Teaching Mathematics with Rubik's Cube

Tom Davis

Tom Davis received a B.S. in mathematics from Caltech and a Ph.D. in mathematics from Stanford University. He is presently a research associate at Stanford in the Electrical Engineering department. His interests include computer graphics, computational geometry, and design systems for very large scale integrated circuits (VLSI).

Introduction. It is impossible to teach a student a subject in which he has little or no interest. This is especially true of mathematics. Very few students care how old Susan was when her mother was three times as old as she is now. Many games and puzzles have been invented which attempt to teach mathematics, but it often seems that the more mathematical content in a game, the less interesting it is to play.

On the other hand, there have been few puzzles which have been as popular as Rubik's Cube. It is fiendishly difficult to solve, yet once you start playing with it, it is almost impossible to put down. Many people who claim to have no interest in mathematics will spend hours twisting its faces. The amazing thing is that there is a tremendous amount of mathematics involved. In fact, some moderately sophisticated mathematics can be "applied" to help generate a solution. This article attempts to illustrate a few rather elementary examples.

Since Rubik's cube is by now fairly well known, it is not worth spending the space here to describe its history, how it works, where to get it, and how hard it is to solve. See [1] for that information. A sort of "spoiler warning" is also in order. If you have not played with the cube, this article could destroy much of the fun of solving the cube for yourself. Ideally, of course, you should stop reading now, work out a complete solution to the cube, and then go on, but if you just wish to skip the worst spoilers, simply do not read the last section, entitled "A (Short) Dictionary of Useful Macros." One macro is used in the text as an example, and could possibly be considered to be a spoiler, but it is not a particularly powerful cube solving tool—it is just a typical example of macro generation.

There are many interesting mathematical questions which are not touched upon here, since they will be mostly of interest to mathematicians. The material in this article is all highly "practical," in the sense that it will help solve the cube. The kind of student who worries about such things as "How many configurations are there?", "What about 4 by 4 by 4 cubes?", "What is the minimal way to do such-and-such?" is usually not someone with math anxiety.

Students with more than average curiosity may wish to investigate the class of positions reachable from the solved position, the subgroups of the complete set of moves, and methods for generating pretty patterns.

Notation. It is critical to have a good notation to describe the moves which can be made on the cube. Rather than describe the pros and cons of the many possibilities,

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we will just present here what seems to be the easiest and most commonly used method. The cube is (apparently) made up of 27 cubes, 26 of which are visible. These little cubes are called "cubies", which come in three flavors: corner cubies (there are 8 of these), edge cubies (12 of these), and face cubies (6). The center cubie is never visible (and in fact does not even exist), so there is no reason to refer to it.

Holding the cube in any orientation, name the six faces as follows: "back", "front", "up", "down", "left", and "right". Luckily, all six words begin with different letters, so each face can be denoted by one of: *B F U D L R.* (This is called the "befuddler notation". The reason that colors of the faces are not used is that different cubes have different colorations, and moves described in terms of "*BFUDLR*" work no matter what color happens to be on top.) Figure 1 shows a view of the cube with the three visible faces labeled.

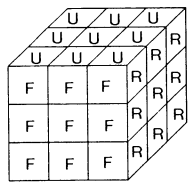

Figure 1.

The cubies themselves can be named as follows: There are B, F, . . . , R center cubies; RF, LF, . . . , UB edge cubies; and RFU, . . . , BLD corner cubies. The edge cubie LF is the unique edge cubie between the left and front faces. The corner cubie LUF is at the corner of the left, up, and front faces, and so on.

A primitive move involves grasping a single face and giving it a quarter turn. *F* is defined to be the transformation of grasping the front face and turning it one-quarter turn clockwise. "Clockwise" means that if the face in question is grasped in the right hand, then it is turned in the direction pointed to by the right thumb. Figure 2 illustrates the effect on a solved cube of the operation *F*.

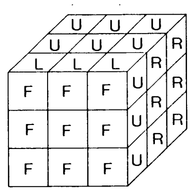

Figure 2.

Turning counterclockwise is the inverse operation and is denoted F^{-1}. (F' is used by some people. The F' notation is convenient for computer output, and computers are often involved in cube calculations. Note also that F^{-1} could have been written F^3, but this would tend to hide the fact that it behaves as an inverse.) A half-turn of the upper face is denoted by U^2, and repeating the sequence "half-turn right followed by a half-turn up" three times is denoted by $(R^2U^2)^3$. You may find it convenient to extend this notation, but we shall stick to it in this article.

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If you have a cube available, you may at first wish to write down a correspondence of the form "Yellow = U, Red = B, . . . " to help keep track of things during your experiments. You can then freely turn the cube to look at the effects of a transformation without fear of losing track of where you are.

Solving the Cube. First, notice that each cubie is unique—assuming that the face cubies are fixed in space, there is only one place where each cubie could belong in the solved cube. Also notice that it could be in the correct position, but "flipped" (if it is an edge cubie) or "twisted" (if it is a corner cubie). The word "toppled" is sometimes used instead of "twisted".

The usual method to restore a jumbled cube to its original condition with all sides solid colors is by means of a series of "macros". A macro is a set of moves which has some specific effect, usually leaving most of the cube exactly as it was, but altering the positions or orientations of a small number of cubies. One commonly used macro takes three edge cubies (say $\boxed{\text{UR}}$, $\boxed{\text{UL}}$, $\boxed{\text{UF}}$), and moves the $\boxed{\text{UR}}$ cubie to where the $\boxed{\text{UL}}$ cubie was, the $\boxed{\text{UL}}$ cubie to the $\boxed{\text{UF}}$ position, and the $\boxed{\text{UF}}$ cubie back to the $\boxed{\text{UR}}$ position. It leaves every other cubie in exactly the same position and orientation. Two other examples of macros include one which flips two edge cubies, leaving all others as they were, and one which topples three corner cubies (again leaving all others alone).

Assuming that you know some of these macros, any time the cube can be improved (i.e., brought closer to being solved) by one, you perform it, and the net effect is a slightly less disordered cube.There are many ways to solve the cube, but most people solve it the same way each time. Some people do the corners first and then the edges, and some do it in the other order. One particular method is the following:

1. Solve the upper face.
2. Get the bottom corners in the correct positions (possibly twisted).
3. Twist the bottom corners into the correct orientation.
4. Get the edge cubies into position.
5. Flip any edges which are backwards.

Most of the rest of this article describes methods for finding, modifying, and improving macros.

Groups and Permutations. If we define a move to be any sequence of twists, then the set of all moves forms a group, where the group operation is composition of moves. The group is large (43,252,003,274,489,856,000 elements), and includes many interesting subgroups. For a course in group theory, the cube obviously makes an ideal prop. In this paper, however, we are mostly interested in studying moves as permutations.

Basically, a permutation is a rearrangement of objects. Suppose that you have three boxes called A, B, and C, each containing an object. Suppose that a move is any rearrangement of the objects in the boxes, subject only to the condition that, when you are done, each box contains exactly one object. A typical move might be denoted as follows: (ACB). This means that whatever was in A was moved to C, C's contents were moved to B, and B's contents back to A. (AC) denotes the exchange of the contents of boxes A and C (the object in B was left where it was).

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If the move (AC) is followed by the move (BC) (denoted "$(AC)(BC)$"), the net result is (ABC). The null move is denoted by ( ). If there are more boxes, say A, B, C, D, and E, a single move might look like $(ABD)(CE)$ or $(AE)(CD)$.The cycles above should be disjoint (have no letters in common). For example, $(ABC)(CDE) = (ABDEC)$. The notation above is called the cycle notation for permutations. Check your understanding on the following example:

$$(ABD)(CE) \quad \text{followed by} \quad (AE)(BC) \quad \text{yields} \quad (AC)(BDE).$$

Note that in the example above, the permutation $(AE)(BC)$ could also have been written in the form $(AE)(BC)(D)$. The (D) simply indicates that D is left fixed by the permutation.

We can think of cube moves in the same way. If each cubie position is thought of as a box, a move is certainly a permutation of the boxes. In particular, the move we have called U can be denoted by the permutation

$$\left(\boxed{\text{UL}}\,\boxed{\text{UB}}\,\boxed{\text{UR}}\,\boxed{\text{UF}}\right)\left(\boxed{\text{ULF}}\,\boxed{\text{ULB}}\,\boxed{\text{URB}}\,\boxed{\text{URF}}\right).$$

This is fine for moving the cubies around, but remember that the orientation of the cubies in their slots also makes a difference. Another way to view the cube is as 54 boxes, each corresponding to one of the cubie faces. It should be clear that the move U corresponds to a rearrangement of 20 cube faces (8 on top, and 12 around the top edge—the center top cube face does not appear to move). Both of these ways of viewing the cube have advantages.

Another useful way to represent permutations is to give a complete list of where each object goes. Thus, the permutation

$$\begin{pmatrix} A & B & C & D \\ C & D & B & A \end{pmatrix}$$

means that A moves to C, B moves to D, C moves to B, and D moves to A. In cycle notation, this permutation would be represented as $(ACBD)$. This alternative representation is perhaps easier to understand, but takes more room, and hides some information.

Note that both representations of permutations are easy to invert. In the case of cycle notation, simply reverse the order of all the items in the cycles, and in the column notation above, simply turn it upside down. Thus, $[(ACD)(EBG)]^{-1} = (DCA)(GBE)$, and

$$\begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix}^{-1} = \begin{pmatrix} A & C & D & B \\ A & B & C & D \end{pmatrix}.$$

An elementary fact about permutations of a finite number of objects is that each one has a finite order. The order of a permutation is the smallest number of times it needs to be applied to bring all of the objects back into their original arrangement. The order of $(ABCD)$ is 4: $(ABCD)^2 = (AC)(BD)$, $(ABCD)^3 = (ADCB)$, and finally, $(ABCD)^4 = (\ )$. It should be clear that the order of any cycle is equal to the length of the cycle.

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A permutation can be made up of more than one cycle. What is the order of the permutation $(ABC)(DEFG)(HIJKLMN)$? If we repeat this permutation a multiple of three times, the first cycle will leave its objects fixed; if it is repeated a multiple of four times, the second cycle will leave its objects fixed, and so on. Thus the order of such a permutation must be the least common multiple of the lengths of the cycles —in this case, $3 \cdot 4 \cdot 7 = 84$. Note that for this to work, the cycles must be disjoint.

One of the first things which should be pointed out to the budding group theorists in your class about permutations (and hence about the cube moves which can be made) is that they do not necessarily commute. The move RF leaves the cube in a completely different configuration from FR. Thus, if you are carefully writing down your moves so that you can retrace your steps, to undo the sequence $UR^2LD^{-1}R$, you would do $R^{-1}DL^{-1}R^2U^{-1}$.

Generating Macros Using Cycle Notation. The knowledge of the order and cycle structure of a permutation can be used to generate some useful cube manipulation macros. The following example illustrates this. Consider the move $M = RUR^{-1}U^{-1}$. (This move was not chosen totally at random—see the section on commutators which follows.) If we ignore the twisting of the corners and the flipping of edges, this move permutes the cubies in the following way:

$$M = \left(\boxed{\text{RUF}}\,\boxed{\text{RDF}}\right)\left(\boxed{\text{ULB}}\,\boxed{\text{URB}}\right)\left(\boxed{\text{RF}}\,\boxed{\text{RU}}\,\boxed{\text{UB}}\right).$$

M has order 6, but the cycle notation tells us more. If M is repeated an even number of times, the effects of the 2-cycles will be eliminated, and similarly, any multiple of three repeats gets rid of the 3-cycle, giving:

$$M^2 = \left(\boxed{\text{RF}}\,\boxed{\text{UB}}\,\boxed{\text{RU}}\right),$$

and

$$M^3 = \left(\boxed{\text{RUF}}\,\boxed{\text{RDF}}\right)\left(\boxed{\text{ULB}}\,\boxed{\text{URB}}\right).$$

Remember that we have ignored twisting of corners here, and if you try these moves on a cube, you will find that M^2 leaves the corners twisted. M^3 does not, however, and is a useful macro for exchanging two pairs of corners. Figure 3 illustrates the effect of applying the macro $RUR^{-1}U^{-1}$ to a solved cube, both in terms of the cubies involved and in terms of the cycles. Figure 4 shows the result of applying the same operation three times in a row.

Any move can be used to generate macros as above, although the macro generated may be too long to be useful. Simply work out the cycle structure and eliminate the cycles you do not want by repeating the move some number of times which is a multiple of the length of the cycle. A useful application of computers arises here—it is not hard to write a program to determine the cycle structure of a permutation, and this can be determined for many short combinations of moves.

Conjugates. If A and P are any two permutations, then the conjugate of A relative to P is defined to be PAP^{-1}. This mathematical concept can be used as a

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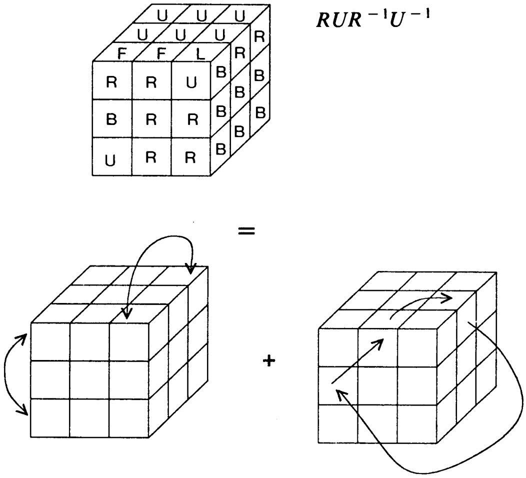

Figure 3.

$(RUR^{-1}U^{-1})^3$

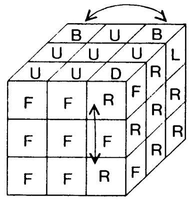

Figure 4.

very powerful tool for solving the cube. P can be thought of as a transformation of coördinates, and P^{-1} will transform back. PAP^{-1} therefore corresponds to the instructions: "Change to a different coördinate system, perform transformation A, and then return to the original coördinate system."

Theorem. *The cycle structure of A is identical with the structure PAP^{-1}. In other words, it has the same number of disjoint cycles having the same lengths.*

We will not present a proof of this theorem, but the following example should make it obvious how such a proof could be constructed.

Let $P = (QRTW)(SVU)$, and $A = (QWVT)(RS)$ (or $(QWVT)(RS)(U)$). We will write P in the alternate form:

$$\begin{pmatrix} Q & R & S & T & U & V & W \\ R & T & V & W & S & U & Q \end{pmatrix},$$

so

$$PAP^{-1} = \begin{pmatrix} Q & R & S & T & U & V & W \\ R & T & V & W & S & U & Q \end{pmatrix} (QWVT)(RS)(U) \begin{pmatrix} R & T & V & W & S & U & Q \\ Q & R & S & T & U & V & W \end{pmatrix}.$$

To expand the product above, follow each letter through the three transformations: P takes Q to R, then A takes R to S, and finally P^{-1} takes S to U. In the same way, U moves eventually to Q, so the cycle (QU) is part of the product. Continuing in this way, we find

$$PAP^{-1} = (WTSR)(QU)(V).$$

The following observation should make a proof of the Theorem obvious. Replace each letter in A as follows: find it in the top row of P^{-1} and use the letter below it. The replacement of all the letters of A yields PAP^{-1}.

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In terms of the cube, this theorem means that if A is any macro fixing all but three cubies, and P is any combination of moves whatever, then PAP^{-1} will fix all but three cubies. Thus if you know a macro which cycles three cubies on the front face, then you can cycle any three edge cubies by doing a combination of moves P to bring them to the front face, performing the macro, and then undoing P (performing P^{-1}). This principle applies to all macros.

Commutators. If P and Q are any two permutations, then the commutator of P and Q is defined to be $PQP^{-1}Q^{-1}$. The notion of a commutator can be used to generate a number of powerful macros.

Suppose that you would like a macro which flips two edge cubies on a face and leaves the rest of the cube as it was. Suppose further that you can find a sequence P of moves which flips a single edge cubie on a face and leaves all the other cubies on that face fixed (this is not too hard to do). Let Q be a rotation of the face which moves the other cubie to be flipped into the position occupied by the original flipped cubie. Then $PQP^{-1}Q^{-1}$ is the desired macro.

This is easy to show. Whatever "damage" P does to the cubies not on the top face, P^{-1} undoes. The Q^{-1} at the end undoes the rotation. The same idea can be used to generate a macro which twists one corner clockwise and another on the same face counterclockwise. Note also that, by means of conjugation, there is no need for the two cubies to share the same face.

A (Short) Dictionary of Useful Macros. This is the section which truly deserves a spoiler warning. Four powerful macros are presented which almost allow a complete solution of a jumbled cube, although they must be used in conjunction with various conjugations which were discussed earlier. Make a move to get the cubies in question into the positions operated upon by the macro, apply the macro, and then undo the move. The first and second macros involve twisting and flipping of cubies. For a complete solution of the cube, one more macro is needed which will exchange exactly two corners or exactly two edges. It can do anything to the other edges or corners, respectively. A transformation which exchanges exactly two corners, for example, is $RD^{-1}R^{-1}D^{-1}B^{-1}DB$. It performs the permutation $(\boxed{\text{RBD}}\,\boxed{\text{LBD}})(\boxed{\text{RB}}\,\boxed{\text{FD}}\,\boxed{\text{BD}}\,\boxed{\text{LD}})$, and topples $\boxed{\text{LFD}}$.

The macros below are in a sense too powerful—each leaves all of the rest of the cube exactly as it was. When you are solving the first face, for example, you can presumably use a much weaker macro because you do not care what happens to the cubies below that face when you are first working on it.

Finally, note that all of the macros presented except for the first one have the form of a commutator.

$$(\boxed{\text{LD}}\,\boxed{\text{FD}}\,\boxed{\text{RD}}) = R^{-1}LBRL^{-1}D^2R^{-1}LBRL^{-1}$$

$$(\boxed{\text{LUF}}\,\boxed{\text{RUB}}\,\boxed{\text{LUB}}) = URU^{-1}L^{-1}UR^{-1}U^{-1}L.$$

To flip $\boxed{\text{UB}}$ and $\boxed{\text{UL}}$:

$$R^{-1}LB^2RL^{-1}D^{-1}R^{-1}LBRL^{-1}ULR^{-1}B^{-1}L^{-1}RDLR^{-1}B^2L^{-1}RU^{-1}.$$

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To topple URB and URF:

$$FD^2F^{-1}R^{-1}D^2RUR^{-1}D^2RFD^2F^{-1}U^{-1}.$$

Note about the cover: The pattern shown on the cover results from applying a conjugate of the first useful macro in a different orientation. For artistic purposes the front face, i.e., the "2" face, is inaccurate. The center cube should be upside down.

Stanley Isaacs deserves credit for suggesting the use of symbols on the faces and formulating the conjugate.

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Aftermath

It worries me some: whatever'll become
    of my papers and books when I go?
They'd bring nothing-at-all at a secondhand stall—
    it's a pity to leave them below.

Now Hardy's a treasure and Banach a pleasure
    and the Knopps a delight for the mind.
There's Pólya *und* Szegö—well, *I* go where *they* go—
    couldn't bear it to leave them behind.

My Titchmarsh I'll need, also Hausdorff—agreed!
    and Courant *und* Hilbert—oh, yes!
And reprints of theses of various species
    and copies of *Bull. A.M.S.*

If I should be early in reaching the pearly,
    or tardily answer the call,
could gracious St. Peter do anything sweeter
    than let in my bookcase and all?

—Katharine O'Brien

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