}}{1-x^{k+1}} \to \sum_{k=0}^{r} \frac{b_k}{1+k}$$

$$= \int_0^1 q(x)dx < \varepsilon \text{ as } x \to 1- .$$

Here we have used the fact that $1 - x^n \leqq n(1-x)$ and that $(1-x^n)/(1-x) \to n$ as $x \to 1$. Since $p_1(x) \in \mathscr{F}$, $\sum_{n=1}^{\infty} a_n p_1(x^n) \to 0$ as $x \to 1-$ so that $\Phi(x) < \varepsilon$ for x near 1. Consideration of $p_2(x) - \phi(x)$ shows similarly that $\Phi(x) > -\varepsilon$ if x is near enough to 1. Thus $\Phi(x) \to 0$ as $x \to 1$, so that $\phi \in \mathscr{F}$.

Continuing with the proof of the theorem, let

$$\phi^*(x) = \begin{cases} 0 & 0 \leqq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leqq x \leqq 1 \end{cases}$$

so that $\Phi(x) = \sum_{n=1}^{\infty} a_n \phi^*(x^n) = \sum_{2x^n \geqq 1} a_n = \sum_{n=1}^{N} a_n = s_N$, where

$$N = \left[\log 2/\log \frac{1}{x}\right].$$

It suffices to show that $\phi^*(x) \in \mathscr{F}$, for then $s_N \to 0$ as $N \to \infty$, whence $\sum_{n=0}^{\infty} a_n = 0$ as required. Now ϕ^* clearly satisfies (a), so it is enough to show that the conditions of the lemma are fulfilled. Since continuous functions are dense in the integrable functions, we can find continuous functions $g_1(x)$ and $g_2(x)$ such that

$$g_1(x) < \frac{\phi^*(x) - x}{x(1-x)} < g_2(x) \qquad \int_0^1 [g_2(x) - g_1(x)]dx < \varepsilon. \tag{10}$$

The functions g_1 and g_2 may then be approximated uniformly by polynomials q_1 and q_2 in such a way that (10) still holds with the g_i's replaced by the q_i's. Putting $p_i(x) = x + x(1-x)q_i(x)$, $q(x) = q_2(x) - q_1(x)$, we obtain polynomials satisfying the hypothesis of the lemma. This completes the proof.

The subject of Tauberian theorems extends far beyond questions concerning the convergence or divergence of a power series on its circle of convergence. One of the central results in the harmonic analysis of the real line is Wiener's Tauberian theorem, which states that if $f \in L^1(\mathbb{R})$ and the Fourier transform of f never vanishes, then linear combinations of translates of f are dense in $L^1(\mathbb{R})$. The relation between the theorems of Wiener and Littlewood is far from obvious, and it has become customary to deduce the latter from the former by way of explicating the Tauberian character of Wiener's theorem. This deduction is standard and may be found, for instance, in [**60**, pp. 104–106]. The proof involves the function $K(x) = e^{-x}\exp(-e^{-x})$ and uses the fact that the gamma function $\Gamma(z)$ has no zeroes on the line $\operatorname{Re} z = 1$.

Unfortunately, the *deduction* of Littlewood's theorem from Wiener's is longer and significantly more complicated in both conception and detail than Wielandt's proof of the (more general!) Hardy-Littlewood theorem: it is a little like proving that the medians of a triangle are concurrent by invoking the fact that a nested sequence of compact sets has nonvoid intersection. Of course, Wiener's powerful methods have applications in many situations where the simple approximation theory argument we have given does not apply.

One such instance concerns the so-called high indices theorem.

HIGH INDICES THEOREM. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ *be analytic in* $|z| < 1$ *and suppose that* $n_{k+1}/n_k \geqq q > 1$ *for all k. If* $\lim_{r\to 1-} f(r) = L$ *exists, then* $\sum_{k=0}^{\infty} a_k = L$.

This theorem lies considerably deeper than Littlewood's theorem or its extension proved above; it was first proved, by Hardy and Littlewood, in 1925 [**23**], having been conjectured by Littlewood as early as 1910. The novelty of the result lies in the fact that the Tauberian condition (the lacunarity of the sequence of coefficients) involves no bound on the size of the coefficients. It is most instructive to try to apply the ideas used in proving the Tauberian theorems of Fejér and Hardy-Littlewood to the high indices theorem: they all fail miserably. In fact, I am aware of no really simple proof of this result. A particularly attractive argument, marked by considerable ingenuity in the use of such tools as the Phragmén-Lindelöf principle and Blaschke products, has been given by Halász [**21**], following some ideas of the German mathematician Dieter Gaier.

In concluding this section we should like to mention an amusing sidelight. Wielandt's proof of the Hardy-Littlewood theorem shares, with Mordell's proof of the Hadamard gap theorem, the property of being a gem of complex analysis mined by a mathematician whose central interests lay altogether outside analysis. The late Professor Mordell was, of course, one of the world's leading number theorists; Professor Wielandt is a group theorist of international repute. Is there a moral to be drawn here?

11. Category. The usual theorems on convergence of sequences of analytic functions, such as Vitali's convergence theorem [**54**, p. 168], require the uniform boundedness of the sequence in question on compact subsets of the domain. There is, however, a sometimes useful result, due to Osgood, which avoids altogether hypotheses other than simple pointwise convergence.

OSGOOD'S THEOREM [**40**]. *Let D be a domain and let* $\{f_n\}$ *be a sequence of functions analytic in D. Suppose* $f_n(z) \to f(z)$ *for each* $z \in D$. *Then f is analytic in an open set* $D_1 \subset D$ *which is dense in D, and convergence is uniform on compact subsets of* D_1.

This result has been rediscovered countless times and has on innumerable other occasions brought the experts to grief. Indeed, the question as to whether f must be analytic *anywhere*, appears (happily, with a correct solution) in the problem section of a recent symposium [**29**, p. 543]. The present formulation suggests — correctly — the use of the Baire category theorem.

Proof of Osgood's Theorem. Let $F_m = \{z: |f_n(z)| \leqq m, n = 1,2,3,\cdots\}$. The F_m are clearly relatively closed in D and $\cup F_m = D$. By the Baire category theorem, some F_m must have interior. For this m, the sequence $\{f_n\}$ is uniformly bounded on F_m^0 hence by Vitali's theorem converges uniformly on compact subsets of F_m^0 to an analytic function. Thus f is analytic on F_m^0. Since the argument can be applied to any subdomain R of D — in particular, to an arbitrary disc — it follows that f must be analytic on a dense open subset D_1 of D. That convergence is uniform on compacta contained in D_1 is a standard argument, which we suppress.

A comment is perhaps in order on our use of the Baire category theorem, which states that a complete metric space is not the countable union of closed nowhere dense sets. Obviously, D is *not* complete in the Euclidean metric. However, it is easy to see that D can be given a new metric which induces the same (Euclidean) topology, under which D is complete. Alternatively, one may replace D by a slightly smaller *compact* set K and relativize the argument to K. We should also mention that category arguments appear elsewhere in complex analysis as well. A notable example is the proof of Hartogs' theorem, mentioned earlier in Section 2.

A nice complement to Osgood's theorem is provided by an example of a sequence of entire functions $f_n(z)$ with the property that

$$\lim_{n\to\infty} f_n(z) = \begin{cases} 0 & z \neq 0 \\ 1 & z = 0. \end{cases} \tag{11}$$

There are (at least) two essentially distinct ways of constructing such a sequence. One method is to construct an entire function $F(z)$ such that $F(0) = 1$ and $F(z) \to 0$ as $|z| \to \infty$ on each ray through the origin. Such functions were first ex-

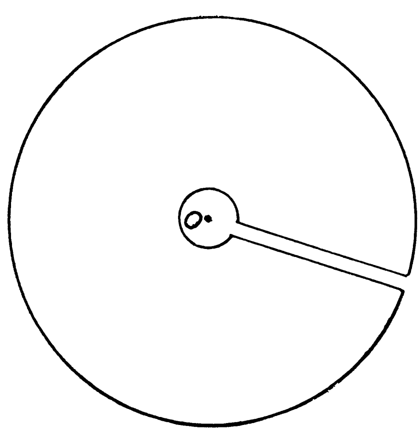

FIG. 3

hibited by Mittag-Leffler; see [**34**] for a detailed discussion and some surprising extensions. The construction of F is, with appropriate hints, a nice and doable exercise and occurs as such in Rudin's text [**49**, pp. 326–327]. Once F has been obtained, one observes that the sequence $f_n(z) = F(nz)$ satisfies (11).

Alternatively, one can apply Runge's theorem to "notched annuli" to construct polynomials satisfying (11). To be explicit, consider the set indicated in figure 3. A moment's reflection will reveal that one can choose a sequence K_n of such sets, with the property that for each $z \in \mathbb{C} \setminus \{0\}$ there exists an integer N such that $z \in K_n$ for all $n \geqq N$. (The inner circle contracts, the outer circle expands, and the notch gets thinner and rotates, tending toward but never reaching its limiting line.) The function

$$g_n(z) = \begin{cases} 0 & z \in K_n \\ 1 & z = 0 \end{cases}$$

clearly extends to a function analytic in a neighborhood of the (disconnected) set $K_n \cup \{0\}$. Since this set does not separate the plane, g may be approximated uniformly (to within $1/n$, say) on $K_n \cup \{0\}$ by a polynomial p_n. These polynomials clearly satisfy (11).

12. Miscellany. The interactions between real and complex analysis are by no means limited to the areas mentioned above. To keep the discussion within manageable limits, we have restricted ourselves to (a subset of) those applications, examples, and aspects of the theory that have not found sustained treatment in the "popular" literature of texts and survey articles. Subjects which are treated elsewhere at adequate length but which deserve mention here by virtue of their interdisciplinary nature include the following:

(a) *The evaluation of real integrals and sums by residue techniques.* This is surely one of the most striking applications of complex function theory to real analysis. Fortunately, any good text on complex analysis will contain a fairly detailed discussion.

(b) *Complex methods in harmonic analysis.* This is a substantial area, which includes topics as diverse as interpolation theorems (see, for instance, [**28**, pp. 93–98]) and theorems of Paley-Wiener type [**43**]. Two of the most attractive recent texts in harmonic analysis [**13**], [**28**] devote whole chapters to this aspect of the theory. Further developments are discussed in the survey article of Weiss [**57**].

(c) *Functional analysis.* Complex variable methods appear here perhaps most notably in the construction of functional calculi for operators on Hilbert space or Banach space. The applications to commutative Banach algebras are particularly substantial; indeed, parts of this last-named subject are virtually coextensive with certain aspects of several complex variable theory. For further references, see [**17**] and [**58**]. In the opposite direction, techniques of functional analysis can be used to establish many results in function theory; this is the programme of [**48**]. Finally, an honest partnership between complex variables and functional analysis occurs in the study of certain Banach spaces of analytic functions, especially H^p spaces [**12**], [**26**].

(d) *Function theoretic methods in differential equations.* Complex methods occur rather naturally in the study of ordinary differential equations [**8**]. Their appearance in the study of partial differential equations is perhaps more surprising. Yet there are substantial applications, and more than one book [**2**], [**19**] has been devoted to this area. Further applications of function theory to problems in partial differential equations will be found in [**18**]. In a rather different direction, the theory of linear partial differential equations with constant coefficients is intimately connected with the study of certain spaces of entire functions of several complex variables; see [**14**] for an exhaustive treatment.

13. Monodromy. No excursion onto the bypaths of complex analysis would be complete without some mention of the monodromy theorem.

MONODROMY THEOREM. *Let D be a simply connected domain and let $f(z)$ be analytic in a neighborhood of $z_0 \in D$. Then if $f(z)$ can be continued analytically from z_0 along every path lying in D, the continuation gives rise to a single-valued unction analytic on all of D.*

A more general version states that analytic continuation along paths is a homotopy invariant; see, for instance, [**53**]. Like the reflection principle, the monodromy theorem is an essential ingredient in the short proof of Picard's little theorem; in its extended form, it is the central result in the subject of analytic continuation. Yet no theorem of basic complex analysis is more abused or less understood. Indeed, it has been misapplied more than once even by mathematicians of the first rank (and specialists in complex analysis, at that!). One may speculate that a source of at least some of the confusion surrounding this result is the essentially topological, rather than function-theoretic, nature of the theorem.

The sort of error into which one may lapse is best indicated by an explicit example. Let D be a simply connected domain and f a function analytic in D which satisfies $f'(z) \neq 0$ on D. Suppose $R = f(D)$ is also simply connected. Question: Must f be univalent (one-one)? An affirmative answer may be found in [**56**, p. 243] and in other references as well. The argument is as follows. At each point $w_0 \in R$ one may define a local inverse $f_{w_0}^{-1}(w)$ of f, analytic in a neighborhood of w_0. Since R is simply connected, the totality of these functions defines a single-valued analytic function f^{-1} on R, which is a global inverse for f. Thus f must be univalent. Note further that the simple-connectivity of D is quite extraneous to the demonstration.

Unfortunately, the argument given above is altogether incorrect, since the essential hypothesis of the monodromy theorem, that analytic continuation be possible along every path in R, has not been verified. Can the proof be salvaged?

The answer is no. In fact, consider the function $f(z) = \int_0^z e^{\zeta^2} d\zeta$. This f is analytic (entire) on the simply connected domain $\mathbb{C}$ and $f'(z) = e^{z^2}$ is nowhere zero. Clearly, f is not univalent. So it suffices to prove that $f(\mathbb{C})$ is simply connected. We claim $f(\mathbb{C}) = \mathbb{C}$. Indeed, suppose $f(z) \neq w$. Since e^{z^2} is an even function, $f(z)$ is odd, so that if f fails to take on the value w it also misses the value $-w$. If $w \neq 0$, this contradicts Picard's (small) theorem. Since $f(0) = 0$, f takes on every value in the complex plane.

For $D = \mathbb{C}$, any function which satisfies $f'(z) \neq 0$ must be transcendental and hence must (by Picard's theorem) take on most values infinitely often. One can, however, construct a non-univalent, locally univalent function mapping the disc $\Delta = \{z : |z| < 1\}$ onto itself, which takes on no value more than three times. The extremely elegant example given above is due to D. S. Greenstein and appears as a solution to MONTHLY Problem 4740. It is an appropriate note on which to end this survey.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, U.S.A.