NOTES

Kepler's Spheres and Rubik's Cube

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"How many spheres of radius r may simultaneously be tangent to a fixed sphere of the same radius?" This question goes back at least to the year 1694, when it was considered by Isaac Newton and his contemporary David Gregory. Johann Kepler had already shown in 1611 [**8**] that twelve outer spheres could be arranged around a central sphere, and now Gregory claimed that a thirteenth could be added. Newton disagreed, but neither man proved his claim, and it was not until 1874 that a proof was found, substantiating Newton's conjecture. (R. Hoppe's original proof is described in [**2**]; more recently, proofs have been found by Günter [**7**] and by Schütte and van der Waerden [**10**]. Perhaps the most elegant proof known is the one given by John Leech [**9**]. For a history of the problem, see [**6**].)

It's worth noting that the 4-dimensional version of this problem is still unsolved; no one knows how to arrange more than twenty-four hyperspheres around a central hypersphere, but neither has it been shown that a twenty-fifth cannot be added. In higher dimensions the situation is even less well understood, with the remarkable exception of dimensions 8 and 24, for which the maximum contact numbers are precisely known [**1**].

The source of difficulty in the original Gregory-Newton problem (and the reason it took 180 years to be solved) is that there is almost room for a thirteenth sphere; the twelve spheres of Kepler can be pushed and pulled in all sorts of ways, and it's credible that some sort of fiddling could create a space big enough to accommodate an extra sphere.

This note will deal with a related question, in the spirit of Ernö Rubik: **if we label the twelve spheres and roll them over the surface of the inner sphere at will, what permutations are achievable?** The answer is surprising, and the proof requires only the rudiments of analytic geometry and group theory. One fringe benefit of this enterprise is that it leads to a natural coordinatization of the vertices of the regular icosahedron.

An equivalent formulation of the problem is gotten by considering only the centers of the spheres: we imagine twelve vertices, free to move on a sphere of fixed radius $R = 2r$ about a fixed origin but subject to the constraint that no two vertices may ever be closer together than R. We will find it convenient to switch back and forth between the two formulations. For definiteness, put $r = \frac{1}{2}$, $R = 1$.

To begin, we must prescribe an initial configuration for the outer spheres. One possibility that springs to mind is to arrange the twelve vertices to form a regular icosahedron. As we'll see, any two distinct vertices of the icosahedron inscribed in the unit sphere are at least

$$\sqrt{2 - 2/\sqrt{5}} \approx 1.01$$

units apart, so not only is this configuration possible, but no two of the outer spheres touch (see FIGURE 1). It is clear that any rigid motion of the icosahedron about its center can be achieved by a joint motion of the spheres; and so, since the rotational symmetries of the icosahedron comprise the alternating group A_5 [5, pp. 49–50], the number of icosahedral arrangements of the spheres accessible from this starting configuration is at least $|A_5| = 60$.

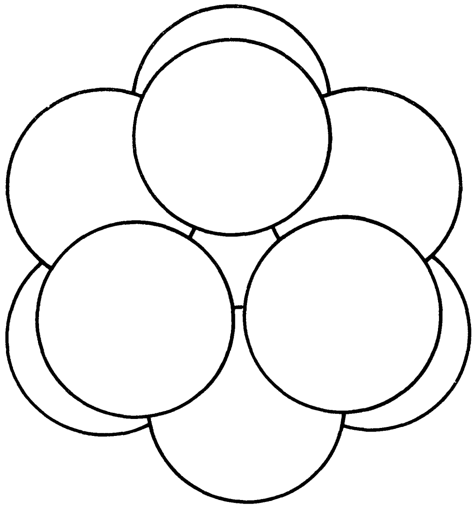

FIGURE 1

But there is another way of placing the twelve spheres around one, and moreover, this arrangement extends to a sphere-packing of three-space. Let the lattice Λ consist of all points $(a/\sqrt{2}, b/\sqrt{2}, c/\sqrt{2})$ with a, b, c integers and $a+b+c$ even. Then no two points of Λ are closer than 1, and so the points of Λ may serve as the centers of non-overlapping spheres of radius $\frac{1}{2}$ (see FIGURE 2). This is the *cubic close-packing* discovered by Kepler. Each vertex has twelve nearest neighbors, and in particular the

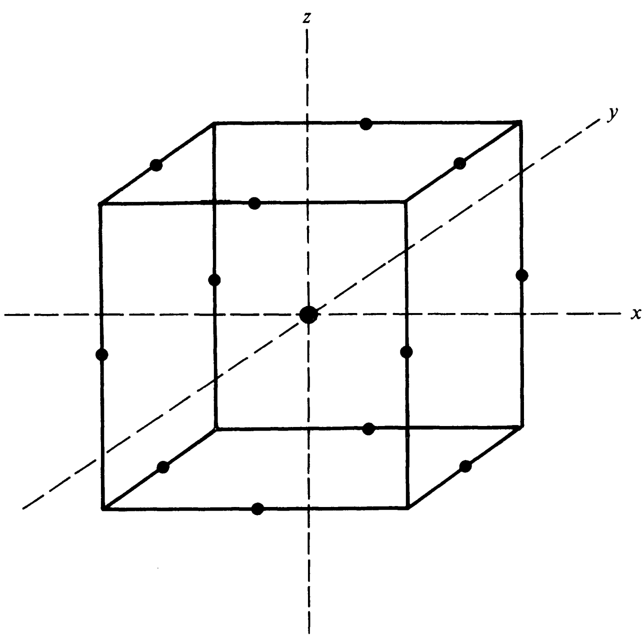

FIGURE 2

neighbors of $(0,0,0)$ are

$$\begin{gathered}(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0)\\(\pm 1/\sqrt{2}, 0, \pm 1/\sqrt{2})\\(0, \pm 1/\sqrt{2}, \pm 1/\sqrt{2}).\end{gathered}$$

These vertices form a *cuboctahedron* (see FIGURE 3(a)). Its rotational symmetry group, like that of the cube and the octahedron, is S_4; one may concretely picture the S_4 action as permuting the four pairs of antipodal triangular faces. Hence at least $|S_4| = 24$ cuboctahedral arrangements are accessible from the initial position. The S_4 action on the vertices, like the A_5 action, performs only even permutations on the twelve vertices/spheres.

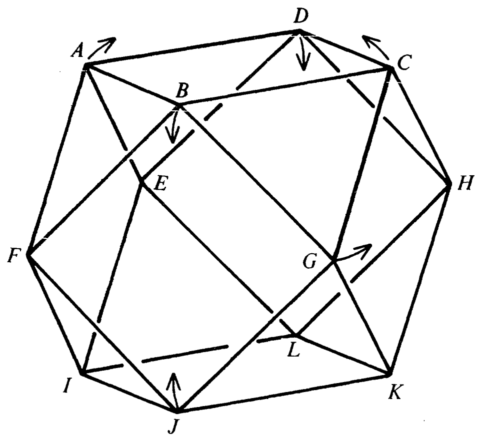

FIGURE 3(a)

Now comes a pleasant surprise: the two arrangements of the twelve spheres (icosahedral and cuboctahedral) may be deformed into one another! This means that in our Rubik-like manipulations, we can avail ourselves of both the A_5 and S_4 symmetries.

To deform the cuboctahedron into the icosahedron, move each point along a great circle (as indicated in FIGURE 3(a) for selected points) so that each square face of the cuboctahedron becomes a pair of triangular faces of the icosahedron. For instance, the points

$$A = \left(-\sqrt{\tfrac{1}{2}}, 0, \sqrt{\tfrac{1}{2}}\right)$$

and

$$C = \left(\sqrt{\tfrac{1}{2}}, 0, \sqrt{\tfrac{1}{2}}\right)$$

both move toward $(0,0,1)$, while the point

$$B = \left(0, -\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$$

moves toward $(0, -1, 0)$. (This is a variation on the construction given in [**5**, pp. 51–53].) We may parametrize this joint motion with a variable t, whose initial value is $\pi/4$:

$$A(t) = (-\cos t, 0, \sin t)$$
$$B(t) = (0, -\sin t, \cos t)$$
$$C(t) = (\cos t, 0, \sin t)$$

and so on. The distance between $A(t)$ and $B(t)$ is

$$\sqrt{\cos^2 t + \sin^2 t + (\sin t - \cos t)^2} = \sqrt{2 - \sin 2t},$$

which increases from 1 to $\sqrt{2}$ as t runs from $\pi/4$ to $\pi/2$. The distance between $A(t)$ and $C(t)$ is

$$2\cos t,$$

which decreases from $\sqrt{2}$ to 0 as t approaches $\pi/2$. Let us intervene and stop the process when the lengths of AB and AC are equal, which occurs at some unique time $t = \theta$, and write $A' = A(\theta)$, $B' = B(\theta)$, etc. The symmetry of the procedure now guarantees that $B'C' = A'B' = B'F' = F'A'$, so that triangles $A'B'C'$ and $A'B'F'$ are congruent equilateral triangles. In fact, the vertices $A', \ldots, L'$ form twenty such triangles on the sphere of radius R, and this implies that they are arranged icosahedrally, with labelings as shown in FIGURE 3(b).

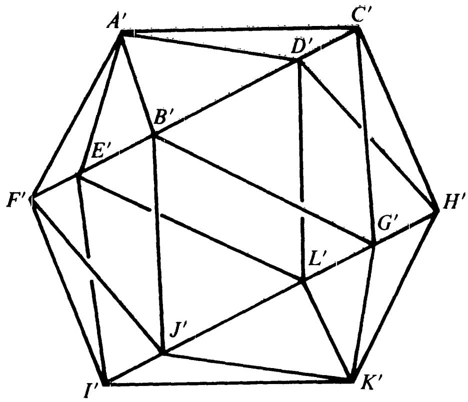

FIGURE 3(b)

(By equating

$$(A'B')^2 = 2 - \sin 2\theta$$

and

$$(A'C')^2 = 4\cos^2\theta = 2 + 2\cos 2\theta$$

one obtains

$$\cos 2\theta = \frac{-1}{\sqrt{5}} \text{ and } \sin 2\theta = \frac{2}{\sqrt{5}},$$

so that the common length of the segments is

$$\sqrt{2 - 2/\sqrt{5}}.$$

This is the side-length of an icosahedron of circumradius 1. Also, the distance between A' and I' is $2\sin\theta$, whose ratio to $A'C' = A'B' = 2\cos\theta$ is $\tan\theta$; this may easily be shown to equal the golden ratio $(1+\sqrt{5})/2$. Hence $A'I'K'C'$ is a golden rectangle, as is any rectangle formed by two diagonals of the icosahedron.)

Since it does not matter whether our initial configuration is icosahedral or cuboctahedral, we may as well assume the latter. Let G be the group of permutations of the twelve spheres that can be achieved by the operations of rotating the cuboctahedron, switching between cuboctahedral and icosahedral formation, and rotating the icosahedron, in all combinations. (We don't claim that G contains *all* the achievable permutations.) Note that points which were antipodal at the start remain antipodal throughout. Also note that since G contains copies of both S_4 and A_5, its order is at least 120.

If we rotate the labeled icosahedron of FIGURE 3(b) by 72 degrees clockwise about its axis $A'K'$, the points are permuted by

$$(A')(B'C'D'E'F')(G'H'L'I'J')(K')$$

which corresponds to the permutation

$$\gamma=(A)(BCDEF)(GHLIJ)(K)$$

of the vertices of the cuboctahedron. On the other hand, if we rotate the cuboctahedron by 90 degrees clockwise about the vertical axis, the points are permuted by

$$\alpha=(ABCD)(EFGH)(IJKL).$$

Since each of these permutations belongs to G, so does their product

$$\alpha\gamma=(ABDFC)(EG)(HIKLJ).$$

But $(\alpha\gamma)^5=$ the two-cycle (EG). Hence, in general, any two antipodal points may be exchanged without affecting the final position of the other ten.

The preceding observation tells us that the group G is a wreath product of the form $2^6.H$, where H is a group acting on *pairs* of antipodal points. (For an explanation of this notation, see [3].) That is, every element of G is uniquely determined by its action on the six principal diagonals of the polyhedron, with a choice of orientation; if you like, think of H as a permutation group acting on six coins, and $2^6.H$ as the set of actions one gets by allowing not only permutations taken from H but also arbitrary flips. Let $\bar{A}$, $\bar{B}$, etc., denote the diagonals AK, BL, etc. Then the image of γ under the natural homomorphism from G to H is

$$\bar{\gamma}=(\bar{A})(\bar{B}\bar{C}\bar{D}\bar{E}\bar{F}),$$

and similarly, the image of α is

$$\bar{\alpha}=(\bar{A}\bar{B}\bar{C}\bar{D})(\bar{E}\bar{F}).$$

To find generators of the group H, we need to know the generators of G. α and γ aren't enough, but we'll only need one more. The S_4 subgroup of G is generated by

$$\alpha=(ABCD)(EFGH)(IJKL)$$

(90 degree rotation about the axis through face $ABCD$) and

$$\beta=(ABF)(CJE)(DGI)(HKL)$$

(120 degree rotation about the axis through face ABF), while the A_5 subgroup is generated by

$$\gamma = (A)(BCDEF)(GHLIJ)(K)$$

(72 degree rotation about axis $A'K'$) and

$$\delta = (ABF)(CJE)(DGI)(HKL) = \beta$$

(120 degree rotation about the axis through face $A'B'F'$). The images of those elements in H are

$$\bar{\alpha} = (\bar{A}\,\bar{B}\,\bar{C}\,\bar{D})(\bar{E}\,\bar{F})$$
$$\bar{\beta} = (\bar{A}\,\bar{B}\,\bar{F})(\bar{C}\,\bar{D}\,\bar{E})$$
$$\bar{\gamma} = (\bar{A})(\bar{B}\,\bar{C}\,\bar{D}\,\bar{E}\,\bar{F})$$

and they must generate H.

Since all three permutations are even, H must be a subgroup of the alternating group A_6, and in fact H is A_6 itself. We prove this by successively examining stabilizer subgroups:

1. The action of H on $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}$ is clearly transitive, so all of its 1-point stabilizer subgroups are isomorphic, and have order $|H|/6$.

2. $\bar{\gamma}$ (which belongs to the stabilizer of $\bar{A}$) acts transitively on $\bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}$, so that all 2-point stabilizer subgroups are isomorphic, and have order $|H|/6 \cdot 5$.

3. Since

$$\bar{\alpha}^2 = (\bar{A}\,\bar{C})(\bar{B}\,\bar{D})(\bar{E})(\bar{F})$$

and

$$\bar{\beta}\bar{\alpha}\bar{\beta} = (\bar{A}\,\bar{D})(\bar{B}\,\bar{C})(\bar{E})(\bar{F})$$

(which belong to the stabilizer of $\bar{E}, \bar{F}$) generate a transitive action on $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, all the 3-point stabilizer subgroups of H are isomorphic, and have order $|H|/6 \cdot 5 \cdot 4$.

4. But since

$$\bar{\alpha}\bar{\gamma}^{-1} = (\bar{A}\,\bar{B}\,\bar{E})(\bar{C})(\bar{D})(\bar{F})$$

belongs to the stabilizer of $\bar{C}, \bar{D}, \bar{F}$, this 3-point stabilizer must contain the cyclic group A_3. In fact, it must be A_3 itself (S_3 is impossible, since $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are all even). Hence $|H|/6 \cdot 5 \cdot 4 = 3$, so that $|H| = 360$ and $H = A_6$, as claimed.

We have now shown that the S_4 and A_5 symmetry groups (of the cuboctahedron and icosahedron, respectively) combine to give a group $G = 2^6.A_6$ of order $2^6 \cdot (6!/2) = 23{,}040$. But might there be even more feasible permutations of the twelve spheres?

Here's an idea, inspired by the twisting operation one performs on a Rubik's cube. Perhaps one can pull six of the spheres tightly to one side (or "hemisphere") of the central sphere, and six to the other side, in such a way that the two groups can be twisted past one another. Specifically, we arrange two of the twelve spheres antipodally, and around each we cluster five spheres as snugly as possible, forming two "caps" (top view shown in FIGURE 4(a)). The question is, can the caps be twisted past each other?

This is equivalent to asking whether the sphere in the middle of such a cap (at the "north pole", as it were) can simultaneously touch all five surrounding spheres. For, if such perfectly snug caps are possible, then six spheres can be restricted to the northern hemisphere, and the five that border on the "equator" will graze but not block the other six spheres, which are all restricted to the southern hemisphere (see FIGURE 4(b)). So now we need only check whether five points, all with latitude 30 degrees North on a sphere of radius $R = 1$, can all be at least distance 1 from each other. (Or equivalently: Can five non-overlapping regular tetrahedra share an edge? To see the connection, let the common edge be the segment joining the origin to the north pole.)

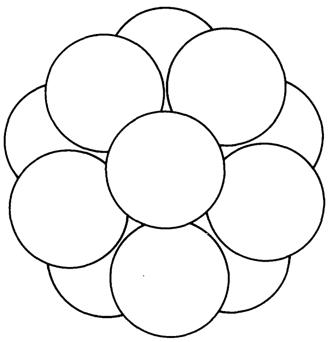
FIGURE 4(a)

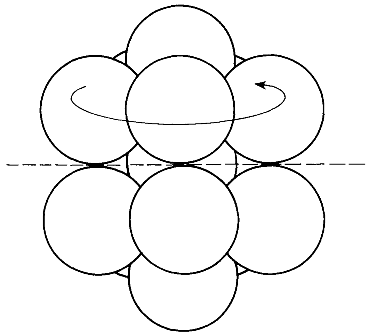
FIGURE 4(b)

But this is simple trigonometry: A regular pentagon inscribed in a circle of radius 1 has side

$$\sqrt{2 - 2\cos\frac{2\pi}{5}} = \sqrt{\frac{5-\sqrt{5}}{2}},$$

so one inscribed in a latitude circle of radius $\sqrt{3}/2$ has side

$$\sqrt{\frac{5-\sqrt{5}}{2}}\,\frac{\sqrt{3}}{2} \approx 1.02.$$

This is just barely bigger than 1, but it's enough to ensure that the caps can indeed be made snug and then twisted past each other.

The "twisting" move we have described is a 5-cycle that may be performed on any 5 vertices of the icosahedron that form a regular pentagon. Since such permutations are even, the group they generate must lie in A_{12}. To see what this group is, consider the situation shown in FIGURE 5(a), with two pentagons intersecting in two points. Let

$$\alpha = (ABGHD)$$

and

$$\beta = (BFIKG).$$

Then

$$\alpha^{-1}\beta^{-1}\alpha\beta = (AB)(GK).$$

By conjugating this permutation through elements of the 2-point stabilizer of A, B, one may show that every permutation $(AB)(XY)$ (with $X,\ Y \neq A, B$) can be achieved. (Look at FIGURE 5(b). Take a pencil and mark those edges to which edge GK can be moved by rotating a pentagon that doesn't contain A or B; it will be seen that all edges XY not involving A and B eventually get marked, and the moves that

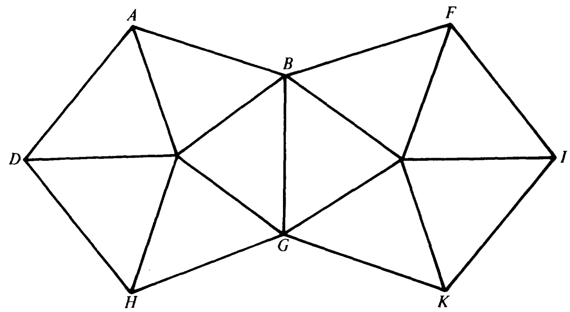

FIGURE 5(a)

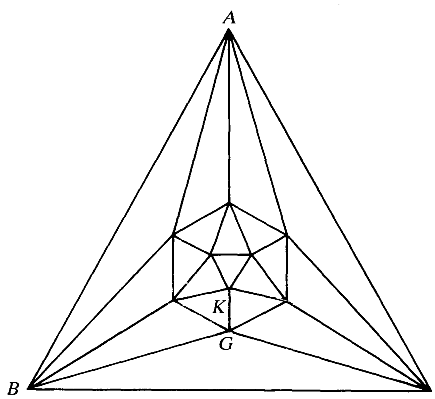

FIGURE 5(b)

carry edge GK to the respective edges XY give precisely the conjugations we need to obtain all permutations of the form $(AB)(XY)$.) The same sort of argument now shows that in fact any permutation of the shape $(VW)(XY)$ can be achieved. Since A_{12} is simple, the normal subgroup generated by these pairs of transpositions must be A_{12} itself. Hence, the twisting moves allow any even permutation on the twelve spheres to be performed.

What happens if we now allow both the cuboctahedron-to-icosahedron maneuver and the twist operation? Recall that the group $G = 2^6 H$ contains two-cycles that exchange antipodes. Even one such two cycle, if adjoined to the group A_{12}, suffices to generate the full symmetric group S_{12}. (Alternatively, one can regard G as primary, and adjoin to it a few 5-cycles so that every pair of spheres can be made antipodal; for we already know how to exchange antipodes at will.) In either case, we see that *every* permutation of the twelve spheres is feasible, and the "Rubik group" of Kepler's spheres is S_{12}.

Some concluding remarks:

1. It may interest the reader to know that a related problem leads not to S_{12} but to the Mathieu group M_{12}, one of the sporadic finite simple groups; see [4].

2. There are many more possible arrangements of the twelve spheres than the cuboctahedral and icosahedral formations, yet the theorem proved above says nothing about twelve-spheres-around-one in general position. It seems likely that every position can be reached from every other; to prove this, it would suffice to show that every configuration can be "tidied" into icosahedral formation. Perhaps some reader will be able to find a proof.

3. If anyone built a working model of "Kepler's Spheres," he or she would probably give the outer spheres slightly smaller radius than the inner sphere to prevent jamming, since both the icosahedron-to-cuboctahedron maneuver and the twist operation cause spheres to graze one another. This suggests the following question: if the outer spheres in an icosahedral arrangement were given slightly *larger* radius than the inner sphere, would any permutations be possible aside from the "trivial" ones obtained from A_5? Perhaps engineers have already developed a theory for handling such problems of constrained motion in space—if so, the author would be glad to hear of it.

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A Characterization of Infinite Dimension for Vector Spaces

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Several different arguments have been used to establish that a finite-dimensional vector space over the real or complex number fields cannot have a pair of linear transformations whose commutator is the identity; i.e., $[A, B] = AB - BA = 1$ is impossible for such spaces. The oldest argument is implicit in the comments of Max Born and Pacual Jordan in their seminal work on a matrix version of quantum mechanics [**1**]. They noted that for finite matrices (over the complex numbers), applying the trace function to the equation

$$PQ - QP = \frac{h}{2\pi i} 1$$