Hamilton's Discovery of Quaternions

Contemporary sources describe Hamilton's trail from repeated failures at multiplying triplets to the intuitive leap into the fourth dimension.

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Introduction

The ordinary complex numbers $(a+ib)$ (or, as they were formerly written, $a+b\sqrt{-1}$) are added and multiplied according to definite rules. The rule for multiplication reads as follows:

> First multiply according to the rules of high school algebra:
>
> $$(a+ib)(c+id)=ac+adi+bci+bdi^2$$
>
> and then replace i^2 by (-1):
>
> $$(a+ib)(c+id)=(ac-bd)+(ad+bc)i.$$

Complex numbers can also be defined as couples (a,b). The product of two couples (a,b) and (c,d) is defined as the couple $(ac-bd, ad+bc)$. The couple $(1,0)$ is called 1, the couple $(0,1)$ is called i. Then we also have the result

$$i^2=(0,1)(0,1)=(-1,0)=-1.$$

By means of this definition the "imaginary unit" $i=\sqrt{-1}$ loses all of its mystery: i is simply the couple $(0,1)$.

The **quaternions** $a+bi+cj+dk$ which William Rowan Hamilton discovered on the 16th of October, 1843, are multiplied according to fixed rules, in analogy to the complex numbers; that is to say:

$$i^2=j^2=k^2=-1,$$

$$ij=k, \qquad jk=i, \qquad ki=j,$$

$$ji=-k, \qquad kj=-i, \qquad ik=-j.$$

They can also be defined as quadruples (a,b,c,d). Quaternions form a **division algebra;** that is, they cannot only be added, subtracted, and multiplied, but also divided (excluding division by zero). All rules of calculation of high school school algebra hold; only the commutative law $AB=BA$ does not hold since ij is not the same as ji.

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How did Hamilton arrive at these multiplication rules? What was his problem and how did he find the solution? We are accurately informed about these matters in documents and papers which appear in the third volume of Hamilton's collected *Mathematical Papers* [**3**]:

> *First*, through an entry in Hamilton's Note Book dated 16 October 1843 [**3**, pp. 103–105];
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> *Second*, through a letter to John Graves of the 17th of October 1843 [**3**, pp. 106–110];
>
> *Third*, through a paper in the Proceedings of the Royal Irish Academy (**2** (1844) 424–434) presented on the 13th of November 1843 [**3**, pp. 111–116];
>
> *Fourth*, through the detailed Preface to Hamilton's "Lectures on Quaternions", dated June 1853 [**3**, pp. 117–155, in particular pp. 142–144];
>
> *Fifth*, through a letter to his son Archibald which Hamilton wrote shortly before his death, that is shortly before the 2nd of September 1865 [**3**, pp. xv–xvi].

We can follow exactly each of Hamilton's steps of thought through all of these documents. This is a rare occurrence in which we can observe what flashed across the mind of a mathematician as he posed the problem, as he approached the solution step by step and then through a lightning stroke so modified the problem that it became solvable.

A brief history of complex numbers

Expressions of the form $A + \sqrt{-B}$ had already been encountered in the middle ages in the solution of quadratic equations. They were called "impossible solutions" or *numeri surdi*: absurd numbers. The negative numbers too were called "impossible." Cardan used numbers $A + \sqrt{-B}$ in the solution of equations of the third degree in the *casus irreducibilis* in which all three roots are real. Bombelli showed that it was possible to calculate with expressions such as $A + \sqrt{-B}$ without contradiction, but he did not like them: he called them "sophistical" and apparently without value. The expression "imaginary number" stems from Descartes.

Euler had no scruples about operating altogether freely with complex numbers. He proposed formulas such as $\cos\alpha = (1/2)(e^{i\alpha} + e^{-i\alpha})$. The geometric representation of complex numbers as vectors or as points in a plane stems from Argand (1813), Warren (1828) and Gauss (1832).

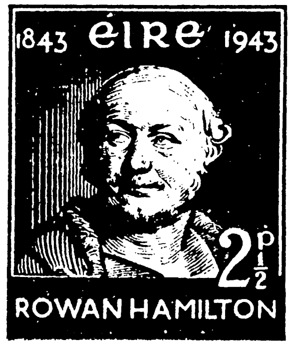
The first named, Argand, defined the complex numbers as directed segments in the plane. He took the basis vectors 1 and i as mutually perpendicular unit vectors. Addition is the usual vector addition, with which Newton made us familiar (the parallelogram law of velocities or of forces). The length of a vector was denoted at that time by the term "modulus", the angle of the vector with the positive x-axis as the "argument" of the complex number. Multiplication of complex numbers, according to Argand, then takes place so that the moduli are multiplied and the arguments are added. Independently of Argand, Warren and Gauss also represented complex numbers geometrically and interpreted their addition and multiplication geometrically.

"Papa, can you multiply triplets?"

Hamilton knew and used the geometric representation of complex numbers. In his published papers, however, he emphasized the definition of complex numbers as the couple (a, b) which followed definite rules for addition and multiplication. Related to that, Hamilton posed this problem to himself: *To find how number-triplets* (a, b, c) *are to be multiplied in analogy to couples* (a, b).

For a long time Hamilton had hoped to discover the multiplication rule for triplets, as he himself stated. But in October 1843 this hope became much stronger and more serious. He put it this way in a letter to his son Archibald [**3**, p. xv]:

SIR WILLIAM ROWAN HAMILTON, a child prodigy whose maturity was all that his childhood promised, was born in Dublin, Ireland, in 1805. He was literate in seven languages and knowledgeable in half a dozen more. In 1827 while still an undergraduate Hamilton was appointed Andrews Professor of Astronomy and Superintendent of the Observatory, and soon afterwards Astronomer Royal, a position he held for the rest of his extraordinarily productive life. His work in dynamics is probably most well-known today. "The Hamiltonian principle has become the cornerstone of modern physics", said Erwin Schrödinger, "the thing with which a physicist expects *every* physical phenomenon to be in conformity." Hamilton's other major discovery is the system of quaternions. The flash of insight which produced this discovery occurred in 1843 and is described in the accompanying article. A century later the Irish government commemorated this achievement with the stamp pictured at the right.



"... the desire to discover the law of multiplication of triplets regained with me a certain strength and earnestness, ..."

In analogy to the complex numbers $(a+ib)$ Hamilton wrote his triplets as $(a+bi+cj)$. He represented his unit vectors $1, i, j$ as mutually perpendicular "directed segments" of unit length in space. Later Hamilton himself used the word vector, which I also shall use in the following. Hamilton then sought to represent products such as $(a+bi+cj)(x+yi+zj)$ again as vectors in the same space. He required, first, that it be possible to multiply out term by term; and second, that the length of the product of the vectors be equal to the product of the lengths. This latter rule was called the "law of the moduli" by Hamilton.

Today we know that the two requirements of Hamilton can be fulfilled only in spaces of dimensions 1, 2, 4 and 8. This was proved by Hurwitz [**5**]. Therefore Hamilton's attempt in three dimensions had to fail. His profound idea was to continue to 4 dimensions since all of his attempts in 3 dimensions failed to reach the goal.

In the previously mentioned letter to his son, Hamilton wrote about his first attempt:

"Every morning in the early part of the above-cited month [October 1843], on my coming down to breakfast, your brother William Edwin and yourself used to ask me, 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head, 'No, I can only add and subtract them.'"

From the other documents we learn more precisely about Hamilton's first attempts. To fulfill the "law of the moduli" at least for the complex numbers $(a+ib)$, Hamilton set $ii=-1$, as for ordinary complex numbers, and similarly so that the law would also hold for the numbers $(a+cj)$, $jj=-1$. But what was ij and what was ji? At first Hamilton assumed $ij=ji$ and calculated as follows:

$$(a+ib+jc)(x+iy+jz)=(ax-by-cz)+i(ay+bx)+j(az+cx)+ij(bz+cy).$$

Now, he asked, what is one to do with ij? Will it have the form $\alpha+\beta i+\gamma j$?

First attempt. The square of ij had to be 1, since $i^2=-1$ and $j^2=-1$. Therefore, wrote Hamilton, in this attempt one would have to choose $ij=1$ or $ij=-1$. But in neither of these two cases will the law of the moduli be fulfilled, as calculation shows.

Second attempt. Hamilton considered the simplest case

$$(a+ib+jc)^2=a^2-b^2-c^2+2iab+2jac+2ijbc.$$

Then he calculated the sum of the squares of the coefficients of 1, i, and j on the right hand side and found

$$(a^2-b^2-c^2)^2+(2ab)^2+(2ac)^2=(a^2+b^2+c^2)^2.$$

Therefore, he said, the product rule is fulfilled if we set $ij = 0$. And further: if we pass a plane through the points 0, 1, and $a + ib + jc$, then the construction of the product according to Argand and Warren will hold in this plane: the vector $(a + bi + cj)^2$ lies in the same plane and the angle which this vector makes with the vector 1 is twice as large as the angle between the vectors $(a + bi + cj)$ and 1. Hamilton verified this by computing the tangents of the two angles.

Third attempt. Hamilton reports that the assumption $ij = 0$, which he made in the second attempt, subsequently did not appear to be quite right to him. He writes in the letter to Graves [**3**, p. 107]:

> "Behold me therefore tempted for a moment to fancy that $ij = 0$. But this seemed odd and uncomfortable, and I perceived that the same suppression of the term which was *de trop* might be attained by assuming what seemed to me less harsh, namely that $ji = -ij$. I made therefore $ij = k$, $ji = -k$, reserving to myself to inquire whether k was 0 or not."

Hamilton was entirely right in giving up the assumption $ij = 0$ and taking instead $ij = -ji$. For example, if $ij = 0$ then the modulus of the product ij would be zero, which would contradict the law of the moduli.

Fourth attempt. Somewhat more generally, Hamilton multiplied $(a + ib + jc)$ and $(x + ib + jc)$. In this case the two segments which are to be multiplied also lie in one plane, that is, in the plane spanned by the points 0, 1, and $ib + jc$. The result of the multiplication was $ax - b^2 - c^2 + i(a + x)b + j(a + x)c + k(bc - bc)$. Hamilton concluded from this calculation [**3**, p. 107] that:

> "... the coefficient of k still vanishes; and $ax - b^2 - c^2$, $(a + x)b$, $(a + x)c$ are easily found to be the correct coordinates of the *product-point* in the sense that the rotation from the unit line to the radius vector of a, b, c being added in its own plane to the rotation from the same unit-line to the radius vector of the other factor-point x, b, c conducts to the radius vector of the lately mentioned product-point; and that this latter radius vector is in length the product of the two former. Confirmation of $ij = -ji$; but no information yet of the value of k."

The leap into the fourth dimension

After this encouraging result Hamilton ventured to attack the general case. ("Try boldly then the general product of two triplets, ..." [**3**, p. 107].) He calculated

$$(a + ib + jc)(x + iy + jz) = (ax - by - cz) + i(ay + bx) + j(az + cx) + k(bz - cy).$$

In an exploratory attempt he set $k = 0$ and asked: Is the law of the moduli satisfied? In other words, does the identity

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2$$

hold?

> "No, the first member exceeds the second by $(bz - cy)^2$. But this is just the square of the coefficient of k, in the development of the product $(a + ib + ic)(x + iy + jz)$, if we grant that $ij = k$, $ji = -k$, as before."

And now comes the insight which gave the entire problem a new direction. In the letter to Graves [**3**, p. 108], Hamilton emphasized the insight:

> "And here there dawned on me the notion that we must admit, in some sense, a *fourth dimension* of space for the purpose of calculating with triplets;"

This fourth dimension appeared as a "paradox" to Hamilton himself and he hastened to transfer the paradox to algebra [**3**, p. 108]:

> "... ; or transferring the paradox to algebra, [we] must admit a *third* distinct imaginary symbol k, not to be confounded with either i or j, but equal to the product of the first as multiplier, and the second as multiplicand; and therefore [I] was led to introduce *quaternions* such as $a + ib + jc + kd$, or (a, b, c, d)."

Hamilton was not the first to think about a multi-dimensional geometry. In a footnote to the letter to Graves he wrote:

> "The writer has this moment been informed (in a letter from a friend) that in the Cambridge Mathematical Journal for May last [1843] a paper on Analytical Geometry of n dimensions has been published by Mr. Cayley, but regrets he does not yet know how far Mr. Cayley's views and his own may resemble or differ from each other."

"This moment" can in this connection only mean the same day in which he wrote the letter to Graves. In the Note Book of the 16th of October 1843 there is no mention of the paper by Cayley. Hamilton therefore appears to have arrived at the concept of a 4-dimensional space independently of Cayley.

After Hamilton had introduced $ij = -ji = k$ as a fourth independent basis vector, he continued the calculation [**3**, p. 108]:

> "I saw that we had probably $ik = -j$, because $ik = iij$, and $i^2 = -1$; and that in like manner we might expect to find $kj = ijj = -i$;"

From the use of the word "probably" it can be seen how cautiously Hamilton continued. He scarcely trusted himself to apply the associative law $i(ij) = (ii)j$ because he was not yet certain if the associative law held for quaternions. Likewise Hamilton could have used the associative law to determine ki:

$$ki = -(ji)i = -j(ii) = (-j)(-i) = j.$$

Instead he applied a conclusion by analogy. He wrote [**3**, p. 108]

> "...; from which I thought it likely that $ki = j$, $jk = i$, because it seemed likely that if $ji = -ij$, we should have also $kj = -jk$, $ik = -ki$."

Finally k^2 had to be determined. Hamilton again proceeded cautiously:

> "And since the order of multiplication of these imaginaries is not indifferent, we cannot infer that k^2, or $ijij$, is $= +1$, because $i^2 \times j^2 = (-1)(-1) = +1$. It is more likely that $k^2 = ijij = -iijj = -1$."

This last assumption $k^2 = -1$, asserts Hamilton, is also necessary if we wish to fulfill the "law of the moduli." He carried this out and concluded [**3**, p. 108]:

> "My assumptions were now completed, namely,
>
> $$i^2 = j^2 = k^2 = -1$$
> $$ij = -ji = k$$
> $$jk = -kj = i$$
> $$ki = -ik = j."$$

And now Hamilton tested if the law of the moduli was actually satisfied.

> "But I considered it essential to try whether these equations were consistent with the law of moduli, ..., without which consistence being verified, I should have regarded the whole speculation as a failure."

He therefore multiplied two arbitrary quaternions according to the rules just formulated

$$(a, b, c, d)(a', b', c', d') = (a'', b'', c'', d''),$$

calculated (a'', b'', c'', d'') and formed the sum of the squares

$$(a'')^2 + (b'')^2 + (c'')^2 + (d'')^2$$

and found to his great joy that this sum of squares actually was equal to the product

$$(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2).$$

In Hamilton's letter to his son we learn even more about the external circumstances which befell him at this flash of insight. Immediately after the previously cited words, "No, I can only add and subtract them." Hamilton continued [**3**, p. xx–xvi]:

> "But on the 16th day of the same month [October 1843]—which happened to be a Monday and a Council day of the Royal Irish Academy—I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps been driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should ever be allowed to live long enough distinctly to communicate the discovery. I pulled out on the spot a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols i, j, k;
>
> $$i^2 = j^2 = k^2 = ijk = -1,$$
>
> which contains the solution of the Problem, but of course as an inscription, has long since mouldered away."

The entry in the pocket book is reproduced on the title page of [**3**]: it contains the formulas

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, \qquad jk = i, \qquad ki = j$$
$$ji = -k, \qquad kj = -i, \qquad ik = -j.$$

I assume as likely that before his walk Hamilton had already written on a piece of paper the result of the somewhat tiresome calculation which showed that the sum of squares

$$(ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2$$

still lacked $(bz - cy)^2$ compared with the product

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2).$$

What then happened immediately before and during that remarkable walk along the Royal Canal, he described again on the same day in his Note Book, as follows:

> "I believe that I now remember the order of my thought. The equation $ij = 0$ was recommended by the circumstances that
>
> $$(ax - y^2 - z^2)^2 + (a + x)^2(y^2 + z^2) = (a^2 + y^2 + z^2)(x^2 + y^2 + z^2).$$
>
> I therefore tried whether it might not turn out to be true that
>
> $$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2,$$
>
> but found that this equation required, in order to make it true, the addition of $(bz - cy)^2$ to the second member. This *forced* on me the non-neglect of ij, and *suggested* that it might be equal to k, a new imaginary."

By underscoring the italicized words *forced* and *suggested* Hamilton emphasized that he was concerned with two entirely different facts. The first was a compelling logical conclusion, which came immediately out of the calculation: it was not possible to set ij equal to zero, since then the law of the moduli would not hold. The second fact was an insight which came over him in a flash at the canal ("an electric circuit seemed to close, and a spark flashed forth"), that is, that ij could be taken to be a new imaginary unit.

After the insight was once there, everything else was very simple. The calculations $ik = iij = -j$ and $kj = ijj = -i$ could be made easily enough by Hamilton in his head. The assumptions $ki = -ik = j$ and $jk = -kj = i$ were immediate. And k^2 could be easily calculated too: $k^2 = ijij = -iijj = -1$.

And so during his walk Hamilton also discovered the rules of calculation which he entered into the pocket book. The pocket book also contains the formulas for the coefficients of the product

$$(a+bi+cj+dk)(\alpha+\beta i+\gamma j+\delta k),$$

that is,

$$a\alpha-b\beta-c\gamma-d\delta$$

$$a\beta+b\alpha+c\delta-d\gamma$$

$$a\gamma-b\delta+c\alpha+d\beta$$

$$a\delta+b\gamma-c\beta+d\alpha$$

as well as the sketch for the verification of the fact that in the sum of the squares of these coefficients all mixed terms (such as $ada\delta$) cancel and only $(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)$ remains. In the Note Book of the same day everything was again completely restated.

Octonions

The letter to Graves in which Hamilton announced the discovery of quaternions was written on the 17th of October 1843, one day after the discovery. The seeds, which Hamilton sowed, fell upon fertile soil, since in December 1843 the recipient John T. Graves already found a linear algebra with 8 unit elements 1, *i*, *j*, *k*, *l*, *m*, *n*, *o*, the algebra of *octaves* or *octonions*. Graves defined their multiplication as follows [**3**, p. 648]:

$$i^2=j^2=k^2=l^2=m^2=n^2=o^2=-1$$

$$i=jk=lm=on=-kj=-ml=-no$$

$$j=ki=ln=mo=-ik=-nl=-om$$

$$k=ij=lo=nm=-ji=-ol=-mn$$

$$l=mi=nj=ok=-im=-jn=-ko$$

$$m=il=oj=kn=-li=-jo=-nk$$

$$n=jl=io=mk=-lj=-oi=-km$$

$$o=ni=jm=kl=-in=-mj=-lk.$$

In this system the "law of the moduli" also holds:

$$(a_1^2+\cdots+a_8^2)(b_1^2+\cdots+b_8^2)=(c_1^2+\cdots+c_8^2) \tag{1}$$

Hamilton answered on the 8th of July 1844 [**3**, p. 650]. He noted to Graves that the associative law $A\cdot BC=AB\cdot C$ clearly held for quaternions but not for octaves.

Octaves were rediscovered by Cayley in 1845; because of this they are also known as *Cayley numbers*. Graves also made an attempt with 16 unit elements but it was unsuccessful. It could not succeed since we know today that identities of the form (1) are only possible for sums of 1, 2, 4 and 8 squares. I should like to close with a brief comment about the history of these identities.

Product formulas for the sums of squares

It is likely that the "law of the moduli" for complex numbers was already known to Euler:

$$(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2.$$

A similar formula for the sum of 4 squares

$$(a_1^2+\cdots+a_4^2)(b_1^2+\cdots+b_4^2)=(c_1^2+\cdots+c_4^2)$$

was discovered by Euler; the formula is stated in a letter from Euler to Goldbach on May 4th, 1748 [**4**]. The formula (1) for 8 squares, which Graves and Cayley proved by means of octonions, was previously found by Degen (1818) [**6**]. Degen erroneously thought that he could generalize the theorem to 2^n squares.

The problem, which started with Hamilton, reads: can two triplets (a, b, c) and (x, y, z) be so multiplied that the law of the moduli holds? In other words: is it possible so to define (u, v, w) as bilinear functions of (a, b, c) and (x, y, z) that the identity

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(u^2+v^2+w^2) \tag{2}$$

results?

The first to show the impossibility for this identity was Legendre. In his great work *Théorie des nombres* he remarked on page 198 that the numbers 3 and 21 can easily be represented rationally as sums of three squares:

$$3=1+1+1,$$

$$21=16+4+1,$$

but the product $3\times 21=63$ cannot be so represented, since 63 is an integer of the form $(8n+7)$. It follows from this that an identity of the form (2) is impossible, to the extent that it is assumed that (u, v, w) are bilinear forms in (a, b, c) and (x, y, z) with rational coefficients. If Hamilton had known of this remark by Legendre he would probably have quickly given up the search to multiply triplets. Fortunately he did not read Legendre: he was self-taught.

The question for which values of n a formula of the kind

$$(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)=(c_1^2+\cdots+c_n^2)$$

is possible, was finally decided by Hurwitz in 1898. With the help of matrix multiplication he proved (in [**5**]) that $n=1,2,4$ and 8 are the only possibilities. For further historical accounts the reader may refer to [**1**] or [**2**].

References

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The sun never sets on mathematics

> The discoveries of Newton have done more for England and for the race, than has been done by whole dynasties of British monarchs; and we doubt not that in the great mathematical birth of 1843, the Quaternions of Hamilton, there is as much real promise of benefit to mankind as in any event of Victoria's reign.
>
> — THOMAS HILL