ARTICLES

The New Polynomial Invariants of Knots and Links

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The theory of knots and links is the analysis of disjoint simple closed curves in ordinary 3-dimensional space. It is the consideration of a collection of pieces of string in 3-space, the two ends of each string having vanished by being fastened together as in a necklace. Many examples can be seen in the diagrams that follow. If the strings can be moved around from one position to another those two positions are the same link or 'equivalent' links. Of course, during the movement no part of a string is permitted to pass right through another part in some supernatural fashion; the string is regarded as being extremely thin and pliable; it can stretch and there is no friction nor rigidity to be considered. As an example, FIGURE 1 shows two pictures of the same link, a famous link called the Whitehead link. Thus the problem of understanding knots and links is one of geometry and topology, and within those disciplines the subject has received considerable study during the last hundred or more years. Knot theory has been a real inspiration to both algebraic and geometric topology, and, conversely, the theoretical machinery of topology has been used to make vigorous attacks on knot theory. The principal problem has always been to find ways of deciding whether or not two links are equivalent. Confronted with two heaps of intertwined strings, how is one to know if one can move the first to the configuration of the second (without cheating and breaking the strings)? Algebraic topology provides some 'invariants,' but recently some entirely new methods have been discovered which are extraordinarily effective (though not infallible), and which, judged by the standards of most modern mathematics, are breathtakingly simple.

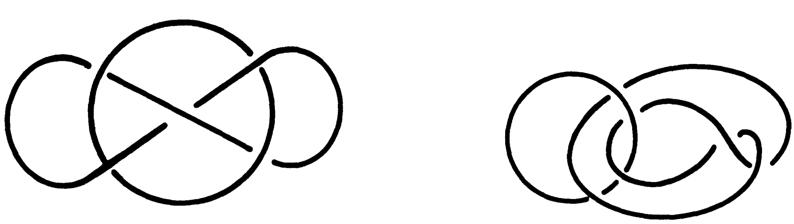


FIGURE 1

*Supported in part by National Science Grant DMS8503733.

The story begins in the spring of 1984. Professor V. F. R. Jones, now of the University of California at Berkeley, had for some years been studying operator algebras and trace functions on these algebras. It was pointed out to him that some of the formalism of his work closely resembled that of the well-known *braid group* of E. Artin [3]. This braid group can be used to study knots, and eventually Jones realized that, using his trace functions, he could define polynomials for knots and links which are *invariants* [9]. This means that to each configuration of pieces of string is associated a polynomial, and that if the string is moved (as described above) to a new position it still has the same polynomial. Thus, if calculation shows two heaps of string have two distinct polynomials, then it is not possible to move the strings from one position to the other. To get the idea of an invariant, consider what is probably the easiest of them all, namely the number of strings that make up a link; a link of two strings can never be deformed to one of three strings. Another polynomial invariant for links (discovered in about 1926 by J. Alexander [1]) was well known so, for a while, it was suspected that Jones' polynomial might be but some elementary manipulation of that polynomial. Soon however it was established that the Jones polynomial was entirely new, independent of all other known invariants. Strenuous efforts to understand the Jones polynomial have since been made by many mathematicians scattered around the world. The most amazing things about it are its simplicity and the fact that it exists at all. In retrospect it seems that several mathematicians during the last thirty years came exceedingly close to discovering Jones' polynomial and would surely have done so had they dreamt there was anything there to discover. A very simple complete proof of the existence of this polynomial appears in §3 below. By now, Jones' polynomial has been generalized two or three times, lengthy computer generated tabulations of examples have been produced, proofs have been explored and simplified, some correlations with algebraic topology have been found and a few geometric applications have been produced. Nevertheless, at the time of writing, there is still a feeling that these new ideas are not really understood, that they do not really fit in with more established theories, and that more generalizations and applications may be possible. Intense investigation continues.

§1. Basic Background

A little basic information about knots and links may allay some misunderstandings, but the confident will proceed to the next section. There are several excellent surveys of the subject prior to Jones' discovery; [2], [15], [17], [16] and [5] are accounts in (approximate) order of increasing mathematical sophistication. As already stated, a link is a finite collection of disjoint simple closed curves in 3-dimensional space $\mathbb{R}^3$, the individual simple closed curves being called the *components* of the link. A link of just one component is a *knot*. It is tacitly assumed that the closed curves are *piecewise linear*, that is that they consist of a finite number (probably very large) of straight line segments placed end to end. This is a technical restriction best ignored in practice; it does however ensure that an infinite number of kinks of any sort, possibly converging to zero size, never occurs. Restricting the components to being differentiable would do equally well. The orthogonal projection of $\mathbb{R}^3$ onto a plane $\mathbb{R}^2$ in $\mathbb{R}^3$ maps a link to a diagram of the type seen frequently in the pages that follow. The direction of that projection is always chosen so that, when in $\mathbb{R}^2$ projections of two distinct parts of the link meet, they do so transversally at a *crossing* as in FIGURE 2(i), never as in FIGURE 2(ii), (iii), or (iv). At a crossing it is indicated which of the two arcs corresponds to the

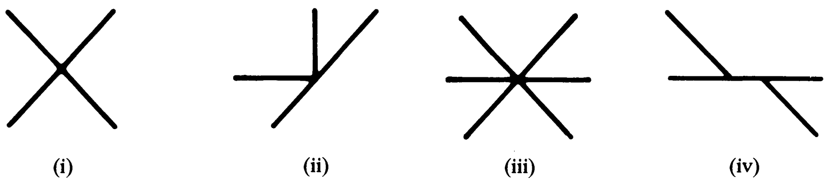


FIGURE 2

upper string, and which to the lower, by breaking the line of the lower one at the crossing. Such a planar diagram will be called a *projection* of the link.

A movement of a link from one position in $\mathbb{R}^3$ to another is called an *ambient isotopy*; that idea defines when two links are the same or '*equivalent*.' Such a movement changes the planar projections of a link. An established theorem states that two links are equivalent if and only if (any of) their projections differ by a sequence of the *Reidemeister moves* [**16**]. These moves, of types I, II and III, are those shown in FIGURE 3 (and their reflections), where, for each type, a small part of the projection is shown before and after the move; the remainder of the projection remains unchanged. It is clear that if two link projections so differ then the links are equivalent; the converse is established by a routine and inelegant proof. Thus to show that the two diagrams of FIGURE 1 represent the same link one can construct a sequence of these moves that allow the first diagram to evolve to the second. However, for two general link projections one has no idea whether many million moves may be required.

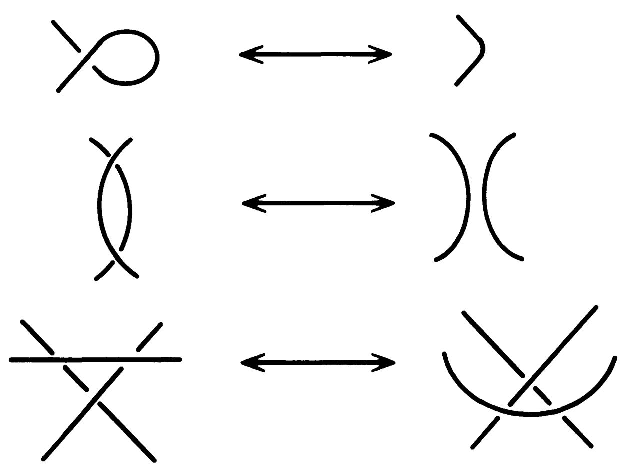


FIGURE 3

An *oriented* link is a link with a direction (usually indicated by an arrow) assigned to each component, so that each acquires a preferred way of travelling around it. Thus a link with n components has 2^n possible orientations. The two oriented links of FIGURE 4 are distinct, for one cannot be moved to the other sending the directions on the one link to those on the other (this is proved in what follows).

A knot is *unknotted* if it is equivalent to a knot that has a projection with zero crossings. Two oriented knots can be *summed* together as indicated in FIGURE 5; they are placed some way apart and a 'straight' band joins one to the other so that in the resultant sum the orientations match up. A knot, other than the unknot, is *prime* if it

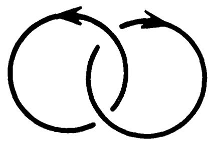


FIGURE 4(a)

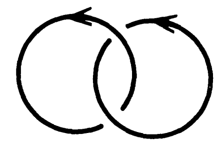


FIGURE 4(b)

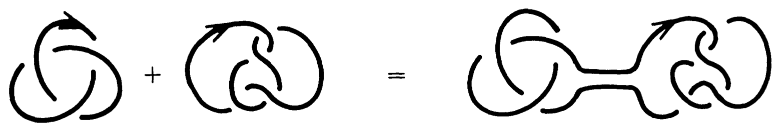


FIGURE 5

cannot be expressed as a sum of two knots neither of which is unknotted. Note that the sum of two oriented links of more than one component is not well defined unless it is specified which two components are to be banded together. As it is known that any knot is uniquely expressible as the sum of prime knots, listings of knots usually include only the prime knots.

If L is an oriented link, let ρL denote L with all its directions reversed, and let $\overline{L}$ be the reflection of L. When considering projections, this reflection is usually thought of as reflection in the plane of the paper, so that the projection of $\overline{L}$ comes from that of L by changing all underpasses to overpasses and *vice versa*. Thus from L can be created ρL, $\overline{L}$, and $\rho\overline{L} = \overline{\rho L}$, and these may be four distinct links, they may be the same in pairs, or all four may be the same. The trefoil knot 3_1 (see FIGURE 6) creates two pairs in this way, the figure of eight knot 4_1 has all four the same whilst 9_{32} has all four distinct.

Knot 3_1 Knot 9_{32}

(a) (b) (c)

FIGURE 6

Inherent in the idea that reflection can change a knot is the convention that the enveloping three-dimensional space $\mathbb{R}^3$ is oriented; it is equipped with a distinction between left-hand and right-hand screwing motions. Knot tables have traditionally listed prime knots according to the minimum number of crossings in a projection of the knot. Thus 7_4 denotes the fourth knot, in some traditional order, that needs seven and no more than seven crossings. The tables have deliberately ignored reflections and reversals, so that an entry may stand for as many as four knots if these orientations are taken into account. With these conventions knots have been classified up to thirteen crossings [19] with the help of computers and the following table has been produced.

#crossings	3	4	5	6	7	8	9	10	11	12	13
#knots	1	1	2	3	7	21	49	165	552	2176	9988

This totals 12965 knots.

There now follows a discussion of the new polynomial invariants of knots and links. It is not possible to restrict the discussion to knots alone, for many-component links are an integral part of the theory. The ideas will not here be developed in the order of their discovery but in an order that now seems simpler to understand.

§2. The Oriented Polynomial

The polynomials to be considered here are Laurent polynomials with *two* variables ℓ and m and with integer coefficients. A Laurent polynomial differs from the usual polynomials of high school only inasmuch as negative as well as positive powers of the variables may occur. One such polynomial $P(L)$ will be associated to each oriented link L. For example, to the link of FIGURE 1 will be assigned the polynomial

$$(-\ell^{-1}-\ell)m^{-1}+(\ell^{-1}+2\ell+\ell^{3})m-\ell m^{3}.$$

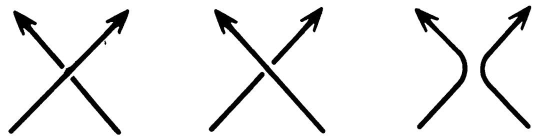
The result that encapsulates this was discovered almost simultaneously by four sets of authors [7] in the wake of Jones' first announcement ($P(L)$ *generalises* Jones' polynomial, see §3). It can be stated as follows:

THEOREM 1. *There is a unique way of associating to each oriented link L a Laurent polynomial P(L), in the variables ℓ and m, such that equivalent oriented links have the same polynomial and*

(i) $P(unknot)=1$,

(ii) *if L_+, L_-, and L_0 are any three oriented links that are identical except near a point where they are as in* FIGURE 7, *then*

$$\ell P(L_+)+\ell^{-1}P(L_-)+mP(L_0)=0.$$



L_+ L_- L_0

FIGURE 7

In a projection of an *oriented* link the crossings are of two types; that of L_+ in FIGURE 7 is called *positive*, that of L_- is *negative*. This idea will be exceedingly important. In L_+, the direction of one segment can be thought of as pointing in the direction dictated by a right-hand screw motion along the direction of the other segment. All the orientations are needed to make this important distinction. Of course, the choice of which type of crossing is given which sign is but a convention.

The meaning of Theorem 1 becomes apparent as one uses it to make a few calculations. Shown in FIGURE 8 is a very elementary example of a triple of links L_+, L_-, and L_0.

FIGURE 8

The first two links are just pictures of the unknot, the first with a single positive crossing, the second with just a negative one. Formulae (i) and (ii) imply that $\ell 1 + \ell^{-1} 1 + mP(L_0) = 0$, from which one deduces that $P(L_0)$, the polynomial for the trivial link of two separated unknots, is $-(\ell^{-1} + \ell)m^{-1}$. Consider now the triple of links in FIGURE 9 (where it is the uppermost crossing that is to be considered).

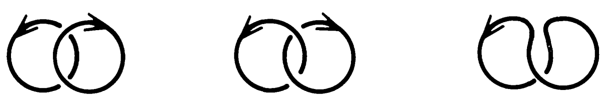


FIGURE 9

Here L_+ is the link whose polynomial has just been calculated, or at least it is equivalent to it and so has that same polynomial; L_0 is the unknot. Thus

$$-\ell(\ell^{-1} + \ell)m^{-1} + \ell^{-1}P(L_-) + m1 = 0,$$

so that the polynomial for the simple link L_-, the link of FIGURE 4(a), is $(\ell + \ell^3)m^{-1} - \ell m$. FIGURE 10 shows a third triple (look at the right-hand crossing). This yields the equation

$$\ell 1 + \ell^{-1}P(L_-) + m\{(\ell + \ell^3)m^{-1} - \ell m\} = 0.$$

Hence the polynomial of the left-hand version of the trefoil knot 3_1 (seen also in FIGURE 6(a)) is $-2\ell^2 - \ell^4 + \ell^2 m^2$.

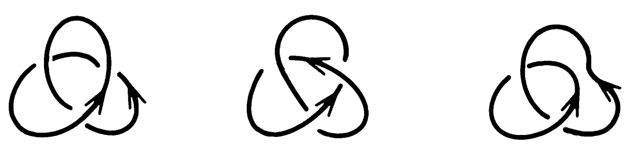


FIGURE 10

This proves that the trefoil is indeed knotted, for were it equivalent to the unknot it would, by Theorem 1, have 1 for its polynomial. Similarly the link of FIGURE 4(a) is not equivalent to the link consisting of two separated unknots.

Consider the method of the preceding calculation of the trefoil's polynomial. Attention was given to one crossing. Switching that crossing produced the unknot, nullifying it (to get 'L_0') produced a link with fewer crossings that had already been considered. This procedure works in general. Suppose that one is confronted with an oriented link L of n crossings and c components. Assume that one has already calculated the polynomials of all (relevant) oriented links of $n-1$ crossings; there are only finitely many of them. Then formula (ii) calculates $P(L)$ in terms of the polynomial of a modified L, namely L with some chosen crossing switched. However it is always possible to change L to U^c, the unlink of c unknots, by switching a subset of some s of the crossings. Thus, performing the switches one by one, using formula (ii) each time, $P(L)$ is calculated in terms of the polynomials of s links of fewer crossings (the 'L_0's') and of $P(U^c)$. However, it is an easy exercise to show, by induction on c, that $P(U^c) = (-(\ell^{-1} + \ell)m^{-1})^{c-1}$. This method of calculation will always work, and it is essentially the only known method of calculating these polynomials. It is nevertheless not a very welcome method for the length of calculation increases exponentially with the number of crossings of the link presentation.

Note that if *all* the diagrams in the above calculations were reflected in the plane of the paper this would change each positive crossing to a negative crossing and *vice versa*. Each L_+ would become an L_-. This would simply exchange the rôles played in the calculation by ℓ and ℓ^{-1}. Thus reflecting a link has the effect on its polynomial of interchanging ℓ and ℓ^{-1}. The left-hand trefoil of FIGURE 6(a) has polynomial $-2\ell^2 - \ell^4 + \ell^2 m^2$, so the right-hand trefoil's polynomial (FIGURE 6(b)) is $-2\ell^{-2} - \ell^{-4} + \ell^{-2}m^2$. These polynomials are obviously different so, by Theorem 1, the trefoils are inequivalent (this fact was tricky to prove until 1984 when Jones produced a version of this proof). Similarly the two oriented links of FIGURE 4 are distinct. For any polynomial P, let $\bar{P}$ denote P with ℓ and ℓ^{-1} interchanged (c.p. complex conjugation). The above discussion has demonstrated the following result.

PROPOSITION 1. *For any oriented link* L, $\overline{P(L)} = P(\bar{L})$.

Consider now the triple of FIGURE 11. This yields

$$\ell P(4_1) + \ell^{-1}1 + m\left\{(\ell + \ell^3)m^{-1} - \ell m\right\} = 0$$

so that

$$P(4_1) = -\ell^{-2} - 1 - \ell^2 + m^2.$$

4_1 U

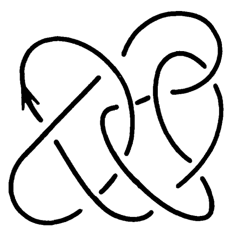
FIGURE 11

Here then the polynomial *is* symmetric with respect to ℓ and ℓ^{-1} and so it does not show 4_1 and $\bar{4}_1$ to be different, and, in fact, a little experimentation shows them to be the same. In practice the P-polynomial provides a very good test as to whether or not $L = \bar{L}$, but any hope that it might be an infallible test is dashed by the knot 9_{42} shown in FIGURE 12. It is known that $9_{42} \neq \bar{9}_{42}$ because a certain 'signature' invariant, from algebraic topology, is nonzero. However,

$$P(9_{42}) = (-2\ell^{-2} - 3 - 2\ell^2) + (\ell^{-2} + 4 + \ell^2)m^2 - m^4$$

and this is a self-conjugate polynomial.

It should be remarked that other *notations* can be used in the whole of this theory of polynomials for knots and links. For example, $P(L)$ can be taken to be a polynomial in three variables x, y, z with the vital defining formula being $xP(L_+) + yP(L_-) + zP(L_0) = 0$. However, the three variables are homogeneous variables as in projective planar geometry (there are still really only two variables), and the balance between ℓ and ℓ^{-1} is lost. Some authors also have a strong preference for some negative signs in the defining formula.



Knot 9_{42} FIGURE 12

Recall that ρL is obtained by L by reversing all its arrows. Unfortunately, $P(\rho L) = P(L)$, for changing L to ρL leaves the signs of all its crossings unchanged. Thus any calculation for $P(L)$ induces exactly the same calculation for $P(\rho L)$. (This means that for a knot, a link of one component, it is not really necessary to specify an orientation at all when thinking about the polynomial.) If, however, the directions of some, but not all, of the components of L are changed, then $P(L)$ can change in a rather drastic way that is not well understood. Examples occur in some of the polynomials of two-component links listed in the table at the end of this article.

A result concerning the behaviour of the polynomial under sums and 'distant' unions is as follows:

PROPOSITION 2. (i) $P(L_1 + L_2) = P(L_1)P(L_2)$;
(ii) $P(L_1 \cup L_2) = -(\ell + \ell^{-1})m^{-1}P(L_1)P(L_2)$.

In (i) $L_1 + L_2$ denotes the sum (see FIGURE 5) of oriented links using *any* component of L_1 to add to *any* component of L_2. As different choices may be made for these components, this leads easily to examples of distinct links having the same polynomial. For example, the two links in FIGURE 13 are distinct as their individual components are different knots. However, by Proposition 2(i) they both have polynomial

$$\{-2\ell^2 - \ell^4 + \ell^2 m^2\}\{-\ell^{-2} - 1 - \ell^2 + m^2\}\{(\ell^{-1} + \ell^{-3})m^{-1} - \ell^{-1}m\}.$$

In (ii) $L_1 \cup L_2$ denotes the union of L_1 and L_2 placed some distance apart from each other so that no part of L_1 crosses over or under part of L_2. Proposition 2 is significant because it relates simple geometry to the *product* of polynomials. This uses the multiplicative structure of polynomials; $P(L)$ is not just an array of coefficients but is a polynomial that may be used to multiply another polynomial! The proposition is easy to prove from Theorem 1.

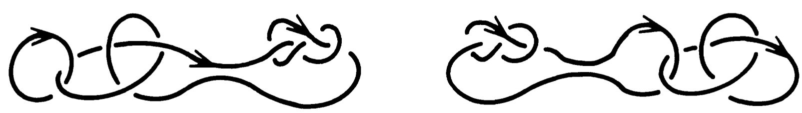


FIGURE 13

There is another way in which it is known that two oriented links will have the same polynomial. Deep in the geometric structure of link theory is the simple idea of decomposing a link using spheres that cut the link at four points. The dotted sphere of FIGURE 14 is an example. If the inside of that sphere is rotated through angle π (about the polar axis) the second diagram results. Such an operation is called *mutation*. Mutation never changes the P-polynomial of a link though it can well change the link, as indeed it does in FIGURE 14.

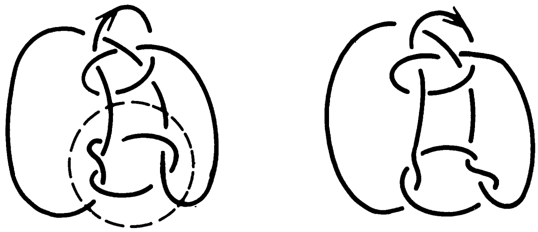


FIGURE 14

For a further example consider a *pretzel* knot as shown in FIGURE 15. The ith circle contains a twist of c_i crossings as indicated, each c_i being odd (to side-step any orientation difficulties). Mutation, with respect to the indicated ellipse, interchanges c_2 and c_3. As any permutation is the result of a sequence of such adjacent interchanges, the P-polynomial of the pretzel knot is independent of the ordering of the c_i's; in general the knot does change if that ordering changes.

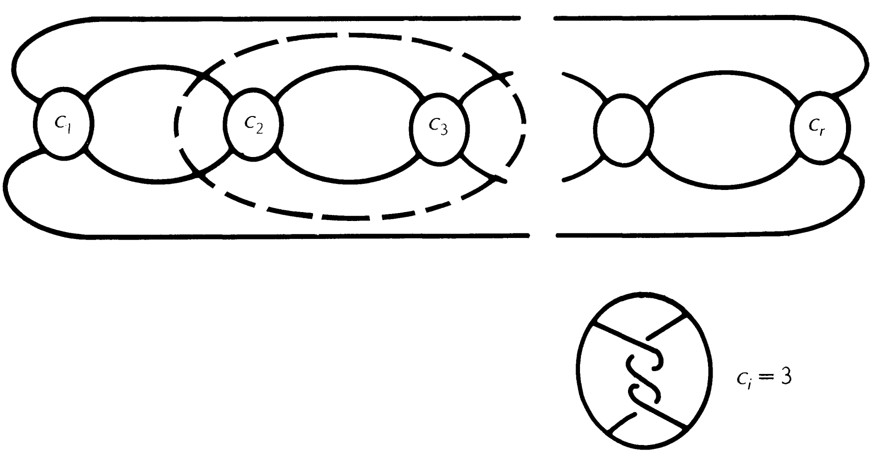


FIGURE 15

So far nothing has been said about a proof for Theorem 1. The proof in [**13**] consists of defining the polynomial with a lengthy argument of induction on the number of crossings of a presentation, showing that however a calculation (like those already discussed) is made the same polynomial results, and checking invariance under the Reidemeister moves. It is thus entirely combinatoric, but the induction argument needs delicate handling. Although other proofs differ in style and emphasis they all seem to use essentially the same combinatorics.

§3. The Jones Polynomial

Whereas the proof that $P(L)$ exists is a little arduous, an almost trivial proof of the existence of the polynomial of Jones has been found by L. H. Kauffman [**11**]. This proof, which must, in recent years, be one of the most remarkable discoveries of readily accessible mathematics, is outlined below.

The original polynomial of V. F. R. Jones associated with an oriented link L is denoted $V(L)$. It is a Laurent polynomial in the variable $t^{1/2}$, that being simply a symbol whose square is the symbol t. It satisfies

$$V(\text{unknot}) = 1$$
$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0,$$

where L_+, L_-, and L_0 are oriented links related as before. Thus $V(L)$ is obtained from $P(L)$ by the substitution

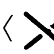
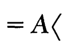
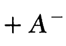
$$(\ell, m) = \left(it^{-1}, i(t^{-1/2} - t^{1/2})\right),$$

where $i^2 = -1$. As mentioned before, $P(L)$ was conceived as a generalisation of $V(L)$.

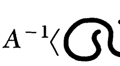
Begin all over again by considering *projections* (pictures) of *unoriented* links. For each such projection L define a Laurent polynomial $\langle L \rangle$ in one variable A by the following three rules that will shortly be explained:

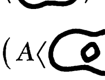
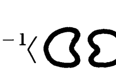
(a) $\langle$  $\rangle = 1$,

(b) $\langle L \cup$  $\rangle = -(A^{-2} + A^2)\langle L \rangle$,

(c) $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$.

This $\langle L \rangle$ is called the *bracket polynomial* of L. Rule (a) states that 1 is the polynomial of the particular projection of the unknot that has no crossing at all. In rule (b) $L \cup \bigcirc$ denotes the projection that consists of L plus an extra component that contains no crossing. Rule (c) refers to three projections exactly the same, except near one point where they are as shown. The first projection of this triple shows a crossing, and in the other two that crossing has been destroyed. It should be noted that, given the picture of the crossing, one can distinguish between the two other pictures using the orientation of space: If, when moving along the underpass towards the crossing one swings to the right, up on to the overpass, one creates the picture of the link whose polynomial is multiplied by A in rule (c). No arrows are required for that. A simple example involving the use of all three rules is as follows:

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= A(A\langle$  $\rangle + A^{-1}\langle$  $\rangle) + A^{-1}(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$

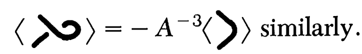
$= -(A^2 + A^{-2})^2 + 2.$

Here when calculating $\langle L \rangle$ there are no problems about making judicious choices of crossings to switch in order to maneuver towards an unknotted situation (as there were with the P-polynomial). Each use of rule (c) *reduces* the number of crossings in the projections until there are no crossings at all; then rules (b) and (a) finish the job of calculation. It is evident that the choice of the order in which the crossings are attacked is irrelevant, so that these rules do indeed define unambiguously a polynomial for each unoriented link projection. What remains to be done is to see if $\langle L \rangle$ is unchanged by the Reidemeister moves I, II and III of §1; if it is, then it is an invariant of real links in $\mathbb{R}^3$:

Move I. $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

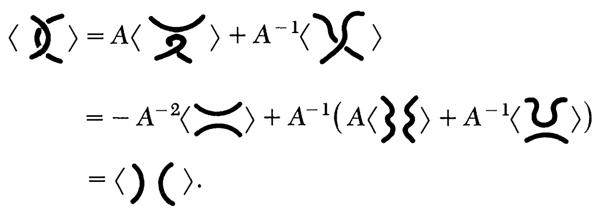
$= (-A(A^{-2} + A^2) + A^{-1})\langle$  $\rangle$

$= -A^3\langle$  $\rangle$.



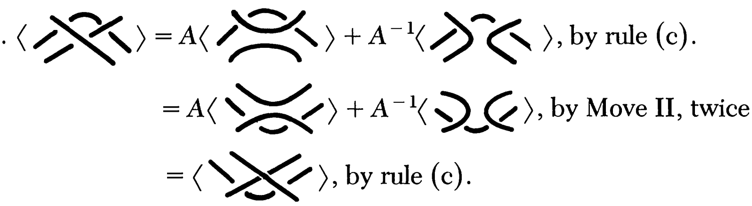
Thus the bracket polynomial *fails* to be invariant under Move I, and that is an exceedingly important observation.

Move II.



Hence the bracket polynomial *is* invariant under Move II.

Move III.



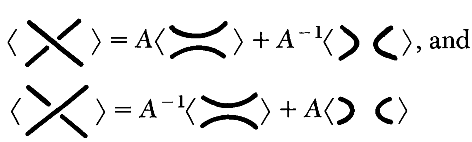
Hence there is also invariance under Move III.

Now give L an *orientation*. Let $w(L)$, the *writhe* of L, be the algebraic sum of the crossings of L, counting $+1$ for a positive crossing, and -1 for a negative crossing (for example, w(left-hand trefoil) $= -3$). Move I adds or subtracts one to $w(L)$, so $w(L)$ is certainly not invariant under that move, but it is (clearly) invariant under Moves II and III. Thus any combination of $w(L)$ and $\langle L\rangle$ will be invariant under Moves II and III, and their non-invariant behaviours under Move I *cancel* in the expression

$$X(L) = (-A)^{-3w(L)}\langle L\rangle.$$

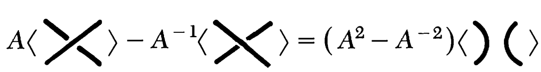
The above is a *complete proof* that $X(L)$ is a well defined invariant of oriented links.

For projections related in the usual way, rule (c) gives



$(*)$.

Thus



Suppose that orientations can be chosen for these last three projections so that the arrows point approximately upwards (c.f. FIGURE 7); call them L_+, L_- and L_0. Then $w(L_\pm) = w(L_0) \pm 1$. Hence, substitution gives

$$A(-A)^3X(L_+) - A^{-1}(-A)^{-3}X(L_-) = (A^2 - A^{-2})X(L_0).$$

Writing $t^{-1/4} = A$ this becomes

$$t^{-1}X(L_+) - tX(L_-) + (t^{-1/2} - t^{1/2})X(L_0) = 0,$$

so that, under the substitution $A = t^{-1/4}$, $X(L)$ is the original Jones polynomial $V(L)$ for they satisfy the same defining formula.

No analogously simple proof is known for the existence of the P-polynomial; in a proof, the difficulty is to show that different chains of calculations *never* give different

FIGURE 16

polynomials. In terms of distinguishing links the P-polynomial is more powerful than is the V-polynomial; two variables are better than one. For example, the knots in FIGURE 16 have the same V-polynomial but different P-polynomials.

There is a property of the Jones polynomial, a *reversing result*, that *seems* to have no analogue in terms of the P-polynomial. Suppose that k is one component of an oriented link L and that a new oriented link L^* is formed from L by reversing just the orientation of k. Of course, $\langle L \rangle = \langle L^* \rangle$, for the bracket polynomial disregards all orientations. Thus $V(L)$ and $V(L^*)$ are the same up to multiplication by some power of t (for each is $\langle L \rangle$ multiplied by a power of $A = t^{-1/4}$). The precise result is:

PROPOSITION 3. $V(L^*) = t^{-3\lambda}V(L)$, *where* 2λ *is the sum of the signs of the crossings of* k *with the other components of* $L - k$.

This λ is called the *linking number* of k and $L - k$. It is noteworthy that, though this result follows trivially from the approach of the bracket polynomial, it is by no means obvious when working from the (L_+, L_-, L_0)-definition of the V-polynomial.

The simplicity of Kauffman's approach to the V-polynomial has led to a much better understanding of that polynomial and to a most pleasing application ([11], [14], [18]) concerning alternating knots. A projection of a link is alternating if, when travelling along any part of the link, the crossings are encountered alternately over, under, over, under,.... In FIGURE 16 the four-crossing projection is alternating, the other is not alternating. The first thirty-one knots in the classical knot tables have alternating projections. A crossing in a link projection will be called *removable* if it is like the crossing in FIGURE 17; it could be removed by rotating the part of the link in the box labelled Y.

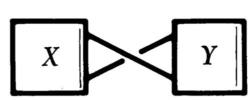


FIGURE 17

PROPOSITION 4. (See [11], [14] and [18].) *Let* L *be a connected oriented link projection of* n *crossings, then*

(i) $n \geqslant$ Spread $V(L)$, *where* spread $V(L)$ *is the difference between the maximum and the minimum degrees of* t *that appear in* $V(L)$;

(ii) $n =$ Spread $V(L)$ *if* L *is also alternating and has no removable crossing.*

It is easy to see, by inspection, if a knot projection is alternating and has no removable crossing. If it has these properties, and n crossings, the proposition implies that the knot can have no projection with fewer crossings. If there were a projection with $n-1$ crossings, then, by (i), $n-1 \geq$ Spread $V(L)$, which contradicts (ii). This solves a very old problem in knot theory; it has always been suspected that an alternating projection was the simplest available.

For the record, this section should include mention of the *Alexander polynomial* $\Delta(L)$ of an oriented link L. Like the Jones polynomial it is a Laurent polynomial in $t^{1/2}$, it can be defined in a similar way by $\Delta(\text{unknot}) = 1$, and

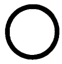
$$\Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0) = 0.$$

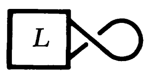
Thus $\Delta(L)$ is obtained from $P(L)$ by a substitution for the variables (though it was the similarity between this formula and that defining $V(L)$ that lead to the discovery of the P-polynomial). The Alexander polynomial has been known and developed for about sixty years [**1**]. It is discussed in the text books of knot theory, usually being defined in terms of the determinant of a certain matrix (though see [**6**]). The value of $\Delta(L)$ when $t = -1$ is called the *determinant* of the link, and this integer was one of the first link invariants to be studied. Although the Alexander polynomial is quite good at distinguishing knots, there do exist knots that it cannot distinguish from the unknot; an example is the pretzel knot (see FIGURE 15) for which $(c_1, c_2, c_3) = (-3, 5, 7)$; for this the Jones polynomial is certainly non-trivial. The Alexander polynomial is fairly well understood in terms of the machinery of algebraic topology (homology groups, fundamental groups, covering spaces, etc.). The same cannot be said for the Jones polynomial and its generalizations. Is there a knot K, other than the unknot, for which $V(K) = 1$? The answer to this is not known. If there is no such K then the Jones polynomial is the long sought elementary method of determining knottedness. There is no reason to suppose that it is so powerful an invariant, but computer searches have revealed no example of such a K, neither has understanding given any clue to finding a method by which such a K might be constructed.

§4. The Semioriented Polynomial

Although it may seem that the preceding sections contain many polynomials, only the P-polynomial and some specialisations of it occur. There has been discovered, however, another polynomial, the F-polynomial, that is similar in concept to the P-polynomial though the two are quite distinct. The way to define this F-polynomial is rather like the way in which $V(L)$ was derived from $\langle L \rangle$.

Firstly, for a *projection* L of an *unoriented* link, define a Laurent polynomial $\Lambda(L)$ in two variables a and x by the rules

(a) $\Lambda($  $) = 1$;

(b) $\Lambda($  $) = a\Lambda(L)$, $\Lambda($  $) = a^{-1}\Lambda(L)$, and $\Lambda(L)$ does not change when L is changed by a Reidemeister move of type II or type III;

(c) $\Lambda(L_+) + \Lambda(L_-) = x(\Lambda(L_0) + \Lambda(L_\infty))$

where L_+, L_-, L_0 and L_∞ are projections of unoriented links that are exactly the same except near a point where they are as shown in FIGURE 18.

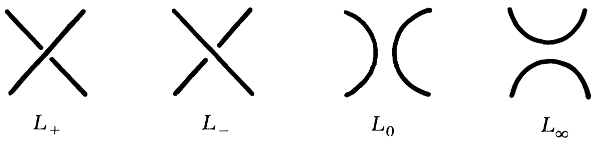


FIGURE 18

Notes. (a) This means that Λ is 1 for the projection of just one component which has no crossing.

(b) If a positive kink is removed, the Λ-polynomial is multiplied by a (or by a^{-1} for a negative kink). Thus the Λ-polynomial is not invariant under Reidemeister Move I.

(c) In the absence of orientations it is not clear which picture in FIGURE 18 should be called L_+ and which L_-, nor which is L_0 and which L_∞. Those ambiguities are irrelevant in the light of the symmetry of the formula of Rule (c).

It should be clear that the methods of calculation developed for the P-polynomial will work equally well in this new situation. The new situation is easier in that orientations do not (yet) appear, but more troublesome in that Rule (c) uses four pictures instead of three. As an excercise check that the Λ-polynomial of the two-component link projection with no crossing is $((a^{-1}+a)x^{-1}-1)$ while that of the usual projection of the left-hand trefoil knot is

$$-2a^{-1}-a+(1+a^2)x+(a^{-1}+a)x^2.$$

A proof that, for a given projection, different schemes of calculation always give the same polynomial requires a more complicated version of the inductive method mentioned at the end of §2 for the P-polynomial.

The Λ-polynomial is, as stated, invariant under the second and third of Reidemeister's moves. Its failure to be invariant under move I can easily be corrected (as was done for $\langle L \rangle$) if L is *now oriented*. As before let $w(L)$ be the sum of the signs of the crossings of the oriented link L.

THEOREM 2. *For any oriented link L, let*

$$F(L)=a^{-w(L)}\Lambda(L).$$

This $F(L)$ is a well-defined invariant of oriented links in 3-space.

Tables of the P-polynomial and of this second two-variable polynomial have been produced by M. B. Thistlethwaite for the 12,965 knots in his tabulation of knot projections up to thirteen crossings. He works, of course, with a computer, that being all the more desirable for the F-polynomial; the occurrence of four rather than three diagrams in the defining formula does make calculations for F much more arduous than for P. The F-polynomials contain very many more terms than do the P-polynomials; for example for either knot in FIGURE 14 the P-polynomial has 14 terms, the F-polynomial has 45 terms. A few more examples appear in the tables at the end of this paper. The greater number of terms seems to mean, in practice, that two knots are more likely to be distinguished by F than by P.

The F-polynomial has a right to be called 'semioriented' because, although L must be oriented to define $F(L)$, changing the orientation only changes $F(L)$ by multiplication by a power of a. The relevant result is:

PROPOSITION 5. *Suppose L^* is obtained from L by reversing the orientation of a component k, then $F(L^*)=a^{4\lambda}F(L)$, where λ is the linking number of k with the other components of $L-k$.*

Compare this with Proposition 3; the proof here is much the same.

An interesting specialisation of $F(L)$ is obtained by the substitution $a=1$. The resultant Laurent polynomial $Q(L)$, in the one variable x, is called the *absolute* polynomial. This substitution makes all the subtleties of the above definition disappear, no orientations of any sort are required (hence the name 'absolute') and one can work entirely with links in $\mathbb{R}^3$ rather than projections in the plane. The Q-polynomial is simply defined by $Q(\text{unknot})=1$, and

$$Q(L_+)+Q(L_-)=x(Q(L_0)+Q(L_\infty)).$$

Chronologically this polynomial was discovered by Brandt, Lickorish, and Millett [4] and Ho [8] as an extension of the ideas of the P-polynomial, and Kauffman [**10**] explained how to insert the second variable 'a' to create the F-polynomial. The fact that the Q-polynomial uses no arrows, and $Q(L)=Q(\bar{L})$, makes this something of a recommended polynomial for beginners. Unfortunately the proofs that the Q and F polynomials are unambiguously defined are of almost the same form and complexity. In FIGURE 19 are examples that show that the P and F polynomials are independent in the sense that neither is hidden within the other, to be revealed by some subtle change of the variables.

8_8 10_{129}

These have the same P but different Q (and F) polynomials.

11_{255} 11_{257}

These have the same F but different Δ (and P) polynomials.

FIGURE 19

As might be expected, some basic properties of the F-polynomial are similar to those of the P-polynomial. This is summarized in the next result; it uses some of the notations from §2.

PROPOSITION 6. (i) $\overline{F(L)}=F(\bar{L})$, *where* $\bar{a}=a^{-1}$ *and* $\bar{x}=x$;

(ii) $F(L_1+L_2)=F(L_1)F(L_2)$;

(iii) $F(L_1\cup L_2)=((a^{-1}+a)x^{-1}-1)F(L_1)F(L_2)$;

(iv) *F is unchanged by mutation.*

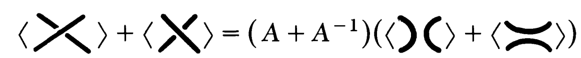
A result that came as something of a surprise [12] was that *both* the P-polynomial *and* the F-polynomial contain the original polynomial of Jones, the V-polynomial. The result, which now has an easy proof, is:

PROPOSITION 7. *For any oriented link L, the substitution*

$$(a, x) = \left(-t^{-3/4}, (t^{-1/4} + t^{1/4})\right)$$

reduces $F(L)$ to $V(L)$.

Proof. Adding together the two equations (*) from §3 gives



Thus $\langle L \rangle$ satisfies exactly the same defining equations as $\Lambda(L)$ when $x = (A + A^{-1})$ and $a = -A^3$ (the latter arising from comparison of the effects of the first Reidemeister move on the two polynomials). Then just recall that it is the substitution $A = t^{-1/4}$ that produces the Jones polynomial.

§5. Calculations, Problems and Tables

Calculations of any of the polynomials mentioned in previous sections can be performed 'by hand' for links with few crossings or for those with some simple pattern. Sometimes the linear nature of the formulae that define these polynomials can be exploited in a most pleasing way. That idea can be illustrated using the Q-polynomial to avoid orientation complications. Suppose that a link T_n contains as part of it the n-crossing twist as shown in FIGURE 20 (where by convention $n = \infty$ is also permitted, as illustrated); if n is negative the twist goes the other way.

FIGURE 20

Focussing on one of these crossings produces a quadruple of links L_+, L_-, L_0 and L_∞ (as in the defining formula for the Q-polynomial), namely, T_n, T_{n-2}, T_{n-1} and T_∞. The defining formula gives

$$Q(T_n) + Q(T_{n-2}) = x\left(Q(T_{n-1}) + Q(T_\infty)\right).$$

Thus

$$\begin{bmatrix} Q(T_n) \\ Q(T_{n-1}) \\ Q(T_\infty) \end{bmatrix} = M \begin{bmatrix} Q(T_{n-1}) \\ Q(T_{n-2}) \\ Q(T_\infty) \end{bmatrix} = M^n \begin{bmatrix} Q(T_0) \\ Q(T_{-1}) \\ Q(T_\infty) \end{bmatrix}$$

where M is the matrix

$$\begin{bmatrix} x & -1 & x \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As an exercise use this idea to calculate, in terms of the matrix M, the Q-polynomial of the link of FIGURE 21, where the twists have m and n crossings respectively.

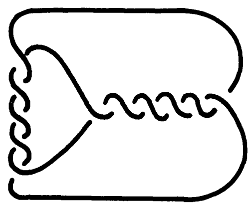


FIGURE 21

In the following exercises L is an oriented link with $c(L)$ components. The proofs are all performed using, in the relevant defining formulae, induction on the number of crossings in a projection.

Exercises

(i) $P(L) = 1$ when $m = -(\ell + \ell^{-1})$.
(ii) In $P(L)$ the least power of m is $m^{1-c(L)}$.
(iii) $F(L) = (-2)^{c(L)-1}$ when $(a, x) = (1, -2)$.
(iv) $F(L) = (-1)^{c(L)-1}$ when $(a, x) = (i, x), x \neq 0$.
(v) $V(L) = \Delta(L)$ when $t = -1$.
(vi) If $c(L) = 1$, $F(L) = P(L)$ when $x = 0 = m$, and $a = \ell$.

Unanswered questions abound. Here are some of them.

Questions. (1) Can the new polynomials be defined without reference to diagrams in the plane?

(2) Can the P and F polynomials be defined in as simple a way as the bracket polynomial?

(3) Can any of the new polynomials for a link be calculated in one step (e.g., by means of a determinant) without working out many polynomials of simpler links?

(4) Is there some simple characterisation of what polynomials can arise as the P or F or V or Q polynomial of some link?

(5) Is there a nontrivial link of c components with the same polynomial (in the P, V, F, or Q sense) as the trivial unlink of c components?

(6) Does any of the new polynomials give information about the number of crossing switches needed to undo a knot (this is called its *unknotting number*)?

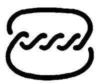
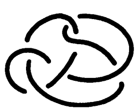
(7) Does there exist some grand master polynomial in which particular substitutions produce both P and F?

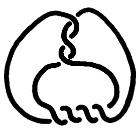
(8) Is there a "coloured" theory for P or F? This would be a theory that had more variables and which could distinguish, for example, a red trefoil linked with a blue unknot from a blue trefoil linked with a red unknot. There is such a variant of the Alexander polynomial.

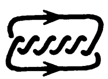
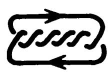
(9) Are there polynomials other than P and F that can be defined along the same lines as they are defined? Several attempts have been made but all have turned out to be subtle variants of the original two polynomials.

TABLES. Below are given tables of the P and F polynomials for a few knots and links of low numbers of crossings to give a feeling for what is involved.

F & P for knots $\leqslant 7$ crossings, for links $\leqslant 6$ crossings.

		P	F
3_1		$(-2\ell^2-\ell^4)+\ell^2m^2$	$(-2a^2-a^4)+(a^3+a^5)x+(a^2+a^4)x^2$
4_1		$(-\ell^{-2}-1-\ell^2)+m^2$	$(-a^{-2}-1-a^2)+(-a^{-1}-a)x+(a^{-2}+2+a^2)x^2+(a^{-1}+a)x^3$
5_1		$(3\ell^4+2\ell^6)+(-4\ell^4-\ell^6)m^2+\ell^4m^4$	$(3a^4+2a^6)+(-2a^5-a^7+a^9)x+(-4a^4-3a^6+a^8)x^2+(a^5+a^7)x^3+(a^4+a^6)x^4$
5_2		$(-\ell^2+\ell^4+\ell^6)+(\ell^2-\ell^4)m^2$	$(-a^2+a^4+a^6)+(-2a^5-2a^7)x+(a^2-a^4-2a^6)x^2+(a^3+2a^5+a^7)x^3+(a^4+a^6)x^4$
6_1		$(-\ell^{-2}+\ell^2+\ell^4)+(1-\ell^2)m^2$	$(-a^{-2}+a^2+a^4)+(2a+2a^3)x+(a^{-2}-4a^2-3a^4)x^2+(a^{-1}-2a-3a^3)x^3$ $+(1+2a^2+a^4)x^4+(a+a^3)x^5$
6_2		$(2+2\ell^2+\ell^4)+(-1-3\ell^2-\ell^4)m^2+\ell^2m^4$	$(2+2a^2+a^4)+(-a^3-a^5)x+(-3-6a^2-2a^4+a^6)x^2+(-2a+2a^3)x^3$ $+(1+3a^2+2a^4)x^4+(a+a^3)x^5$
6_3		$(\ell^{-2}+3+\ell^2)+(-\ell^{-2}-3-\ell^2)m^2+m^4$	$(a^{-2}+3+a^2)+(-a^{-3}-2a^{-1}-2a-a^3)x+(-3a^{-2}-6-3a^2)x^2+(a^{-3}+a^{-1}+a+a^2)x^3$ $+(2a^{-2}+4+2a^2)x^4+(a^{-1}+a)x^5$

7_1		$(-4\ell^6-3\ell^8)+(10\ell^6+4\ell^8)m^2$ $+(-6\ell^6-\ell^8)m^4+\ell^6m^6$	$(-4a^6-3a^8)+(3a^7+a^9-a^{11}+a^{13})x+(10a^6+7a^8-2a^{10}+a^{12})x^2+(-4a^7-3a^9+a^{11})x^3$ $+(-6a^6-5a^8+a^{10})x^4+(a^7+a^9)x^5+(a^6+a^8)x^6.$
7_2		$(-\ell^2-\ell^6-\ell^8)+(\ell^2-\ell^4+\ell^6)m^2$	$(-a^2-a^6-a^8)+(3a^7+3a^9)x+(a^2+3a^6+4a^8)x^2+(a^3-a^5-6a^7-4a^9)x^3+(a^4-3a^6-4a^8)x^4$ $+(a^5+2a^7+a^9)x^5+(a^6+a^8)x^6.$
7_3		$(-2\ell^{-8}-2\ell^{-6}+\ell^{-4})+(\ell^{-8}+3\ell^{-6}-3\ell^{-4})m^2$ $+(-\ell^{-6}+\ell^{-4})m^4$	$(-2a^{-8}-2a^{-6}+a^{-4})+(-2a^{-11}+a^{-9}+3a^{-7})x+(-a^{-10}+6a^{-8}+4a^{-6}-3a^{-4})x^2$ $+(a^{-11}-a^{-9}-4a^{-7}-2a^{-5})x^3+(a^{-10}-3a^{-8}-3a^{-6}+a^{-4})x^4$ $+(a^{-9}+2a^{-7}+a^{-5})x^5+(a^{-8}+a^{-6})x^6.$
7_4°		$(-\ell^{-8}+2\ell^{-4})+(\ell^{-6}-2\ell^{-4}+\ell^{-2})m^2$	$(-a^{-8}+2a^{-4})+(4a^{-9}+4a^{-7})x+(2a^{-8}-3a^{-6}-4a^{-4}+a^{-2})x^2+(-4a^{-9}-8a^{-7}-2a^{-5}+2a^{-3})x^3$ $+(-3a^{-8}+3a^{-4})x^4+(a^{-9}+3a^{-7}+2a^{-5})x^5+(a^{-8}+a^{-6})x^6.$
7_5		$(2\ell^4-\ell^8)+(-3\ell^4+2\ell^6+\ell^8)m^2$ $+(\ell^4-\ell^6)m^4$	$(2a^4-a^8)+(-a^5+a^7+a^9-a^{11})x+(-3a^4+a^8-2a^{10})x^2+(-a^5-4a^7-2a^9+a^{11})x^3$ $+(a^4-a^6+2a^{10})x^4+(a^5+3a^7+2a^9)x^5+(a^6+a^8)x^6.$
7_6		$(1+\ell^2+2\ell^4+\ell^6)+(-1-2\ell^2-2\ell^4)m^2$ $+\ell^2m^4$	$(1+a^2+2a^4+a^6)+(a+2a^3-a^7)x+(-2-4a^2-4a^4-2a^6)x^2+(-4a-6a^3-a^5+a^7)x^3$ $+(1+a^2+2a^4+2a^6)x^4+(2a+4a^3+2a^5)x^5+(a^2+a^4)x^6.$
7_7		$(\ell^{-4}+2\ell^{-2}+2)+(-2\ell^{-2}-2-\ell^2)m^2+m^4$	$(a^{-4}+2a^{-2}+2)+(2a^{-3}+3a^{-1}+a)x+(-2a^{-4}-6a^{-2}-7-3a^2)x^2+(-4a^{-3}-8a^{-1}-3a+a^3)x^3$ $+(a^{-4}+2a^{-2}+4+3a^2)x^4+(2a^{-3}+5a^{-1}+3a)x^5+(a^{-2}+1)x^6.$

		P	F
0^2_1		$(-\ell^{-1}-\ell)m^{-1}$	$(a^{-1}+a)x^{-1}-1$
2^2_1		$(\ell+\ell^3)m^{-1}-\ell m$	$(-a-a^3)x^{-1}+a^2+(a+a^3)x$
4^2_1		$(-\ell^3-\ell^5)m^{-1}+(3\ell^3+\ell^5)m-\ell^3m^3$	$(a^3+a^5)x^{-1}-a^4+(-3a^3-2a^5+a^7)x+(a^4+a^6)x^2+(a^3+a^5)x^3$
4^2_1		$(-\ell^{-5}-\ell^{-3})m^{-1}+(\ell^{-3}-\ell^{-1})m$	$(a^{-5}+a^{-3})x^{-1}-a^{-4}+(-3a^{-5}-2a^{-3}+a^{-1})x+(a^{-4}+a^{-2})x^2+(a^{-5}+a^{-3})x^3$
5^2_1		$(-\ell^{-1}-\ell)m^{-1}+(\ell^{-1}+2\ell+\ell^3)m-\ell m^3$	$(a^{-1}+a)x^{-1}-1+(-2a^{-1}-4a-2a^3)x+(-1+a^4)x^2+(a^{-1}+3a+2a^3)x^3+(1+a^2)x^4$
6^2_1		$(\ell^5+\ell^7)m^{-1}+(-6\ell^5-3\ell^7)m+(5\ell^5+\ell^7)m^3$ $-\ell^5m^5$	$(-a^5-a^7)x^{-1}+a^6+(6a^5+4a^7-a^9+a^{11})x+(-3a^6-2a^8+a^{10})x^2+(-5a^5-4a^7+a^9)x^3$ $+(a^6+a^8)x^4+(a^5+a^7)x^5$
6^2_1		$(\ell^{-7}+\ell^{-5})m^{-1}+(-\ell^{-5}+\ell^{-3}-\ell^{-1})m$	$(-a^{-7}-a^{-5})x^{-1}+a^{-6}+(6a^{-7}+4a^{-5}-a^{-3}+a^{-1})x+(-3a^{-6}-2a^{-4}+a^{-2})x^2$ $+(-5a^{-7}-4a^{-5}+a^{-3})x^3+(a^{-6}+a^{-4})x^4+(a^{-7}+a^{-5})x^5$
6^2_2		$(\ell^{-7}+\ell^{-5})m^{-1}+(-\ell^{-7}-2\ell^{-5}+2\ell^{-3})m$ $+(\ell^{-5}-\ell^{-3})m^3$	$(-a^{-7}-a^{-5})x^{-1}+a^{-6}+(-2a^{-9}+3a^{-7}+3a^{-5}-2a^{-3})x+(-a^{-8}-2a^{-6}-a^{-4})x^2$ $+(a^{-9}-2a^{-7}-2a^{-5}+a^{-3})x^3+(a^{-8}+2a^{-6}+a^{-4})x^4+(a^{-7}+a^{-5})x^5.$
6^2_3		$(-\ell^3-\ell^5)m^{-1}+(2\ell^3-\ell^5-\ell^7)m$ $+(-\ell^3+\ell^5)m^3$	$(a^3+a^5)x^{-1}-a^4+(-2a^3-a^5-a^9)x+(-3a^6-3a^8)x^2+(a^3+a^9)x^3$ $+(a^4+3a^6+2a^8)x^4+(a^5+a^7)x^5$
6^2_3		$(-\ell^{-5}-\ell^{-3})m^{-1}+(2\ell^{-3}+\ell^{-1}+\ell)m-\ell^{-1}m^3$	$(a^{-5}+a^{-3})x^{-1}-a^{-4}+(-2a^{-5}-a^{-3}-a)x+(-3a^{-2}-3)x^2+(a^{-5}+a)x^3$ $+(a^{-4}+3a^{-2}+2)x^4+(a^{-3}+a^{-1})x^5$

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Four Mathematical Clerihews

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1. Pythagoras
   Did stagger us
   And our reason encumber
   With irrational number.

2. Kurt Gödel
   Created a hurdle
   For the truths of a system:
   You just can't list'em!

3. Wily Fermat propounded,
   "Many will be confounded
   At the thought that my theorem
   Is really quite near'em."

4. Said Alfred Tarski:
   "Talk of 'truth' is a farce. Key
   To getting it right
   Is to know 'Snow is white.'"