MAA Reports

Algebra
Gateway to a Technological Future

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Contents

Preface

"The United States takes deserved pride in the vitality of its economy, which forms the foundation of our high quality of life, our national security, and our hope that our children and grandchildren will inherit ever greater opportunities. That vitality is derived in large part from the productivity of well-trained people and the steady stream of scientific and technical innovations they produce." Thus begins the executive summary of *Rising Above the Gathering Storm*, a report published by the National Academy of Sciences (NAS, 2006) in response to a request by Congress to determine the "top 10 actions … that federal policymakers could take to enhance the science and technology enterprise so that the United States can successfully compete, prosper, and be secure in the global community of the 21st century."

The NAS report is one of many in recent years designed to help the United States strengthen its mathematics and science education to compete in our increasingly "flat" world, a world in which our competitors are not just our neighbors, but anyone who has access to the marvelous technology which now ties together our entire world. Among the central recommendations of the report was to "strengthen the skills of 250,000 current K–12 teachers", particularly those teaching mathematics and science. As an ancillary to that report, the authors recommended the creation of "K–12 curriculum materials modeled on a world-class standard," this to be accomplished by a national panel and made available free of charge as a voluntary national curriculum. Other recent reports that have responded to the flat-world challenge include *Foundations for Success*, from the Mathematics Achievement Partnership of achieve.org (Achieve, 2004) and the *Guidelines for Assessment and Instruction in Statistics Education* of the American Statistical Association (ASA, 2005). Both of these reports recommended specific levels of mathematics that students in K–8 should achieve to be prepared for algebra and beyond. Of course, NCTM's 1989 *Curriculum and Evaluation Standards for School Mathematics* and its 2000 *Principles and Standards for School Mathematics* also recommend specific goals for mathematical understanding in grades K–12, while NCTM's recent *Focal Points* identifies particular topics that should be emphasized at each grade level, PreK–8. Finally, the recently created National Mathematics Advisory Panel (NMAP, 2006) is attempting to organize and interpret the results of research studies over the past few decades to determine the important mathematics that leads to success in algebra, what is known about learning, teaching, and assessing mathematics, and what teachers must know and be able to do

In the midst of all this activity, the Mathematical Association of America, with funding from the National Science Foundation, convened representatives from the mathematics and mathematics education communities across the K–16 spectrum to survey what has been learned about the teaching of algebra, to identify common principles that can serve as models for improvement, and to recommend directions for future research. This conference was held in November of 2006. Approximately 50 invited participants were divided into five working groups corresponding to five different levels of algebra instruction: (1) Early Algebra, (2) Introductory Algebra, (3) Intermediate Algebra, (4) Algebra for Prospective Teachers, and (5) College Algebra. Each group reviewed research on what was known about the teaching of algebra on that level and made suggestions for future directions that would improve both the knowledge base and the actual teaching and learning of algebra. We first summarize the basic recommendations of the five working groups and then present some general conclusions. Then we give the full reports (each written by the group's facilitator) and conclude with some historical background to set the reports in context.

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National Mathematics Advisory Panel. For current information, visit http://www.ed.gov/about/bdscomm/list/mathpanel/index.html.

I
Executive Summary

Victor Katz

Each of the five working groups produced a report, given in sections II–VI, in which they make explicit recommendations for research and action. But since many of these recommendations are related, we have grouped them into four main areas.

1. **Research on determining the central ideas of an algebra curriculum for the 21st century in each of the levels of instruction.**
2. **Research on understanding the nature of students' algebraic thinking and how they react to differing types of instruction.**
3. **Intensive professional development to make sure that the teaching staff can communicate the ideas of 1 using the results of 2.**
4. **Collaborative efforts among all the stakeholders, including teachers, mathematicians, mathematics educators, administrators, public officials, and parents, to make sure that the previous three recommendations can be implemented.**

In each area, we have summarized the relevant recommendations of the working groups. (EA – Early Algebra; In – Introductory Algebra; Im – Intermediate Algebra; APT – Algebra for Prospective Teachers; CA – College Algebra). In reading these summaries and, later, in reading the full reports of the groups, it is important to keep in mind that the first three groups make the general assumption that students will be introduced to algebraic ideas in the elementary grades and then proceed to a formal introductory algebra course in secondary school followed (perhaps after a year of geometry) by an intermediate algebra course aimed at preparation for college mathematics. The aim, then, of algebra in these three levels is not only to provide everyone with the algebra needed to succeed in virtually any career, but also to prepare an increasing number of students for careers in science and technology. It is those students who will contribute to "the steady stream of scientific and technical innovations" necessary for the U.S. to continue to prosper. Similarly, the working group on Algebra for Prospective Teachers is dealing with the education of teachers who will work at those three levels of instruction. On the other hand, the college algebra working group is aiming its recommendations at improving instruction for the large numbers of students in our community colleges and four-year colleges who never learned algebra in secondary school, even though they were exposed to it. Research has shown that, in general, these students do not enter scientific or technical careers. Nevertheless, these students also need algebra skills so they can succeed in whatever career they do enter.

1. Research on determining the central ideas of algebra
(Recommendations EA1, In1, Im1, APT2)

EA1. Develop coherent, connected early algebra content. Early algebra is an approach to educating elementary students that cultivates habits of mind that focus on the deeper, underlying structure of mathematics. These "habits of mind" include two central features: (1) *generalizing*, or identifying, expressing and justifying mathematical structure, properties, and relationships and (2) reasoning and actions based on the forms of generalizations. Preliminary research shows that when this approach is used, the potential payoffs are tremendous. But we still need a much fuller picture of the essential early algebra ideas, how these ideas are connected to the existing curriculum, how they develop in children's thinking, how to scaffold this development, and what are the critical junctures in this development. This work requires an examination of the mathematics topics at the PreK-8 levels that are truly foundational to algebra (e.g. whole numbers, rational numbers, and a few other topics), making sure that the building blocks to algebra within those topics are emphasized throughout the grade levels.

In1. Decide on the core concepts and procedures that should form the content of introductory algebra. It is frequently said of the U.S. mathematics curriculum, and the algebra curriculum in particular, that it is "a mile wide and an inch deep." In other words, there are so many topics in the curriculum that it is not possible to cover the important ones in depth, if indeed they can be identified. Interestingly, there is a close similarity among all the standard algebra texts, because no author wants to omit anything that might be important to some particular teacher or school system. Therefore, it is crucial that, through a series of research conferences, mathematicians, mathematics educators, classroom teachers, and textbook publishers should build a consensus on the content of the core of algebra for the twenty-first century. In particular, these conferences should result in identification of the underlying ideas of algebra as a way of providing a common set of principles to guide the development of curricula. (Note: The National Mathematics Panel intends to offer an explicit listing of the central ideas of algebra.)

Im1. Identify the ideas of intermediate algebra. For intermediate algebra, as well as for introductory algebra, there is an absence of a central core of the subject as conceptualized by textbook writers, administrators, policy makers, and even instructors. Although there are numerous topics that could be included in this core, it is more important to focus on the underlying ideas of algebra as a way of providing a common set of principles to guide the development of curricula. It is also important to inculcate in secondary students the same habits of mind mentioned in EA1, including especially "productive disposition," the habit of seeing mathematics as useful in solving problems.

APT2. Determine the relationship between the content and design of the abstract algebra course typically taken by future teachers of algebra and their later teaching of school algebra. New types of abstract algebra courses for teachers have been developed in recent years, but there is a need for solid research studies on their effect. The results will then enable the creation of college courses in abstract algebra more useful to prospective teachers.

2. Research on understanding the nature of students' algebraic thinking and how they react to differing types of instruction.
(Recommendations EA2, In2, In3, In6, CA4)

EA2. Understand the pervasive nature of children's algebraic thinking. Research has provided us with "existence proofs" of the kinds of algebraic thinking of which children are capable. But we still need an understanding of how pervasive this knowledge can become. That is, to what extent does it represent a norm in children's abilities? In particular, we need to understand the critical junctures at which particular kinds of learning can occur.

In2. Investigate the transition to symbolization and how teachers can effectively facilitate it. The historical transition to symbolization as we know it took about a hundred years during the sixteenth and seventeenth centuries, even though there had been various earlier attempts at symbolization in several different cultures. So we should not be surprised that our students sometimes have trouble with this transition. Thus, a study to investigate how this process takes place and what could be done to make it easier would be a significant addition to the mathematics education literature.

In3. Investigate models that promote learning for students with different needs, preparation, and backgrounds in the same classroom. New pedagogical methods including community building, group work, and inquiry learning can help all students, but we do not know the best balance between such methods and more traditional ones such as direct instruction or individual work. Projects should study the use of these methods in different ways and pay particular attention to their use in diverse classrooms.

In6. Determine what use of technology is appropriate in the introductory algebra classroom. Research has shown that graphing calculators and computer algebra systems can enhance learning and provide useful practice. We need to compile evidence of what actually happens when these are used, including what students learn with this technology that they do not learn without it and what they fail to learn when they use the technology that they learn without its use.

CA4. Determine the impact of a refocused college algebra course on student learning. The curriculum committees of both AMATYC and MAA, basing their ideas on what has been learned in recent years about college algebra through NSF support, have called for replacing the current college algebra course with one in which students address problems represented as real world situations by creating and interpreting mathematical models. There have been several studies of the effect of college algebra courses designed to meet the new MAA and AMATYC guidelines, with very positive results. Nevertheless, it is important that there also be a few extended longitudinal studies of student learning in the refocused college algebra courses, in connection with the large scale implementation envisioned in project CA1 below.

3. Intensive professional development to make sure that the teaching staff can communicate the ideas of 1 using the results of 2.
(Recommendations EA3, In4, In5, Im2, APT3, CA1)

EA3. Develop "Early Algebra Schools". These are schools that integrate a connected approach to early algebra across all grades K–5 and provide all teachers with the essential forms of professional development for implementing early algebra. The systemic change implied by these schools should involve not only elementary teachers, but also middle school teachers, principals, administrators, education officials, math coaches, parents, and even university personnel. A very critical component of such a school would be systematic and long term professional development of elementary teachers leading to sustainable change in their mathematics knowledge and pedagogical practice. These "early algebra schools", whose structure would be based on the early research of items EA1 and 2, would then serve as research sites for building the evidence base of the impact of early algebra education.

In4. Prepare and sustain teachers in implementing good instructional practices and curricular materials. There is a dearth of curricular materials for professional development of new and practicing teachers, especially materials that enable teachers to transfer their own learning into new teaching practices.

In particular, we need to develop materials that focus on deepening teachers' mathematical content knowledge so they can focus their instruction on the central ideas of algebra.

In5. Identify systemic changes needed to support teacher growth. In order to deal with the new materials mentioned in In4, teachers need more structured time during the school day for collaboration and growth. Such time is expensive, although there is evidence that it can pay off in student learning. We need to investigate a variety of models that try alternative approaches to providing such structured time and document the changes they produce.

Im2. Build capacity of the teaching corps. In order to build the capacity of our K–16 teaching corps to foster mathematical ways of thinking in students, we envision an evolutionary approach beginning with several strategic moves that, after an initial phase of testing and refinement, become self-disseminating and self-replicating. Some examples of such moves include

- Establishing collaborative communities of mathematicians, mathematics educators, and teachers that share a focus on student learning of significant mathematical ideas.
- Finding ways of opening discussions about pedagogical techniques among two and four year college faculty.
- Encouraging all college faculty to understand that their course is a potential teacher preparation course,

APT3. Determine how professional development in algebra content and pedagogy affect teachers' classroom practices. This research must begin with a careful analysis of the important algebraic concepts that should inform teachers' understanding of the mathematics they are teaching. Furthermore, we need to learn how teachers' understanding of algebra and its teaching develops from the use of different kinds of instructional materials.

CA1. Establish a large scale program to enable institutions to refocus college algebra. This would help a large number of institutions implement the new guidelines for college algebra recommended by CRAFTY, the MAA's subcommittee on Curriculum Renewal Across the First Two Years. In particular, such a program must involve substantial professional development for instructors of college algebra as well as efforts to incorporate new ideas into the textbooks.

4. Collaborative efforts among all the stakeholders, including teachers, mathematicians, mathematics educators, administrators, public officials, and parents, to make sure that the previous three recommendations can be implemented. (Recommendations Im3, APT4, CA2, CA3)

Im3. Advise policy makers on barriers and avenues to successful implementation of sound instructional practices. Because development of new ways of teaching would be wasted if they run into institutional or systemic barriers, we need to determine those institutional structures and policies that prevent the flow of innovation as well as those that channel it in the right direction. Thus mathematical researchers and practitioners must collaborate with policy makers, administrators, parents, textbook and testing companies, and the wider business community, to learn from each other their respective concerns, to create a sense of shared responsibility, and to encourage creative informed decision making.

APT4. Collaborate in teacher development. There is a need to identify strategies that successfully nurture collaboration between mathematics educators and mathematicians, the types of initiatives that encourage mathematicians to become more involved in teacher preparation, including institutional rewards for faculty engaged in this activity, and ways of quickly disseminating innovative methods of teaching abstract algebra to prospective teachers.

CA2. Establish an electronic library of exemplary college algebra resources. This would provide classroom activities, extended projects, and videos of lessons that would help instructors implement new ideas for student learning.

CA3. Establish a national resource database on college algebra. This resource would include information on funded projects, textbooks, research articles, etc. that could help widely disseminate positive results from exemplary college algebra programs.

Conclusions

This algebra conference is, of course, not the first conference or committee in recent years to consider ways of improving the teaching of algebra. Among the earlier conferences and committees are:

- The Algebra Initiative Colloquium (1993), organized by the U.S. Department of Education's Office of Educational Research and Improvement. This conference brought together many mathematicians and mathematics educators to discuss the improvement of algebra teaching. Reports from the working groups as well as papers from many of the participants were published by the Department of Education as *The Algebra Initiative Colloquium* in 1995 (Lacampagne, et al).
- The Mathematics Learning Study Committee of the National Research Council. Their charge was to explore how students in pre-K through 8th grade learn mathematics and recommend how teaching, curricula, and teacher education should change to improve mathematics learning during these critical years. Their report was published by the National Academies Press in 2001 as *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, et al).
- The Nature and Role of Algebra in the K–14 Curriculum, a conference jointly sponsored by the National Council of Teachers of Mathematics and the Mathematical Sciences Education Board of the National Research Council. The report of this meeting, including papers presented by many of the participants, was published by the National Academies Press in 1998 under the title *The Nature and Role of Algebra in the K–14 Curriculum* (National Research Council).
- The International Commission on Mathematical Instruction Study Group on the Teaching and Learning of Algebra met in 2001 and published its report in 2004 as an ICMI Study: *The Teaching and Learning of Algebra* (Stacey, et al).

Each of these conferences and committees published sets of recommendations. Some of the recommendations of these conferences have been implemented, through NSF funding and otherwise. And certainly, there are many instances where the teaching and learning of algebra has been improved. For example, given the startling increase in the number of students taking Advanced Placement AB and BC calculus exams without any decrease in the quality of the performance, we must assume that these students learned algebra quite well. And since there has been a major increase in student performance at the 4th grade level on NAEP, and a no less significant increase at the 8th grade level, we can further assume that students below high school are getting better prepared for algebra.

But, as usual, we cannot be satisfied with our successes, particularly since we are always reminded of the failures of our schools. Thus, this current conference and its working groups have made some new recommendations, as well as revived earlier ones. All of us in the mathematics education community are working toward the same goals of continued improvement in our students' understanding of algebra, especially in the extension of this understanding to the large numbers of students who in earlier times never had the opportunity of learning.

Historically, algebra was taught to the few, not the many; it was taught to a society's elite, to those who were expected to assume leadership positions. Now, the participants in this conference and in the other recent conferences all want to determine how to teach algebra to everyone. They share the belief that knowledge of algebra is in fact important for all citizens. However, this is certainly not a universal belief in this country. Is it really necessary for a large proportion of our population to be competent in algebra? Many prominent individuals have argued that since they have succeeded in life without knowing algebra, there is no reason why we should continue to "torture" our children with it.

On the other hand, there is certainly a strong argument to be made that the study of algebra not only "trains the mind", but is necessary for everyone desirous of participating in our democracy. For example, Alan Schoenfeld wrote in (Lacampagne, et al, 1995):

> There is a new literacy requirement for citizenship. Algebra today plays the role that reading and writing did in the industrial age. If one does not have algebra, one cannot understand much of science, statistics, business, or today's technology. Thus, algebra has become an academic passport for passage into virtually every avenue of the job market and every street of schooling. With too few exceptions, students who do not study algebra are therefore relegated to menial jobs and are unable often even to undertake training programs for jobs in which they might be interested. They are sorted out of the opportunities to become productive citizens in our society.

Of course, what Schoenfeld meant is that those who never learn algebra are denied entry to many segments of the job market. The "prominent individuals" mentioned above did learn algebra at one point, but believe that they do not need it in their current occupations. It would be interesting to study the work of these people and see if that is really the case. In any case, it is certainly true that the language of algebra has invaded virtually every discipline, including the social sciences and business, while there now exist machines that can automatically accomplish many of the tasks that were traditionally the main goals of teaching algebra. It has therefore become even more important that the "habits of mind" that algebra helps to teach become part of the education of everyone in our democracy.

This report is, of course, rooted in the belief that we can and should make algebra part of the curriculum for all students. But this will require a massive and sustained effort on the part of all the relevant groups: teachers, administrators, college faculty, mathematics educators, mathematicians, business leaders, parents, and so on. In particular, if we are to teach algebra to the "many", we must begin at the earliest possible time, rather than when students are teenagers. Thus, although all of these recommendations are important for the future of our country, the most important, I believe, are the recommendations of the early algebra working group. We trust that the recommendations made here will be taken seriously and will be implemented in the near future. There is a lot of work to be done.

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II
Early Algebra

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Understanding Early Algebra

Historically, school mathematics has focused on arithmetic and computational fluency in the elementary grades, followed by a largely procedural approach to algebra from middle grades onward. However, this approach has been unsuccessful in terms of student achievement (U.S. Department of Education & National Center for Education Statistics, 1998a; 1998b; 1998c) and has compromised the ability of US schools to compete internationally in mathematics (Hiebert, et al, 2005; Stigler, et al, 1999). As a result, it is now widely understood that preparing elementary students for the increasingly complex mathematics of this century requires a different approach, specifically, one that cultivates habits of mind that attend to the deeper, underlying structure of mathematics (Kaput, 1999; Romberg & Kaput, 1999) and that embeds this way of thinking longitudinally in students' school experiences, beginning with the elementary grades (NCTM 1989, 2000). This approach to elementary grades mathematics has come to be known as early algebra and, while it is difficult to define because of variations in the culture of algebra across communities and countries, there is general agreement that it comprises two central features: (1) generalizing, or identifying, expressing and justifying mathematical structure, properties, and relationships and (2) reasoning and actions based on the forms of generalizations (Lins & Kaput, 2004; Kaput, 2007). While these features transcend a particular subject area in mathematics and, indeed, are critical to all areas of mathematics in the elementary grades—arithmetic, geometry, statistics, and so forth—in practice, early algebra research has often focused on (1) the use of arithmetic as a domain for expressing and formalizing generalizations (generalized arithmetic) and (2) generalizing numerical or geometric patterns to describe functional relationships (functional thinking).

Understanding early algebra also requires understanding what it is not. In particular, it is not an 'add-on' to the existing curriculum. That is, it is not intended to be viewed as a separate set of activities that teachers (might) teach after arithmetic skills and procedures have been mastered. On the contrary, early algebra is a way of thinking that brings meaning, depth and coherence to children's mathematical understanding by delving more deeply into concepts already being taught so that there is opportunity to generalize relationships and properties in mathematics. Additionally—and equally important—early algebra is not a re-packaging of algebra skills and procedures, typically taught as a 'pre-algebra' course in the middle grades, for elementary grades. While children in elementary grades may develop some skill at symbolic manipulation, the goal is instead that they learn to reason algebraically and that they begin to acquire a symbolic, 'algebraic' language for expressing and justifying their ideas.

Importance of Early Algebra in Elementary School Mathematics

When early algebra is treated as an organizing principle of elementary grades mathematics, the potential payoffs are tremendous: (1) It addresses the five competencies needed for children's mathematical proficiency (Kilpatrick, Swafford & Findell, 2001); (2) It gives children an understanding of more advanced mathematics in preparation for concepts taught in secondary grades; and (3) It democratizes access to mathematical ideas so that more students understand more mathematics and, thus, have increased opportunity for lifelong success.

Early algebra builds the five strands of mathematical proficiency identified by the National Research Council (Kilpatrick, et al, 2001): conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. A fundamental aspect of early algebra is developing a deep conceptual understanding of operations and relationships. Through generalized arithmetic, for instance, early algebra provides opportunities for students to generalize mathematical properties such as commutativity, to understand how operations affect numbers, and to develop a relational view of equality. Through functional thinking, students become proficient at expressing how quantities co-vary in relation to each other. Moreover, these activities both require and support the development of children's procedural fluency. That is, children need arithmetic skills in order to find functional relationships or explore computations that allow them to develop generalizations about operations. Conversely, algebraic reasoning tasks provide meaningful contexts in which children can practice important arithmetic skills and procedures. Moreover, these contexts enrich children's understanding of basic operations because they allow children to build different conceptions for operations and how they behave.

Finally, because early algebra is problem-based, it develops children's strategic competence and capacity for adaptive reasoning. That is, the purpose of early algebra is not to develop isolated skills or procedures (either arithmetic or algebraic), but to explore mathematical situations that draw on students' knowledge of skills and procedures, that require active reflection, and that involve building arguments and justifying and explaining ideas. Moreover, researchers and practitioners have found that coordinating these processes leads to mathematical activity that, while more challenging, children from all academic abilities enjoy and find engaging and worthwhile.

While early algebra does require children to understand arithmetic concepts and skills, it extends beyond this to include concepts, skills and processes historically addressed in secondary grades. Thus, as young children reason algebraically, they develop important habits of mind and knowledge for mathematical success in later grades. In particular, children develop an increasingly formal, 'algebraic' language system—rooted with meaning in their natural language expressions—for describing generalizations. They acquire the early language of proof, including forms of inductive and deductive reasoning and an appreciation for general arguments and the limitations of empirical arguments. They learn to use representational tools such as tables and graphs, and they learn to generate and organize data. They develop an understanding of how to express co-variation in functional relationships (including linear, quadratic, and exponential) and how to interpret multiple representations of functions (e.g., tabular, graphical). Through reasoning algebraically, children also learn to describe, symbolize and justify properties of number and operation, including important axioms such as the commutative and associative properties of addition and multiplication and the distributive property of multiplication over addition, that are foundational to formal algebra courses in secondary grades. A fundamental early algebra conjecture is that when children have these experiences in elementary grades, over sustained periods, they develop a much deeper mathematical foundation than children whose experiences are focused primarily on calculation procedures. As a result, early algebra develops students who are better prepared for formal algebra courses in secondary grades.

Finally, because early algebra involves building mathematical understanding, not the rote acquisition of skills and procedures, it allows for the democratization of mathematical ideas so that more students understand more mathematics. Researchers and practitioners have found that children who have typically been marginalized in traditional classrooms, leading to their limited participation and success in mathematics in later grades, are often readily engaged by early algebra because it develops mathematical understanding in the context of worthwhile activity. The result is the potential for more students, including those often disenfranchised by mathematics, to be mathematically successful and, thus, have increased opportunity for lifelong success.

Results of Early Algebra Research

We now know a great deal about the kinds of mathematics that young children are capable of in the elementary grades, and we also know that early algebra challenges the developmental constraints previously placed on young learners. In particular, a number of scholars have documented the capacity of students from diverse socioeconomic and educational backgrounds to reason algebraically (see, e.g., Bastable & Schifter, 2007; Blanton & Kaput, 2004; Carpenter, Franke, & Levi, 2003; Carpenter & Levi, 2000; Carpenter, Levi, Berman, & Pligge, 2005; Carraher, & Blanton, 2007; Carraher, Brizuela, & Schliemann, 2000; Carraher & Earnest, 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006; Doherty, 2003; Falkner, Levi, & Carpenter, 1999; Kieren, 1981; Lins & Kaput, 2004; Moses & Cobb, 2001; NCISLA, 2000; NCTM, 1997; Russell, Schifter, & Bastable, 2006; Schifter, 1999; Schifter, 2006; Schifter, Bastable, Russell, Riddle, & Seyferth, 2008; Schifter, Monk, Russell, & Bastable, 2007; Tierney & Monk, 2007; Usiskin, 1997). As a result of this work, we know that, to varying degrees, children can

(1) Describe, symbolize and justify arithmetic properties and relationships;
(2) Develop an algebraic, relational view of equality;
(3) Use appropriate representational tools, as early as first grade, that will support the exploration of functional relationships in data;
(4) Identify and symbolize functional relationships;
(5) Progress from building empirical arguments to building justifications using problem contexts and learning to reason with generalizations to build general arguments; and
(6) Learn to compare abstract quantities of physical measures (e.g., length, area, volume), in order to develop general relationships (e.g., transitive property of equality) about these measures.

In addition, we have gained insight into the nature of teachers' algebraic knowledge and the forms of professional development needed to transform that knowledge as well as teachers' knowledge of practice (see, e.g., Blanton & Kaput, 2003; Blanton & Kaput, 2005a, 2005b; Blanton & Kaput, 2007; Carpenter, et al, 2004; Doerr, 2004; Franke, Carpenter & Battey, 2007; Kaput & Blanton, 2005; Kieren, 1992; Schifter, Bastable & Russell, 2008-a; Schifter, Bastable & Russell, 2008-b; Stephens, et al, 2004; Stein, Baxter & Leinhardt, 1990).

However, like research on children's algebraic thinking, teacher research on the teaching and learning of algebraic thinking has given us important understanding, but classrooms have yet to experience the types of widespread implementation that will lead to significant impacts on children's thinking in typical schools throughout the country. From this background of a young, emerging research base, we describe next several critical areas where early algebra research is needed.

Critical Needs for Future Early Algebra Research

We have identified three critical areas for future research and development in early algebra: (1) systemic change through the development of "early algebra schools", (2) identifying core algebra ideas and how they are connected to the curriculum across K–12 algebra education, and (3) understanding the pervasive nature of children's algebraic thinking.

EA1. Systemic Change: The Development of "Early Algebra Schools"

In spite of the growing knowledge base in early algebra research, much of the evidence is local in nature and based on research that has occurred in restricted settings. There are still enormous tasks ahead for implementing early algebra across all elementary grades in all schools and for documenting the full impact of a consistent, sustainable K–5 early algebra education. To this end, one major need is for systemic investments that lead to the development of "early algebra schools". By this, we mean schools that integrate a connected approach to early algebra across all grades K–5 and that provide all teachers — within a school and across a district — with the essential forms of professional development for implementing early algebra (Kaput & Blanton, 2005; Carpenter, et al, 2004). This type of systemic change is complex and should involve not only elementary teachers, but also middle school teachers who will need to understand how to access the knowledge of incoming students who have had rich early algebra experiences, as well as principals, administrators, education officials, math coaches, parents, and even university/school partnerships in which pre-service teachers learn with and are mentored by in-service teachers. Moreover, the systemic initiatives should reflect a comprehensive approach based on whole-school and whole-district participation (Blanton & Kaput, 2007).

We emphasize that elementary teachers are at the heart of this type of systemic change and in the most critical area of need. Most have little experience with the kinds of mathematics we know as early algebra and are often products of the type of school algebra instruction we need to replace. Thus, we must provide the appropriate forms of professional support that will effect sustainable change in teachers' knowledge of mathematics and knowledge of practice. This requires professional development to be embedded in systemic change, not based solely on efforts in which teachers work in isolation in their schools.

There are two fundamental reasons we see this as an urgent need in our field: (1) the need to build a comprehensive evidence base, and (2) the need to sustain early algebra education in schools for viability in children's learning. While there is much research that shows the depth of children's capacity for algebraic reasoning, this research is often based on restricted settings (e.g., specific grades) that do not reflect the full potential of K–5 early algebra education. We need qualitative and quantitative evidence of the impact of early algebra education in terms of students' mathematical success in elementary grades and later grades, but this type of evidence requires children to receive comprehensive early algebra education across grades K–5 from teachers who have received the necessary professional development and who have the necessary support systems in place. Early algebra schools, built through systemic change initiatives, would serve as critical research sites for building this evidence base.

Systemic change would also lead to sustainability in early algebra education so that children's experiences are viable, not fragmented. The potential of early algebra is lessened if its implementation is random and not connected across grades in children's thinking. Conversely, schools and districts that adopt a philosophy of early algebra education and implement that philosophy with fidelity can transform children's school mathematics experience.

EA2. Developing a Coherent, Connected Early Algebra Content

It is critical for early algebra to be deeply and explicitly connected to what children are learning so that it is not treated as an isolated topic. While much is known about early algebra content, most of it focusing on generalized arithmetic or functional thinking, what is still needed is a coherent picture of the essential early algebra ideas, how these ideas are connected to the existing curriculum, how they develop in children's thinking, how to scaffold this development, and the critical junctures in this development. For example, although research on children's functional thinking in 2nd–4th grades has led to an important understanding of appropriate forms of tasks and trajectories in children's thinking at these grades, we need to know how these ideas can be developed in earlier grades and how they support and are supported by the learning of particular arithmetic concepts. Similarly, we need to understand how to extend arithmetic, beginning in Kindergarten, so that children learn to generalize arithmetic concepts and acquire the symbolic language over a progression of grades to express their thinking.

The development of a coherent picture of early algebra is not limited to grades K–5: Critical work is needed to connect early algebra education to the mathematics children learn in middle and secondary grades. In particular, while a K–5 scope and sequence of ideas is essential, it must be framed within the context of K–12 algebra education. Although there is important ongoing work at both elementary grades and middle/secondary grades, an urgent need in algebra education is the development of common ideas, purposes, and goals through collaborative efforts by researchers who work in these different areas so that the work of early algebra informs efforts in later grades.

Finally, the results of this investment may take various forms: collaborative conferences among researchers across K–12 algebra education, the development of teacher professional development materials that would address early algebra but could also span K–12 algebra education (this would also provide a content basis for the work of (1)), or even new curricula that build algebraic thinking progressively, beginning with the elementary grades and ultimately spanning grades K–12.

EA3. Understanding the Pervasive Nature of Children's Algebraic Thinking

Early algebra research has provided us with important existence proofs of the kinds of algebraic thinking young children from diverse academic backgrounds can do. What is still needed, however, is an understanding of how pervasive that knowledge can become. That is, to what extent does it represent a norm in children's abilities? As with any learning, the knowledge children glean from a particular 'algebraic conversation' in a moment of instruction will vary. Understanding the kinds of knowledge children take from these conversations, the trajectories in that knowledge, and the critical junctures in those trajectories are important steps in identifying the pervasive nature of algebraic knowledge. In this sense, (3) can inform the development of a coherent and connected set of early algebra ideas – (2) – and, together, they can provide a research-based early algebra "curriculum" that supports the work of systemic change – (1) – with teachers, principals, and administrators.

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III
Introductory Algebra

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Established Principles

Our working group began its task by agreeing on principles that have been well established by research and that can be used as a basis for further work to improve the learning of introductory algebra. Each principle is followed by one or two citations. In general, each citation refers to a number of studies, but there is not attempt at a full review of the literature.

1. Introductory algebra has too many topics. The attempt to *cover* them all impedes learning core concepts in depth. (NCTM 2006, p. 3)
2. Lack of logical connections between core concepts and procedures detracts from learning. (NCTM 2000, p. 20 and Bransford et al. p. 236.)
3. Transition to symbolization is complex, difficult, and facilitating the transition to symbolization is not well understood. (Knuth et al. 2006, pp. 297–312.)
4. Balance is needed between procedural fluency and conceptual understanding. (NCTM 2000, p. 35, Resek and Kirshner 2006, p. 539.)
5. In order to develop deep understanding of mathematical content and student learning, teachers need more structured time for collaboration both during the school day and in sustained professional development. (NCTM 2000, p. 370 and Garet et al. 2001, p. 936.)
6. Curriculum should engage students through the use of challenging problems that are meaningful (both mathematically and in context) and substantial questions that motivate students to persevere in substantial mathematical investigations. Teachers must manage the frustration that can come with genuine thinking. (NRCIM 2004, p. 4 and Hiebert et al. 1997, pp. 8–9.)
7. Discourse among students facilitates learning through questioning, explaining, and expanding upon each other's reasoning. Writing promotes reflection, informal proof, and consolidation of learning. (NCTM 2000, pp. 60–63)
8. Appropriate use of technology can enhance learning. (NCTM 2000, p. 25, Koedinger and Corbett 2006, pp. 72–74, and Ellington 2003, p. 457.)
9. Assessment drives content. (NRC 1999, p. 3)

Important Questions

We continued our discussion by selecting some questions central to the teaching and learning of algebra. We could not answer these questions, and we felt that their answers could lead to major improvements in the learning of algebra.

1. What are the small number of core concepts and procedures that should form the content of introductory algebra?
2. How does the transition to symbolization occur, and how can we effectively facilitate it? Does this differ for children and adults?
3. What is the appropriate balance in introductory algebra between
   a. direct instruction and inquiry-based methods?
   b. procedural fluency and conceptual understanding?
   c. functions, modeling, and algebraic structures?
   d. supporting low-achieving students and challenging high-achieving students ?
4. Can community-building, inquiry-based teaching, and group learning reach students in the same classroom who are at different levels, including those who lack command of basic concepts? How do we teach algebra to special education students?
5. What use of technology is appropriate in the introductory algebra classroom? What exactly do students learn with various kinds of technology that they don't learn without, and conversely?
6. How can policymakers be convinced that they are asking too much of teachers and schools when they mandate change without providing appropriate resources and time?
7. What are appropriate ways for teachers and schools to be held accountable for student learning?

Future Research

Based on our two lists, we came up with future research directions that should be pursued that could lead to significant improvement of the learning of introductory algebra. The lists of principles and questions suggest other good areas for funding that we are not mentioning. We wanted to have a short list of key directions.

In1. Decide on the core concepts and procedures that should form the content of introductory algebra.

As noted in our first principle, there are too many topics that teachers try to cover. Research shows us that teaching students to understand and apply a smaller number of core concepts and procedures is a more effective way for them to learn. At the same time, there is no consensus on the content of the core. That consensus needs to be built through a series of conferences. Educators need to know which topics are no longer as important as they once were and which topics have risen in importance. Organizing introductory algebra around a core should make the study more coherent. The conferences must include people from partner disciplines and stakeholders from industry, as well as teachers, mathematicians and mathematics educators. The working group is especially concerned that teachers be well represented at the conferences.

The working group was not under the illusion that a curriculum built around such a core could by itself give coherence to introductory algebra. Teachers themselves must recognize the centrality of the ideas, and they must keep asking students "why" so that the students become aware that most ideas arise in a number of contexts. However, without consensus on one or possibly several usable 'cores', the majority of introductory algebra courses will remain a disconnected list of topics. And as long as teachers feel the need to spend time on each item on the list, they will not have the time needed to foster in-depth learning.

In2. Investigate the transition to symbolization, and how teachers can effectively facilitate it. Does this differ for children and adults?

In introductory algebra students must use symbols in meaningful ways. A great deal of research has looked at developing students understanding and facility with numbers, but relatively little work has been done investigating what occurs in the transition from number work to work with symbols. These investigations would be part of the basis for investigating ways teachers could effectively facilitate students' transitions.

At the community college level, we work with adults in introductory algebra. A number of teachers, both pre-service and in-service, are not able to work meaningfully with symbols. We do not know whether the transition to symbolization differs in adults and in children. We do not know whether teachers should work in different ways with older students.

In3. Investigate models that promote learning for students with different needs, preparations, and backgrounds in the same classroom.

We have seen that pedagogical methods such as community building, group work, and inquiry learning can help all students in classrooms with diverse populations grow and learn algebra. We do not know how these methods work and the best ways to use them, nor what proportions of these activities to use with more traditional ones such as direct instruction and individual work. Since these practices seem promising and teachers are asking for better ways to work in diverse classrooms, projects should be funded that study the use of these methods in different ways. Special attention should be paid to special education students and how they can best learn in diverse classrooms.

In4. Prepare and sustain teachers in implementing good instructional practices and curricular materials.

There is a dearth of curricular materials for professional development of new and practicing teachers. There are a few encyclopedic texts, but not the sort of materials that engage teachers in activities close to classroom practice. Teachers need to be able to transfer the content they are learning into their actual teaching. Materials are needed for secondary teachers at the in-service, pre-service and graduate levels. NSF was able to spark a culture that produced mathematics curricula at the K–12 level and for calculus. The same type of effort that will spur a culture of change is needed here.

Work on the college level suggests that extended professional development is needed for professors, teaching assistants, and adjuncts in order to change their teaching style and to improve the success rates of their students. Materials need to be developed that can be used with these populations also.

Teachers need to be learning key mathematical ideas and acquiring deep understanding of the mathematics they teach. It is unclear what content knowledge teachers need to support algebra learning. This needs to be investigated. It is also unclear how to enable teachers to work effectively with students from diverse cultures, languages and backgrounds. So this is another issue that requires investigation.

In5. Identify systemic changes needed to support teacher growth.

Principle 3, above, states that teachers need more structured time during the school day for collaboration and growth. Such time is expensive, although there is evidence that it can pay off in student learning. For school districts to fund such time, they need strong evidence that it will pay off and that there are no cheaper alternatives. NSF has funded projects which gave teachers structured time, but research is needed to compare a variety of models that try alternative approaches to providing such structured time and document the changes made. Also, we need studies that compare the efficacy and sustainability of mathematics specialists, mathematics coaches, teacher leaders, and study groups.

In6. Determine what use of technology is appropriate in the introductory algebra classroom. What exactly do students learn with various kinds of technology that they do not learn without, and conversely?

Research has shown that graphing calculators can enhance learning and computers can provide useful practice. Many teachers and schools remain skeptical about whether they should be used and, if so, to what extent. Studies and publications should document evidence of what actually happens when they are used. What, if anything, do students learn when they use calculators or computer algebra systems that they do not learn without them? Also, what, if anything, do students fail to learn when they do use this technology that they learn when the technology is not used?

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IV
Intermediate Algebra

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The Role of Intermediate Algebra in the High School Curriculum

Although many school districts now require a second year of algebra for all students, regardless of college plans, the working group took the position that the primary purpose of an intermediate algebra course is to prepare college-intending students for further work in mathematics. Thus the course includes both advanced topics important for later success in freshman college courses (primarily calculus), and a deeper study of topics already covered. Traditionally it covers many of the symbolic manipulation skills that students need to do well in calculus and in science and engineering courses. More recent reform versions of this course focus on understanding the properties of elementary functions through graphical and numerical observations in addition to symbolic representations, and on using functions to model phenomena. In either version the course can sometimes neglect important skills in parsing algebraic structure, interpreting symbolic form, and understanding the purpose and reasoning behind manipulations of symbols (McCallum, 2004). These skills have always been important, but are particularly needed now that many engineers and scientists use symbolic manipulation programs.

The study group did not make an attempt to give a detailed description of the content of this course, which in any case varies widely. Comprehensive studies of algebra have been done before, for example (Lacampagne, et al, 1995) and (NCTM, 2000). Rather we focused on describing the underlying ideas of algebra, and proposed an action plan to identify and implement the best approach to preparing students for further study in college mathematics.

Action Plan

We envision a three-tiered program of action:

- Identify the basic ideas of algebra
- Build capacity to focus algebra instruction on these ideas
- Advise policy makers on barriers and avenues to successful implementation of sound instructional practices.

Enactment of the program should be sensitive to the culture and contours of the U.S. educational terrain, including

- the imperative of maintaining the nation's economic competitiveness

- the pervasiveness of technology
- the mathematical requirements of other fields of academic study
- a commitment to good education for all
- the significant number of adult learners
- high stress working conditions in the nation's schools.

Im1. Identify the Ideas of Algebra

Weak coherence of algebra curricula results in part from the absence of a central core of the subject as conceptualized by textbook writers, administrators, policy makers, and even instructors. Various authors have attempted to conceptualize the components of this central core, for example (Cuoco, 1996] and (Harel, 1998). We propose a focus on the underlying ideas of algebra, rather than behavioral performance measures, as a way of providing a common set of principles to guide the development of curricula, while allowing diversity in selection of topics as appropriate for different audiences. We envision this as a research and development program involving collaboration between education researchers, mathematicians, and teachers.

The following are some examples of underlying ideas that the working group came up with during its two days of deliberations. This list is not intended to be either foundational or complete, but rather to serve as a starting point for further research. Section 3 gives examples of activities and exercises designed to foster or assess these ideas.

The idea of an algebraic expression: Recognizing that the symbols in an expression stand for numbers, and that expressions represent calculations with numbers.

The idea of an equation: Understanding that an equation is an assertion of equality between two expressions. For example, when students attempt to solve $3x + 5 = 2x - 3$ by first changing it to $x + 5 = (2/3)x - 3$, they appear to be following the mantra of "doing the same thing to both sides" and to be unaware that it is a sum and a difference that are equal.

The relation between form and function of an algebraic expression: Recognizing that different forms of algebraic expressions and equations reveal different properties of the objects they represent (functions, graphs, solutions). Performing algebraic manipulations as a strategic choice rather than obedience to a command (simplify, expand, factor). See below.

Solving equations as a process of reasoning: Understanding that the steps in solving an equation constitute a series of mathematical deductions with logical justifications; understanding the difference between transformations on expressions that preserve value, and operations on equations that preserve equality.

The following items describe habits of mind or ways of thinking that characterize an understanding of algebra.

The habit of finding algebraic representations, particularly when faced with a problem or situation that does not necessarily look mathematical.

The habit of parsing algebraic expressions. This includes understanding that algebraic expressions have a structure that they inherit from order of operations, and learning to look for structure in symbolic representations by viewing sub-parts as units whose internal structure can temporarily be ignored. For example, being able to say what is being added by the plus sign in $5 - 3/7 + 2/5$, seeing $27u^6 - 8v^3$ as a difference of two cubes, or noticing that $P\left(1+\frac{r}{12}\right)^{12n}$ is linear in P.

Anticipating the results of a calculation without doing it. For example, one should be able to predict the coefficient of x^4 in $(x + 1)^5$ without expanding it out.

Abstracting regularity from repeated calculation. See next section.

Making connections between representations and moving flexibly between different types of representation and different forms of the same type. For example, students should be able to connect a quadratic expression with a table of its values or a verbal description of exponential growth with its graph.

The next two items represent two contrasting views of the algebra curriculum: as preparation for the study of functions or as universal arithmetic.

Imagining a quantity's value varying continuously and imagining invariant relationships between covarying quantities. See next section.

Decontextualization of algebraic expressions. See next section.

Im2. Build Capacity

Our second action item is to build the capacity of our K–16 teaching corps to foster mathematical ways of thinking in students. We envision an evolutionary approach seeded by smart, strategic moves that, after an initial phase of testing and refinement, become self-disseminating and self-replicating. We give some possible examples of such moves, understanding that part of the program would be to investigate them and discover others:

(1) Equipping instructors with a framework for putting to new uses the materials they have, and for supplementing those materials with problems and assessments in ways that support an approach based on the ideas of algebra.

(2) Establishing collaborative communities of mathematicians, mathematics educators, and teachers that
- thrive through a common focus on mathematical ideas
- share a focus on student learning of significant mathematical ideas.

(3) Finding ways of opening discussions about pedagogical techniques among two and our year college faculty, for example through a focus on teacher preparation courses or on the use of technology.

(4) Encouraging faculty in two and four year colleges to understand that their course is a potential teacher preparation course.

(5) Approaching teacher professional development programs with a focus on
- deepening mathematical content knowledge and encouraging mathematical ways of thinking and habits of mind
- examining why students aren't learning, rather than what teachers are doing wrong.

Im3. Advise Policy-makers

Capacity-building is wasted if new ways of teaching run into institutional or systemic barriers. The final piece of our action program is policy-oriented scholarship on institutional structures and policies that dam up the flow of innovation and which structures and policies channel it in the right direction. In order to be more effective at influencing policy, researchers and practitioners must collaborate with policy makers, administrators, parents, textbook and testing companies, and the wider business community. The purpose of this collaboration is for the various communities to learn from one another about their respective concerns, to create a sense of shared responsibility, and to encourage creative informed decision making. Examples include collaborations among these stake-holders to

- work with social scientists to identify organizational structures that effectively influence policy
- work with government or business oriented policy organizations that have succeeded in forging large coalitions
- conduct neutral and authoritative examination of standards and their influence on curriculum design and instructional practice
- conduct mathematically focused professional development for administrators and policy makers
- investigate ways of influencing various stake-holders so that they come to recognize the need to better understand the complexities of mathematics teaching and learning
- help policy makers develop a notion of collateral damage of the decisions they make, and instructors' perceived constraints that are unintentional byproducts of policy decisions
- provide evidence to policy makers in a form they can use, for example, white papers from MSEB or the National Academies.

Examples of the Ideas of Algebra

(1) The relation between form and function

The following examples are from (McCallum, 2004).

1. After a container of ice-cream has been sitting in a room for t minutes, its temperature in degrees Fahrenheit is $a - b2^{-t} + b$, where a and b are positive constants. Write this expression in a form that
   - shows that the temperature is always greater than a. [Answer: $a + b(1 - 2^{-t})$]
   - shows that the temperature is always less than $a + b$. [Answer: $a + b - b2^{-t}$]
2. If R_1 is fixed, write the expression

$$\frac{R_1 + R_2}{R_1 R_2}$$

in a form that shows whether it is increasing or decreasing as a function of R_2. [Answer: $\frac{1}{R_1} + \frac{1}{R_2}$]

(2) Abstracting regularity from repeated calculations

This includes developing expressions or equations that represent quantities or quantitative relationships by reflecting on the numerical reasoning in which one engages to understand a situation and to answer questions about it. This sort of work should start before algebra. For example, what follows is a series of questions aimed at scaffolding students' thinking so that they reflect on the language of arithmetic to represent quantitative relationships in a situation (Pat Thompson). Each question has the same premises:

> Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B. The two fellows argued about which one stood taller over his sister. It turned out that Brother A won by 17 centimeters.
>
> Question 1: Brother A was 186 cm tall. Sister A was 87 cm tall. Brother B was 193 cm tall. How tall was Sister B?
>
> Question 2: Brother A was 186 cm tall. Sister A was 87 cm tall. Brother B was ___ cm tall. Sister B was ___ cm tall.
>
> Fill in the blanks so that everything works out.

1. What does it mean for "everything to work out"?
2. Find two pairs of numbers (i.e., four numbers) that work.
3. What must be true about ANY set of numbers so that everything works out? How do you know this?

Question 3: Brother A was ___ cm tall. Sister A was ___ cm tall. Brother B was ___ cm tall. Sister B was ___cm tall.

Fill in the blanks so that everything works out.

1. What does it mean for "everything to work out"?
2. Find two sets of four numbers (i.e., eight numbers) that work.
3. What must be true about ANY set of numbers so that everything works out? How do you know this?

As another example, the following problem [Cuoco, 2008-b] causes no difficulty with prealgebra students who understand the connection between rate, time, and distance:

> Mary drives from Boston to Washington, a trip of 500 miles. If she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back, how many hours does her trip take?

But the next problem is baffling to many of the same students a year later in algebra class:

> Mary drives from Boston to Washington, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes 18 1/3 hours, how far is Boston from Washington?

Analysis of this situation led to a method of exploiting students' ability to solve the first (prealgebra) problem to help them construct the equation whose solution settles the second (algebra) problem.

The first step is to guess at an answer, *not* in order to stumble on a right answer, but rather to focus on the steps one takes to check the guess. So, if the guess is 500 miles, then Mary takes 500/60 = 8 2/3 hours to drive down and 500/50 = 10 hours to get home. The total trip is 18 2/3 hours, so 500 is not the right answer, but we have begun to construct an algorithm for checking any guess. We ask students to be explicit about what they did to check the guess. If they are not sure, they take another guess, and another, and another, until they are able to articulate something like

> "You take the guess, divide it by 60, then divide it by 50, add your answers and see if you get 18 1/3."

The generic "guess checker" is then

$$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} = 18\frac{1}{3}$$

This gives them the equation that models the problem:

$$\frac{x}{60} + \frac{x}{50} = 18\frac{1}{3}$$

For further discussion of this topic, see (Cuoco & Manes, 2001).

(3) Covarying quantities

One of the primary roles of intermediate algebra in the secondary school is to prepare students for the study of further mathematics, especially calculus. An important way it does this is by having students build process conceptions of function definitions (Breidenbach, et al, 1992, Carlson et al, 2008, Harel & Dubinsky, 1991, Thompson, 1994c). This means that students view the function definition as producing numbers "instantaneously", which enables students once they conceive of specific functions or genres of functions from that perspective to reason about their behaviors over intervals of their domain. For example, they might reason about the behavior of $f(x)=(x-1)(x-2)(x-3)(x-4)$ in this way:

> As $x \to 1-$, each factor is negative, so the product is positive, and the product approaches 0 as x gets very close to 1.
>
> As x passes 1, $(x - 1)$ is positive while the other three terms are negative, so the product is negative and decreasing.
>
> As $x \to 2-$, the product is still negative, but it is approaching 0, so the product must have changed from decreasing to increasing somewhere between 1 and 2.
>
> And so on

Another way in which intermediate algebra can prepare students for calculus is by having them thoroughly conceptualize the idea of constant rate of change and average rate of change. If students develop the meaning that the average rate of change of a function over the interval is the constant rate of change by which a function changes over the same interval and which produces the same net change as produced by the original function, then they are positioned to understand the idea of derivative (Thompson, 1994b, Thompson, 1994a). They are also positioned to understand accumulation as happening by some function changing over its domain at some rate of change (Carlson et al, 2003, Thompson & Silverman, 2008).

(4) Decontextualization

Decontextualization comes up in a few ways (Cuoco, 2008-a]:

- When you use algebraic expressions as book-keeping mechanisms, so that the meaning lies not in the letters but in the coefficients. For example, the coefficients of $(t + h)^5$ give you the theoretical distribution when five coins are tossed. The t and h don't stand for numbers when you think this way. So too when you use

  $$\left(x+x^2+x^3+x^4+x^5\right)^3$$

  to get the distribution of sums when three dice are thrown. So, this area—generating functions—uses formal calculations to keep track of things.
- As formal identities. It's important for students to know that any identity that is established formally remains "true under any substitution" as long as the system from which you substitute has enough structural similarity to the system of polynomials in which you establish the identity (formally, polynomial calculations are "universal" in some sense). So, for example, you can show that

  $$(x-1)\left(1+x+x^2+\cdots+x^{n-1}\right)=x^n-1$$

  without assigning any meaning to x. That means that it's true when x is replaced by any real or complex number, any algebraic expression, or even by certain functions. This identity can help students learn about geometric series and about the algebra of the regular n-gon in the complex plane.
- Sometimes polynomial rings are useful as systems all by themselves, with their own internal logic and rules for calculation. This again goes back to their universal nature. For example, Intermediate Algebra students often work with complex numbers as if they are polynomials in "i", except

when you are done, you have an additional simplification rule—replace i^2 by -1. This is often considered a bad thing, but in fact, it can lead to a very general-purpose method that uses formal polynomials and that allows you to construct a system in which a polynomial equation has a root.

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V
Preparation and Professional Development of Algebra Teachers

James Fey, Helen Doerr, Robert Farinelli, Rueben Farley,
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Strategies for improving the algebra achievement of middle and high school students depend in fundamental ways on improving the content and pedagogical knowledge of their teachers. Effective efforts toward that goal depend on answers to three basic questions:

- What algebraic knowledge is required for effective teaching of the subject?
- How can pre-service teachers be given a good start on developing essential knowledge of algebra for teaching?
- How can practicing mathematics teachers advance their knowledge of algebra for teaching through reflection on their classroom experiences and participation in systematic professional development activities?

Unfortunately, there are few answers to those questions in the existing mathematics education research literature. While knowledge for teaching elementary mathematics has been studied extensively over the past decade, there are few comparable studies focused on middle and high school mathematics and algebra in particular. While there are scattered examples of innovative algebra courses for prospective teachers, we know little about the effects of those courses on teachers' knowledge, their practices in early career teaching of algebra, or the learning of their students. While there is a consistent general finding that the most effective professional development focuses on curriculum materials that in-service teachers actually use, we know little about how that finding applies to high school mathematics and algebra in particular.

Given the limited research basis for improving the preparation and professional development of middle and high school algebra teachers, we believe that highest priority should be given to research and development projects in four areas:

APT1. Identifying Knowledge Used in Algebra Teaching

Designing effective strategies for preparation and professional development of algebra teachers depends on a deep understanding of the work for which the teachers are being prepared. At the present time we know very little about the ways that teacher knowledge of algebra, student learning, pedagogy, curriculum, and assessment are used in the practice of effective teachers. Teachers' algebraic content knowledge has been the focus of research that has emphasized an important but limited number of topics including linear equations, functions, and slope. Such research rarely examines the structural path from arithmetic operations through algorithms and advanced operations that are expanded and explored in an abstract algebra course. There is more work to be done that directly examines teachers' algebraic content knowledge. Thus an important priority for further research is to examine the application of algebraic content knowledge (as well as other aspects of teachers' knowledge) in practice. How is it that effective teachers draw on their knowledge in the acts of teaching?

Important work in this direction has begun with the work of Ferrini-Mundy and colleagues (Ferrini-Mundi et al, 2005), Chazan and Herbst, and Heid and Wilson, examining aspects of the knowledge of algebra that is needed for effective teaching. In our view, further studies should include the following:

- We need observational studies of algebra teachers in action across settings that have large variations in curricular materials and student demographics. It will be useful to learn how teachers deal with the algebraic structure of number systems as they teach about symbol manipulation, how they use ideas of mathematical modeling to help students connect formal algebra to its applications, and how they help students develop symbol sense and algebraic reasoning.
- We need studies that examine curricular choices that teachers make as they are teaching algebra. It will be useful to learn how teachers identify and implement algebraic focal points in their instructional efforts.
- We need studies of teachers' strategies for assessing student learning of algebra and of the character and role of externally mandated algebra tests.
- We need studies of the ways that teachers respond to students' algebraic thinking as it occurs in diverse classroom settings.
- We need studies of the ways that teachers use technology to enhance student learning of algebra.

The array of such studies should focus explicitly on the role of teachers' algebraic knowledge for teaching—including mathematical content knowledge, pedagogical content knowledge, curricular knowledge, and knowledge of students' learning—as it shapes their instructional practices. In each area of investigation we need to develop a system that connects research, development, practice, and evaluation.

APT2. Improving Preparation of Algebra Teachers

For students preparing to teach algebra in middle or high school, the centerpiece of their content studies is a course in abstract algebra. This standard undergraduate course potentially offers an opportunity to develop deep understanding of school algebra, as well as the connections between arithmetic ideas and school algebra on the one hand and more advanced algebraic concepts, including linear algebra, on the other. Abstract algebra is also a course where serious links between algebraic computation and geometry might be used to illustrate the power of algebraic processes in all areas of mathematics. The course in abstract algebra is also an important opportunity for students to develop understanding and skill in mathematical reasoning and proof.

Unfortunately, there is substantial anecdotal evidence that current approaches to abstract algebra for future teachers commonly fall short of the ambitious expectations. Mathematics teacher candidates seldom report gaining illuminating insights into school algebra from their study of abstract algebra. Furthermore, when teacher candidates participate in internship experiences, there are few times when the structural perspective of abstract algebra is effectively brought to bear on teaching problems.

Traditional senior/graduate level abstract algebra courses focus on algebraic structures through an axiomatic approach that places a great emphasis on learning how to develop rigorous proofs. This focus was designed to serve students who will continue to pursue a Ph.D. in mathematics, but it may not best serve those who are preparing to teach algebra in grades 6–12. In recent years there have been a number of exploratory studies involving the development of abstract algebra/algebraic structures courses for future middle and high school mathematics teachers. These courses frequently focus on specific content knowledge that is appropriate for future teachers and they make explicit connections between college level abstract algebra and school algebra.

At this point, the field needs a coherent program of research and development projects addressing questions that are critical to the design and implementation of such courses for future teachers.

- We need studies that identify and examine the mathematics content specifically needed for middle and secondary teaching. Such an effort is a prerequisite for design of algebraic structure courses that concentrate on abstract algebra concepts underlying algebra in the middle and secondary school curriculum.
- We recommend supporting initiatives that focus on development of alternative algebraic structures courses/materials for prospective middle and secondary teachers. Such courses can still provide opportunities for students to learn how to construct precise mathematical arguments, but prove to be more beneficial in the preparation of prospective middle and high school mathematics teachers.
- The effectiveness of traditional and innovative algebraic structure courses in preparation of teachers for algebra instruction in grades 6–12 has not been systematically studied. Understanding the impact of such courses/materials on teachers' knowledge relative to teaching algebra is critical. So it is important to develop careful research studies evaluating the efficacy of such courses.

All three kinds of efforts—content analysis, design of innovative courses, and study of pre-service teacher learning—are projects in which collaboration among mathematicians, mathematics teacher educators, and mathematics education researchers will be most productive. Furthermore, it will be useful to initiate exploratory studies of ways that other courses in the undergraduate preparation of secondary school teachers could also contribute better to their content preparation.

APT3. Professional Development of Algebra Teachers

It is now widely recognized that the content and pedagogical preparation provided to teachers in pre-service studies is only the beginning of their development as professional teachers. To assure that middle and high school students receive the best possible algebra instruction, we need better understandings of the ways that teachers' knowledge of algebra grows and changes as a result of classroom experience and planned professional development. Progress on this problem seems most likely to result from two particular strands of inquiry.

- The first strand of inquiry relates to the effects of different algebra-oriented professional development strategies on teachers' classroom practices and student achievement. We need to understand the effects of experiences like content coursework that focuses on the abstract algebra topics underlying school algebra or the variety of "high school mathematics from an advanced standpoint" courses that have emerged recently. We also need to understand the effects of professional development strategies like lesson study and other similar work-embedded experiences.

  For studies of algebra-oriented professional development strategies it will be important to study teachers before, during, and after the professional development program and to look at the effects of experiences on learning by their students.
- The second strand of inquiry relates to what teachers learn about algebra and its teaching from their experience in the practice of teaching. For example, it would be valuable to know more about how teachers' understanding of algebra and its teaching develops from use of different kinds of instructional materials.

An important starting point for research on development of teacher knowledge about algebra and its teaching would be a careful analysis of the key algebraic concepts that should inform teachers' understanding of the mathematics they are teaching. This analysis work should involve both mathematicians and mathematics educators looking at the connections between university courses in algebra and the school mathematics curriculum. The hypothesis is that higher-level understanding of the algebraic concepts should subsequently allow teachers to develop more interesting problems, provide better explanations, make connections within the mathematics, and more effectively evaluate whether students are progressing.

Inquiry related to this question should be broad in its approach, beyond a sole focus on student achievement. The dispositions and views of mathematics of both teachers and students should also be considered. For example, how do they view the role of definitions? Do they have an appreciation of how algebra has developed across time and cultures? Do they recognize multiple representations of the same mathematical structure in different contexts? Moreover, we must recognize that algebraic ideas permeate the entire school mathematics curriculum, not just courses titled algebra. In general, the differing needs of middle school and high school teachers should be accounted for.

Case studies of teachers might be particularly fruitful, comparing their mathematical knowledge and stated beliefs with what is actually observed in classroom practices (mathematical goals, instructional methods, assessment). It also might be useful to track the ways that teachers' knowledge acquired in their pre-service preparation sets the stage for and interacts with knowledge of algebra acquired from subsequent professional development efforts and classroom experiences.

APT4. Collaboration in Teacher Development

The variety of efforts required to improve preparation and professional development of middle and high school algebra teachers will be most effective if they draw together mathematicians, mathematics educators, classroom teachers, and others who have complementary interest and expertise. They need to work collaboratively in the teaching of algebra courses for pre-service teachers and in professional development for in-service teachers — to address issues of reasoning, proof, applications, rigor, and connections to school mathematics.

Thanks to MSP grants, many mathematics departments have collaborated on the development of new courses in algebraic structures or have revised their current abstract algebra courses to meet the needs of current and future teachers of algebra. Unfortunately, at many colleges and universities there are few mathematicians who are actively engaged with teacher preparation issues. This leads us to recommend support of work in three areas:

- First, we need to identify strategies that successfully nurture collaboration between mathematics educators and mathematicians. There are anecdotal reports of existence proofs, but the field needs careful study and dissemination of what works in those situations.
- Second, we need to identify types of initiatives that encourage mathematicians to become more involved in teacher preparation, more active in providing professional development opportunities for teachers, and more knowledgeable about the environment that students preparing to be mathematics teachers will experience upon graduation.
- Third, in order to make the type of collaboration envisioned a long-term reality, it is important to find ways to encourage institutional rewards for faculty engaged in teacher development activity. Studies that identify effective strategies for changing the relationships of mathematics and education faculty will be particularly useful.
- Fourth, we should encourage the intercollegiate spread and revision of innovative new abstract algebra courses for teachers, shadow courses, and capstone courses based on collaborative development, teaching, and evaluation.

There are some prototypes of the kinds of collaborative activity called for in these recommendations —most notably the recent PMET work and the efforts of Project NEXT to develop a new generation of collegiate mathematicians. But the problem is large and especially critical in relation to teaching of school algebra.

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VI
College Algebra

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Approximately 700,000 students annually enroll in College Algebra courses, most of which focus on algebraic manipulation. Students are told to learn the procedures for factoring polynomials, "simplifying" radicals, "solving" equations with absolute values, and "solving" inequalities. Students are expected to learn to follow the same procedure demonstrated to them by the instructor. In these courses, students are not expected to use the solutions in any context outside of mathematics.

The course typically serves as a terminal course for students in many majors as well as a prerequisite to courses such as pre-calculus, statistics, business calculus, finite mathematics, and mathematics for elementary education majors. While the goals of the course are based on what some think are the necessary prerequisites for calculus, fewer than 10% of the students ultimately enroll in calculus. When businesses and other employees discuss the quantitative skills of their non-technical workforce they are, to a large degree, discussing individuals who have enrolled in College Algebra. The course fails to emphasize those abilities that faculty from other disciplines and representatives from industry expect from these students. Furthermore, by the most conservative estimates, fewer than 50% of the students who enroll in the course receive a grade of A, B, or C.

Fortunately, with support from the National Science Foundation, a great deal has been learned about College Algebra and about materials and approaches that can be used to create a course that is more valuable to students. This refocused College Algebra course can result in a greater retention and in a higher success rate. The national mathematics organizations have endorsed the need for such a change. A great deal of inertia exists, and it is recognized that offering a renewed course must take into account the fact that most sections of this course are taught by part time instructors and graduate teaching assistants. However, the opportunity exists to make a profound change for this large body of students. Based upon what is known concerning College Algebra, the working group proposes a long-term program that would produce a dramatic change in College Algebra nationwide. Although seemingly expensive in total, such a program would cost very little for each enrolled student.

What is known about College Algebra

With support from the NSF for studies, workshops and pilot projects, and by professional organizations such as the Mathematical Association of America (MAA) and American Mathematical Association of Two Year Colleges (AMATYC), a great deal has been learned about the College Algebra experience in the United States:

- **Annually 650,000 to 750,000 college students enroll in College Algebra**. This number is firm, subject only to varying names of the course at different institutions. The definitive Conference Board of the Mathematical Sciences study provides detailed information on these enrollment numbers (Lutzer et al, 2002).
- **Fewer than 10% of the students who enroll in College Algebra intend to prepare for technical careers and a much smaller percentage end up entering the workforce in technical fields**. Again, there is definitive evidence to support this situation that occurs at all types of institutions. Typical findings are that even at research universities, only 9% of College Algebra students ultimately register for calculus and about 1% enroll in third semester calculus (Dunbar, 2006). Comparable data exists for community colleges and other types of institutions (Herriott, 2006, Agras, 2005, McGowen, 2006).
- **Nationwide more than 45% of students enrolled in College Algebra either withdraw or receive a grade of D or F**. For example, a community college in Georgia has a DWF rate of 49% among the 2,300 students who enrolled in College Algebra (Herriott, 2006); at a two-year college in Florida the DFW rate was 45% (Agras, 2005); at a university in Virginia the DFW rate was 63% for students enrolled in large lecture sections and 51% for students enrolled in classes of 35 students (Ellington et al, 2006).
- **When given an opportunity, faculty from other disciplines and representatives from business, industry, and commerce have consistently called for mathematics departments to make major changes in the content of College Algebra**. Particularly noteworthy are the sentiments voiced by prospective employers (Steen et al, 2007) and the results of 11 weekend workshops of representatives of 17 different partner disciplines (Ganter et al, 2004). They uniformly recommend: that algebraic techniques not be the focus; that there be a strong emphasis on conceptual understanding; that communication skills should be stressed; and that the courses focus on mathematical modeling and realistic problem solving. (See for example Gordon, S., 2006).
- **The curriculum committees of national mathematics organizations have uniformly called for replacing the current College Algebra course with one in which students address problems presented as real world situations by creating and interpreting mathematical models**. These recommendations have been consistent and persistent, coming from AMATYC (Wood et al, 2006) and MAA (Pollatsek, 2004 and Ganter et al, 2004). Most recently MAA's Curriculum Renewal Across the First Two Years committee has approved *CRAFTY's College Algebra Guidelines,* which describe the renewed College Algebra course that is the recommended offering for all colleges and universities. A copy of the guidelines is attached to this report.
- **There is widespread interest in mathematics departments concerning offering modeling-based College Algebra and also an expressed need for support for instructors, adjuncts and graduate assistants who teach most sections of courses at this level.** When department liaisons were asked if they would like to participate in a proposed NSF-supported MAA research/pilot study project to offer modeling based and control sections of College Algebra, more than 210 departments responded positively within six days. Of the 11 institutions that were invited to participate in the project, 10 offered the modeling based sections in both the Spring and Fall semesters of 2006 and all 10 determined, based upon their experiences, that they would continue to offer modeling sections beyond the period of the grant. They also expressed the view that more professional development was needed for their instructional staffs and that an extended period of time would be needed to fully implement modeling based courses.
- **With support from NSF a large number of exemplary materials have been developed and put in place, although on a very small scale. The materials address the areas stressed by faculty from**

other disciplines and representatives from industry and, in addition, the student success rate has increased. These materials are described and the successes documented on these small scale efforts in a number of publications: Hastings, 2006; Ellington, 2005a; Fox et al, 2001; Gordon, F., 2006; and Johnson, 2004. For example, at one community college (Agras, 2005) 71% of students (n = 159) who took a modeling course received a grade of A, B, or C, while only 55% of students in traditional courses did so. At one university (Ellington, 2005b), 20.3 % of students (n = 989) withdrew from the traditional College Algebra course while only 5.6% of students (n = 284) withdrew from the modeling course. In a companion study (Ellington et al, 2006), 32% more students enrolled in a modeling course completed that course *and* a subsequent course than was the case for students enrolled in a traditional course. In a four year institution (Oty et al, 2000), 84% of students (n = 73) reported that they found a modeling based course to be interesting while only 42% of students (n = 79) found their traditional course interesting. It is reported that 55% of students (n = 178) in a modeling course maintained or improved their grade in a subsequent course, whereas only 10% of students (n = 212) in the control group did so (Norwood, 1995).

Despite all of these findings, the overall majority of students nationwide are enrolled in College Algebra courses that focus on algebraic manipulation. A large scale effort by the mathematics community would produce a huge change with profound implications nationwide. Such an effort should consist of four components:

CA1. Sustained Support to Enable Large Numbers of Institutions to Refocus College Algebra

We recommend that support be given to large numbers of institutions to change their College Algebra program. Each participating institution would engage in a four year implementation period that would include participation in an initial workshop followed by on-going mentoring, site visits, faculty development, material and curriculum development, presentations, publications and research. It is essential that this on-going professional development fully involve those individuals who actually teach College Algebra at the institution; typically this includes graduate teaching assistants and adjunct instructors.

The four year period allows for a phased implementation of the various components of refocused College Algebra: content changes including use of data and modeling; small-group in-class activities; and out-of-class projects. Based upon the evidence described above, the working group is confident that there would be widespread participation by mathematics departments. Participating institutions would provide a significant portion of the overall costs.

Grants could be provided to two to four professional organizations or consortiums of institutions enabling 300 colleges and universities to develop and fully implement refocused College Algebra courses. Assuming these efforts are successful, the materials, experience and research generated by this number of institutions would tip the balance and change common practice nationwide.

CA2. Research on Impact of Refocused College Algebra on Student Learning

We recommend two or three in-depth, multi-year, longitudinal research projects to study all aspects of the development and implementation of refocused College Algebra with an emphasis on determining the impact of well-designed and well-supported refocused College Algebra courses on student achievement and understanding as well as persistence in future mathematically-related coursework. Each project would include a research team and a consortium of colleges and universities that are committed to refocusing College Algebra in a comprehensive manner by refining curriculum and professional development materials for use at their institution and providing extensive development and support of those teaching the sections of the course: full-time faculty, adjunct instructors and/or graduate teaching assistants. In short, each institution would be attempting to offer a well-conceived refocused College Algebra program. While some support would be provided for the curriculum and faculty development components, the major focus would be on research assessing the effects of change on faculty and institutions and on student learning.

CA3. Electronic Library of Exemplary College Algebra Resources

We recommend the support of projects to provide departments and individual instructors with resources (electronic and video) to enable and equip them to teach refocused College Algebra. Resources would include classroom activities (examples and homework problems, data sets, spreadsheet files), extended projects (designed for collaboration and outside class work), samples of student work, and videos of lessons that clearly show how an instructor might use these resources to create a student centered classroom in which students are actively engaged in learning that promotes critical thinking, communication skills, and higher order thinking skills.

Instructors often attend sessions and workshops on the teaching of College Algebra. There is much written on the subject. However, instructors, because of their own experiences regarding how they were taught (usually in a lecture format, large classes, and little engagement) are not able to internalize and easily implement the ideas they either hear about or read. Examples of exemplary lessons that demonstrate best practices, classroom activities (with teacher notes), and long-term projects (with notes on implementation) could enable large numbers of departments and instructors to put the new ideas about a refocused College Algebra into practice. By utilizing technology, we can provide these to instructors who might use them to prepare for the next day's lesson and/or get ideas on how they can improve their own lessons. The videos would demonstrate the best practices using a lesson on content that is appropriate for College Algebra. Working with the projects that develop and deliver the workshops designed to change College Algebra, the creators of the videotaped lessons will first create models to demonstrate the important ingredients to changing teaching habits. As the resource expands, it will help to define content for College Algebra

We recommend that five to ten small curriculum development projects be supported as well as two major projects: one to collect, revise, catalogue and disseminate exemplary lessons, classroom activities, and long-term projects and a second to develop the videotaped models.

CA4. National Resource Database on College Algebra

We recommend a long-term project to prepare and maintain a national resource database that would include summary information on funded projects, textbooks, research articles, etc. An evaluation of College Algebra as related to retention and other student successes would be a central component of the database and all projects would be required to provide this data as a part of their funding requirements. The project could be based on a TIMSS-like Model with several features:

- Curriculum Analysis (the Intended College Algebra Curriculum); the textbook content analysis, and the implemented curriculum.(Robitaille, 1993; Schmidt, McKnight et al, 1997; Schmidt, McKnight et al, 2001).
- Description of the nature of the faculty, the students, the courses, retention and success rates using qualitative and quantitative tools.
- Information that is known and that is being developed in research and evaluation of such research and implementation projects, e.g., NSF projects, Title III projects, Title V1 (traditionally Hispanic colleges), Tribal colleges programs, and related projects.

The knowledge of positive results and dissemination of these results is currently not widespread. There needs to be an ongoing identification of exemplary programs based on this meta-style analysis.

After conclusion of the funding period, the work should be continued for another five years through the competitive grant process.

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Addendum: College Algebra Guidelines

These guidelines represent the recommendations of the MAA/CUPM subcommittee, Curriculum Renewal Across the First Two Years, concerning the nature of the College Algebra course that can serve as a terminal course as well as a pre-requisite to courses such as pre-calculus, statistics, business calculus, finite mathematics, and mathematics for elementary education majors.

Fundamental Experience

College Algebra provides students a college level academic experience that emphasizes the use of algebra and functions in problem solving and modeling, provides a foundation in quantitative literacy, supplies the algebra and other mathematics needed in partner disciplines, and helps meet quantitative needs in, and outside of, academia. Students address problems presented as real world situations by creating and interpreting mathematical models. Solutions to the problems are formulated, validated, and analyzed using mental, paper and pencil, algebraic, and technology-based techniques as appropriate.

Course Goals

- Involve students in a meaningful and positive, intellectually engaging, mathematical experience;
- Provide students with opportunities to analyze, synthesize, and work collaboratively on explorations and reports;
- Develop students' logical reasoning skills needed by informed and productive citizens;
- Strengthen students' algebraic and quantitative abilities useful in the study of other disciplines;
- Develop students' mastery of those algebraic techniques necessary for problem-solving and mathematical modeling;
- Improve students' ability to communicate mathematical ideas clearly in oral and written form;
- Develop students' competence and confidence in their problem-solving ability;
- Develop students' ability to use technology for understanding and doing mathematics;
- Enable and encourage students to take additional coursework in the mathematical sciences

Competencies

1. Problem Solving

Goals for students include

- solving problems presented in the context of real world situations with emphasis on model creation and interpretation;
- developing a personal framework of problem solving techniques (e.g., read the problem at least twice; define variables; sketch and label a diagram; list what is given; restate the question asked; identify variables and parameters; use analytical, numerical and graphical solution methods as appropriate; determine plausibility of and interpret solutions);
- creating, interpreting, and revising models and solutions of problems.

2. Functions and Equations

Goals for the students include

- understanding the concepts of function and rate of change;
- effectively using multiple perspectives (symbolic, numeric, graphic, and verbal) to explore elementary functions;
- investigating linear, exponential, power, polynomial, logarithmic, and periodic functions, as appropriate;
- recognizing and using standard transformations such as translations and dilations with graphs of elementary functions;
- using systems of equations to model real world situations;
- solving systems of equations using a variety of methods;
- mastering algebraic techniques and manipulations necessary for problem-solving and modeling in this course.

3. Data Analysis

Goals for the students include

- collecting (in scientific discovery or activities, or from the Internet, textbooks, or periodicals), displaying, summarizing, and interpreting data in various forms;
- applying algebraic transformations to linearize data for analysis;
- fitting an appropriate curve to a scatter plot and using the resulting function for prediction and analysis;
- determining the appropriateness of a model via scientific reasoning.

Emphasis in Pedagogy

Goals for the instructor include

- facilitating the development of students' competence and confidence in their problem-solving abilities;
- utilizing and developing algebraic techniques as needed in the context of solving problems;
- emphasizing the development of conceptual understanding of the mathematics by the students;
- facilitating the improvement of students' written and oral mathematical communication skills;
- providing a classroom atmosphere that is conducive to exploratory learning, risk-taking, and perseverance;

- providing student-centered, activity-based instruction, including small group activities and projects;
- using technology (computer, calculator, spreadsheet, computer algebra system) appropriately as a tool in problem-solving and exploration;
- conducting ongoing assessment activities designed to determine when mid-course adjustments are warranted.

Assessment

- Assessment tools will measure students' attainment of course competencies, including:
  - solving problems and interpreting results using algebraic tools;
  - building and interpreting models and predicting results;
  - communicating processes and solutions orally and in writing;
  - making quantitative and algebraic arguments;
  - reading and interpreting data presented in various forms.
- Assessment tools will include
  - individual quizzes;
  - individual examinations;
  - additional activities or assignments, such as
    - individual or group homework, projects, and activities;
    - individual or group oral presentations;
    - portfolios that demonstrate student growth;
    - group quizzes and exams.
- The course will be assessed by analyzing its effectiveness in:
  - facilitating student achievement of the course competencies;
  - positively affecting student attitudes about mathematics;
  - preparing students for subsequent courses in mathematics and mathematics-dependent disciplines; preparing students for subsequent endeavors in and outside academia.

VII
Learning Algebra: An Historical Overview

Victor Katz

As a background to the working groups' recommendations, we need to remember that algebra and its teaching have a very long history. From its beginning, in the Egypt and Mesopotamia of 2000 BCE, algebra was taught chiefly to people in the (very small) educated classes of each society in which it was used. Algebra was taught through problem solving, using primarily artificial problems, many of which were geometric. Yet the societies always believed that those who learned how to solve these problems would somehow be able to solve the various kinds of problems that arose in the day-to-day work of the society's leaders, in other words, that learning algebra would "train the mind," or, perhaps, help "inculcate mathematical habits of mind." Although there are obvious differences between the societies where algebra originated and the United States in the 21st century, it is certainly worthwhile to look at the history of algebra and its teaching and consider how this history could have implications for how we teach algebra today.

As a beginning, we need to clarify what it is we mean by the term "algebra", since today it is used with numerous meanings. Historically, at least through the nineteenth century, there has been a rather consistent definition of the term. And since we are now in the year of Euler, it is probably best to take that definition from his *Complete Introduction to Algebra* of 1767: "Algebra [is] the science which teaches how to determine unknown quantities by means of those that are known." In other words, algebra deals with the solution of equations. And since the late sixteenth century, these equations have always been written using symbols. Thus, we could also define algebra as "the study of the 24th letter of the alphabet." Of course, algebra is certainly more than that, since the very process of solving equations must involve manipulations that can be thought of as part of "generalized arithmetic." Or, as Colin Maclaurin wrote in his *Treatise of Algebra* of 1748, algebra is "a general method of computation by certain signs and symbols which have been contrived for this purpose and found convenient."

Naturally, the above definitions apply to what we now call "school algebra" and not to the more advanced "abstract algebra." But for now, let us briefly trace the history of algebra as defined above. In many history texts, algebra is considered to have three stages in its historical development: the rhetorical stage, the syncopated stage, and the symbolic stage. By the rhetorical, we mean the stage where are all statements and arguments are made in words and sentences. In the syncopated stage, some abbreviations are used when dealing with algebraic expressions. And finally, in the symbolic stage, there is total symbolization — all numbers, operations, relationships are expressed through a set of easily recognized symbols and manipulations on the symbols take place according to well-understood rules.

These three stages are certainly one way of looking at the history of algebra. But besides these three stages of expressing algebraic ideas, there are three conceptual stages that have happened along side of these changes in expressions. The conceptual stages are the geometric stage, where most of the concepts of algebra are geometric; the static equation-solving stage, where the goal is to find numbers satisfying certain relationships; and the dynamic function stage, where motion seems to be an underlying idea. Naturally, neither these stages nor the earlier three are disjoint from one another; there is always some overlap.

Algebra is present in the earliest mathematical documents we have, in particular in the numerous extant Mesopotamian clay tablets dating from a few centuries around 1800 BCE. Among the central mathematical ideas in these tablets was the solution of what we call quadratic equations, using the geometric notion of completing the square. In general, the problems presented were quite abstract, even though frequently the Mesopotamians wrote their quadratic problems in terms of finding the length and width of a rectangle. But, after all, it cannot be a realistic problem to find the length and width if we know the area and the perimeter. Another way of looking at this was that the scribes had set up rules for playing a kind of game. In particular, they prepared tablets containing long lists of quadratic problems to be solved using the basic rules, even though there were no real situations which required them. What was important, evidently, was for the students to learn how to "play the game" using the rules. In other words, they would use these problems to develop problem solving skills, including the ability both to follow well-established procedures and also to modify the methods when necessary or reduce complicated problems to ones already solved. These problems illustrate what I would call the beginning of algebra, the beginning of a process of solving numerical problems via manipulation of the original data according to fixed rules. And dealing with geometric quantities was the object of this first algebra. (The Egyptians of the same time period solved what we would call linear equations, but their ideas are based on the notion of proportionality, which I would call an arithmetic rather than an algebraic notion.)

In Greece, of course, mathematics was geometry. Yet what we think of as algebraic notions were certainly present in the work of Euclid and Apollonius. There are numerous propositions, particularly in Book II of Euclid's *Elements* and in his *Data* that show how to solve what appear to be algebraic problems for geometric results, such as the position of a particular point on a line. Euclid solved these problems by manipulating geometric figures, but, unlike the Mesopotamians, based these manipulations on clearly stated axioms. For example, proposition 85 of the *Data* solved the same kind of problem that was frequently solved in Mesopotamia: *If two straight lines contain a given area in a given angle, and if the sum of them be given, then shall each of them be given (i.e., determined).* That is, Euclid showed how to find each of two line segments if their sum and their product (the area determined by them) are given.

Of course, even though the underlying rationale for Mesopotamian equation solving was geometric, the scribes still developed algorithms, or procedures, to solve equations. Eventually, the algorithms began to replace the geometry. The history of algebra begins moving to the "equation solving" stage. We see evidence of this in Diophantus's knowledge of the algorithm for solving quadratic equations, solely based on numbers, in the third century. In India, the quadratic formula also appears without any geometric underpinning as early as the sixth century. And in both cases, we also see the beginnings of some symbolism.

But the first true algebra text still extant is the work on *al-jabr* and *al-muqabala* by Muhammad ibn Musa al-Khwarizmi, written in Baghdad around 825. The first part of this book is a manual for solving linear and quadratic equations. Al-Khwarizmi classifies equations into six types, three of which are mixed quadratic equations. For each type, he presents an algorithm for its solution. For example, to solve the quadratic equation of the type "squares and numbers equal to roots" ($x^2 + c = bx$), al-Khwarizmi tells his readers to take half the number of "things", square it, subtract the constant, find the square root, and then add it to or subtract it from the half the roots already found. As in Mesopotamian times 28 centuries earlier, the algorithm is entirely verbal. There are no symbols.

Having written down an algorithm, al-Khwarizmi justifies it using a "cut-and-paste" geometry, very much like the Mesopotamians. But once the justifications are dispensed with, al-Khwarizmi only expects the reader to use the appropriate algorithm. This is different from the Mesopotamian procedure, in which each problem indicates some use of the geometric background. And although al-Khwarizmi promised in the preface of his book to show how to use algebra to solve real problems, in fact virtually all the problems he presents are abstract. He does not even try, as the Mesopotamians did, to phrase these as real-world problems.

Algebra has now moved decisively from the original geometric stage to the static equation-solving stage. Al-Khwarizmi wants to solve equations. And an equation has one or two numerical answers. His successors in the Islamic world do much the same thing. They set up quadratic equations to solve and then solved them by an algorithm to get one or two answers. You may notice that I am only considering quadratic equations. Surely, Islamic mathematicians solved linear equations. In his *al-jabr*, Al-Khwarizmi in fact has numerous problems solvable by linear equations, mostly in his section on inheritance problems. But to a large extent, solving linear equations, as in the case of ancient Egypt, was part of what we would call arithmetic, not algebra. That is, the basic ideas were part of proportion theory, an arithmetical concept.

Over the next few centuries, Islamic mathematicians worked out various ideas in algebra. They developed all the procedures of polynomial algebra, including the rules of exponents, both positive and negative, and the procedures for dividing as well as multiplying polynomials. Yet the goal of these manipulations was to solve equations, and since the Islamic mathematicians could not solve equations of degree higher than two by an algorithm, they developed two alternative methods. First, there was a return to geometry, but a more sophisticated geometry than Euclid's. Namely, Omar Khayyam found a way to solve cubic equations by determining the intersection of particular conic sections. A second alternative, and one that was certainly more useful, was to determine numerical ways of approximating the solution, ways closely related to what has become known as the Horner method. Still, of course, the idea was to find a single answer (or maybe two or three).

The Islamic algebra which was transmitted to Europe in the twelfth and thirteenth centuries was just the static equation-solving algebra. There were several routes that al-Khwarizmi's algebra took into Europe, including the work of Leonardo of Pisa (Fibonacci) in Italy, and Abraham bar Hiyya in Spain, as well as the direct translations made by Robert of Chester and Gerard of Cremona. In all of them, the basic idea of static equation-solving remained. We have a problem to solve, in general a numerical problem, which involves squares. We figure out which algorithm to use and then use it to get our answer (or in some cases, two answers).

In sixteenth century Italy, there was a major breakthrough in mathematics. Several Italian mathematicians figured out how to solve cubic equations and fourth degree equations as well. But as we see in Girolamo Cardano's *Ars Magna*, the basic principle here was the same as in the solution of quadratic equations. First, Cardano classified cubic equations into a large number of different classes. For each class, he presented an algorithm for solution. He often justified the algorithms by some geometric argument, but basically it was the algorithm itself that was important. It is that which enabled one to find an answer (or perhaps two or three) to a very clearly defined equation. But that was it. Algebra was still just about finding solutions to equations. And if you look through Cardano's work, you find that again virtually all the problems he solves by using equations are purely abstract.

However, beginning with Cardano and the other Italian algebraists of his time, there is a very rapid change from the Islamic rhetorical algebra through the syncopated stage of abbreviations into the modern symbolic stage, all in the context of static equation solving. Thus, Cardano himself used abbreviations for operations, including roots. Francois Viète used letters of the alphabet to represent unknowns as well as constants in equations and also our modern plus and minus signs, but still kept to words for powers. It was Thomas Harriott who first extensively used not only our now standard symbols for radicals and equals, but also used juxtaposition to represent multiplication. And then René Descartes replaced Viète's consonants and vowels for constants and unknowns with our standard letters near the beginning of the alphabet and those near the end, respectively. He also introduced our modern exponential notation.

With a new notation coming into place in the seventeenth century, a great change in point of view was also taking place in algebra itself. Mathematicians started asking questions other than "find the solution to that problem expressed as an equation." There are probably many reasons for this, but certainly one of the reasons was increasing interest in astronomy and physics. Johann Kepler was interested in the path of the planets. Galileo Galilei was interested in the path of a projectile. In both of these cases, it was not a "number" that was wanted, but an entire curve. Both Kepler and Galileo realized that the solutions to their problems were conic sections, and the only way they knew how to deal with these was by what they had learned from Apollonius. His mathematics was largely "static" in that he was not concerned with moving points—just with a particular slice of a cone. Nevertheless, Kepler and Galileo were able to pull out of his work the ideas they needed to represent motion.

Now neither Kepler nor Galileo had a useful notation for representing motion. They did not use algebra, but relied on Greek models, including the detailed use of geometric proportion theory. Although Galileo wrote for the educated layman, one still had to fight through lots of verbiage to understand what he was doing, and even more so was this the case with Kepler, writing for experts. It seemed that something had to be done so that the important physical results of Kepler and Galileo could be better disseminated and so that further developments could result from their work.

Then in 1637, the appropriate tools for representing this work appeared. The two fathers of analytic geometry, Fermat and Descartes, produced their first works on the subject. Both were interested in the use of algebra to represent curves, although, interestingly, neither cited any motivations from physics. Descartes' clear motivation was to use the algebra to solve geometric problems, while Fermat was just interested in representing curves through algebra. But since both showed how to represent a curve, however described verbally, through algebra, analytic geometry gave mathematicians a mechanism for representing motion. Fermat generally used Viète's symbolism in algebra, but Descartes developed his own, which is essentially the kind that won out. And Newton, for one, picked up on this, using Descartes' symbolism, as he developed the calculus.

Curiously, Newton, when he wrote the *Principia*, was somewhat hesitant to use algebra. Much of the work is presented using classical geometry, although with the incorporation of infinitesimals. But given the newly developed algebra, which Newton certainly used freely in some of his other works, it is not surprising that one of the major thrusts in the early eighteenth century was to translate Newton's ideas into algebraic language and prove them using algebra and the newly invented calculus. The mathematicians who accomplished this, along with others in that time period, were no longer interested in finding a "number" as answer to a problem. They wanted a curve. They were interested in seeing how objects—be they planets or projectiles—moved, and they moved in curved paths. In fact, the primary goal of mathematicians, it appears, once the calculus was invented, was to determine curves that solved problems, not just points.

Using algebraic notation, a curve in the plane was the graph of an equation of the form $f(x, y) = 0$. Thus, the study of functions began, functions of two variables, as here, and, more easily, functions of one variable. By the middle of the eighteenth century, Leonhard Euler made the notion of function into a central one in mathematics, an idea that has remained to this day.

Although we could take the history of algebra into the twentieth century, for the purposes of "school algebra", it is probably sufficient to stop here. The question then is whether this history gives us any ideas about teaching algebra today. First, by considering the conceptual stages of the development of algebra, one might consider whether the introduction of algebra through geometric concepts could be valuable and then whether the notion of function is better left until after the students have mastered the ideas of equation solving. By considering the notational stages, one might consider that it was a long and difficult road from dealing with algebraic ideas using natural language to dealing with them using symbols. Thus, there should be consideration of how to make the transition from natural language, including the use of numbers, to symbolization. Although we cannot assume that the understanding of mathematical ideas by today's students mirrors the historical development of the subject, there is certainly good reason to explore pedagogical ideas coming from a historical analysis. Research has shown, in particular, that the historical difficulties mathematicians had at critical junctures in the development of their subject are frequently mirrored in the learning difficulties that students have at the same critical junctures. Therefore, it is worthwhile for teachers of algebra to be aware of the history of the subject as they consider how to present the material to their students.