Products of Sines and Cosines

STEVEN GALOVICH
Carleton College
Northfield, MN 55057

In [3], Z. Usiskin presents a number of intriguing identities involving products of specific values of the sine function. Here are some examples:

$$\sin 10°\sin 50°\sin 70° = \frac{1}{8}, \tag{1}$$

$$\sin 6°\sin 42°\sin 66°\sin 78° = \frac{1}{16}. \tag{2}$$

Most of the identities derived by Usiskin have the same form: The product of the sines of k appropriately chosen angles (k is a positive integer), all between 0° and 90°, equals $1/2^k$.

In a recent conversation, Usiskin asked for an "explanation" of these identities. By that he meant more than merely proofs of these equations. His paper contains proofs of (1) and (2) and several other similar formulas. Rather, Usiskin is asking for a general statement which contains his identities as special cases and for both rigorous and heuristic arguments which justify the generalization. This note provides this type of explanation.

We begin by writing the left side of (1) and (2) in terms of complex roots of unity. We then analyze generalizations of the expressions thus obtained and we present two evaluations of the generalized expressions. Both are based on number-theoretic considerations; the first is elementary in nature, while the second, involving ideas from algebraic number theory, is more sophisticated. The second approach also provides the heuristic "explanation" sought by Usiskin for the identities. The principal result of the paper is Theorem 2, which evaluates the products that generalize identities (1) and (2). Theorem 1, a well-known fact of algebraic number theory, is used to prove both Theorem 2 and Theorem 3. The latter theorem describes an interesting property of regular n-gons inscribed in a unit circle.

Formulation of the problem

To set the stage for our formal argument, let us play with the first identity. We begin by rewriting it as

$$(2\sin 10°)\,(2\sin 50°)\,(2\sin 70°) = 1. \tag{3}$$

In the field of complex numbers,

$$[(\cos 10° + i\sin 10°) - (\cos 10° - i\sin 10°)]/i = 2\sin 10°.$$

Let $\zeta = \cos 10° + i\sin 10°$. The complex conjugate and inverse of ζ is $\zeta^{-1} = \cos 10° - i\sin 10°$ and

$$2\sin 10° = (\zeta - \zeta^{-1})/i.$$

Also $2\sin 50° = (\zeta^5 - \zeta^{-5})/i$ and $2\sin 70° = (\zeta^7 - \zeta^{-7})/i$. In terms of ζ and ζ^{-1}, equation (3) becomes:

$$\prod_{j=1,5,7} (\zeta^j - \zeta^{-j})/i = 1.$$

Note that by De Moivre's formula, $\zeta^{36} = 1$, i.e., ζ is a 36th root of unity.

Through analogous manipulations, identity (2) reduces to

$$\prod_{j=1,7,11,13} (\alpha^j - \alpha^{-j})/i = 1, \tag{4}$$

where $\alpha = \cos 6° + i \sin 6°$. By the way, note another identity:

$$\prod_{j=1,7,11,13} 2\cos(6j)° = \prod_{j=1,7,11,13} (\alpha^j + \alpha^{-j}) = 1. \tag{5}$$

Also observe that α is a 60th root of unity.

Lest we develop the impression that products of these kinds always equal 1, consider

$$(2\cos 10°)(2\cos 50°)(2\cos 70°) = \sqrt{3},$$

an interesting identity in its own right.

What is the general pattern which describes the values of these products? Several of the identities presented by Usiskin, including (3) and (4), can be rewritten in the following form: For $n \in \{36, 60, 72, 120, 130, 360\}$ and $r = 2\pi/n$ (henceforth we use radian measure)

$$\prod_{j=1}^{n/4}{}'(2\sin(j \cdot r)) = 1 \tag{6}$$

where $\prod_j'$ denotes the product of those integers j that are relatively prime to n. When $n = 36$ and $n = 60$, one obtains identities (3) and (4), respectively.

Our overall goal (accomplished in Theorem 2) is to describe products of the following form:

$$\prod_{j=1}^{k}{}' 2f(jr) \tag{7}$$

where $f(x) = \sin x$ or $f(x) = \cos x$ and $k = n/4$, $n/2$, or n. As in (4) and (5), these numbers can be represented as products of sums of roots of unity. Thus we turn to a study of basic arithmetic properties of roots of unity (i.e., those involving addition, subtraction, and multiplication).

Roots of unity

The investigation of arithmetic properties of roots of unity, the so-called theory of cyclotomic fields, was pioneered in the mid-19th century by the number theorist E. E. Kummer. The facts about cyclotomic fields which we need are elementary and can be established without recourse to the beautiful and intricate machinery constructed by Kummer. Later we shall derive the key results using ideas and results of algebraic number theory.

Let n be a positive integer such that $n \neq 1, 2, 4$. Let

$$\zeta_n = \cos(2\pi/n) + i\sin(2\pi/n);$$

ζ_n is a primitive nth root of unity, which means that $\zeta_n^n = 1$ and $\zeta_n^j \neq 1$ for $1 \leq j \leq n-1$. Of

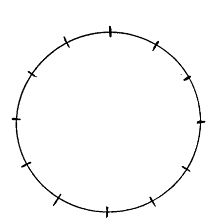

FIGURE 1. **The 12th roots of unity in the complex plane.**

course, for $1 \leq j \leq n-1$, ζ_n^j is also an nth root of unity, hence is a root of the equation $x^n - 1 = 0$. Notice that

$$\zeta_n^j + \zeta_n^{-j} = 2\cos(2\pi j/n)$$

and

$$\left(\zeta_n^j-\zeta_n^{-j}\right)/i=2\sin(2\pi j/n).$$

The primitive nth roots of unity are ζ_n^j for $(j,n)=1$ where (j,n) is the greatest common divisor of j and n. There are $\varphi(n)$ primitive roots of unity where $\varphi(n)$, the Euler phi-function, is defined to be the number of positive integers less than n that are relatively prime to n. The well-known formula for $\varphi(n)$ will come in handy later: Write $n=p_1^{a_1}\cdots p_m^{a_m}$ where $p_1,\cdots,p_m$ are distinct primes. Then

$$\varphi(n)=\prod_{i=1}^{m}p_i^{a_i-1}(p_i-1).$$

Over the field $\mathbb{C}$ of complex numbers, we have the following polynomial factorizations:

$$x^n-1=\prod_{j=0}^{n-1}\left(x-\zeta_n^j\right)$$

and

$$(x^n-1)/(x-1)=1+x+\cdots+x^{n-1}=\prod_{j=1}^{n-1}\left(x-\zeta_n^j\right). \tag{8}$$

When $x=1$ is substituted in (8), the result is the following factorization of n in $\mathbb{C}$:

$$\prod_{j=1}^{n-1}\left(1-\zeta_n^j\right)=n. \tag{9}$$

We wish to investigate the product

$$\rho_n=\prod_{j=1}^{n-1}{}'\left(1-\zeta_n^j\right), \tag{10}$$

since it is closely related to the products of sines and cosines described in (7). To determine the value of ρ_n, we use a standard tactic in number theory: First consider the case in which n is prime, then allow n to be arbitrary and use mathematical induction on the number of prime factors of n.

Suppose n is prime. Then $\rho_n=\prod_{j=1}^{n}(1-\zeta_n^j)=n$. Next, to get a feel for the general case, suppose $n=p^2$ where p is prime; then from (9) and the fact that $\zeta_{p^2}^p=\zeta_p$, it follows that

$$p^2=\rho_n\prod_{k=1}^{p-1}\left(1-\zeta_n^{pk}\right)=\rho_n\prod_{k=1}^{p-1}\left(1-\zeta_p^k\right)=\rho_n\cdot\rho_p=\rho_n\cdot p.$$

Thus $\rho_n=p$, if $n=p^2$.

We now have both the basis step and the gist of the idea for an inductive proof of the next result.

THEOREM 1. *If ρ_n is defined by* (10), *then*

$$\rho_n=\begin{cases}p & \text{if } n=p^a \text{ where } p \text{ is prime}\\ 1 & \text{otherwise.}\end{cases}$$

Proof. We argue by induction on the number, f, of prime factors of n. (We count repeated prime factors, so 12 has 3 prime factors by our reckoning.) We have already established the basis step, $f=1$. Let n be an arbitrary positive integer with f factors. Suppose the theorem holds for all positive integers with fewer than f factors. (Writing $n=p_1^{a_1}\cdots p_m^{a_m}$ where the p_i are distinct primes, we have $f=\sum_{j=1}^{m}a_i$.) Then

$$n=\prod_{j=1}^{n}\left(1-\zeta_n^j\right)=\rho_n\prod_{\substack{j=1\\(j,n)>1}}^{n}\left(1-\zeta_n^j\right)=\rho_n\prod_{\substack{d|n\\d\neq 1}}A_d$$

where the last product is taken over divisors on n that are greater than 1 and for each such divisor d,

$$A_d=\prod_{(j,n)=d}\left(1-\zeta_n^j\right).$$

We claim that $A_d=\rho_{n/d}$. Assuming this claim, the proof of the theorem proceeds as follows:

Case 1. Suppose $m=1$; in other words, $n=p^a$ where p is prime. Then

$$\prod_{\substack{d|n\\d\neq 1}}A_d=\prod_{\substack{d|n\\d\neq 1}}\rho_{n/d}=\prod_{i=i}^{a-1}p=p^{a-1}$$

by inductive hypothesis, hence $\rho_n=n/p^{a-1}=p$.

Case 2. Suppose $m>1$. Let $S_i=\{n/p_i,n/p_i^2,\dots,n/p_i^{a_1}\}$ for $1\le i\le m$. If $d\in S_i$, then $\rho_{n/d}=p_i$ by induction and

$$\prod_{d\in S_i}A_d=\prod_{d\in S_i}\rho_{n/d}=p_i^{a_i}.$$

If

$$d\notin\bigcup_{i=1}^{m}S_i,$$

then $A_d=\rho_{n/d}=1$ by induction. Therefore,

$$n=\rho_n\prod_{\substack{d|n\\d\neq 1}}A_d=\rho_n\cdot n\quad\text{and}\quad\rho_n=1.$$

We now must prove the claim that $A_d=\rho_{n/d}$:

$$\begin{aligned}A_d&=\prod_{(j,n)=d}\left(1-\zeta_n^j\right)\\&=\prod_{l=1}^{n/d}{}'\left(1-\left(\zeta_n^d\right)^l\right)\\&=\prod_{l=1}^{n/d}{}'\left(1-\zeta_{n/d}^l\right)=\rho_{n/d}.\end{aligned}$$

Products of sines and cosines

Let us turn to the evaluation of products of sines and cosines. Let n be a positive integer different from 1, 2, and 4. For $s=1,1/2,1/4$, and with $r=2\pi/n$, let

$$C_n(s)=\prod_{j=1}^{ns}{}'2\cos(jr)$$

and

$$S_n(s)=\prod_{j=1}^{ns}{}'2\sin(jr).$$

Consider first $C_n(1)$:

$$C_n(1)=\prod_{j=1}^{n}{}'\left(\zeta_n^j+\zeta_n^{-j}\right)$$

$$= \prod_{j=1}^{n}{}' \zeta_n^{-j}\left(1 + \zeta_n^{2j}\right)$$

$$= \zeta_n^{-J} \prod_{j=1}^{n}{}' \left(1 + \zeta_n^{2j}\right)$$

where

$$J = \sum_{j=1}^{n}{}' j,$$

the summation taken over integers j such that $(j, n) = 1$. On the other hand:

$$S_n(1) = \prod_{j=1}^{n}{}' \left(\zeta_n^{j} - \zeta_n^{-j}\right)/i$$

$$= i^{-\varphi(n)} \zeta_n^{-J} \prod_{j=1}^{n}{}' \left(\zeta_n^{2k} - 1\right).$$

As an exercise the reader may check that $\varphi(n)$ is even and $J = n\varphi(n)/2$. Thus $i^{-\varphi(n)} = (-1)^{\varphi(n)/2}$ and $\zeta_n^{-J} = 1$, implying that

$$C_n(1) = \prod_{j=1}^{n}{}' \left(1 + \zeta_n^{2j}\right)$$

and

$$S_n(1) = (-1)^{\varphi(n)/2} \prod_{j=1}^{n}{}' \left(\zeta_n^{2j} - 1\right) = (-1)^{\varphi(n)/2} \prod_{j=1}^{n}{}' \left(1 - \zeta_n^{2j}\right).$$

The numbers $C_n(1)$ and $S_n(1)$ look very similar to ρ_n. To establish the exact relationship, we consider several cases.

Case 1. $n = p^a$ where p is an odd prime. Because n is odd, the set $\{\zeta_n^{2j} | 1 \le j \le n, (j, n) = 1\}$ is precisely the set of primitive nth roots of unity, while the set $\{-\zeta_n^{2j} | 1 \le j \le n, (j, n) = 1\}$ is the set of primitive $(2n)$th roots of unity. Therefore,

$$C_n(1) = \prod_{j=1}^{n}{}' \left(1 - \left(-\zeta_n^{2j}\right)\right) = \rho_{2n} = 1$$

and

$$S_n(1) = (-1)^{\varphi(n)/2} \prod_{j=1}^{n}{}' \left(1 - \zeta_n^{2j}\right)$$

$$= (-1)^{\varphi(n)/2} \rho_n = (-1)^{\varphi(n)/2} p.$$

Case 2. $n = 2^a$. The sets $\pm\{-\zeta_n^{2j} | 1 \le j \le n, (j, n) = 1\}$ each consist of the (2^{a-1})st roots of unity with each such root appearing twice. Thus,

$$C_n(1) = \prod_{j=1}^{n}{}' \left(1 - \left(-\zeta_n^{2j}\right)\right)$$

$$= \prod_{j=1}^{n}{}' \left(1 - \zeta_n^{2j}\right) = S_n(1)$$

$$= \prod_{j=1}^{n/2}{}' \left(1 - \zeta_{n/2}^{j}\right)^2$$

$$= 2^2 = 4.$$

Case 3. $n = 2p^a$ where p is an odd prime. As in case 1, $C_n(1) = 1$ and $S_n(1) = p$. The arguments parallel those in case 1.

Case 4. $n = 4p^a$ where p is an odd prime. This time $\{-\zeta_n^{2j} | 1 \le j \le n, (j,n) = 1\}$ is the set of p^ath roots of unity with each such root appearing twice. Therefore

$$C_n(1) = \rho_{p^a}^2 = p^2$$

and

$$S_n(a) = \rho_{2p^a}^2 = 1.$$

Case 5. All other cases. In all other instances, $C_n(1) = S_n(1) = 1$. For example if n is odd, then n is not a prime power and $C_n(1) = S_n(1) = \rho_n = 1$. If n is even, then neither $n/2$ nor $n/4$ is a prime power and $C_n(1) = S_n(1) = 1$. We leave the details for the reader.

Next we consider $C_n(s)$ and $S_n(s)$ for $s = 1/2$ or $1/4$. Since $\varphi(n)$ is even, $|C_n(1)| = C_n(1/2)^2$ and $|S_n(1)| = S_n(1/2)^2$. If $\varphi(n)$ is divisible by 4 (this occurs, for example, in case 1 for $p \equiv 1 \pmod 4$ and in cases 2, 4, and 5), then $|C_n(1/2)| = C_n(1/4)^2$ and $S_n(1/2) = C_n(1/4)^2$. Note that $S_n(1/4)$, $C_n(1/4)$ and $S_n(1/2)$ are always positive. The sign $C_n(1/2)$ is somewhat subtle since $\cos(x) < 0$ if $\pi/2 < x \le \pi$. We carry out the analysis in one case.

Suppose $n = p^a$ where p is an odd prime. Then

$$C_n(1/2) = \prod_{j=1}^{n/2}{}' 2\cos(jr)$$

has the same sign as $(-1)^b$ where b is the number of integers j such that $\pi/2 < jr < \pi$, which equals the number of integers j such that $p^a/4 < j < p^a/2$. A straightforward calculation shows that this number is even if $p \equiv 1,7 \pmod 8$ and odd if $p \equiv 3,5 \pmod 8$. Thus

$$C_n(1/2) = \begin{cases} 1 & \text{if } p \equiv 1,7 \pmod 8 \\ -1 & \text{if } p \equiv 3,5 \pmod 8. \end{cases}$$

We summarize our findings in the following theorem.

THEOREM 2. *Suppose $n \ne 1,2,4$. The values of $C_n(s)$ and $S_n(s)$ are given in* TABLE 1.

n	s	$S_n(s)$	$C_n(s)$
1. $n = p^a$	1	$(-1)^{\varphi(n)/2}p$	1
p an odd prime	1/2	$\sqrt{p}$	1 if $p^a \equiv 1,7 \pmod 8$ -1 if $p^a \equiv 3,5 \pmod 8$
2. $n = 2^a, a > 2$	1	4	4
	1/2	2	2 if $a > 3$ -2 if $a = 3$
3. $n = 2p^a$	1	$(-1)^{\varphi(n)/2}p$	1
p an odd prime	1/2	$\sqrt{p}$	1 if $p^a \equiv 1,3 \pmod 8$ -1 if $p^a \equiv 5,7 \pmod 8$
4. $n = 4p^a$	1	1	p^2
p an odd prime	1/2	1	p if $p^a \equiv 1 \pmod 4$ $-p$ if $p^a \equiv 3 \pmod 4$
	1/4	1	$\sqrt{p}$
5. All other cases	1	1	1
	1/2	1	1
(if 4 divides n)	1/4	1	1

TABLE 1

Algebraic interpretation

Thus far we have responded to part of Usiskin's challenge. By our standards an explanation of a phenomenon consists in part of formulating and proving a general statement which contains the empirical observation as a special case. But what are the heuristic notions from algebraic number theory lurking behind the theorems we have proved?

All the calculations used to prove Theorems 1 and 2 involve linear combinations of nth roots of unity with integer coefficients. The set of all complex numbers of this form constitutes a subring of $\mathbb{C}$, called the **ring of cyclotomic integers:**

$$\mathbf{Z}[\zeta_n] = \left\{ \sum_{j=0}^{m} a_j\zeta_n^j \middle| m \in \mathbf{Z}, m \geq 0, a_j \in \mathbf{Z} \right\}.$$

Actually, since $\zeta_n^n = 1$,

$$\mathbf{Z}[\zeta_n] = \left\{ \sum_{j=0}^{n-1} a_j\zeta_n^j \middle| a_j \in \mathbf{Z} \right\}.$$

It is not difficult to show that

$$\mathbf{Z}[\zeta_n] = \left\{ \sum_{j=0}^{n-2} a_j\zeta_n^j \middle| a_j \in \mathbf{Z} \right\}.$$

The ring $\mathbf{Z}[\zeta_n]$ is a subring of the **cyclotomic field** of nth roots of unity

$$\mathbf{Q}(\zeta_n) = \left\{ \sum_{j=0}^{n-1} a_j\zeta_j \middle| a_j \in \mathbf{Q} \right\},$$

which is an algebraic extension of $\mathbf{Q}$ of degree $\varphi(n)$. Each element α of $\mathbf{Q}(\zeta_n)$ is a root of a polynomial equation of the form $f_\alpha(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0$ where each b_i is a rational number. The elements of $\mathbf{Z}[\zeta_n]$ are precisely those elements α of $\mathbf{Q}(\zeta_n)$ for which the coefficients b_i of $f_\alpha(x)$ are integers. $\mathbf{Z}[\zeta_n]$ is called the **integral closure** of $\mathbf{Z}$ in $\mathbf{Q}(\zeta_n)$ and is the maximal (in the sense of inclusion) subring of $\mathbf{Q}(\zeta_n)$ which contains $\mathbf{Z}$ and which has finite rank as an abelian group.

Let R be a commutative ring with unity. An element u in R is a unit if there exists $v \in R$ such that $uv = 1$. The set of units $U(R)$ of a commutative ring forms a group under ring multiplication. For example, $U(\mathbf{Z}) = \{\pm 1\}$. In $\mathbf{Z}[\zeta_n]$, the roots of unity, $\pm\zeta_n^j$, $0 \leq j \leq n-1$, are all units. The only integers that are units in $\mathbf{Z}[\zeta_n]$ are ± 1. The structure of the full unit group of $\mathbf{Z}[\zeta_n]$ is given by the famous Dirichlet unit theorem: $U(\mathbf{Z}[\zeta_n])$ is the direct product of the set of units $\{\pm\zeta_n^j | 0 \leq j \leq n-1\}$ with $\varphi(n)/2 - 1$ copies of $\mathbf{Z}$. Theorem 1 implies that if n is not a prime power and $(j, n) = 1$, then $1 - \zeta_n^j$ is a unit in $\mathbf{Z}[\zeta_n]$.

Let x be a nonzero nonunit in R. Then x is called **irreducible** if whenever $x = yz$ for $y, z \in R$, either y or z is a unit of R. In $\mathbf{Z}$ the irreducible elements are precisely the prime numbers. As we shall soon see, if n is a prime power, then the element $1 - \zeta_n$ is irreducible in $\mathbf{Z}[\zeta_n]$.

We now introduce the norm function from $\mathbf{Q}(\zeta_n)$ to $\mathbf{Q}$. Let $a = \sum_{i=0}^{n-1} a_i\zeta_n^i \in \mathbf{Q}(\zeta_n)$. We define the polynomial $a(x) = \sum_{i=0}^{n-1} a_i x^i$. (Then $a = a(\zeta_n)$.) Define the complex number $N(a)$, called the **norm of a,** by

$$N(a) = \prod_{j=1}^{n}{}' a(\zeta_n^j).$$

From elementary Galois theory it follows that $N(a)$ is rational. Thus N maps $\mathbf{Q}(\zeta_n)$ into $\mathbf{Q}$. As a function N has the following properties.

1. If $a \in \mathbf{Z}[\zeta_n]$, then $N(a) \in \mathbf{Z}$.
2. For $a \in \mathbf{Q}(\zeta_n)$, $a \neq 0$, $N(a) > 0$.
3. For $a, b \in \mathbf{Q}(\zeta_n)$, $N(ab) = N(a)N(b)$. (In other words, N is a multiplicative function.)

Notice that if $u \in U(\mathbf{Z}[\zeta_n])$, then $N(u) = 1$. For if $uv = 1$ for some $v \in \mathbf{Z}[\zeta_n]$, then $1 = N(1) = N(uv) = N(u)N(v)$. Since $N(u)$ and $N(v)$ are both positive integers, $N(u) = 1$. Conversely, if $N(u) = 1$, then $u \in U(\mathbf{Z}[\zeta_n])$:

$$1 = N(u) = \prod_{j=1}^{n}{}' u(\zeta_n^j) = u(\zeta_n) \prod_{j=2}^{n}{}' u(\zeta_n) = uv \quad \text{where} \quad v = \prod_{j=2}^{n}{}' u(\zeta_n) \in \mathbf{Z}[\zeta_n].$$

Next we show that if $N(a) = q$ is prime in $\mathbf{Z}$, then a is irreducible in $\mathbf{Z}[\zeta_n]$. For if $a = bc$ for some $b, c \in \mathbf{Z}[\zeta_n]$, then $q = N(a) = N(b)N(c)$ which means that $N(b) = 1$ or $N(c) = 1$. If $N(b) = 1$ (or $N(c) = 1$), then b (or c) is a unit in $\mathbf{Z}[\zeta_n]$, hence a is irreducible.

We can now interpret Theorem 1 in the language of algebraic number theory. First consider ρ_n. When $n = p^a$ is a prime power, then $\rho_n = N(1 - \zeta_n) = p$. In this case $1 - \zeta_n$ (and $1 - \zeta_n^j$ for $(j, n) = 1$) is irreducible in $\mathbf{Z}[\zeta_n]$. When n is not a prime power, then $\rho_n = N(1 - \zeta_n) = 1$ and $1 - \zeta_n$ (and $1 - \zeta_n^j$ for $(j, n) = 1$) is a unit in $\mathbf{Z}[\zeta_n]$.

Now the numbers $C_n(1)$ and $S_n(1)$ are closely related to ρ_n. For example, the following table expresses $C_n(1)$ in terms of ρ_n.

n	$p^a, p \neq 2$	2^a	$2p^a$	$4p^a$	all other cases
$C_n(1)$	ρ_{2n}	$(\rho_{n/2})^2$	$\rho_{n/2}$	$(\rho_{n/4})^2$	ρ_n

If $C_n(1) = \rho_k^e$ where k is not a prime power and $e = 1$ or 2, then $C_n(1)$ is the norm of a unit of $\mathbf{Z}[\zeta_n]$, hence $C_n(1) = 1$. Since "most" positive integers are not prime powers, most of the time $C_n(1) = 1$ and $S_n(1) = 1$. The exceptional cases occur when n is a prime power or close to a prime power. Then $C_n(1)$ (or $S_n(1)$) is either the norm of an irreducible of $\mathbf{Z}[\zeta_n]$ or the square of the norm of an irreducible, hence $C_n(1)$ (or $S_n(1)$) is either prime or the square of a prime.

With these remarks the algebraic "explanation" of our principal result is complete. Readers wishing to dig deeper into cyclotomic fields and algebraic number theory may consult [**1**], [**2**], or [**4**].

Chords of regular n-gons

As a final application of this circle of ideas, we establish a result about regular n-gons. Although the next theorem and proof are evidently well known, it is natural to include them in this note.

THEOREM 3. *Let P be a regular n-gon inscribed in a unit circle and let v be a fixed vertex of P. The product of the lengths of the $n - 1$ chords of P drawn from v to the other $n - 1$ vertices is n.*

Proof. Place the circle so that its center is $(0,0)$ and its vertices are $\{\zeta_n^j \mid 0 \leq j \leq n - 1\}$. Without

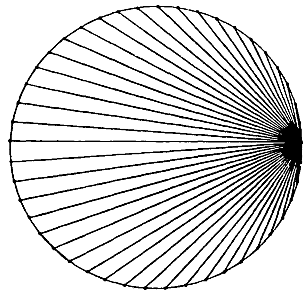

FIGURE 2

loss of generality, suppose $v=1$. For each j, $1 \le j \le n-1$, the vector joining ζ_n^j to 1 is represented by the complex number $1-\zeta_n^j$ and has length $|1-\zeta_n^j|$. The product of these lengths is

$$\prod_{j=1}^{n-1} |1-\zeta_n^j| = \left| \prod_{j=1}^{n-1} \left(1-\zeta_n^j\right) \right| = n.$$

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