Preface

This book provides an inquiry-based introduction to advanced Euclidean geometry. It can be used either as a computer laboratory manual to supplement a course in the foundations of geometry or as a stand-alone introduction to advanced topics in Euclidean geometry. The geometric content is substantially the same as that of the first half of the classic text *Geometry Revisited* by Coxeter and Greitzer [3]; the organization and method of study, however, are quite different. The book utilizes dynamic geometry software, specifically GeoGebra, to explore the statements and proofs of many of the most interesting theorems in advanced Euclidean geometry. The text consists almost entirely of exercises that guide students as they discover the mathematics and then come to understand it for themselves.

Geometric content

The geometry studied in this book is Euclidean geometry. Euclidean geometry is named for Euclid of Alexandria, who lived from approximately 325 BC until about 265 BC. The ancient Greeks developed geometry to a remarkably advanced level and Euclid did his work during the later stages of that development. He wrote a series of books, called the *Elements*, that organize and summarize the geometry of ancient Greece. Euclid's *Elements* became by far the best known geometry text in history and Euclid's name is universally associated with geometry as a result.

Roughly speaking, *elementary* Euclidean geometry is the geometry that is contained in Euclid's writings. Most readers will already be familiar with a good bit of elementary Euclidean geometry since all of high school geometry falls into that category. *Advanced* Euclidean geometry is the geometry that was discovered later—it is geometry that was done after Euclid's death but is still built on Euclid's work. It is to be distinguished from *non-Euclidean geometry*, which is geometry based on axioms that are different from those used by Euclid. Throughout the centuries since Euclid lived, geometers have continued to develop Euclidean geometry and have discovered large numbers of interesting relationships. Their discoveries constitute advanced Euclidean geometry and are the subject matter of this text.

Many of the results of advanced Euclidean geometry are quite surprising. Most people who study them for the first time find the theorems to be amazing, almost miraculous, and value them for their aesthetic appeal as much as for their utility. I hope that users of this book will come to appreciate the elegance and beauty of Euclidean geometry and better understand why the subject has captivated the interest of so many people over the past two thousand years.

The book includes a study of the Poincaré disk model for hyperbolic geometry. Since this model is built within Euclidean geometry, it is an appropriate topic for study in a course on Euclidean geometry. Euclidean constructions, mostly utilizing inversions in circles, are used to illustrate many of the standard results of hyperbolic geometry.

Computer software

This is not the kind of textbook that neatly lays out all the facts you should know about advanced Euclidean geometry. Instead, it is meant to be a guide to the subject that leads you to discover both the theorems and their proofs for yourself. To fully appreciate the geometry presented here, it is essential that you be actively involved in the exploration and discovery process. Do not read the book passively, but diligently work through the explorations yourself as you read them.

The main tool used to facilitate active involvement and discovery is the software package GeoGebra. It enables users to explore the theorems of advanced Euclidean geometry, to discover many of the results for themselves, and to see the remarkable relationships with their own eyes.

The book consists mostly of exercises, tied together by short explanations. The user of the book should work through all the exercises while reading the book. That way he or she will be guided through the discovery process. Any exercise that is marked with a star (*) is meant to be worked on a computer, using GeoGebra, while the remaining exercises should be worked using pencil and paper. No prior knowledge of GeoGebra is assumed; complete instructions on how to use GeoGebra are included in Chapters 1 and 3.

GeoGebra is open source software that can be obtained free of charge from the website www.geogebra.org. That the software is free is important because it means that every student can have a copy. I believe it is essential that all students experience the discovery of geometric relationships for themselves. When expensive software packages are used, there is often only a limited number of copies available and not every student has access to one. Every student can have GeoGebra available all the time.

One of the best features of GeoGebra is how easy it is to use. Even a beginner can quickly produce intricate diagrams that illustrate complicated geometric relationships. Users soon learn to make useful tools that automate parts of the constructions. To ensure that every user of this book has the opportunity to experience that first hand, the reader is expected to produce essentially all the diagrams and illustrations. For that reason the number of figures in the text is kept to a minimum and no disk containing professionally-produced GeoGebra documents is supplied with the book.

GeoGebra is rapidly becoming the most popular and most widely used dynamic software package for geometry, but it is not the only one that can be used in conjunction with this text. Such programs as Geometer's Sketchpad, Cabri Geometry, Cinderella, and Geometry Expressions can also be utilized. The instructions that are included in Chapters 1 and 3 are specific to GeoGebra, but the rest of the book can be studied using any one of the programs mentioned.

Proof

A major accomplishment of the ancient Greeks was the introduction of logic and rigor into geometry. They *proved* their theorems from first principles and thus their results are more certain and lasting than are mere observations from empirical data. The logical, deductive aspect of geometry is epitomized in Euclid's *Elements* and proof continues to be one of the hallmarks of geometry to this day.

Until recently, all those who worked on advanced Euclidean geometry followed in Euclid's footsteps and did geometry by proving theorems, using only pencil and paper. Now that computer programs such as GeoGebra are available as tools, we must reexamine the place of proof in geometry. Some might expect the use of dynamic software to displace the deductive approach to geometry, but there is no reason the two approaches cannot enhance each other. I hope this book will demonstrate that proof and computer exploration can coexist comfortably in geometry and that each can support the other.

The exercises in this book will guide the student to use GeoGebra to explore and discover the statements of the theorems and then will go on to use GeoGebra to better understand the proofs of the theorems as well. At the end of this process of discovery the student should be able to write a proof of the result that has been discovered. In this way the student will come to understand the material to a depth that would not be possible if just computer exploration or just pencil and paper proof were used and should come to appreciate the fact that proof is an integral part of exploration, discovery, and understanding in mathematics.

Not only is proof an important part of the process by which we come to discover and understand geometric results, but the proofs also have a subtle beauty of their own. I hope that the experience of writing the proofs will help students to appreciate this aesthetic aspect of the subject as well.

In this text the word "verify" will be used to describe the kind of confirmation that is possible with GeoGebra. Thus to *verify* that the angle sum of a triangle is 180° will mean to use GeoGebra to construct a triangle, measure its three angles, calculate the sum of the measures, and then to observe that GeoGebra reports that the sum is always equal to 180° regardless of how the size and shape of the triangle are changed. On the other hand, to *prove* that the angle sum is 180° will mean to supply a written logical argument based on the axioms and previously proved theorems of Euclidean geometry.

Two ways to use this book

This book can be used as a manual for a computer laboratory that supplements a course in the foundations of geometry. The notation and terminology used here are consistent with *The Foundations of Geometry* [11], but this manual is designed to be used alongside any textbook on axiomatic geometry. The review chapter that is included at the beginning of the book establishes all the necessary terminology and notation.

A class that meets for one three-hour computer lab session per week should be able to lightly cover most of the text in one semester. When the book is used as a lab manual, Chapter 0 is not covered separately, but serves as a reference for notation, terminology, and statements of theorems from elementary Euclidean geometry. Most of the other chapters can be covered in one laboratory session each. The exceptions are Chapters 6 and 10, which are quite short and could be combined, and Chapter 11, which will require two or three sessions to cover completely.

A course that emphasizes Euclidean geometry exclusively will omit Chapter 14 and probably Chapter 13 as well, since the main purpose of Chapter 13 is to develop the tools that are needed for Chapter 14. On the other hand, most instructors who are teaching a course that covers non-Euclidean geometry will want to cover the last chapter; to do so it will probably be necessary to omit many of the applications of the Theorem of Menelaus. A thorough coverage of Chapter 14 will require more than one session.

At each lab session the instructor should assign an appropriate number of GeoGebra exercises, determined by the background of the students and the length of the laboratory session. It should be possible for students to read the short explanations during the session and work through the exercises on their own. A limited number of the written proofs can be assigned as homework following the lab session.

A second way in which to use the book is as a text for an inquiry-based course in advanced Euclidean geometry. Such a course would be taught in a modified Moore style in which the instructor does almost no lecturing, but students work out the proofs for themselves. A course based on these notes would differ from other Moore-style courses in the use of computer software to facilitate the discovery and proof phases of the process. Another difference between this course and the traditional Moore-style course is that students should be encouraged to discuss the results of their GeoGebra explorations with each other. Class time is used for student computer exploration and student presentations of solutions to exercises. The notes break down the proofs into steps of manageable size and offer students numerous hints. It is my experience that the hints and suggestions offered are sufficient to allow students to construct their own proofs of the theorems. The GeoGebra explorations form an integral part of the process of discovering the proof as well as the statement of the theorem. This second type of course would cover the entire book, including Chapter 0 and all the exercises in all chapters.

The preparation of teachers

The basic recommendation in *The Mathematical Education of Teachers* [2] is that future teachers take courses that develop a deep understanding of the mathematics they will teach. There are many ways in which to achieve depth in geometry. One way, for example, is to understand what lies beneath high school geometry. This is accomplished by studying the foundations of geometry, by examining the assumptions that lie at the heart of the subject, and by understanding how the results of the subject are built on those assumptions. Another way in which to achieve depth is to investigate what is built on top of the geometry that is included in the high school curriculum. That is what this course is designed to do.

One direct benefit of this course to future high school mathematics teachers is that those who take the course will develop facility in the use of GeoGebra. Dynamic geometry software such as GeoGebra will undoubtedly become much more common in the high school classroom in the future, so future teachers need to know how to use it and what it can do. In addition, software such as GeoGebra will likely lead to a revival of interest in advanced Euclidean geometry. When students learn to use GeoGebra they will have the capability to investigate geometric relationships that are more intricate than those studied in the traditional high school geometry course. A teacher who knows some advanced Euclidean geometry will have a store of interesting geometric results that can be used to motivate and excite students.

Do it yourself!

The philosophy of these notes is that students can and should work out the geometry for themselves. But students will soon discover that many of the GeoGebra tools they are asked to make in the exercises can be found on the world wide web. I believe students should be encouraged to make use of the mathematical resources available on the web, but that they also benefit from the experience of making the tools for themselves. Downloading a tool that someone else has made and using it is too passive an activity. Working through the constructions for themselves and seeing how the intricate constructions of advanced Euclidean geometry are based on the simple constructions from high school geometry will enable them to achieve a much deeper understanding than they would if they simply used ready-made tools.

I believe it is especially important that future high school mathematics teachers have the experience of doing the constructions for themselves. Only in this way do they come to know that they can truly understand mathematics for themselves and that they do not have to rely on others to work it out for them.

The question of whether or not to rely on tools made by others comes up most especially in the last chapter. There are numerous high-quality tools available on the web that can be used to perform constructions in the Poincaré disk. Nonetheless I think students should work through the constructions for themselves so that they clearly understand how the hyperbolic constructions are built on Euclidean ones. After they have built rudimentary tools of their own, they might want to find more polished tools on the web and add those to their toolboxes.

Acknowledgments

I want to thank all those who helped me develop this manuscript. Numerous Calvin College students and my colleague Chris Moseley gave useful feedback. Gerald Bryce and the following members of the MAA's Classroom Resource Materials Editorial Board read the manuscript carefully and offered many valuable suggestions: Michael Bardzell, Salisbury University; Diane Hermann, University of Chicago; Phil Mummert, Taylor University; Phil Straffin, Beloit College; Susan Staples, Texas Christian University; Cynthia Woodburn, Pittsburgh State University; and Holly Zullo, Carroll College. I also thank the members of the MAA publications department, especially Carol Baxter and Beverly Ruedi, for their help and for making the production process go smoothly. Finally, I thank my wife Patricia whose patient support is essential to everything I do.

Gerard A. Venema
venema@calvin.edu
April, 2013