Notes on the Sections

Microeconomics (Mary H. Lesser and Warren Page)

Microeconomics is the study of a market's economy. The market for a good or service is said to be in *equilibrium* at a price when the quantity demanded equals the quantity supplied. This precalculus-based article provides an introductory overview of key notions in microeconomics. It shows how consumer behavior and firms' profit-optimizing decisions give rise to a market's demand and supply curves.

Section 2 (Supply and Demand) Algebraic and graphical representations of demand and supply functions are used to examine properties related to a market's equilibrium. The problems illustrate, perhaps surprisingly, what happens at equilibrium when a per-item tax is imposed on the supplier, and when the same tax is levied on the buyer. The notion of elasticity is used to explain why the fraction of the tax borne by the buyer is the same in each scenario, as is the fraction of the tax borne by the supplier.

Section 3 (Production Costs and Profit) and Section 4 (Consumer Behavior and Budget Constraints) show that supply and demand curves are outcomes of optimization processes. In Problem 3.3, for example, a firm's approximate supply curve is determined based on its labor cost, fixed and variable production costs, and profit. Section 4 considers consumers with M dollars to spend on a combination of x units of a good X and y units of a good Y. The most satisfying combination is determined by maximizing their utility $U(x, y) = xy$ subject to the budget constraint $xp_x + yp_y = M$, where p_x and p_y are the respective unit prices of X and Y. In Problem 4.3, the consumer's maximized utility is used to produce an approximate demand curve that, with the approximate supply curve of Problem 3.3, determines the market conditions at equilibrium for good X. Problem 4.4, an added option, requires the partial derivatives of $U(x, y) = \frac{3}{4}xy^4$.

Scenarios Involving Marginal Analysis (Julie Glass, Lynn Paringer, Jane Lopus)

The term "marginal" is used in economics to connote the change in a function due to a one-unit increase in its independent variable. For example, marginal cost means the change in total cost from producing one more unit, and marginal revenue is the change in total revenue from selling one more unit. Since a function's derivative at a value approximates the "marginal" at that value, economists use marginal analysis in situations where exact data is unavailable and where approximations suffice for making decisions. In this article, which requires single-variable calculus, the individual(s) in each scenario use marginal decision-making to solve problems in their economic undertakings.

Section 2 (Economic Industrial Espionage) Ingrid creatively uses analytic and graphical properties of marginal cost to sketch the graphs and determine the equations of a firm's variable cost function, total cost function, and average variable cost function. For another firm, she uses marginal cost to approximate the change in total cost, and then estimates the change in that firm's profit based on its expected marginal revenue.

Section 3 (Monopolistic Competitor) Physician Phyllis uses her demand function (number q of patients who pay p dollars for annual coverage) and total cost function to determine how many patients at what price will maximize her profit. Soon after, Phyllis uses her marginal revenue at maximum profit to decide if she should participate in a government program that pays her \$ 700 for each government patient she accepts. After establishing that she should participate, Phyllis needs to determine how many private and government patients to cover in order to calculate her new maximize profit.

Section 4 (Firm's Start Up) Sam and Stuart use a sampling method to estimate a demand function for their newly patented game. From their data, they create a marginal-cost function to estimate their variable-cost function and at what quantity their average variable cost is minimized. To better appreciate how demand changes with price, they use the price elasticity of demand to determine the effect on demand if they raise the price 3% per unit. Finally, Sam and Stuart explore the merits of increasing their fixed costs (advertising, rent) versus their anticipated corresponding increase in demand.

Intermediate Macroeconomics Theory (Michael K. Salemi)

This overarching survey shows what undergraduate mathematics is used in intermediate macroeconomics. The Assignments show the mathematics used in particular contexts, and the Advanced Projects illustrate additional mathematical skills that advanced students would employ The problems use algebra (mainly) through calculus, most of which can be done by students who have no knowledge of economics. But this doesn't mean that the problems are all easy.

Assignment 2.1 (Growth Accounting) concerns the construction and interpretation of macroeconomic data. Algebraic and graphical properties of exponential and logarithmic functions are used to compute and interpret compound growth rates for time-series data. The Advanced Project uses sigma notation and partial differentiation, in the method of least squares, to derive the linear regression equation for Gross Domestic Product data.

Assignment 2.2 (Production Functions and Labor Demand) introduces the Cobb-Douglas production function $Y = AK^{1-\alpha}L^{\alpha}(A > 0$ and $0 < \alpha < 1)$ that models a firm's output Y based on its capital cost K and labor cost L. Single-variable calculus is required to explore issues such as what quantity of labor at given wages that a profit-maximizing firm would employ. The Advanced Project repeats the Assignment for the production function $Y = A[\alpha L^{-\beta} + (1-\alpha)K^{-\beta}]^{-1/\beta}$, where $0 < \alpha < 1$ and $-1 < \beta \neq 0$.

Assignment 2.3 (Solow Growth Model) considers the long-run growth trajectory of a developed economy. Algebra only is needed to investigate a system of seven equations and four parameters that describes an economy. After expressing the state of the economy by a single equation, the growth rates are determined for capital, labor, output, and consumption. The Advanced Project uses a spreadsheet to exploit the recursive nature of the model. For given initial parameter values, one examines how the capital-labor relation, consumption per worker, and output per worker evolve from arbitrary initial conditions to their equilibrium values.

Assignment 2.4 (Short-run Model) examines the short-term deviations of an economy's variables (X) from their steady-state (long-run) values (X_S). The short-term model consists of two sectors: the IS sector determines balance between investment and savings, and the LM sector determines balance between the money supply and money demand. The IS sector's equations are combined into a single equation (called the IS schedule) that expresses output Y as a decreasing linear function of interest rate r. The LM equation $M = PLY^{e}r^{-f}$ $(L, e, f > 0)$ needs to be linearized, and $M - M_s = \frac{\partial M}{\partial P}(P - P_S) + \frac{\partial M}{\partial Y}(Y - Y_S) + \frac{\partial M}{\partial r}(r - r_S)$ recast as an equation (termed the LM schedule) that expresses r as an increasing linear function of Y. Issues considered include by what amount the interest rate must rise to offset the increase in money demand caused by a one percent increase in price level. The Advanced Project involves solving the IS and LM schedules for the model's equilibrium solution (Y, r), and predicting the changes in Y and r that result from changes in other variables. Everything, other than linearizing the LM equation, can be done using algebra. Presenting the LM's equation in its linearized form would render algebra sufficient for all parts of the Assignment and Advanced Project.

Assignment 2.5 (Business Cycle Model) focuses on monetary policy. The model, a system of three equations in six variables and three parameters, examines what interest rate r_t the Federal Reserve should choose in order to maintain the initial long-term equilibrium value of output Y_t or inflation π_t (or both) in the face of a change in spending or in inflation. In the Advanced Project, the Federal Reserve is assumed to follow a weighted policy $r_t = \theta\pi_t + \phi Y_t$. The Advanced Project also describes how to set up a spreadsheet and explore, for combinations of θ and ϕ, the long-run values of Y_t and π_t due to changes in spending or inflation. Everything in the Assignment and Advanced Project can be done using algebra.

Assignment 2.6 (Monetary Policy game) compares the effects on the economy's output and inflation when the Federal Reserve conducts monetary policy by using discretion (tailored to each situation) and when it is governed by a policy rule (applicable to every situation). Each scenario is treated as a two-person game between the private sector and Federal Reserve, whose objective function describes its level of preference for output and inflation. The Advanced Project repeats the Assignment's analysis for a new objective function. Everything can be done using algebra; where calculus is used to optimize quadratic functions, this can be done algebraically by determining a parabola's vertex. Although the interpretation and analysis of a few calculations would be beyond those with no knowledge of economics, explanations are included in the Solutions section.

Closed Linear Systems (Warren Page and Alan Parks)

An economy governed by linear equations is said to be *linear*. A *closed* economy is one that is self-contained in some sense – as, for example, when production equals consumption, or expenditures equal income. This article describes three closed, linear economies and the manner by which each attains equilibrium. Matrix operations and systems of linear equations in matrix form suffice to understand the text and handle the problems other than Problems 2.7–2.9, whose guided exercises lead to proofs of the theorems on which the text is based. These exercises require knowledge of the Monotone Convergence theorem (a monotone series converges if and only if it is bounded) and the meaning of a convergent infinite series.

Section 2 (Production Adjustment and Price Adjustment Models) The production adjustment model describes how production adjusts to satisfy consumption. The price adjustment model describes how price adjusts to balance income and the cost of labor. The models are combined to establish the *equilibrium principle*: the total spent on consumption equals the total spent on labor.

Section 3 (Normalized Leontief Model) The production adjustment and price adjustment models are incorporated into a single model by treating labor as a good whose total consumption and price are its usage and cost. The model's units are adjusted so that each good's total one-unit production is consumed. The investigation also leads to properties of probability vectors and Markov matrices.

Mathematics in Behavioral Economics (Michael Murray)

Section 2 (Fairness) uses a two-player game to illustrate how attitudes about fairness influence economic behavior, and to challenge standard economic theory's assumption that individuals care only about their benefits. Both players know the game's rules and that they will play only once with their unknown partner. They will divide a fixed amount of money if the second player accepts the fraction of that money offered by the first player. Algebra only is needed to discover what offer the second player would accept, and what the first player would offer knowing what the second player would accept.

Section 3 (Probabilities) Standard economic theory assumes rational behavior and the incorporation of new information in accord with the laws of mathematical probability. However, cognitive scientists have demonstrated that our brains' natural responses to new information are at odds with Bayes' rule. This section discusses systematic ways that people's probability assessments violate the mathematical rules of probability and their possible economic consequences.

Section 4 (Decisions Under Uncertainty) contrasts two models for explaining choices made under uncertainty. Expected utility theory used in standard economics assumes individuals make choices that will maximize the mathematical expectation of their utility functions (which represent preferences among outcomes). Since people's choices are often influenced by their risk-averse perception of immediate circumstances, rather than by maximizing expected well being, behavioral economists describe behavior by prospect theory, which uses a value function whose domain is the losses and gains of outcomes facing an individual. Everything in this section can be explored using algebra and the notion of mathematical expectation $E(z) = (1 - p)z_1 + pz_2$.

Section 5 (Inter-temporal Discounting) Two methods are compared of formalizing how people discount future costs and benefits relative to immediate costs and benefits. Economists using exponential discounting, based on a constant rate of discounting, assume inter-temporal consistence: decision makers will choose the same alternative for the future that they originally chose. Behavioral economists use hyperbolic discounting, which encompasses exponential discounting by including a parameter that characterizes people's observed tendency to be more present biased and cost averse. The problems involving hyperbolic discounting examine whether people will make or regret making the same choices for the future that they made earlier.

The introduction to exponential discounting shows that $dW(t)/dt = rW(t)$ yields $W(t) = W_0 e^{-rt}$, and uses $\int_0^\infty f(t)e^{-rt}\,dt$ to represent the present discounted value of lifetime benefit of $f(t)$. Nevertheless, everything else in Section 5 other than Problem 5.3, which requires simple differentiation, can be done algebraically using exponential functions and finite geometric series.

Econometrics (Ray Jean B. Goodman)

This article considers aspects of linear regression and its applications in econometrics. Section 2 (Linear Regression) uses the method of least squares to determine estimators b_0 and b_1 of β_0 and β_1 in a simple two-variable model. Problem 2.1 requires sigma notation and differentiation to minimize $L = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - b_0 - b_1 x_i)^2$ with respect to b_0 and b_1. Problem 2.2 asks to represents the system of n equations $e_i = y_i - b_0 - b_1 x_i$ in matrix form $\mathbf{e} = \mathbf{y} - X\mathbf{b}$, where $\mathbf{e}$, $\mathbf{y}$, $\mathbf{b}$ are column vectors and X is an $n \times 2$ matrix. Then $L = \mathbf{e^t e}$, and setting the vector derivative $\partial L/\partial \mathbf{b}$ equal to the column vector of zeros yields $X^t\mathbf{y} = X^t X\mathbf{b}$. Thus, $\mathbf{b} = (XX^t)^{-1}X^t\mathbf{y}$ for conditions on X under which matrix $X^t X$ is nonsingular.

Section 3 (Multiple Linear Regression) considers linear regression for k independent variables. Problem 3.1 asks to represent the system of n equations $e_i = y_i - b_0 - b_1 x_{i1} - b_2 x_{i2} - \cdots - b_k x_{ik}$ in matrix form and show that $L = \mathbf{y^t y} - 2\mathbf{b^t}X^t\mathbf{y} + \mathbf{b^t}X^t X\mathbf{b}$.

Section 4 (Elasticity and Functional Forms) The elasticity of a function $y = f(x)$ may be interpreted as the % change in y due to a 1% increase in x. Problem 4.1 requires calculus to show that elasticity can be approximated by $xf'(x)/f(x)$ and expressed as $d(ln\, f(x))/d(ln\, x)$. The section's other problems use only logarithmic and semilogarithmic transformations of functions and linear regression on their transformed data. They investigate relevant economic questions such as the elasticity of output with respect to labor, the compound growth rates of the M1 and M2 money supply, and at what rate the labor force has grown.

Section 5 (Applications in an Econometrics Course) This section of worked-out problems illustrates graphical and statistical applications in an econometrics course, including analyses related to assessing statistical significance of a regression equation's estimators and the equation as a whole.

The Portfolio Problem (Kevin J. Hastings)

Financial economics, in large part, studies the interrelationships of financial variables such as the fair market price of an asset based on its risk, and the balance of a portfolio's risk relative to its expected return.

Section 2 (Average Rate of Return and Risk) uses the daily prices of four stocks to introduce the article's needed concepts. The sample mean and variance of the ith stock's prices are respective estimates for the stock's theoretic expected return (mean μ_i) and risk (variance σ_i^2). If w_i is the fraction of money invested in the ith asset, the portfolio's expected return μ and risk σ^2 are w_i-weighted sums of its individual assets' risk-return measures. The problems consider the maximum rate of return for a three-stock portfolio, and compare the minimized variances of a two-stock portfolio when the stocks' daily returns are independent and when their dependence is characterized by a given correlation coefficient. The need to minimize σ^2, expressed as a quadratic function of w_1, can be done using calculus or by algebraically determining the vertex of a parabola.

Section 3 (Solving the Optimization Problem) The objective is to maximizes a kind of portfolio net value $\mu - a\sigma^2$, where the constant $a \geq 0$ characterizes risk aversion. For a portfolio of four assets, Problem 3.1 uses the Lagrangian to maximize $f(w_1, w_2, w_3, w_4) = \sum_{i=1}^4 w_i\mu_i - a\sum_{i=1}^4 w_i^2\sigma_i^2$ subject to the constraint $g(w_1, w_2, w_3, w_4) = w_1 + w_2 + w_3 + w_4 - 1 = 0$. For the solution's optimally determined weights, $\sum_{i=1}^4 w_i\mu_i$ is called the *portfolio's risk-averse expected return*. The risk-averse expected returns for the other portfolio problems can be obtained by substituting the given data into the expressions for the weights obtained in Problem 3.1's solution.

Section 4 (The Portfolio Separation Theorem) Problem 3.1's solution is used to establish the Portfolio Separation theorem: "For a portfolio with a risk-free asset, the ratios wi/wj of the optimally determined weights are independent of the investor's risk aversion." Problem 3.1's solution can be used to determine if the theorem holds for a portfolio of three stocks and a risk-free asset (e.g., a bond). However, in Problem 4.3, the Lagrangian with two multipliers is needed to determine if the theorem holds for a portfolio of four stocks and a risk-free asset.

Topics in Modern Finance (Frank Wang)

Section 2 (Statistics Background) The annual closing prices of IBM and HP stocks are used to introduce the needed statistical concepts: the mean, variance, covariance, and correlation coefficient. The problems require only numerical computation.

Section 3 (Modern Portfolio Theory) An asset's returns are used to characterize its risk (measured by the variance) and expected return (measured by the mean). The risk and expected return of a portfolio of assets are weighted sums of its individual assets' risk-return measures. Section 3.1 explores a two-asset portfolio and begins to hint at the rational for diversification. The problems, which require minimizing quadratic functions of x, can be done using calculus or by algebraically determining the vertex of a parabola. Thus, Section 2 may be combined with Section 3.1 as a richer exploration for students without calculus. Section 3.2 requires partial derivatives and Lagrange's method to explore a three-asset portfolio, and Section 3.4 requires matrix algebra for dealing with multi-asset portfolios.

Section 4 (The Capital Asset Pricing Model) is used to determine the expected return of an asset based on how risky it is. In Problem 4.1, algebra suffices to derive the Capital Market Line equation that expresses a portfolio's expected return $\mu = r + m\sigma$, where r is the rate of a risk-free asset (e.g., bond) and m is the Market's (expected return – r) per unit of Market risk. Problem 4.2 requires single-variable calculus to derive the Security Market Line equation that expresses a stock's expected return as a positive linear function of β, a measure of the stock's price covariate with the market.

Section 5 (The Black-Scholes Formula) Knowledge of mathematical probability is required to determine the exact value of a risky asset called an "option" (a contract for the right to buy or sell a stock at a fixed price within a specified time interval). In the problems, computer-simulated trajectories of a stock's prices over the year's 254 trading days are used to estimate the stock's future price. The current value of a stock's option obtained from simulation is compared with that from the Black-Scholes formula. The Black-Scholes formula also is used to estimate the implied volatility of a stock based on its option's market price.

Section 6 (Afterwords) examines some of assumptions underlying standard financial models. The section's problem guides the reader to calculate the z-score of the daily change of the Down Jones Industrial Average to illustrate why the normal distribution is inadequate to characterize the behavior of the market. And it shows the assumption that decision makers act rationally needs to be modified in accordance with recent findings by psychologists and behavioral economists. (See, for example, Sections 3 and 5 in Michael Murrays article Mathematics of Behavioral Economics.)

Maximizing Profit with Production Constraints (Jennifer Wilson)

Mathematical economics is the subdiscipline of economics that includes the formulation and derivation of mathematical methods for analyzing and solving problems in economics. Aspects of this are illustrated in Wilson's article that, using multivariable calculus and matrices, discusses mathematical principles and methods for optimizing functions with constraints.

Section 2 (Production Functions) Partial derivatives are used to explore production functions and economically interpret some of their key properties.

Section 3 (Unconstrained Optimization) A function's matrix of second derivatives is used to investigate the relationship between the function's concavity, the existence of a unique extremum, and the second-derivative test.

Section 4 (Constrained Optimization) uses the method of Lagrange multipliers to optimize functions with one or two constraints, and shows how to modify the second-derivative test for such problems. It also discusses the economic interpretation of the Lagrange multiplier at optimal production as the approximate amount of production increase for each additional unit in the total budget.

Section 5 (Optimization with Inequality Constraints) discusses two theorems that give necessary and sufficient conditions for optimization. They are illustrated in an example whose function of three variables is subject to three inequality constraints.

At first glance, this article may appear intimidating because of the densely displayed symbols and equations. This is due in part to messy computations associated with the functions in economics, and because all derivations and examples are worked out for the reader. The sections, therefore, are straightforward to read. And their problems are accessible to students who have the aforementioned background. Although the analysis is carried out in generality, the problems can be made simpler and more user-friendly by substituting specific values for parameters. Computer-algebra software also can be used to handle some of the more involved calculations. Using such technology, instructors can generate a variety of related problems at different levels of sophistication, and students can explore more realistic optimization problems in economics.