RUBIK'S REVENGE: THE GROUP THEORETICAL SOLUTION

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Introduction. Recently Rubik and Ideal Toy have supplied us with an analog to Rubik's Cube with four slices instead of three. It is sold under the name of *Rubik's Revenge*; see Fig. 1.

The advantage is that no solution is offered besides the toy. This is however a disadvantage too, because the "Revenge" is even more difficult to solve without mathematics than the Cube and might hence disappoint a few puzzlers. I shall return to these difficulties at the end.

A general mathematician might think that this is just some more of the same. But such an abstract attitude provides no solution to the problem: fix this mess! And group theorists are used to consider groups individually; non-isomorphic groups are not much "alike".

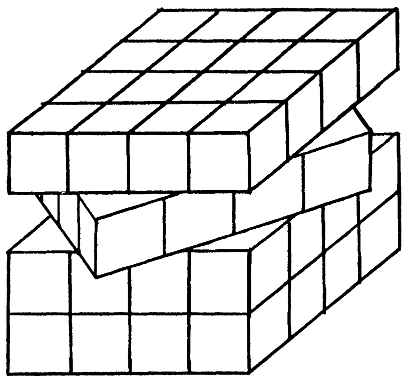

FIG. 1. Rubik's Revenge.

Terminology. In analogy with the Cube, see Singmaster [4], we label the six faces of the Revenge *Front*, *Back*, *Up*, *Down*, *Right*, *Left*, and we abbreviate these designations to the first letter; see Fig. 2.

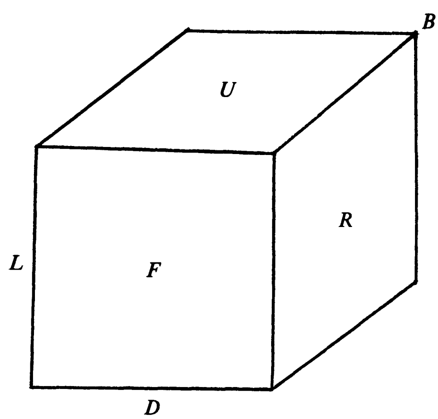

FIG. 2. Labels of faces.

Mogens E. Larsen: I am 42 years old, lektor (latin for "associate professor") at the department of mathematics of the University of Copenhagen, where I teach numerical analysis, applied mathematics, and occasionally complex variables, which last subject I studied at M.I.T. in the year 1969–70. Fond of games and toys I founded the Go Club of Copenhagen in 1972, and I was among the first Rubik addicts, when professor Robert Fossum from Urbana left us with a cube in the tea room of the department in the summer of 1980.

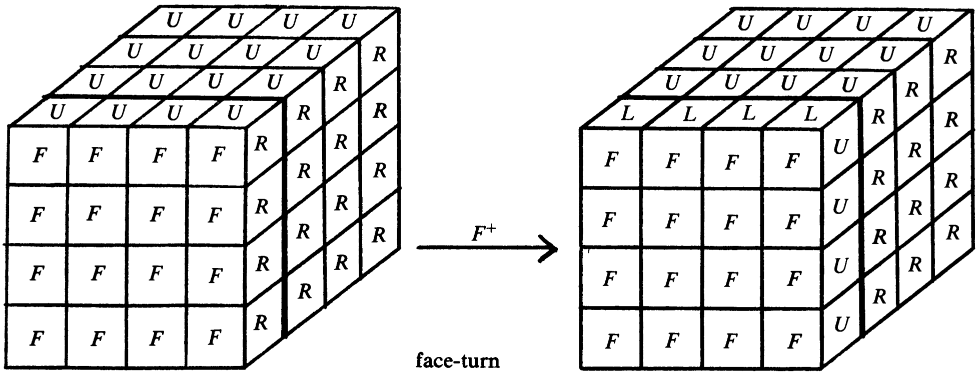

FIG. 3

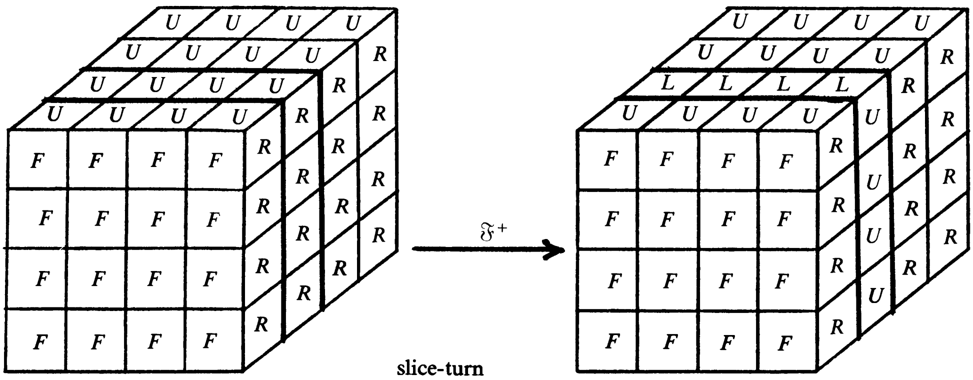

FIG. 4

A move of one of the six faces is denoted with the capital letter beginning the name of that face followed by a $+$, $-$ or 2. The "$+$" stands for turning the face 90° clockwise (seen from the outside), "$-$" for turning it 90° the other way and "2" for turning 180° (any way). Fig. 3 shows an example. This is the same as for the Cube.

For the Revenge there is also the possibility of turning the slice next to the face. We shall denote the turn of an interior slice by the gothic capital letter of the adjacent face, together with a $+$, $-$ or 2 having the same meaning as above. Fig. 4 shows an example.

An operation is denoted in the order in which the turns are supposed to be done. For example

$$F^+L^-\mathfrak{U}^-$$

means: "Turn the front face with the clock, then turn the left face against the clock and last turn the middle upper slice against the clock."

Finally we shall call the small cube-bricks of which the big cube is built *cubinos*. These are divided in three classes known respectively as *corner-cubinos* (8 of them), *edge-cubinos* (24 of them) and *center-cubinos* (24 of them) as shown on Fig. 5.

The groups of possible operations are denoted by large capitals, while abstract groups are denoted by gothic capitals.

The groups. The operations form a group in a natural way. The product of two operations is merely doing these two operations one after the other in the order of writing:

$$(F^+L^-\mathfrak{U}^-)(\mathfrak{R}^+D^2) = F^+L^-\mathfrak{U}^-\mathfrak{R}^+D^2.$$

This product is obviously associative. The identity is the operation E of doing "no turns".

We introduce the rules of reductions. (Symbol "F" may be replaced by any of the other 12 basic symbols.)

$$F^+F^- = E$$
$$F^-F^+ = E$$
$$F^+F^+ = F^2$$
$$F^-F^- = F^2$$
$$F^2F^2 = E.$$

For convenience we shall also write for the inverse

$$(F^+L^-\mathfrak{U}^-)^- = \mathfrak{U}^+L^+F^-.$$

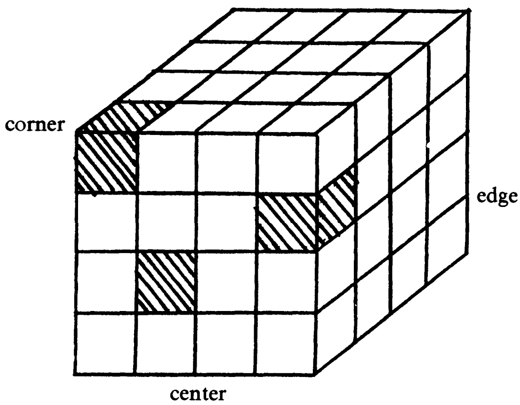

FIG. 5. Cubinos.

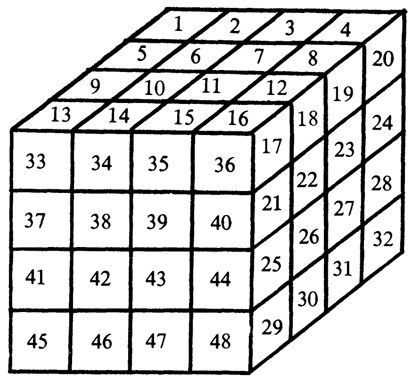

FIG. 6. Labels of cubino-faces.

Note that the inverse is obtained by putting the symbols in reverse order and changing $+\,-$ to $-\,+$. This may easily be checked by reducing as in the following example.

$$(F^+L^-\mathfrak{U}^-)(\mathfrak{U}^+L^+F^-) = E.$$

Let us denote the group of operations by $\mathfrak{G}$. This group is infinite and generated by the twelve elements of order 4:

$$F^+, B^+, U^+, D^+, R^+, L^+, \mathfrak{F}^+, \mathfrak{B}^+, \mathfrak{U}^+, \mathfrak{D}^+, \mathfrak{R}^+, \mathfrak{L}^+$$

Let us label the 96 cubino-faces with numbers from 1 to 96; see Fig. 6. (It does not matter how they are labeled.)

Each operation corresponds to a permutation of these 96 figures, obtained by applying the operation to the labeled Revenge. Now, the product of two operations corresponds to the product of the corresponding permutations. So this correspondence is a homomorphism

$$\varphi: \mathfrak{G} \to \mathfrak{S}_{96},$$

where $\mathfrak{S}_n$ is the symmetric group of permutations of n elements (see **[1]**, p. 54; **[3]**, p. 30).

The problem is to get a hold of φ. For example, *given* $\pi \in \mathfrak{S}_{96}$, which represents a required pattern on the Revenge, *find* an operation $X \in \mathfrak{G}$ with

$$\pi = \varphi(X),$$

means a construction of the required pattern from the solved state. Of course π must be in the range of φ, else X does not exist. Let

$$\mathbf{R} = \varphi(\mathfrak{G}) \subset \mathfrak{S}_{96}.$$

Then **R** is the group of permutations obtainable by the possible operations. The structure of **R** can be investigated in several ways.

One of the problems is to find the order of **R** (which is finite and less than $96! \simeq 10^{150}$).

Another problem is to write down in a convenient way an operation from $\varphi^{-1}(\pi)$ for any given $\pi \in \mathbf{R}$.

Let ε be the identity permutation which fixes everything, and

$$\mathfrak{E} = \varphi^{-1}(\varepsilon) \subset \mathfrak{G}$$

the kernel of φ. Then $\mathfrak{E}$ is the group of operations leaving all cubino-faces unmoved. Thus we have a factor group

$$\mathbf{R} \simeq \mathfrak{G}/\mathfrak{E}.$$

It is not possible by any operation to turn any cubino-face around itself on the place where it is, because both types of operations, F^{+} and $\mathfrak{F}^{+}$, preserve orientation, that is, the corner nearest the center remains nearest the center; see Fig. 7.

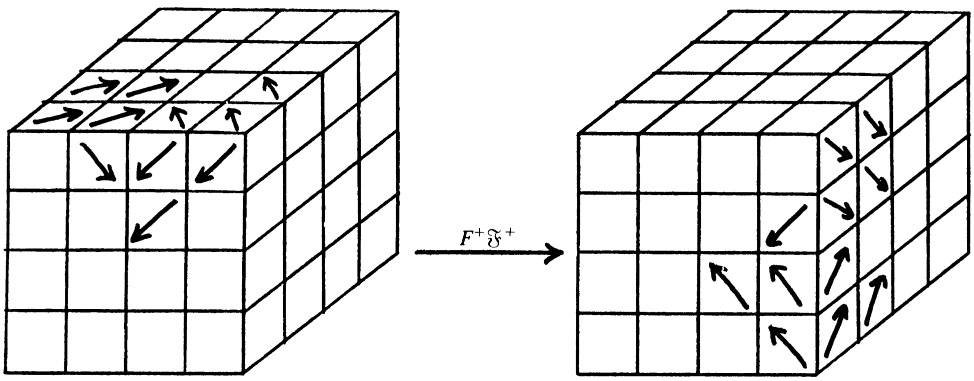

FIG. 7. Preservation of cubino-face orientation.

But it is possible to permute the four equally coloured center-cubinos among themselves. This means that the order of **R** is bigger than the number of visible patterns. Therefore, let $\mathbf{C}_0$ denote the subgroup of **R** permuting each of the six classes of four equal center-cubinos among themselves (i.e., $\mathbf{C}_0$ consists of the invisible operations). This group $\mathbf{C}_0$ is not normal in **R**, but nevertheless, the number of patterns of the Revenge is the number of cosets to $\mathbf{C}_0$ in **R**. (This is to say, two patterns in **R** look the same, if one is obtained from the other by permutation of some equal center-cubinos of like color.)

The structure of the group R. Each operation creates a permutation of the 56 cubinos and sometimes a turn of some of them too. But is it not possible to move any cubino to an arbitrary prescribed position.

The 8 corner-cubinos are permuted among themselves and if possible turned individually.

The 24 edge-cubinos are permuted among themselves and if possible flipped individually (we shall see that this flip is actually impossible).

The 24 center-cubinos are permuted among themselves, but as explained above, they cannot be rotated individually.

This means that each operation creates three simultaneous permutations of the three kinds mentioned above together with some turns and flippings. We shall now examine each of these kinds in more detail.

The corner-cubinos

First we consider the corners ignoring edges and centers. (We can imagine the Revenge with edges and centers painted all black.) Then the most we can hope to do with the corner-cubinos is to obtain any permutation of them from $\mathfrak{S}_8$ together with any 8-tuple of turns from $\mathfrak{Z}_3$ (the cyclic group of order 3). This means a total of $3^8 8!$ patterns.

Group-theoretically we consider these patterns as forming a group of permutations, the so-called wreath product of $\mathfrak{Z}_3$ by $\mathfrak{S}_8$, written

$$\mathfrak{Z}_3 \wr \mathfrak{S}_8$$

(see [**1**], p. 81; [**4**], p. 59).

Intuitively the group $\mathfrak{Z}_3 \wr \mathfrak{S}_8$ contains a subgroup leaving all corner-cubinos on their places, but turning them individually. This subgroup is normal and isomorphic to $\mathfrak{Z}_3^8$. The corresponding factorgroup consists of the permutations of the 8 corner-cubinos ignoring their possible turns, that is

$$(\mathfrak{Z}_3 \wr \mathfrak{S}_8)/\mathfrak{Z}_3^8 \simeq \mathfrak{S}_8.$$

The possible operations create a subgroup $\mathfrak{H}$ of $\mathfrak{Z}_3 \wr \mathfrak{S}_8$. This is the same as it is for the standard Cube, because the corner-cubinos are not affected by the slice-turns. From the analysis of the Cube (see [**2**], p. 50; [**4**], p. 17) we know that $\mathfrak{H}$ has index 3 in $\mathfrak{Z}_3 \wr \mathfrak{S}_8$. Indeed, the sum of all turns of corner-cubinos must be divisible by 3.

Next we consider the subgroup **H** of **R** consisting of those permutations obtainable by operations fixing edge-cubinos and center-cubinos. The group **H** is isomorphic to a subgroup of $\mathfrak{H}$. The group **H** is as above the same for the Revenge as for the Cube and hence of index 2 in $\mathfrak{H}$. It contains the subgroup of turns without permutations, **T**,

$$\mathbf{T} \simeq \mathfrak{Z}_3^7,$$

and the corresponding factorgroup is isomorphic to $\mathfrak{A}_8$, the alternating group of even permutations of 8 elements (see [**1**], p. 59; [**3**], p. 32),

$$\mathbf{H}/\mathbf{T} \simeq \mathfrak{A}_8.$$

Hence the order of **H** is $3^7 \cdot \frac{1}{2} \cdot 8!$

The fact that **H** has index 2 in $\mathfrak{H}$ means that we can only create an odd permutation of corner-cubinos at the expense of some odd permutation of either edge-cubinos or center-cubinos. We shall return to this problem in the following section.

The center-cubinos

First we consider the center-cubinos ignoring corners and edges. Then the most we can hope to do with the center-cubinos is to obtain any permutation of them from $\mathfrak{S}_{24}$, because they cannot be rotated, as already mentioned.

The possible operations create a subgroup $\mathfrak{C}$ of $\mathfrak{S}_{24}$. We shall compute $\mathfrak{C}$ below.

Next we consider the subgroup **C** of **R** consisting of those permutations obtainable by operations fixing edge-cubinos and corner-cubinos. The group **C** is isomorphic to a subgroup of $\mathfrak{C}$, let us write

$$\mathbf{C} \subset \mathfrak{C} \subset \mathfrak{S}_{24}.$$

It turns out that **C** is isomorphic to the alternating group of even permutations of 24 elements, and that $\mathfrak{C}$ is isomorphic to the symmetric group itself.

Before computing these groups we shall recall a couple of notations from group theory.

The *commutator* of two group elements X and Y is defined as

(*commutator*) $$[X,Y] = XYX^- Y^-$$

(see [**1**], p. 138; [**2**], p. 33; [**3**], p. 48; [**4**], p. 17).

REMARK 1. Two group elements X and Y commute exactly when the commutator of them is the neutral element, $[X,Y] = E$.

REMARK 2. The inverse of a commutator is the "reverse"

$$[X,Y]^{-} = [Y,X].$$

The *conjugate* of a group element X *under* the group element Y is

$$(\textit{conjugate}) \qquad YXY^{-},$$

(see [**1**], p. 13; [**2**], p. 31; [**4**], p. 13).

REMARK 3. On the Cube and the Revenge we consider the conjugate of a "nice" operation X as follows. The operation X is for example a cycle of 3 cubinos. The operation Y moves 3 cubinos which we want to cycle to the locations where X operate. The irrelevant mess Y might create is cleared up by Y^{-} at the same time as the 3 wanted cubinos are returned to the original 3 places, but cycled. (This fact is reckoned in mathematics as the theorem that conjugate permutations have the same structure of cycles; see [**1**], p. 54.)

REMARK 4. It is fairly easy to see that there is a possible permutation in **R** which moves 3 distinct center-cubinos to any stipulated locations (no assumption is made about other cubinos). The same is independently true about edge-cubinos.

We shall now return to the computation of the center-groups.

THEOREM 1. $\mathbf{C} \simeq \mathfrak{A}_{24}$ *and* $\mathfrak{C} \simeq \mathfrak{S}_{24}$.

Proof. The group **C** contains the 3-cycle (1), see Fig. 8.

$$(1) \qquad [[\mathfrak{F}^{+},\mathfrak{D}^{+}],U^{-}] \in \mathbf{C}.$$

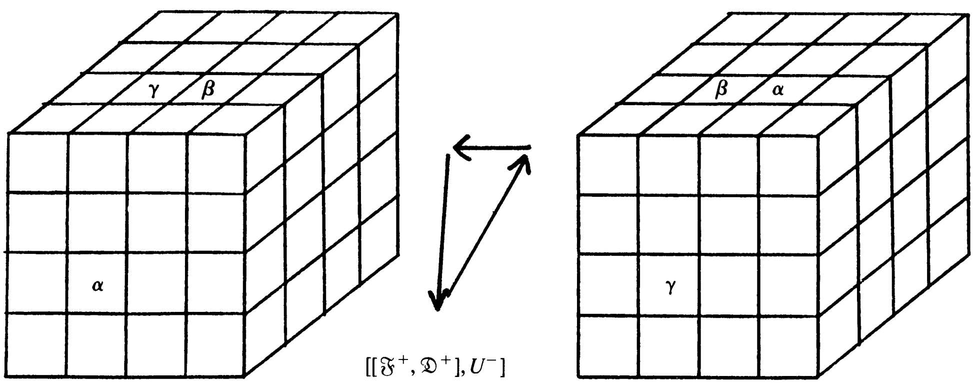

FIG. 8. A 3-cycle of center-cubinos.

It is obvious from Remarks 3 and 4 that we can obtain *any* 3-cycle of center-cubinos by conjugation of this one. Hence **C** contains all the 3-cycles of center-cubinos.

Any alternating group is generated by the set of 3-cycles (see [**1**], p. 61; [**3**], p. 33). Hence we must have the inclusion

$$\mathfrak{A}_{24} \subset \mathbf{C} \subset \mathfrak{C} \subset \mathfrak{S}_{24}.$$

Now, $F^{+} \in \mathfrak{C}$ is a cycle of length 4 and hence an odd permutation. So $\mathfrak{C}$ is greater than $\mathfrak{A}_{24}$ and we must have

$$\mathfrak{C} \simeq \mathfrak{S}_{24}.$$

On the other hand, $F^{+} \in \mathbf{R}$ consists of 4 cycles of length 4 and hence is even, see Fig. 9. Indeed, it contains one 4-cycle of corner-cubinos, one 4-cycle of center-cubinos and two 4-cycles of edge-cubinos. Hence it can only give an odd permutation of center-cubinos simultaneously with an odd permutation of corner-cubinos. So, when corner-cubinos are kept fixed (the even permutation of nothing), then the center-cubinos must be permuted by an even permutation.

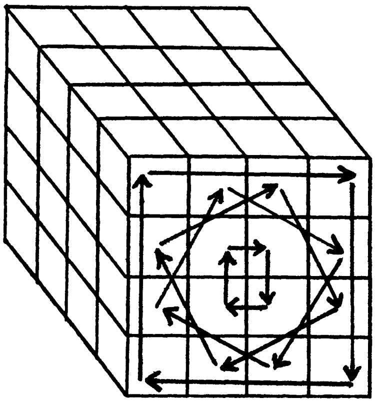

FIG. 9. The 4 4-cycles of F^+.

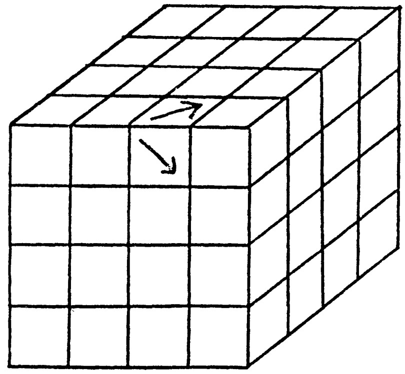

FIG. 10. Preservation of orientation.

And $\mathfrak{F}^+ \in \mathbf{R}$ consists of 3 cycles of length 4, so this is odd. But one of them is a 4-cycle of edge-cubinos and the remaining two are 4-cycles of center-cubinos. So, considered as a permutation of center-cubinos $\mathfrak{F}^+$ is even.

Conclusion. No operation in $\mathbf{C}$ can be an odd permutation. Hence

$$\mathbf{C} \simeq \mathfrak{A}_{24}. \quad \square$$

The order of $\mathfrak{C}$ then is 24!, and for $\mathbf{C}$ it is $\frac{1}{2} \cdot 24!$.

The fact that $\mathbf{C}$ has index 2 in $\mathfrak{C}$ means that we are only able to create an odd permutation of center-cubinos at the expense of some other odd permutation. From the proof it follows that the other odd permutation must be a permutation of the corner-cubinos.

The last fact may be explained further. Consider the group $\mathfrak{P}$ of permutations of corner-cubinos and center-cubinos while ignoring the edge-cubinos. The group $\mathfrak{P}$ must be isomorphic to a subgroup of $\mathfrak{H} \times \mathfrak{C}$. The meaning of the restriction above is then that a pair of permutations $(X, Y) \in \mathfrak{H} \times \mathfrak{C}$ belongs to $\mathfrak{P}$ if and only if the permutations $X \in \mathfrak{H}$ and $Y \in \mathfrak{C}$ have the same parity. This means that $\mathfrak{P}$ is the subgroup of $\mathfrak{H} \times \mathfrak{C}$ of index 2 consisting of all operations that are even as permutations of these 32 cubinos.

Still unsettled is the question of the group $\mathbf{P}$ consisting of those operations from $\mathfrak{P}$ which fix all edge-cubinos. Obviously they include all pairs from $\mathbf{H} \times \mathbf{C}$. We can thus say

$$\mathbf{H} \times \mathbf{C} \subset \mathbf{P} \subset \mathfrak{P},$$

and as $\mathbf{H} \times \mathbf{C}$ has index 2 in $\mathfrak{P}$, we know now that $\mathbf{P}$ is one or the other.

When we have computed the edge-groups, we shall be able to show that

$$\mathbf{P} \simeq \mathfrak{P}.$$

The edge-cubinos

First we consider the edge-cubinos ignoring corners and centers. Then the most we can hope to do with the edge-cubinos is to obtain any permutation of them from $\mathfrak{S}_{24}$ together with any 24-tuple of flips from $\mathfrak{Z}_2$. It turns out that edge-cubinos cannot be flipped, so it is enough to consider the subgroups of $\mathfrak{S}_{24}$ and we can avoid the wreath product.

THEOREM 2. *No operation flips any edge-cubino.*

Proof. If an edge-cubino is flipped, then the orientation from Fig. 7 must look like Fig. 10. But this is not obtainable, because the cubino-faces then do not preserve orientation. $\square$

REMARK. It is also true that if an edge-cubino is physically removed, the hidden "foot" will be revealed to be asymmetric.

This means that for any given location and any given edge-cubino, it can only be oriented in one way. Hence we can consider the group of operations on the edges as a permutation group $\mathfrak{K}$ of the 24 edge-cubinos, i.e., $\mathfrak{K} \subset \mathfrak{S}_{24}$.

Next we consider the subgroup $\mathbf{K}$ of $\mathbf{R}$ of operations on the edge-cubinos fixing the center-cubinos and the corner-cubinos.

We have from Theorem 2, that

$$\mathbf{K} \subset \mathfrak{K} \subset \mathfrak{S}_{24}.$$

They all turn out to be equal.

THEOREM 3.

$$\mathbf{K} \simeq \mathfrak{K} \simeq \mathfrak{S}_{24}.$$

Proof. At first we prove that

$$\mathfrak{A}_{24} \subset \mathbf{K}.$$

The group $\mathbf{K}$ contains the 3-cycle (2); see Fig. 11.

(2) $$[\mathfrak{L}^-,[L^+,U^-]] \in \mathbf{K}.$$

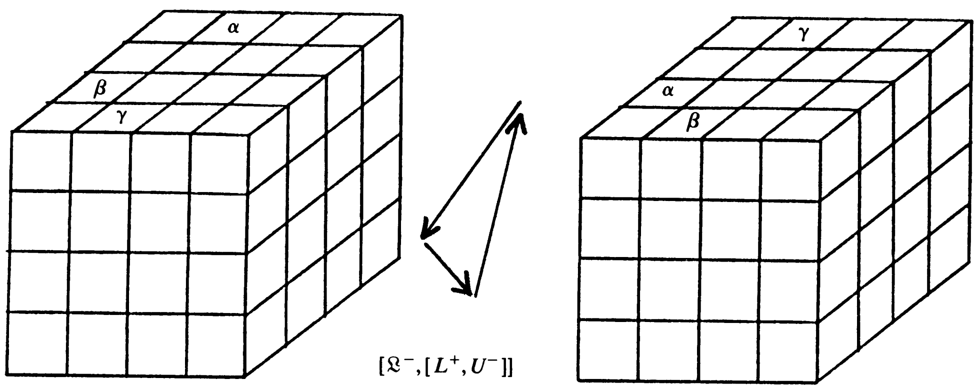

FIG. 11. A 3-cycle of edge-cubinos.

It is obvious from Remarks 3 and 4 that we can obtain *any* 3-cycle of edge-cubinos by conjugation of this one. Hence $\mathbf{K}$ contains all 3-cycles of edge-cubinos and thereby $\mathfrak{A}_{24}$.

It is obvious that $\mathfrak{K}$ contains an odd permutation, $\mathfrak{F}^+ \in \mathfrak{K}$ is a 4-cycle of edge-cubinos and odd.

But $\mathfrak{F}^+$ indicates the existence of a 4-cycle in $\mathbf{K}$ too. $\mathfrak{F}^+$ consists of a 4-cycle of edge-cubinos together with two 4-cycles of center-cubinos. A permutation of center-cubinos made of two 4-cycles is even and hence belongs to $\mathbf{C}$ according to Theorem 1. This means that we can rearrange the center-cubinos with the help of a permutation from $\mathbf{C}$, that is a permutation fixing all other cubinos. Doing this we are left with exactly one 4-cycle of edge-cubinos, which is the wanted odd permutation in $\mathbf{K}$. So $\mathbf{K}$ is bigger than $\mathfrak{A}_{24}$ and hence equal to $\mathfrak{S}_{24}$. □

The orders of $\mathbf{K}$ and $\mathfrak{K}$ are both 24!.

The fact that $\mathbf{K} \simeq \mathfrak{K}$ $(\simeq \mathfrak{S}_{24})$ means that any permutation of edge-cubinos can be done without disturbing the others. This also means that any permutation in $\mathfrak{P}$ can be done without changing the edge-cubinos. So from this follows, that

$$\mathbf{P} \simeq \mathfrak{P}.$$

The structure of $\mathbf{R}$

The result in Theorem 2 says about $\mathbf{R}$, that

$$\mathbf{R} \simeq \mathbf{K} \times \mathbf{P}.$$

Any possible permutation can be executed independently on the edge-cubinos and on the other cubinos. And from Theorem 1 we have

$$\mathbf{H} \times \mathbf{C} \subset \mathbf{P} \subset \mathfrak{H} \times \mathfrak{C},$$

where $\mathbf{P}$ consists of the even permutations. This is to say, that of the 3 subgroups containing $\mathbf{H} \times \mathbf{C}$ and of index 2 in $\mathfrak{H} \times \mathfrak{C}$, the group $\mathbf{P}$ is the one different from $\mathbf{H} \times \mathfrak{C}$ and $\mathfrak{H} \times \mathbf{C}$. See Fig. 12.

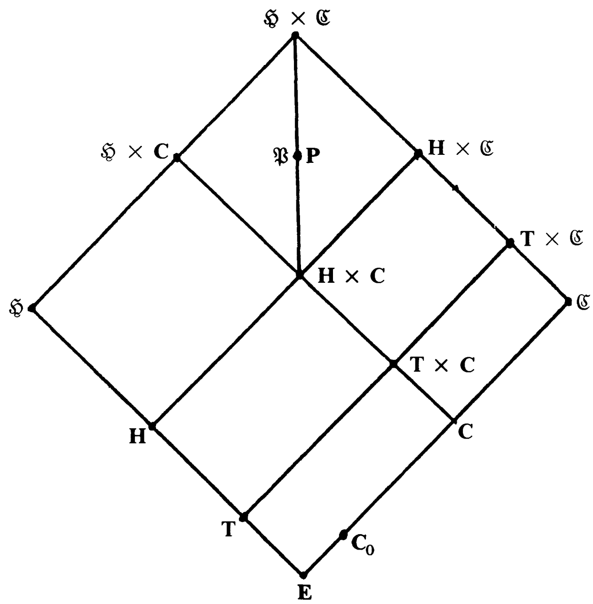

FIG. 12. Diagram of center- and corner-groups.

The order of $\mathbf{R}$ then is

$$\text{order}(\mathbf{R}) = \text{order}(\mathbf{K}) \cdot \text{order}(\mathbf{P}),$$

and

$$\text{order}(\mathbf{P}) = 2 \cdot \text{order}(\mathbf{H}) \cdot \text{order}(\mathbf{C});$$

all together

$$\text{order}(\mathbf{R}) = 24! \cdot 2 \cdot 3^7 \cdot \frac{1}{2} \cdot 8! \cdot \frac{1}{2} \cdot 24!.$$

To compute the number of patterns recall the group $\mathbf{C}_0$ of invisible permutations. As $\mathbf{C}_0$ is a subgroup of $\mathbf{C}$, it must consist of the even permutations of each of the six 4-tuples by itself. There are $4!^6$ permutations of 6 sets of 4 elements each, and of these, half are even. The order of $\mathbf{C}_0$ hence is

$$\frac{1}{2} \cdot 24^6.$$

The number of patterns of visible difference hence is

$$\frac{3^7 \cdot 8! \cdot 24!^2}{24^6} = 177\ 62872\ 41975\ 57644\ 87697\ 82553\ 87965\ 78406\ 40000\ 00000.$$

(This number includes a factor 24 from the group of physical turns of the cube (the *hexahedron* group; see [3], p. 37).)

How to solve a mess? The principle of solution may be based on a descending series of subgroups, each normal in the preceding one with a nice factor group. The use of the series shall be explained below. Let us define $\mathbf{K}\ \simeq \mathfrak{A}_{24}$ as the subgroup of $\mathbf{K}$ of even permutations, and $\mathbf{E}$ as the group of one element, the unit. We shall then give an example of such a series (with factor groups below):

$$\begin{array}{ccccccccccccc} \mathbf{R} & \supset & \mathbf{K}\times\mathbf{H}\times\mathbf{C} & \supset & \mathbf{K}\times\mathbf{T}\times\mathbf{C} & \supset & \mathbf{K}\times\mathbf{E}\times\mathbf{C} & \supset & \mathbf{K}\ \times\mathbf{E}\times\mathbf{C} & \supset & \mathbf{E}\times\mathbf{E}\times\mathbf{C} & \supset & \mathbf{E}\times\mathbf{E}\times\mathbf{E} \\ & \mathfrak{Z}_2 & & \mathfrak{A}_8 & & \mathbf{T}\simeq\mathfrak{Z}_3^7 & & \mathfrak{Z}_2 & & \mathfrak{A}_{24} & & \mathfrak{A}_{24} & \end{array}$$

We consider the pattern (the mess) of the Revenge as a permutation obtained from the nice start position with one-coloured faces by some operation. This permutation belongs to $\mathbf{R}$, and we want to find a way to write it as a series of the twelve generators of $\mathbf{R}$, F^+, $\mathfrak{F}^+$, etc.

If the permutation of corner-cubinos is odd, then we start with any face-turn, for example F^+. This is to be considered as a representation of the non-unit of the first factor group $\mathfrak{Z}_2$. Thus we reduce the problem to a permutation in $\mathbf{K}\times\mathbf{H}\times\mathbf{C}$.

Then we use the generators of $\mathbf{H}/\mathbf{T}\simeq\mathfrak{A}_8$, the 3-cycles of corners known from the Cube; see [**2**], p. 32; [**4**], p. 44, e.g.,

$$[F^-, U^+B^+U^-],$$

to arrange the corner-cubinos on right places relative to each other. This means we have reduced the problem to a permutation in the group $\mathbf{K}\times\mathbf{T}\times\mathbf{C}$.

Now the generators of $\mathbf{T}\simeq\mathfrak{Z}_3^7$ turn the corners in the right directions. They are also obtainable from the Cube; see [**2**], p. 38; [**4**], p. 44, e.g.,

$$\left[[F^+, D^+]^2, U^+\right].$$

This reduces the problem to a permutation in the group $\mathbf{K}\times\mathbf{E}\times\mathbf{C}$, and having arranged the corners we rejoice that permutations from this group keep corners fixed.

If the necessary permutation of edge-cubinos is odd, then we need a generator of $\mathbf{K}/\mathbf{K}'\simeq\mathfrak{Z}_2$; i.e., any odd permutation, e.g.,

$$\mathfrak{F}^+.$$

After this reduction to the group $\mathbf{K}'\times\mathbf{E}\times\mathbf{C}$, we can replace the edge-cubinos completely by help of formula (2) and its conjugates. This replacement reduces our permutation to the group $\mathbf{E}\times\mathbf{E}\times\mathbf{C}$.

This group is generated by the formula (1) and its conjugates. Using this we don't need to reach $\mathbf{E}$; it is enough to reduce to any permutation from $\mathbf{C}_0$ (the difference is invisible).

The descending series used here is vivid graphically, because more and more cubinos fall into place. The lower subgroups tend to have more complicated operations. A series can be constructed on the opposite principle, where the last stages use simpler operations. Watching the result is like magic. Nothing seems to be happening until the last few moments, when PRESTO! everything falls into place.

The problems in the case of the Revenge are due to the two factor groups $\mathfrak{Z}_2$. They are so easy to overlook, but to postpone these two "corrections" might double the work. Besides, they are hard to invent by oneself —a mathematical benefit from having heard about odd and even!

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