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How Euler Did It

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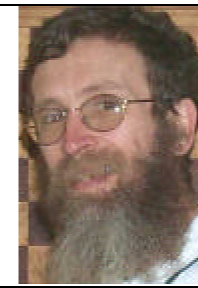

Divergent series

June 2006

Today we are fairly comfortable with the idea that some series just don't add up. For example, the series

$$1-1+1-1+1-1+\text{etc.}$$

has nicely bounded partial sums, but it fails to converge, in the modern meaning of the word. It took mathematicians centuries to resolve the paradoxes of diverging series, and this month's column is about an episode while we were still confused.

In the 1700's, though, many mathematicians were more optimistic, or perhaps more naïve, about the limitations of mathematics, thinking that with enough brilliance and enough work they could solve any differential equation and sum any series. Daniel Bernoulli, for example, thought that the series above ought to have value ½, not for the usual reason that involves geometric series and $\frac{1}{1-x}=1+x+x^2+x^3+\cdots$, but because of a probabilistic argument. He thought that since half of the partial sums of the series are +1 and half of them are zero, the correct value of the series would be the expected value $\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 0=\frac{1}{2}$.

By a similar argument, Bernoulli concluded that 1 + 0 – 1 + 1 + 0 – 1 + 1 + 0 – 1 + etc. would have value 2/3, and that, by a judicious interpolation of zeroes into the series, it could be argued to have any value between 0 and 1.

It is sometimes frustrating to us that the writers of the time usually did not distinguish between a series and a sequence. The words "progression" and "series" took both meanings. At the same time, though, they made a now-obsolete distinction between the *value* of a series and the *sum* of the same series. Bernoulli and his contemporaries would give the series $1-1+1-1+1-1+\text{etc.}$ the *value* ½, since they could make reasonable calculations and analyses that supported this value, but they would be reluctant to call that value a *sum*. They seemed to think that sums required convergence of some sort.

Euler entered the conversation with a paper *De seriebus divergentibus* (On divergent series) [E247], written in 1746, but not read to the Academy until 1754, nor published until 1760. Euler wrote this paper about two years after he finished his great precalculus textbook the *Introductio in analysin infinitorum*, in which he devotes a good deal of time to issues of series. The *Introductio* does not deal with divergent series directly, but they are often near to his thoughts.

Euler's intent in E247 is to give a value to a series he calls the "hypergeometric series of Wallis:"

$$1 - 1 + 2 - 6 + 24 - 120 + 720 - 5040 + \text{etc.}$$

A century earlier, John Wallis had introduced the numbers he called "hypergeometric numbers" and we call "factorial numbers." Euler's series is the alternating sum of those numbers.

If Euler is going to sum such a series, he first has to convince his reader, who may not be as optimistic as he is, that such series can have a meaningful value. He states his case beginning with an uncontroversial example:

> "Nobody doubts that the geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\text{etc.}$ converges to 2. As more terms are added, the sum approaches 2, and if 100 terms are added, the difference between the sum and 2 is a fraction with 30 digits in its denominator and a 1 in its numerator.
>
> The series $1+1+1+1+1+\text{etc.}$ and 1 + 2 + 3 + 4 + 5 + 6 + etc. whose terms do not tend toward zero, will grow to infinity and are divergent."

This is based on an idea of convergence, but since it lacks logical quantifiers, it is doomed to far short of modern standards of rigor.

On the other hand, Euler has interesting things to say about the alternating series of 1's. He tells us that in 1713 Leibniz said in a letter to Christian Wolff that $1-1+1-1+1-1+\text{etc.}$ should have the value ½, "based on the expansion of the fraction $\frac{1}{1+1}$." Instead of starting with a geometric series, as we usually see the calculation done today, Leibniz started with the series expansion of $\frac{1}{1-x}$, that is $1+x+x^2+x^3+x^4+\text{etc.}$ and evaluated it at $x = -1$. Likewise, Leibniz took $1-2+3-4+5-6+\text{etc.}$ to be ¼ by expanding $\frac{1}{(1+1)^2}$. Leibniz thought that all divergent series could be evaluated.

Euler gives us four pairs of examples of divergent series:

I. $1 + 1 + 1 + 1 + 1 + 1 + \text{etc.}$

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\text{etc.}$$

II. $1-1+1-1+1-1+\text{etc.}$

$$\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\frac{5}{6}-\frac{6}{7}+\text{etc.}$$

III. $1 + 2 + 3 + 4 + 5 + 6 + \text{etc.}$

$1 + 2 + 4 + 8 + 16 + 32 + \text{etc.}$

IV. $1 - 2 + 3 - 4 + 5 - 6 + \text{etc.}$

$1 - 2 + 4 - 8 + 16 - 32 + \text{etc.}$

He explains that such series have been "the cause of great dissent among mathematicians of whom some deny and others affirm that such a sum can be found." He says that the first series on the list ought to be infinite, both from the nature of his understanding of infinite numbers and because, as a geometric series it has value $\frac{1}{1-1}=\frac{1}{0}$, "which is infinite."

Euler is trying to be fair in his account of this divisive controversy, and he presents the other side of the argument, writing

> "One could object to this argument by saying that $\frac{1}{1+a}$ is not equal to the infinite series $1-a+a^2-a^3+a^4-a^5+a^6-\text{etc.}$ unless a is a fraction smaller than 1, because, if we work out the division, we get
>
> $\frac{1}{1+a}=1-a+a^3-a^3+\cdots\pm a^n \mp \frac{a^{n+1}}{1+a}$, and if n stands for an infinite number, then the fraction $\mp\frac{a^{n+1}}{1+a}$ cannot be omitted because it doesn't vanish unless $a < 1$, in which case the series converges."

Euler's sympathies, though, are with the side of the argument that sums even divergent geometric series, and he states what he hopes will be the final word on the issue of the existence of divergent series:

> "Defenders of the idea of summing divergent series resolve this paradox by devising a rather subtle means of discriminating among quantities that become negative, some that stay less than zero and others that become more than infinity. Of the first sort is -1, which by adding it to the number a + 1 leaves the smaller

> number a. Of the second sort is the -1 that arises as 1 + 2 + 4 + 8 + 16 + etc., which is equal to the number one gets by dividing +1 by -1. In the first case, the number is less than zero, and in the second case it is greater than infinity.
>
> This can be confirmed by the following example of a sequence of fractions:
>
> $$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{0}, \frac{1}{-1}, \frac{1}{-2}, \frac{1}{-3} \text{ etc.}$$
>
> where the first four terms are seen to grow, then grow to infinity, and beyond infinity they become negative. Thus the apparent absurdity is resolved in a most ingenious way."

Euler is claiming that numbers greater than infinity are the same as numbers less than zero.

Having resolved (he hopes) the question of existence, Euler turns to summing some series. He needs some ground-work. Euler asks us to consider an arbitrary (alternating) series, s:

$$s = a - b + c - d + e - f + g - h + \text{etc.}$$

Neglecting signs, the first differences are

$$b - a,\ c - b,\ d - c,\ e - d, \text{ etc.}$$

The second differences are

$$c - 2b + a,\ d - 2c + b,\ e - 2d + c, \text{ etc.}$$

and so forth for fourth, fifth, etc. Euler denotes the first value in each sequence of differences by a Greek letter. He takes $a = b - a$, $b = c - 2b + a$, $g = d - 3c + 3b - a$, etc., and tells us that

$$(1) \qquad s = \frac{a}{2} - \frac{a}{4} + \frac{b}{8} - \frac{g}{16} + \frac{d}{32} - \text{etc.}$$

This is Euler's key tool for much of the rest of this paper. Euler doesn't prove his formula (1), but most readers should be able to justify his claim. My own "proof" depends on an only slightly obscure identity about binomial coefficients:

$$\sum_{k=n}^{\infty} \frac{1}{2^{k+1}} \binom{k}{n} = 1.$$

Of course, the proof is valid by modern standards of rigor only if the series s satisfies certain conditions of convergence. Still, it is a remarkable formula. If the series that defines s converges rather slowly, then this new series is likely to accelerate the convergence considerably, because the differences a, b, g etc. are likely not to be very large, and the formula introduces a geometric series into the denominators.

To show us how this works, Euler looks at his alternating series of ones. For that series, a = 1, and all the differences, a, b, g etc. are zero. Formula (1) gives us

$$s = \frac{1}{2} - \frac{0}{4} + \frac{0}{8} - \text{etc.}$$
$$= \frac{1}{2},$$

as promised.

If we take

$$s = 1 - 2 + 3 - 4 + 5 - 6 = \text{etc.},$$

then all the first differences are 1, and all subsequent differences are 0, so that

$$s = ½ - ¼ = ¼.$$

If we take

$$s = 1 - 4 + 9 - 16 + 25 - 36 + \text{etc.}$$

the alternating sum of perfect squares, then our first differences are

3, 5, 7, 9, 11

and the second differences are all 2, so that

$$s = \frac{1}{2} - \frac{3}{4} + \frac{2}{8} = 0\,.$$

The alternating geometric sum of powers of 3 is a little bit trickier.

$$s = 1 - 3 + 9 - 27 + 81 - 243 + \text{etc.}$$

First differences are	2,	6,	18,	54,	162
Second differences are		4,	12,	36,	108
Third differences are			8,	24,	72
Fourth				16,	48
		etc.			

So,

$$\begin{aligned} s &= \frac{1}{2} - \frac{2}{4} + \frac{4}{8} - \frac{8}{16} + \text{etc.} \\ &= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \text{etc.} \\ &= \frac{1}{4}. \end{aligned}$$

In the last step, he uses his result about 1 – 1 + 1 – 1 + etc.

Now we turn to the main problem, Wallis's hypergeometric series

$$A = 1 - 1 + 2 - 6 + 24 - 120 + 720 - 5040 + 40320 - \text{etc.}$$

Taking 1 – 1 = 0, and dividing by 2, we get the series

$$\frac{A}{2} = 1 - 3 + 12 - 60 + 360 - 2520 + 20160 - 181440 + \text{etc.}$$

and differences

2	9	48	300	2160	17640	161280
	7	39	252	1860	15480	143640
		32	213	1608	13620	128160
			181	1395	12012	114540
				1214	10617	102528
					9403*	91911
						82508

* A footnote in the *Opera Omnia* edition says that the original has a typographical error here, 9407, and that all subsequent numbers that depended on this are also in error, but are corrected in the *Opera Omnia*.

This makes

$$\frac{A}{2} = \frac{1}{2} - \frac{2}{4} + \frac{7}{8} - \frac{32}{16} + \frac{181}{32} - \frac{1214}{64} + \frac{9403}{128} - \frac{82508}{256} + \text{etc.}$$

This may not look like progress, but note that the numerators are smaller than they were in the original series for *A*, and that there are rapidly growing denominators. There is a sense in which this series isn't "as divergent" as the original one was.

Moreover, it is still alternating, so the same trick will work again (and again, and again). Take the differences again (switching back to the series for A rather than for $A/2$ that he used in the first step) , as before, and find that they are

$$\alpha = 18/8, \quad \beta = 81/16, \quad \gamma = 456/32, \quad \delta = 3123/64, \quad \text{etc.}$$

Using these, we can get yet another series for A:

$$A = \frac{7}{8} - \frac{18}{32} + \frac{81}{128} - \frac{456}{512} + \frac{3123}{2048} - \frac{24894}{8192} + \text{etc.}$$

The next iteration of this process gives

$$A - \frac{5}{16} = \frac{81}{256} - \frac{132}{2048} + \frac{771}{16384} - \frac{4122}{131072} + \text{etc.}$$

so, telescoping a bit, and then neglecting the terms represented by the "etc.", he claims

$$\begin{aligned} A &= \frac{5}{16} + \frac{512}{2048} + \frac{2046}{131072} + \text{etc.} \\ &= \frac{38015}{65536} \\ &= 0.580. \end{aligned}$$

This is an example of what we now call an "asymptotically convergent" series. For a while, the partial sums seem to be converging, but then they swerve away from what seemed to be the limit and diverge. Modern readers may have seen them in other contexts like Euler-Maclaurin series, and they were of great interest to important 20th century mathematicians like G. H. Hardy. [H]

With some more work, not shown in the article, Euler tells us that the series can be shown to to have the value 0.59634739, but the editors of the *Opera Omnia* tell us that this last 9 should be a 6.

This is the key result of this paper, but Euler understands that some readers might not be convinced that he hasn't made any mistakes. So, he solves the same problem several other ways. For example, he finds diverging series for $1/A$ and $\log A$, and finds that similar methods also lead to a value of A near 0.59. He finds ways to write A and $1/A$ as continued fractions and evaluates those continued fractions to get still more estimates consistent with the ones before.

One of his more interesting solutions involves differential equations and infinite series. He writes

$$s = x - 1x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \text{etc.}$$

and plans to evaluate it in the case $x = 1$. Differentiating gives

$$\frac{ds}{dx} = 1 - 2x + 6xx - 24x^3 + 120x^4 - \text{etc.}$$
$$= \frac{x-s}{xx}.$$

Euler loves differential equations. He rewrites this as $ds + \frac{sdx}{xx} = \frac{dx}{x}$, solves for s, and (reminding us that e is the number whose hyperbolic logarithm equals 1, that is e is what we expect it to be) gets

$$s = e^{\frac{1}{x}} \int \frac{e^{\frac{-1}{x}} dx}{x}.$$

Substituting $x = 1$ into the definition of s gives us our alternating hyperbolic series on the left. On the right hand side, we see that the variable x is being over-used, typical in the 18th century, and the substitution $x = 1$ should only be made outside the integral, not inside. He gets

$$1 - 1 + 2 - 6 + 12 - 120 + \text{etc.} = e \int \frac{e^{\frac{-1}{x}} dx}{x}$$

where the integral is taken to be between 0 and 1. Euler applies elementary numerical methods, evaluating the integrand at ten values between 0 and 1 and adding them up, and estimates that $A = 0.59637255$, consistent with his other estimates of A, but containing some small calculation errors explained in the *Opera Omnia*.

By the end of the article, Euler has estimated A at least six very different ways, and every time he gets the same estimate. When such different analyses all lead to the same conclusion, it is easy to understand why mathematicians of Euler's time believed in the utility of interesting numbers, and could believe that numbers "beyond infinity" might be negative.

References:

[H] Hardy, G. H., *Divergent Series*, Oxford University Press, New York, 1949.

[E247] Euler, Leonhard, De seriebus divergentibus, *Novi Commentarii academiae scientiarum Petropolitanae* **5**, (1754/55) 1760, p. 205-237, reprinted in *Opera Omnia* Series I vol 14 p. 585-617. Available through The Euler Archive at www.EulerArchive.org.

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How Euler Did It is updated each month.