2022 Session A

A1. Determine all ordered pairs of real numbers (a, b) such that the line $y = ax+b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

Answer: Those (a, b) for which

- $|a| \geq 1$, or
- $a = b = 0$, or
- $0 < |a| < 1$ and $b < \ln[2(1 - \sqrt{1 - a^2})/a^2] + \sqrt{1 - a^2} - 1$, or
- $0 < |a| < 1$ and $b > \ln[2(1 + \sqrt{1 - a^2})/a^2] - \sqrt{1 - a^2} - 1$.

Solution: Notice that upon reflection about the y-axis, the curve $y = \ln(1+x^2)$ is unchanged, and the line $y = ax + b$ becomes the line $y = -ax + b$. Thus, (a, b) is a solution if and only if $(-a, b)$ is a solution, so we need only verify the claim for $a \geq 0$.

Define $h(x) = \ln(1 + x^2) - ax - b$, with derivative $h'(x) = 2x/(1 + x^2) - a$; we must determine the pairs (a, b) for which $h(x) = 0$ for exactly one x.

By L'Hôpital's rule, $\lim_{x\to\pm\infty}[\ln(1 + x^2) - b]/x = 0$, so it follows that $\lim_{x\to\pm\infty} h(x)/x = -a$. Thus, if $a > 0$, then $\lim_{x\to-\infty} h(x) = \infty$ and $\lim_{x\to\infty} h(x) = -\infty$. Also, $x^2 - 2x + 1 = (x - 1)^2 \geq 0$ implies that $h'(x) \leq 1 - a$, with equality only when $x = 1$.

If $a \geq 1$, then $h'(x) \leq 0$, with equality only if $a = 1$ and $x = 1$. It follows that h is strictly decreasing, and from this, the limits we proved, and the continuity of h, we conclude that h takes on every value (including 0) exactly once. Thus, (a, b) is a solution for all $a \geq 1$ and all b.

If $0 < a < 1$, then $h'(x) = 0$ for two values of x, which from the quadratic formula are $x_\pm = (1 \pm \sqrt{1 - a^2})/a$. Notice that $h'(x) < 0$ for $x < x_-$ and $x > x_+$, and $h'(x) > 0$ for $x_- < x < x_+$. Thus, h takes on every value between $h(x_-)$ and $h(x_+)$ three times, takes on the values $h(x_-)$ and $h(x_+)$ twice each, and takes on all other values once. Since $h(x_-) < h(x_+)$, the solutions are those (a, b) for which $h(x_-) > 0$ or $h(x_+) < 0$; equivalently, $b < \ln(1 + x_-^2) - ax_-$ or $b > \ln(1 + x_+^2) - ax_+$, which upon substitution yield the formulas claimed above.

If $a = 0$, then $h'(x) < 0$ for $x < 0$ and $h'(x) > 0$ for $x > 0$; also, $\lim_{x\to\pm\infty} h(x) = \infty$. It follows that $h(x)$ takes on every value greater than $h(0)$ twice, takes on the value $h(0)$ once, and does not take on values less than $h(0)$. Since $h(0) = -b$, we conclude that $(0, 0)$ is the only solution with $a = 0$.

A2. Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?

Answer: $2n-2$.

Solution: Let R be a large real number and define

$$p(x) = Rx^n - x^{n-1} - x^{n-2} - \cdots - x^2 - x + R = R(x^n+1) - \frac{x^n - x}{x-1}.$$

Then

$$p(x)^2 = R^2(x^{2n} + 2x^n + 1) - 2R(x^n+1)\frac{x^n - x}{x-1} + \frac{(x^n-x)^2}{(x-1)^2}.$$

Thus we see that the coefficients of x^{2n}, x^n and x^0 will be quadratic polynomials in R with positive leading coefficient, and hence will be positive for all sufficiently large R. Since

$$(x^n+1)\frac{x^n-x}{x-1} = \frac{x^{2n}-x}{x-1} - x^n = x^{2n-1} + x^{2n-2} + x^{n+1} + x^{n-1} + \cdots + x,$$

we see that the coefficients of all powers of x other than 0, n, and $2n$ take the form $-2R+c$, where c depends on the power but not on R. Thus for large enough R (in fact $2R > n-2$ suffices), these coefficients will all be negative. Thus, $p(x)$ has $2n-2$ negative coefficients for large R.

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where without loss of generality we assume $a_n > 0$, and suppose $p(x)^2$ has $2n-1$ negative coefficients. Since

$$p(x)^2 = a_n^2 x^{2n} + 2a_n a_{n-1} x^{2n-1} + \cdots + 2a_0 a_1 x + a_0^2,$$

the coefficients of x^{2n} and x^0 must both be nonnegative. Thus all other coefficients must be negative. In particular a_0 and a_1 must have opposite signs (and be nonzero). Thus, since $n \geq 2$, there is a largest integer k with $k < n$ such that $a_k > 0$. Then the coefficient of x^{n+k} is

$$2a_k a_n + \sum_{m=k+1}^{n-1} a_m a_{n+k-m}.$$

The first term is positive since a_k and a_n are both positive, and all the terms in the sum are nonnegative since maximality of k implies a_m and a_{n+k-m} are both nonpositive. Thus, the coefficient of x^{n+k} is also positive, a contradiction.

Thus, the maximum is $2n-2$.

A3. Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \ldots$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$ for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 (mod 5).

Solution 1: For $n \geq 1$, we must have $a_{n+1} \not\equiv -1 \pmod{p}$, since a_n and a_{n+2} are not divisible by p. Therefore, we can cancel factors of a_n and (for $n \geq 2$) $a_n + 1$ to derive the following (mod p) congruences:

$$\begin{aligned}
a_3 &\equiv \frac{1+a_2}{a_1}, \\
a_4 &\equiv \frac{1+a_3}{a_2} \equiv \frac{1+a_1+a_2}{a_1 a_2}, \\
a_5 &\equiv \frac{1+a_4}{a_3} \equiv \frac{1+a_1}{a_2}, \\
a_6 &\equiv \frac{1+a_5}{a_4} \equiv a_1, \\
a_7 &\equiv \frac{1+a_6}{a_5} \equiv a_2.
\end{aligned}$$

Since a_n and a_{n+1} determine a_{n+2}, by induction the sequence is periodic modulo p with period 5, and hence periodic with period 5; in other words, $a_{n+5} = a_n$ for all $n \geq 1$.

Next, we assert that since 5 is prime, the period-5 infinite sequences $(a_1, a_2, a_3, \ldots)$, $(a_2, a_3, a_4, \ldots)$, $(a_3, a_4, a_5, \ldots)$, $(a_4, a_5, a_1, \ldots)$, and $(a_5, a_1, a_2, \ldots)$ are all distinct unless $a_1 = a_2 = a_3 = a_4 = a_5$. For example, if the third and fifth sequences are identical, then $a_3 = a_5 = a_2 = a_4 = a_1$. Therefore, each non-constant period-5 cycle that meets the conditions of the problem corresponds to 5 infinite sequences that meet the conditions, so the total number of allowed sequences that are non-constant is a multiple of 5.

It remains only to show that the number of allowed constant sequences – those with $a_n = c$ for some $c \in \{1, 2, \ldots, p-1\}$ and all $n \geq 1$ – is congruent to 0 or 2 modulo 5. The number of such sequences is the number of such c for which $c^2 \equiv 1 + c \pmod{p}$. Multiplying by 4 and then adding $1 - 4c$ to each side yields $(2c-1)^2 \equiv 5 \pmod{p}$. Since $p > 5$ is prime, we can divide by 4 modulo p, so every solution of the latter congruence is a solution of the former congruence. And by the same properties of p, there are either 0 or 2 square roots of 5 modulo p, and if there are 2, each yields a corresponding value of c. (Each of these values of c is nonzero because $(-1)^2 \not\equiv 5 \pmod{p}$.)

Solution 2: We say that an ordered pair of integers (a, b) has property P if $a, b \in \{1, 2, \ldots, p-2\}$ and $a + b \neq p - 1$. We claim that if (a_1, a_2) has property P, then it is part of a unique infinite sequence $a_1, a_2, a_3, \ldots$ that satisfies the conditions of the problem. The justification uses the following lemma.

Lemma. *The conditions in the problem statement on a_n, a_{n+1}, and a_{n+2} imply that*

$$\begin{aligned}
a_{n+2} \equiv -1 \pmod{p} &\iff a_n + a_{n+1} \equiv -1 \pmod{p} \quad \textit{and} \\
a_n \equiv -1 \pmod{p} &\iff a_{n+1} + a_{n+2} \equiv -1 \pmod{p}.
\end{aligned}$$

Proof. The first equivalence follows from $a_n(a_{n+2}+1) \equiv 1 + a_{n+1} + a_n \pmod{p}$, and the fact that we can divide by a_n modulo p since a_n is not a multiple of p. The second equivalence is proved similarly. □

If (a_n, a_{n+1}) has property P, then $a_n + a_{n+1} \not\equiv -1 \pmod{p}$. Since a_n and $1 + a_{n+1}$ are not multiples of p, there is a unique $a_{n+2} \in \{1, \ldots, p-1\}$ such that $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$, and $a_{n+2} \neq p-1$ by the first part of the lemma. Further, by the second part of the lemma, $a_{n+1} + a_{n+2} \neq p-1$. Thus, (a_{n+1}, a_{n+2}) has property P, and is uniquely determined by (a_n, a_{n+1}) and the conditions in the problem statement. Our claim follows by induction on n.

Next, suppose that the sequence $a_1, a_2, a_3, \ldots$ that satisfies the conditions of the problem. For $n \geq 1$, since a_n and a_{n+2} are not multiples of the prime number p, neither is $a_n a_{n+2}$, so $a_{n+1} \not\equiv -1 \pmod{p}$. In particular, a_2, a_3, and a_4 are not congruent to -1 modulo p. Then by the first part of the lemma, $a_1 + a_2$ and $a_2 + a_3$ are not congruent to -1 modulo p. Then by the second part of the lemma, $a_1 \not\equiv -1 \pmod{p}$. In particular, (a_1, a_2) must have property P.

Therefore, $f(p)$ is the number of ordered pairs (a_1, a_2) that have property P. There are $p-2$ possible values of a_1, and for each such value, there are $p-3$ values of a_2 consistent with property P. Then $f(p) = (p-2)(p-3) = p^2 - 5p + 6$, and $f(p) \equiv p^2 + 1 \pmod{5}$. Since $p \neq 5$ and p is prime, p is congruent to 1, 2, 3, or 4 modulo 5. In all cases, $p^2 \equiv \pm 1 \pmod{5}$, and the conclusion of the problem follows.

A4. Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.

Answer: $2\sqrt{e} - 3$.

Solution 1: The approach is a "first-step analysis" that expresses the expected value for the sum starting with X_1 in terms of the expected value for the sum starting with X_2. Let $F(x)$ denote the expected value of the quantity that is S if $X_1 \le x$, and 0 if $X_1 > x$. The problem statement asks for $F(1)$. Since $0 \le S \le 1$, it follows that $|F(y) - F(x)| \le |y - x|$, and in particular F is continuous.

Conditioning on $X_1 = u$,

$$F(x) = (1-x)\cdot 0 + \int_0^x (u/2 + F(u)/2)\, du = \frac{x^2}{4} + \frac{1}{2}\int_0^x F(u)\, du.$$

Since F is continuous, the right side is differentiable, and then so is F. This implies the differential equation

$$F'(x) - F(x)/2 = x/2, \quad F(0) = 0.$$

The simplest particular solution for the nonhomogeneous component is $F_p(x) = -x - 2$ (which is found by substituting $ax + b$ and solving). The general homogeneous solution to $F' - F/2 = 0$ is $F_h(x) = ce^{x/2}$, so in all the solution has the form

$$F(x) = -x - 2 + ce^{x/2}.$$

Plugging in the initial condition gives $c = 2$, so finally, $F(1) = -3 + 2\sqrt{e}$.

Solution 2: This approach calculates the expected value by summing an explicit series conditioned on the position of the first increase in the sequence (note that such a position exists with probability 1). We will also need the order statistics for uniform random variables: the expected value of the j-th largest among $X_1, \ldots, X_n$ is $\frac{n+1-j}{n+1}$.

Now suppose that the first increase occurs at position k, so $X_1 \ge X_2 \ge \cdots \ge X_k$ and $X_k < X_{k+1}$. Denote this event by I_k. The probability of this occurring is $P(I_k) = \frac{1}{k!}\cdot\frac{k}{k+1}$ due to symmetry and independence; the first factor imposes the monotonic order on $X_1, \ldots, X_k$, and the second factor comes from the fact that X_{k+1} can lie in any of the $k+1$ possible order positions (relative to $X_1, \ldots, X_k$) except for the smallest (i.e. the $(k+1)$-st largest).

We next calculate the expected value of each X_j for $1 \le j \le k$, conditional on the event I_k. The claim is that

$$E[X_j \mid I_k] = 1 - \frac{j(k+1)}{k(k+2)}.$$

This is shown by conditioning on the k possible order positions for X_{k+1}. In particular, if it is in one of the j largest positions, then the expected value of X_j is shifted down to the $(j+1)$-st largest position, namely $\frac{k+2-j-1}{k+2}$, but if it is in a smaller position, the expected value of X_j is $\frac{k+2-j}{k+2}$. Overall,

$$E\left[X_j \mid I_k\right] = \frac{j}{k}\cdot\frac{k+1-j}{k+2} + \frac{k-j}{k}\cdot\frac{k+2-j}{k+2},$$

which simplifies to the claimed expression.

We can now evaluate the expected value of the sum in the case of I_k. Using the geometric series, and its relative

$$\sum_{j=1}^{k} jx^j = \frac{x\left(kx^{k+1} - (k+1)x^k + 1\right)}{(x-1)^2},$$

we find that

$$\begin{aligned}
E[S \mid I_k] &= \sum_{j=1}^{k} \frac{1}{2^j} E\left[X_j \mid I_k\right] = \sum_{j=1}^{k} \frac{1}{2^j}\left(1 - \frac{j(k+1)}{k(k+2)}\right) \\
&= 1 - \frac{1}{2^k} - \frac{k+1}{k(k+2)} \cdot \frac{\frac{1}{2}\left(\frac{k}{2^{k+1}} - \frac{k+1}{2^k} + 1\right)}{\frac{1}{4}} \\
&= 1 + \frac{1}{k2^k} - \frac{2(k+1)}{k(k+2)}.
\end{aligned}$$

Finally, summing over k gives

$$\begin{aligned}
E[S] &= \sum_{k\geq 1} P(I_k) E[S \mid I_k] \\
&= \sum_{k\geq 1} \frac{1}{k!}\frac{k}{k+1}\left(1 + \frac{1}{k2^k} - \frac{2(k+1)}{k(k+2)}\right) \\
&= \sum_{k\geq 1} \frac{1}{(k+1)!}\left(k + \frac{1}{2^k} - \frac{2(k+1)}{k+2}\right) \\
&= \sum_{k\geq 1}\left(\frac{(k+1)(k+2) - 3(k+2) + 2}{(k+2)!} + \frac{1}{2^k(k+1)!}\right) \\
&= (e-1) - 3(e-2) + 2\left(e - \frac{5}{2}\right) + 2\left(\sqrt{e} - 1 - \frac{1}{2}\right) = 2\sqrt{e} - 3.
\end{aligned}$$

Solution 3: Let $c_j = 2^{-j}$ if $X_1 \geq \cdots \geq X_j$, and let $c_j = 0$ otherwise. Then $S = \sum_{j=1}^{\infty} c_j X_j$. The probability that $c_j > 0$ is $1/j!$, and the expected value of X_j conditioned on $c_j > 0$ is the (unconditioned) expected value of the minimum of $X_1, \ldots, X_j$, which is $1/(j+1)$ [because the cumulative distribution function of the minimum is $1 - (1-x)^j$]. By Tonelli's theorem (which applies because $c_j X_j$ is nonnegative), the expected value of the sum is the sum of the expected values:

$$E[S] = \sum_{j=1}^{\infty} E[c_j X_j] = \sum_{j=1}^{\infty} \frac{2^{-j}}{(j+1)!} = 2\sum_{n=2}^{\infty} \frac{(1/2)^n}{n!} = 2(e^{1/2} - 1 - 1/2) = 2\sqrt{e} - 3.$$

A5. Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

Answer: 290.

Solution: After k tiles have been placed, $2k$ squares will be covered by tiles. The uncovered squares will form at most $k+1$ blocks of one or more consecutive squares, whose total length will therefore be $2022 - 2k$ squares.

Claim 1. Alice can always ensure that there are at least 290 uncovered squares at the end.

Proof. Alice can use the following strategy when there is at least one uncovered block of length $L \geq 3$ squares: Alice picks such a block, and covers the second and third squares of that block, breaking it into a block of length 1 and (if $L > 3$) a block of length $L - 3$.

Suppose Alice is able to place m tiles according to this strategy, but not $m+1$. Then after $2m-1$ (if Bob is unable to place an mth tile) or $2m$ tiles have been placed, all remaining uncovered blocks have length 1 or 2. At this point, Alice has created at least m blocks of length 1, and there are at most $2m+1$ uncovered blocks. Thus, the total number of uncovered squares is at most $2(2m+1) - m = 3m + 2$. Since at most $4m$ squares are covered at this point, $7m + 2 \geq 2022$, and hence $m \geq 2020/7 > 288$. Thus, the game reaches a point with at least 289 uncovered blocks of length 1, none of which can be covered subsequently. Since the number of uncovered squares is always even, at the end of the game there are at least 290 uncovered squares.

Claim 2. Bob can always ensure that there are at most 290 uncovered squares at the end.

Proof. Bob can use the following strategy when there is at least one uncovered block of length $L \geq 4$ squares: Bob picks such a block, and covers the third and fourth squares of that block, breaking it into a block of length 2 and (if $L > 4$) a block of length $L - 4$

Let D be the difference between the number of uncovered blocks with length other than 2 and the number of uncovered blocks with length 2. At the start of the game, $D = 1$. Placing a tile can increase the number of uncovered blocks by 1, or it can cover a block of length 2, but not both. Thus, neither player can increase D by more than 1 by placing a tile. If Bob is able to place a tile according to the strategy above, then D decreases by at least 1. Therefore, $D \leq 2$ for as long as Bob is able to follow the strategy.

Suppose Bob is able to place m tiles according to this strategy, but not $m+1$. Then after $2m$ (if Alice is unable to place an $(m+1)$st tile) or $2m+1$ tiles have been placed, all remaining uncovered blocks have length at most 3. At this point, let n_1, n_2, and n_3 be the number of uncovered blocks with lengths 1, 2, and 3, respectively. Then the total number of uncovered blocks is $n_1 + n_2 + n_3 \leq 2m + 2$, and since either $4m$ or $4m+2$ squares are covered by tiles, $n_1 + 2n_2 + 3n_3 + 4m \leq 2022$. Double the first inequality in the previous sentence, add it to the second inequality, and eliminate m to get $3n_1 + 4n_2 + 5n_3 \leq 2026$. Also, at this point $n_1 + n_3 - n_2 = D \leq 2$. Multiply this inequality by 4 and add it to the previous inequality to get $7n_1 + 9n_3 \leq 2034$. Since uncovered blocks of length 2 or 3 will have a tile placed in them before the game ends, the number of uncovered squares at the end will be $n_1 + n_3 \leq (7n_1 + 9n_3)/7 \leq 2034/7 < 291$.

A6. Let n be a positive integer. Determine, in terms of n, the largest integer m with the following property: There exist real numbers $x_1, \ldots, x_{2n}$ with $-1 < x_1 < x_2 < \cdots < x_{2n} < 1$ such that the sum of the lengths of the n intervals

$$[x_1^{2k-1}, x_2^{2k-1}],\ [x_3^{2k-1}, x_4^{2k-1}], \ldots,\ [x_{2n-1}^{2k-1}, x_{2n}^{2k-1}]$$

is equal to 1 for all integers k with $1 \le k \le m$.

Answer: $m = n$.

Solution: Note that the given condition can be rewritten as

$$\sum_{j=1}^{2n} (-1)^j x_j^{2k-1} = 1.$$

We will show that $x_j = -\cos(j\pi/(2n+1))$ works for k up to $m = n$. To see this, let $\omega = e^{2i\pi/(2n+1)}$. Then ω is a primitive $(2n+1)$-st root of unity, so for all integers a that are not multiples of $2n+1$,

$$\sum_{j=0}^{2n} \omega^{aj} = 0.$$

It follows that for $k = 1, \ldots, n$,

$$\sum_{j=0}^{2n} \left(\frac{\omega^j + \omega^{-j}}{2} \right)^{2k-1} = 0,$$

since the binomial expansion of the $(2k-1)$-st power is a linear combination of ω^{aj} for odd integers j from $-2k+1$ to $2k-1$, none of which are multiples of $2n+1$. Since $(\omega^j + \omega^{-j})/2 = \cos(2j\pi/(2n+1))$, we compute

$$\begin{aligned}
1 &= 1 - \sum_{j=0}^{2n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) \\
&= -\sum_{j=1}^{n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) - \sum_{j=n+1}^{2n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) \\
&= -\sum_{j=1}^{n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) + \sum_{j=n+1}^{2n} \cos^{2k-1}\left(\frac{(2n+1-2j)\pi}{2n+1} \right) \\
&= -\sum_{j=1}^{n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) + \sum_{j=1}^{n} \cos^{2k-1}\left(\frac{(2j-1))\pi}{2n+1} \right) \\
&= \sum_{\ell=1}^{2n} (-1)^{\ell-1} \cos^{2k-1}\left(\frac{\ell\pi}{2n+1} \right) \\
&= \sum_{\ell=1}^{2n} (-1)^{\ell} x_\ell^{2k-1},
\end{aligned}$$

for $k = 1, \ldots, n$.

We will give two proofs that n is the maximum possible value of m.

Proof 1. Define $x_0 = -1$ and $x_{2n+1} = 1$. Let $f(x)$ be the $\{-1,1\}$-valued function that equals 1 on the intervals $[x_0, x_1], [x_2, x_3], \ldots, [x_{2n}, x_{2n+1}]$ and -1 on the intervals $(x_1, x_2), (x_3, x_4), \ldots, (x_{2n-1}, x_{2n})$. Write $f(x) = f_e(x) + f_o(x)$, where f_e and f_o are the even and odd parts: $f_e(x) = (f(x)+f(-x))/2$ and $f_o(x) = (f(x)-f(-x))/2$. Both of these are therefore $\{-1,0,1\}$-valued functions. Note that $f_e(x) = 1$ for $x \in [x0, \min(x_1, -x_{2n})]$ and $x \in [\max(-x_1, x_{2n}), x_{2n+1}]$. The hypothesis implies that

$$\int_{-1}^{1} x^\ell f(x)\,dx = 0$$

for even ℓ up to $2m-2$. Since the contribution of f_o cancels by symmetry, we see that

$$\int_{-1}^{1} x^\ell f_e(x)\,dx = 0$$

for even ℓ up to $2m-2$, and by symmetry this also holds for all odd ℓ. Thus, it holds for $\ell = 0, 1, \ldots, 2m-1$. By a sign change of f_e, we will mean a transition as we increase x from an interval where $f_e = \pm 1$ to one where $f_e = \mp 1$, possibly with an interval where $f_e = 0$ between them. For any sign change, say it occurs at the upper endpoint of the first interval. We claim that the integral condition above implies that the function $f_e(x)$ has at least $2m$ sign changes. (Since $f_e(\pm 1) = 1$, it has an even number of sign changes. If it has fewer then $2m$ sign changes, let $P(x)$ be the monic polynomial with simple roots exactly where the sign changes occur. Then $P(x)f_e(x) \geq 0$ for all x, and it is strictly positive near $x = \pm 1$, but we compute $\int_{-1}^{1} P(x)f_e(x)\,dx = 0$, a contradiction.) But since f_e is $-1,0,1$-valued, each sign change requires at least 2 "jumps" of size 1 in f_e, and f_e can jump by 1 only at the points $\pm x_k$ for $k = 1, \ldots, 2n$. Thus, the number of jumps of size 1 is at least $4m$ and at most $4n$, and hence $m \leq n$.

Proof 2. Look at the polynomial $p(x) = (x+1)\prod_{j=1}^{2n}[x - (-1)^j x_j]$. The condition in the problem statement is that the sum of the $(2k-1)$-st powers of the roots of this polynomial is zero for $k = 1, 2, \ldots, m$. By induction on Newton's identities, it follows that the $(2k-1)$-st elementary symmetric function of the $2n+1$ roots is zero for $k = 1, 2, \ldots, \min(m, n+1)$. If $m > n$, this would imply that $p(x) = xq(x^2)$ for some polynomial q. Then since -1 is a root of p, so would be 1, which would violate the hypothesis that $x_1, \ldots, x_{2n}$ are strictly between -1 and 1.

2022 Session B

B1. Suppose that $P(x) = a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial with integer coefficients, with a_1 odd. Suppose that $e^{P(x)} = b_0 + b_1x + b_2x^2 + \cdots$ for all x. Prove that b_k is nonzero for all $k \geq 0$.

Solution: Call a power series $\sum_{k=0}^{\infty} \frac{c_k}{k!}x^k$ even-ish if $c_0 = 1$ and all other c_k are even integers, and odd-ish if $c_0 = 1$ and all other c_k are odd integers. (If the series converges for x near 0, these definitions say that the associated function $f(x)$ has $f(0) = 1$ and for $k > 0$ all $f^{(k)}(0)$ even, respectively odd, integers.) Note that we have

$$\sum_{k=0}^{\infty} \frac{c_k}{k!}x^k \cdot \sum_{k=0}^{\infty} \frac{d_k}{k!}x^k = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} c_r d_{k-r} x^k.$$

Therefore a product of even-ish series is again even-ish (for any $k \geq 1$ and any r at least one of c_r and d_{k-r} is even) and a product of an even-ish series and an odd-ish series is odd-ish (the only term contributing to the coefficient of x^k that is not even is $\binom{k}{0}c_0d_k$, which is odd). Since

$$e^{a_1x} = \sum_{k=0}^{\infty} \frac{a_1^k}{k!}x^k$$

is odd-ish, and for $j > 1$

$$e^{a_jx^j} = \sum_{k=0}^{\infty} \frac{a_j^k (jk)!/k!}{(jk)!} x^{jk}$$

is even-ish (since for $k \geq 1$, $(jk)!/k!$ is an even integer). Thus, by the remarks above, $e^{P(x)} = e^{a_1x} \cdot e^{a_2x^2} \cdot \cdots \cdot e^{a_nx^n}$ is odd-ish, and in particular its coefficients cannot be zero.

B2. Let $\times$ represent the cross product in $\mathbb{R}^3$. For what positive integers n does there exist a set $S \subset \mathbb{R}^3$ with exactly n elements such that

$$S = \{v \times w \colon v, w \in S\}?$$

Answer: $n = 1$ or $n = 7$.

Solution: Since we only care about positive n, we can assume S is nonempty. For any $v \in S$, we must have $v \times v = 0 \in S$. If $S = \{0\}$, then we have a solution with $n = 1$. Otherwise, let v_1 be a nonzero vector in S. Since v_1 must be in $\{v \times w \colon v, w \in S\}$, there must be vectors $v_2, v_3 \in S$ with $v_1 = v_2 \times v_3$. For $n \geq 3$, define $v_{n+1} = v_1 \times v_n \in S$. Since $v_1 \perp v_n$ for all $n \geq 3$, by induction on these n, we have $|v_n| = |v_1|^{n-3}|v_3|$. Since S is finite, this implies $|v_1| = 1$. Thus, all nonzero vectors in S have unit length.

Now choose a particular nonzero $v_1 \in S$, and as before, choose $v_2, v_3 \in S$ with $v_1 = v_2 \times v_3$. Then since $|v_2| = |v_3| = |v_2 \times v_3| = 1$, we know that v_2 and v_3 are orthogonal to each other as well as to v_1. Thus, $\{v_1, v_2, v_3\}$ forms an orthonormal basis of $\mathbb{R}^3$, and S contains all the cross products of two of them in either order, which means $\{0, \pm v_1, \pm v_2, \pm v_3\} \subset S$. Finally, if w is a nonzero vector in S, it has length 1, and its cross product with each v_i is either 0 or of length 1. It follows that $w = \pm v_i$ for some i. Hence the only possibilities are $n = 1$ and $n = 7$.

B3. Assign to each positive real number a color, either red or blue. Define a recoloring process as follows. First, let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Second, recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?

Answer: Yes.

Solution 1: We first prove the following lemma.

Lemma. After a recoloring d and $2d$ cannot both be blue.

Proof. If d is blue after the recoloring, then in the original coloring any two points at distance d are opposite colors. But then for any x the sequence $x, x+d, x+2d, \ldots$ must alternate between red and blue. Hence there are pairs of points $2d$ apart of the same color and after the recoloring $2d$ will be red.

Now suppose that d is blue after two recolorings. Then after one recoloring the sequence $d, 2d, 3d, 4d$ had to alternate between red and blue. Since $2d$ and $4d$ cannot both be blue after one recoloring, they must both be red after one recoloring. Thus after one recoloring d and $3d$ must both be blue and $2d$, $4d$ must both be red.

Hence in the original coloring $d, 2d, 3d, 4d$ also alternate between red and blue. Thus the point $5d/2$ must have been the same color as one of $(2d, 3d)$ and as one of $(d, 4d)$. Thus both $d/2$ and $3d/2$ must be red after the first recoloring, and hence d must be red after the second recoloring. This is a contradiction. Thus all the numbers are red after two recolorings.

Solution 2:

We claim that all numbers are colored red after two recolorings.

Lemma. After a recoloring, if $d > 0$ is blue, then $d/2$ and $3d/2$ are both red.

Proof. By hypothesis, in the previous coloring, d and $2d$ must have had different colors. Also, $2d$ and $3d$ must have had different colors, and likewise for $3d$ and $4d$. Then d and $4d$ must have had different colors, so $5d/2$ must have had the same color as either d or $4d$, and since it is $3d/2$ away from each, $3d/2$ must be red after the recoloring. Similarly, $5d/2$ must have had the same color as either $2d$ or $3d$, and since it is $d/2$ away from each, $d/2$ must be red after the recoloring.

One consequence of the lemma is that d cannot be blue in two consecutive recolorings, since $d/2$ and $3d/2$ are distance d apart. Thus, if d is blue after two recolorings, is must have been red after one recoloring. Then by the reasoning in the proof of the lemma, $2d$ must have been blue after one recoloring, so $3d$ must have been red, so $4d$ must have been blue. But by the lemma, since $4d$ was blue after one recoloring, $2d = 4d/2$ must have been red, a contradiction. Therefore, no d can be blue after two recolorings.

B4. Find all integers n with $n \geq 4$ for which there exists a sequence of distinct real numbers $x_1, \ldots, x_n$ such that each of the sets

$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots, \{x_{n-2}, x_{n-1}, x_n\}, \{x_{n-1}, x_n, x_1\}, \text{ and } \{x_n, x_1, x_2\}$$

forms a 3-term arithmetic progression when arranged in increasing order.

Answer: $n = 9, 12, 15, 18, \ldots$; more precisely, all multiples of 3 that are strictly greater than 6.

Solution: Since $\{x_j, x_{j+1}, x_{j+2}\}$ forms an arithmetic progression after reordering, we must have $x_{j+1} + x_{j+2} = 2x_j$, or $x_j + x_{j+2} = 2x_{j+1}$, or $x_j + x_{j+1} = 2x_{j+2}$. Thus, x_{j+2} must be equal to one of $2x_j - x_{j+1}$, $2x_{j+1} - x_j$, or $(x_j + x_{j+1})/2$. Hence $x_{j+2} - x_{j+1}$ must be equal to one of $-2(x_{j+1} - x_j)$, $x_{j+1} - x_j$, or $-(x_{j+1} - x_j)/2$. Thus, by an easy induction, there will be a sequence of integers k_j such that $x_{j+1} - x_j = (-2)^{k_j}(x_2 - x_1)$ for $1 \leq j \leq n-1$.

The statements in the previous paragraph are also true for the triples $\{x_{n-1}, x_n, x_1\}$ and $\{x_n, x_1, x_2\}$, and in particular, $x_1 - x_n = (-2)^{k_n}(x_2 - x_1)$ for some integer k_n. Notice also that $|k_{j+1} - k_j| \leq 1$ for $1 \leq j \leq n-1$, that $|k_1 - k_n| \leq 1$, and that $k_1 = 0$. Cyclically rotating the sequence, and rescaling (possibly by a negative number), we may assume that $x_2 - x_1 = 1$ and that all the k_j are nonnegative. Then

$$0 = (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) + (x_1 - x_n) = \sum_{j=1}^{n} (-2)^{k_j} \equiv \sum_{j=1}^{n} 1 = n \pmod 3.$$

Hence n must be a multiple of 3.

For $m \geq 2$ and $n = 3m + 3$, an example of such a sequence is

$$(1, 3, 5, \ldots, 4m-3, 4m-1, 4m-2, 4m, 4m-4, \ldots, 8, 4, 0, 2).$$

Notice that this sequence consists, aside from $4m - 2$ and 2, of an increasing subsequence of odd numbers and a decreasing subsequence of multiples of 4. Thus, the elements of the sequence are distinct unless $4m - 2 = 2$, which happens only for $m = 1$. In fact, there is no example for $n = 6$. (If, as above, we arrange that $x_2 - x_1 = 1$ and all the differences $x_{j+1} - x_j$ and $x_1 - x_n$ are in $\{1, -2, 4, \ldots\}$, then by parity we must have an even number equal to 1. We cannot have three consecutive differences be $1, 1, -2$ or $1, -2, 1$, since either would give two equal terms three apart. Since a 1 can only be adjacent to another 1 or a -2, it follows that we cannot have differences of 1 that are adjacent or two apart. Then we must have exactly two differences of 1, three apart, and the only remaining possible difference sequence is $1, -2, -2, 1, -2, -2$, which does not sum to 0.) Therefore, the answer is that n can be any multiple of 3 that is strictly greater than 6.

B5. For $0 \le p \le 1/2$, let $X_1, X_2, \ldots$ be independent random variables such that

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } p, \\ 0 & \text{with probability } 1-2p, \end{cases}$$

for all $i \ge 1$. Given a positive integer n and integers $b, a_1, \ldots, a_n$, let $P(b, a_1, \ldots, a_n)$ denote the probability that $a_1X_1 + \cdots + a_nX_n = b$. For which values of p is it the case that $P(0, a_1, \ldots, a_n) \ge P(b, a_1, \ldots, a_n)$ for all positive integers n and all integers $b, a_1, \ldots, a_n$?

Answer: $0 \le p \le 1/4$.

Solution: For each $0 \le p \le 1/4$, there is a q between 0 and $1/2$ (inclusive) for which $q(1-q) = p$. For $1 \le k \le n$, let Y_k and Z_k be independent random variables such that $Y_k = 1/2$ with probability q and $Y_k = -1/2$ with probability $1-q$, and Z_k has the same distribution as Y_k. Then X_k can be expressed as $Y_k - Z_k$. Let $Y = a_1Y_1 + \cdots + a_nY_n$, and let $Z = a_1Z_1 + \cdots a_nZ_n$; then $a_1X_1 + \cdots + a_nX_n = Y - Z$, and Y and Z have the same distribution. This distribution has probabilities $p_{-m}, p_{-m+1}, \ldots, p_m$ where $m = (|a_1| + \cdots + |a_n|)/2$; notice that the indices are half-integers if m is a half-integer. Define $p_j = 0$ for every index j with $|j| > m$. Then for every integer b, the Cauchy-Schwarz inequality implies that

$$P(b, a_1, \ldots, a_n) = \sum_{j=-m}^{m} p_{j+b}p_j \le \sum_{j=-m}^{m} p_j^2 = P(0, a_1, \ldots, a_n).$$

To show that the inequality need not be true for $p > 1/4$, let $a_k = 2^{k-1}$ for $k \ge 1$. Then $a_1X_1 + \cdots + a_nX_n = 0$ only when $X_1 = \cdots = X_n = 0$, which occurs with probability $P(0, a_1, a_2, \ldots, a_n) = (1-2p)^n$. For a particular choice of $X_1, \ldots, X_n$ that are not all zero, let m be the largest index for which $X_m \ne 0$. Then if $a_1X_1 + \cdots + a_nX_n = 1$, we must have $X_m = 1$ and $X_k = -1$ for $1 \le k < m$. The probability of this event (including the fact that $X_{m+1} = \cdots = X_n = 0$) is $p^m(1-2p)^{n-m}$. Thus, $P(1, a_1, a_2, \ldots, a_n) = p(1-2p)^{n-1} + p^2(1-2p)^{n-2} + \cdots + p^n$.

If $p > 1/4$, we claim that $P(1, a_1, a_2, \ldots, a_n) > P(0, a_1, a_2, \ldots, a_n)$ for some n. Indeed this is true for $n = 1$ if $p > 1/3$ and for $n = 2$ if $p = 1/3$, so henceforth we assume $1/4 < p < 1/3$. Then

$$\begin{aligned} P(1, a_1, a_2, \ldots, a_n) &= p(1-2p)^{n-1} \sum_{k=0}^{n-1} [p/(1-2p)]^k \\ &= p(1-2p)^{n-1} \frac{1 - [p/(1-2p)]^n}{1 - p/(1-2p)} \\ &= (1-2p)^n \frac{p}{1-3p}(1 - [p/(1-2p)]^n). \end{aligned}$$

Since $p > 1/4$, we have $p/(1-3p) > 1$, and since $p < 1/3$, we have $p/(1-2p) < 1$. Thus, for n sufficiently large, $1 - [p/(1-2p)]^n > (1-3p)/p$, and hence $P(1, a_1, a_2, \ldots, a_n) > (1-2p)^n = P(0, a_1, a_2, \ldots, a_n)$ as claimed.

B6. Find all continuous functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(xf(y)) + f(yf(x)) = 1 + f(x+y)$$

for all $x, y > 0$.

Answer: $f(x) = 1/(1+cx)$ where c is a nonnegative real number.

Solution 1: Suppose we have $f(a) = f(a') = 1$, then plugging $(x, y) = (a, a')$ into the functional equation, we find $f(a + a') = 1$. In particular, if there is any a with $f(a) = 1$, we find that $f(na) = 1$ for all $a \in \mathbb{N}$.

Suppose there is some closed interval $[a, a']$ with $f(x) = 1$ for all $a \le x \le a'$. Using the remark above we see that $f(x) = 1$ for all $na \le x \le na'$ and any positive integer n. Choosing n large enough that $n(a' - a) > a$, we find that successive intervals $[na, na']$ and $[(n+1)a, (n+1)a']$ overlap. Hence there is some $b > 0$ such that $f(x) = 1$ for all $x > b$. For any x choose y large enough that $yf(x) > b$ and $x + y > b$. Then the functional equation gives $f(x) = 1$. Thus f is identically equal to 1.

Now suppose that there is some $c > 0$ with $f(c) > 1$. Define $u = \frac{c}{f(c)-1}$, so that $uf(c) = u + c$. Then the functional equation for $(x, y) = (u, c)$ gives $f\left(cf\left(\frac{c}{f(c)-1}\right)\right) = 1$. In particular, setting $a = cf\left(\frac{c}{f(c)-1}\right)$ we have $f(a) = 1$ and hence $f(na) = 1$ for all n. Since we can choose $na > c$, there is a least $d > c$ such that $f(d) = 1$. On the interval $[c, d)$ the expression $\frac{x}{f(x)-1}$ goes from u (at $x = c$) to infinity (as $x \to d^-$). Thus we can choose $x_m \in [c, d)$ with $\frac{x_m}{f(x_m)-1} = ma$ for all m with $ma > u$. Hence at $x = x_m$, $xf\left(\frac{x}{f(x)-1}\right)$ takes on the value x_m. But this says that $xf\left(\frac{x}{f(x)-1}\right)$ is a non-constant continuous function on $[c, d)$, hence it takes on every value in some open interval I. Since $f > 1$ on $[c, d)$, using the argument at the beginning of this paragraph with c replaced by x shows that $f(a) = 1$ for all $a \in I$. Hence by the previous paragraph, we have f identically 1, and a contradiction. Thus $f(x) \le 1$ for all x.

Suppose $f(a) = 1$ for some a. Then for any $x \in (0, na)$ (with $n \ge 3$) we have $f(xf(na - x)) + f((na - x)f(x)) = 1 + f(na) = 2$ and hence $f(xf(na - x)) = 1$ for all x. However, taking $x = a, 2a$ we find that $xf(na - x)$ takes on the values a and $2a$ and hence again this gives an interval on which f equals 1. Hence if $f(a) = 1$ for any a, then f is identically equal to 1.

Thus we may suppose $f(x) < 1$ for all $x \in (0, \infty)$. Note that this implies $f(x + y) < f(yf(x))$ for all $x, y > 0$. For any $x_0 \in (0, \infty)$ define a sequence by $x_{n+1} = \frac{x_n}{2} f\left(\frac{x_n}{2}\right)$ for $n \ge 1$. Then $x_{n+1} < \frac{x_n}{2}$, so this sequence decreases to 0, and the functional equation gives $f(x_{n+1}) = \frac{1+f(x_n)}{2}$, so $f(x_n)$ converges to 1.

Now we will show that f is strictly decreasing. Suppose $c < d$. The expression $(d-x)f(x)$ tends to d if we take $x = x_n$ and $n \to \infty$ and tends to 0 as $x \to d$. Thus there is some $x \in (0, d)$ with $(d - x)f(x) = c$. Taking $y = d - x$ in the functional equation, we get $x + y = d$ and $yf(x) = c$, and hence $f(c) > f(d)$. Thus f is strictly decreasing. Note that this argument shows that for any $c < d$, we can choose x, y such that $x + y = d$ and $yf(x) = c$, and hence we will get $f(xf(y)) + f(c) = 1 + f(d)$.

Now fix any x_0 and look at the sequence defined by $x_{n+1} = \frac{x_n}{2} f\left(\frac{x_n}{2}\right)$. Then we have $f(x_{n+1}) = \frac{1+f(x_n)}{2}$ and hence $f(x_n) = 1 - 2^{-n}(1 - f(x_0))$. Since f is decreasing it follows that $\lim_{x\to 0^+} f(x) = 1$, and hence we can (and will) extend f continuously to $[0,\infty)$. Further since $f < 1$ is decreasing, we have

$$\frac{x_n}{2} > x_{n+1} > \frac{x_n}{2} f(x_n) = \frac{x_n}{2}\left(1 - 2^{-n}(1 - f(x_0))\right),$$

and hence

$$2^{-n}x_0 > x_n > 2^{-n}x_0 \prod_{k=0}^{n-1}\Big(1 - 2^{-k}(1-f(x_0))\Big) > 2^{-n}x_0 \prod_{k=0}^{\infty}\Big(1 - 2^{-k}(1-f(x_0))\Big).$$

Since the product converges, we get a positive constant C with $2^{-n}x_0 > x_n > C2^{-n}x_0$. Since the graph of $y = f(x)$ lies in the union of the rectangles $[x_{n+1}, x_n] \times [f(x_n), f(x_{n+1})]$, we have that for any $x \in [x_{n+1}, x_n]$

$$\frac{1 - f(x)}{x} \leq \frac{1 - f(x_n)}{x_{n+1}} \leq \frac{2^{-n}(1 - f(x_0))}{2^{-n-1}Cx_0} = \frac{2(1 - f(x_0))}{Cx_0}.$$

Hence the (negated) slopes of secant lines to $y = f(x)$ through $(0,1)$ are bounded and we can write

$$0 \leq \alpha = \liminf_{x\to 0^+} \frac{1 - f(x)}{x} \leq \limsup_{x\to 0^+} \frac{1 - f(x)}{x} = \beta < \infty.$$

Suppose the derivative at 0 does not exist, so that $\alpha < \beta$. Choose an x_0 small enough that $f(x_0) > 2/3$ and $\frac{1-f(x_0)}{x_0} = \beta - \epsilon$ for some small ϵ, and consider the sequence above. Then we have $x_{n+1} > \frac{x_n}{2} f(x_0) > \frac{x_n}{3}$ and since the slopes $\frac{1-f(x_n)}{x_n}$ are increasing, we have $\frac{1-f(x_n)}{x_n} \geq \beta - \epsilon$. Now for any sequence c_n tending to 0 with $\frac{1-f(c_n)}{c_n} < \gamma = \frac{\beta+\alpha}{2}$, we can choose d_n to be the next larger term in the sequence (x_k) and have $d_n/3 < c_n < d_n$. By the above we can choose x'_n, y'_n (which will tend to 0 since $x'_n, y'_n < d_n < 3c_n$) with $x'_n + y'_n = d_n$ and $y'_n f(x'_n) = c_n$ and hence, we get

$$\begin{aligned}\frac{1 - f(x'_n f(y'_n))}{x'_n f(y'_n)} &= \frac{(1 - f(d_n)) - (1 - f(c_n))}{\left(d_n - \frac{c_n}{f(x'_n)}\right) f(y'_n)} \geq \frac{(\beta - \epsilon)d_n - \gamma c_n}{\left(d_n - \frac{c_n}{f(x'_n)}\right) f(y'_n)} \\ &= \frac{\beta - \epsilon}{f(y'_n)} + \frac{\left(\frac{\beta-\epsilon}{f(x'_n)} - \gamma\right) c_n}{\left(d_n - \frac{c_n}{f(x'_n)}\right) f(y'_n)} \geq \frac{\beta - \epsilon}{f(y'_n)} + \frac{\frac{\beta-\epsilon}{f(x'_n)} - \gamma}{\left(3 - \frac{1}{f(x'_n)}\right) f(y'_n)}.\end{aligned}$$

As n tends to infinity, this lower bound tends to $\beta - \epsilon + \frac{\beta-\gamma-\epsilon}{2}$. But for sufficiently small ϵ this exceeds β, contradicting the definition of β. Thus $f'(0) = -\beta$ exists.

Now for any $c < d$ with $d - c$ small, we again choose x, y with $x + y = d$ and $yf(x) = c$. Hence $x = d - y < d - yf(x) = d - c$ is small. Then

$$\frac{f(c) - f(d)}{d - c} = \frac{1 - f(xf(y))}{d - c} = \frac{1 - f(xf(y))}{xf(y)} \cdot \frac{f(y)}{1 + y\frac{1-f(x)}{x}}.$$

Taking the limit as c, d both converge to some fixed t, we see that y also converges to t and x converges to 0. We conclude that f is differentiable at t and

$$f'(t) = -\frac{\beta f(t)}{1 + \beta t}.$$

Thus f is continuously differentiable and satisfies the differential equation above with $f(0) = 1$. Thus $f(x) = \frac{1}{1+\beta x}$ for some $\beta \geq 0$ and these are indeed solutions.

Solution 2: The only solutions are $f(x) = 1/(1+cx)$ for all $x \in \mathbb{R}^+$, where c is a nonnegative real number. Checking that these functions are solutions is straightforward. To prove that these are the only solutions, we start with some notation and a technical lemma. Let D_- and D_+ denote the derivatives of a function from the left and right, respectively.

Lemma 1. *If $g : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function for which $D_-g(z) = \lim_{w \to z^-}[g(z) - g(w)]/(z-w)$ exists and equals 0 for all $z \in \mathbb{R}^+$, then g is a constant function.*

Proof. If g is not constant, choose $a, b \in \mathbb{R}^+$ such that $g(a) \neq g(b)$, and without loss of generality assume that $a < b$ and $g(a) < g(b)$. Let $s = [g(b) - g(a)]/(b-a)$. The continuous function $g(x) - sx$ has a maximum value on $[a, b]$, and since $g(a) - sa = g(b) - sb$, this maximum value is attained at at least one point in $[a, b]$ other than a; thus, there exists $z \in (a, b]$ such that $g(w) - sw \leq g(z) - sz$ for all $w \in [a, b]$. Then $[g(z) - g(w)]/(z - w) \geq s > 0$ for all $w \in [a, z)$, contradicting the hypothesis. □

The next lemma moves the goalposts closer. This and all lemmas below assume the hypotheses of the problem in addition to the hypotheses stated in the lemma.

Lemma 2. *If both of the following limits exist, $\lim_{x \to 0^+} f(x) = 1$, and $\lim_{x \to 0^+}[1 - f(x)]/x = c \geq 0$, then $f(x) = 1/(1+cx)$ for all $x \in \mathbb{R}^+$.*

Proof. Choose $x, z \in \mathbb{R}^+$ with $x < z$, and let $y = z - x$ and $w = yf(x)$. Then

$$\lim_{x \to 0^+} \frac{z-w}{x} = \lim_{x \to 0^+} \frac{z - (z-x)f(x)}{x} = \lim_{x \to 0^+} \frac{z[1-f(x)] + xf(x)}{x} = cz + 1.$$

In particular, $z - w$ is positive and a one-to-one function of x for $x > 0$ sufficiently small, so taking a limit as $x \to 0+$ is equivalent to taking a limit as $w \to z^-$.

Next, from the identity in the problem statement, $f(z) - f(w) = f(x+y) - f(yf(x)) = f(xf(y)) - 1 = f(xf(z-x)) - 1$. Then, using the fact that $f(z-x)$ is a continuous function of (sufficiently small) x,

$$D_-f(z) = \lim_{w \to z^-} \frac{f(z) - f(w)}{z - w} = \lim_{x \to 0+} \frac{f(xf(z-x)) - 1}{xf(z-x)} \cdot \frac{xf(z-x)}{z-w} = -c\frac{f(z)}{1+cz},$$

Let $g(x) = \log f(x) + \log(1+cx)$ for all $x \in \mathbb{R}^+$. The chain rule applies to derivatives from the left when the outer function is two-sided differentiable, so $D_-g(z)$ exists for all $z \in \mathbb{R}^+$, and $D_-g(z) = -c/(1+cz) + c/(1+cz) = 0$. By Lemma 1, g is constant, and since $\lim_{x \to 0+} g(x) = 0$, we have $g(x) = 0$ for all $x \in \mathbb{R}^+$. If follow that $f(x) = \exp[g(x) - \log(1+cx)] = 1/(1+cx)$ as claimed. □

The remainder of the proof establishes the two limits in the hypothesis of Lemma 2.

Lemma 3. *Given the hypotheses of the problem,* $\limsup_{x \to 0^+} f(x) \geq 1$.

Proof. If $\limsup_{x \to 0^+} f(x) < 1$, then for some $x_0 > 0$ and some $r < 1$, we have $f(x) \leq r$ for $0 < x \leq x_0$. Let $x_{n+1} = x_n f(x_n/2)/2$ for $n \geq 0$. Then, by induction, $x_0 > x_1 > \cdots$ and $f(x_n) \leq r$ for all $n \geq 0$. But with $x = y = x_n/2$, the identity in the problem statement yields $2f(x_{n+1}) = 1 + f(x_n)$, which implies that $f(x_n) \to 1$ as $n \to \infty$, a contradiction. □

Lemma 4. *For all $a, b \in \mathbb{R}^+$ with $a < b$, there exists $z(a,b) \in \mathbb{R}$ such that $f(z(a,b)) = 1 + f(b) - f(a)$.*

Proof. By Lemma 3, $\limsup_{x\to 0^+}(b-x)f(x) \geq b$, and $(b-x)f(x)$ is a continuous function of f with value 0 at $x = b$. Since $0 < a < b$, it follows from the intermediate value theorem that $(b-x)f(x) = a$ for some $x \in (0,b)$. Let $y = b - x$ and apply the identity in the problem statement to get $f(xf(b-x)) + f(a) = 1 + f(b)$. Thus, the lemma is proved with $z(a,b) = xf(b-x)$. □

Lemma 5. *If $f(z) = 1$ for some $z \in \mathbb{R}^+$, then $f(x) = 1$ for all $x \in (0,z)$.*

Proof. If $f(x) \neq 1$ for some $x \in (0,z)$, then let $z_0 = z$ and $z_1 = x$, and define z_{n+1} for $n \geq 1$ in terms of $z_0, z_1, \ldots, z_n$ as follows. Let $M_n = \max_{0\leq n} f(z_k)$ and $m_n = \min_{0\leq n} f(z_k)$. Choose integers i and j with $0 \leq i,j < n$ such that $f(z_i) = m_n$ and $f(z_j) = M_n$, and let $a_n = \min(z_i, z_j)$ and $b_n = \max(z_i, z_j)$. Then $|f(b_n) - f(a_n)| = M_n - m_n$. Notice that $f(z_0) \neq f(z_1)$ implies that $f(z_i) < f(z_j)$, which implies that $z_i \neq z_j$, which implies that $a_n < b_n$. Let $z_{n+1} = z(a_n, b_n)$.

We will prove by induction that $M_n - m_n \geq n|f(x) - 1|$. Notice that $M_0 - m_0 = 0$ and that $M_1 - m_1 = |f(z_1) - f(z_0)| = |f(x) - f(z)| = |f(x) - 1|$. Next, since $z_1 = x < z = z_0$, we have $a_1 = x$ and $b_1 = z$, so by Lemma 4, $f(z_2) = f(z(a_1, b_1)) = 1 + f(z) - f(x) = 2 - f(x)$. Thus, $|f(z_2) - 1| = |f(x) - 1|$; also, $f(z_2)$ and $f(z_1) = f(x)$ lie on opposite sides of 1. Then $M_2 - 1 = 1 - m_2 = |f(x) - 1|$, from which it follows immediately that $M_n - 1 \geq |f(x) - 1|$ and $1 - m_n \geq |f(x) - 1|$ for $n \geq 2$. Also, $M_2 - m_2 = 2|f(x) - 1|$.

Assume now for some $n \geq 2$ that $M_n - m_n \geq n|f(x)-1|$. Recall that $f(b_n)$ is either M_n or m_n. If $f(b_n) = M_n$ then by Lemma 4, $f(z_{n+1}) - f(b_n) = f(z(a_n, b_n)) - f(b_n) = 1 - f(a_n) = 1 - m_n$. Then $m_{n+1} = m_n$ and $M_{n+1} = f(z_{n+1}) = M_n + 1 - m_n \geq M_n + |f(x) - 1|$, so $M_{n+1} - m_{n+1} \geq M_n - m_n + |f(x) - 1| \geq (n+1)|f(x) - 1|$, as desired. If $f(b_n) = m_n$, a similar argument completes the induction.

We have now proved that $M_n - m_n$ becomes arbitrarily large as n increases, and since $m_n > 0$, in fact M_n becomes arbitrarily large. In particular, $M_n \geq 2$ for some n; then since M_n is a value of f and $f(z) = 1$, by the intermediate value theorem there exists $w \in \mathbb{R}^+$ such that $f(w) = 2$. Then letting $x = y = w$ in the identity in the problem statement, we have $2f(2w) = 1 + f(2w)$, so $f(2w) = 1$. Then applying Lemma 4, we have that $f(z(w, 2w)) = 1 + 1 - 2 = 0$. But this contradicts the hypotheses of the problem. □

Next, suppose that $f(x) > 1$ for some $x \in \mathbb{R}^+$. Let $y = x/(f(x) - 1)$, so that $x + y = yf(x)$. Then by the identity in the problem statement, $f(xf(y)) = 1$, so Lemma 5 applies. Furthermore, if Lemma 5 applies, then the hypotheses of Lemma 2 apply with $c = 0$. Since it suffices to verify the hypotheses of Lemma 2, we can assume for the rest of the proof that $f(x) < 1$ for all $x \in \mathbb{R}^+$.

Lemma 6. *In the case that $f(x) < 1$ for all $x \in \mathbb{R}^+$, we have $\lim_{x\to 0^+} f(x) = 1$, and f is strictly decreasing on $\mathbb{R}^+$.*

Proof. The first statement follows immediately from the hypothesis and Lemma 3. Next, if $f(a) = f(b)$ for some $a < b$, then by Lemma 4, $f(z(a,b)) = 1$, contradicting the hypothesis. Thus, f is strictly monotonic, and because of the limit at 0, it cannot be increasing, so it is decreasing. □

Lemma 6 verifies the first limit in the hypotheses for Lemma 2. Let $f(0) = 1$, so that f is continuous for $x \geq 0$. It remains to show that $D_+f(0)$ exists; then it must be nonpositive since f is decreasing, making c in Lemma 2 nonnegative.

Lemma 6 and $f(0) = 1$ imply that $h(u) = f^{-1}(1-u)$ is well defined and strictly increasing for $u \geq 0$ sufficiently small, with $h(0) = 0$. It suffices to show that $D_+h(0)$ exists and is positive, whereupon $D_+f(0) = -1/D_+h(0)$.

Lemma 7. *For $v \geq u \geq 0$ with v in the domain of h,*

$$(1-v)h(v) \leq h(u) + h(v-u) \leq h(v).$$

Proof. The inequalities follow directly from $h(0) = 0$ in the cases $u = 0$ and $u = v$, so assume henceforth that $0 < u < v$. Let $b = h(v)$. As in the proof of Lemma 4, consider $(b-x)f(x)$ as a function of $x \in [0, b]$. By Lemma 6, this is a strictly decreasing continuous function with values b at $x = 0$ and 0 at $x = b$. Then $f((b-x)f(x))$ is strictly increasing and continuous in x, with values $f(b) = 1-v$ at $x = 0$ and $f(0) = 1$ at $x = b$. Thus, there is a (unique) $x \in (0, b)$ such that $f((b-x)f(x)) = 1-u$. Then $h(u) = (b-x)f(x)$. Applying the identity in the problem statement with $y = b-x$, we have $f(xf(b-x)) = 1+f(b)-f((b-x)f(x)) = 1-v+u$, so $h(v-u) = xf(b-x)$. Then $h(u)+h(v-u) = (b-x)f(x)+xf(b-x) < b-x+x = b = h(v)$, as desired, and since $f(x)$ and $f(b-x)$ exceed $f(b) = 1-v$, we have $h(u) + h(v-u) > (b-x+x)(1-v) = (1-v)h(v)$, finishing the proof. □

We claim that for all $w > 0$ in the domain of h and all $u \in (0, w]$,

$$(1-13w)h(w)/w = h(w)/w - 13h(w) < h(u)/u < h(w)/w + 13h(w) = (1+13w)h(w)/w.$$

This implies that $\ell = \liminf_{u\to 0^+} h(u)/u \geq (1-13w)h(w)/w$ and $L = \limsup_{u\to 0^+} h(u)/u \leq (1+13w)h(w)/w$. For $w < 1/13$, the lower bound on ℓ is positive, and $L/\ell \leq (1+13w)/(1-13w)$. Letting $w \to 0$, we conclude that $0 < \ell = L = \lim_{u\to 0^+} h(u)/u = D_+h(0)$, as desired. It remains only to prove our claim.

For $k \geq 0$, applying Lemma 7 with $v = w/2^k$ and $u = w/2^{k+1} = v/2$, we have $(1-w/2^k)h(w/2^k) \leq 2h(w/2^{k+1}) \leq h(w/2^k)$. By induction, $[h(w)/2^k]\prod_{j=0}^{k-1}(1-w/2^j) \leq h(w/2^k) \leq h(w)/2^k$ for $k \geq 0$. Using concavity of the logarithm,

$$\begin{aligned}\prod_{j=0}^{k-1}(1-w/2^j) &= \exp\left(\sum_{j=0}^{k-1}\log(1-w/2^j)\right) \geq \exp\left(\sum_{j=0}^{k-1}\log(1-w)/2^j\right) \\ &> \exp\left(2\log(1-w)\right) > \exp\left(\log(1-2w)\right) = 1-2w.\end{aligned}$$

Thus,

$$(1-2w)h(w)/2^k \leq h(w/2^k) \leq h(w)/2^k.$$

If our claim is false, we can choose $u_0 \in (0, w]$ such that $|h(u_0)/u_0 - h(w)/w| \geq 13h(w)$. We will construct a sequence $\{u_j\}$ and prove by induction that for $j \geq 0$,

$$|h(u_j) - u_jh(w)/w| \geq (1 + 3\cdot 2^{2-j})h(w)u_0.$$

For $j = 0$, this inequality is equivalent to the assumption for u_0. Notice that the right side is bounded below by $h(w)u_0$ for all j, while the left side approaches 0 as $u_j \to 0$. We will obtain a contradiction by showing as part of the induction that u_j becomes arbitrarily small.

To proceed inductively, assume for some $n \geq 0$ that real numbers $u_0 > u_1 > \cdots > u_n > 0$ and integers $0 \leq k_0 < k_1 < \ldots < k_n$ have been chosen so that for $j = 0, 1, \ldots, n$, the inequality displayed above holds and $w/2^{k_j} \geq u_j > w/2^{k_j+1}$. We can choose such a k_j for each $u_j \in (0, w]$, and choosing k_0 finishes the base case; the monotonicity of the sequences will be established as part of the induction. Let $u_{n+1} = w/2^{k_n} - u_n$. Then $0 \leq u_{n+1} < w/2^{k_n} - w/2^{k_n+1} = w/2^{k_n+1} < u_n$. Applying Lemma 7 with $u = u_n$ and $v = w/2^{k_n}$ yields $(1 - w/2^{k_n})h(w/2^{k_n}) \leq h(u_n) + h(u_{n+1}) \leq h(w/2^{k_n})$. Then using the inequalities displayed above,

$$\begin{aligned}
|h(u_{n+1}) - u_{n+1}h(w)/w| &= |h(u_{n+1}) - (w/2^{k_n} - u_n)h(w)/w| \\
&= |h(u_{n+1}) + h(u_n) - h(w)/2^{k_n} - [h(u_n) - u_n h(w)/w]| \\
&\geq |h(u_n) - u_n h(w)/w| - |h(u_{n+1}) + h(u_n) - h(w)/2^{k_n}| \\
&\geq (1 + 3 \cdot 2^{2-n})h(w)u_0 - |h(u_{n+1}) + h(u_n) - h(w/2^{k_n})| \\
&\qquad - |h(w/2^{k_n}) - h(w)/2^{k_n}| \\
&\geq (1 + 3 \cdot 2^{2-n})h(w)u_0 - wh(w/2^{k_n})/2^{k_n} - 2wh(w)/2^{k_n} \\
&\geq (1 + 3 \cdot 2^{2-n})h(w)u_0 - 3wh(w)/2^{k_n}.
\end{aligned}$$

Notice that $k_n \geq k_0 + n$ and $w/2^{k_0} < 2u_0$, so that $w/2^{k_n} < 2^{1-n}u_0$. Thus, $|h(u_{n+1}) - u_{n+1}h(w)/w| > (1 + 3 \cdot 2^{2-n} - 3 \cdot 2^{1-n})h(w)u_0 = (1 + 3 \cdot 2^{2-(n+1)})h(w)u_0$, as desired. Next, we observe that $u_{n+1} = 0$ would contradict the inequality we just proved, so $u_{n+1} > 0$, whence k_{n+1} is well-defined. Finally, we have already shown that $u_{n+1} < w/2^{k_n+1} < u_n$, so $k_{n+1} \geq k_n + 1$, and monotonicity of both sequences continues. With the induction complete, observe that $u_j \leq w/2^{k_j} \leq 2^{1-j}u_0$, which approaches 0 as $j \to \infty$, which contradicts the inequality the induction just established. Thus, our claim is true, and our solution is complete.

Solution 3: For positive numbers x and z, let $y = z/f(x)$. Then

$$f\left(xf\left(\frac{z}{f(x)}\right)\right) + f(z) = 1 + f\left(x + \frac{z}{f(x)}\right). \tag{*}$$

Since f is continuous, the right side approaches $1 + f(x)$ as $z \to 0^+$, and in particular is bounded for (positive) z in a neighborhood of 0. Then since f is positive-valued, both terms on the left side are bounded, and in particular $f(z)$ is bounded, for z near 0.

Let $g(z) = zf(z/2)/2$ for $z > 0$; boundedness of f near 0 implies that $g(z) \to 0^+$ as $z \to 0^+$. Then since g is continuous, $f(g(z))$ and $f(z)$ have the same $\liminf$ L_- and the same $\limsup$ L_+ as $z \to 0^+$, both of which are finite according to the preceding paragraph. Letting $x = y = z/2$ in the original functional equation and dividing by 2 yields

$$f(g(z)) = \frac{1 + f(z)}{2}.$$

Taking the $\liminf$ and $\limsup$ of this equation as $z \to 0^+$, it follows that $L_- = L_+ = 1$, so $f(z) \to 1$ as $z \to 0^+$. Define $f(0) = 1$, so that f is now continuous at 0.

Claim. *Further, f is differentiable from the right at* 0.

Proof. Choose z_0 sufficiently small that $|f(z) - 1| \leq 1/3$ for $z \in (0, z_0]$. For such z, we have $2/3 \leq f(z/2) \leq 4/3$, and hence

$$z/3 \leq g(z) \leq 2z/3.$$

Let $z_n = g(z_{n-1})$ for $n \geq 1$; then $z_0 > z_1 > z_2 > \cdots$, and $z_n \to 0^+$ as $n \to \infty$. By continuity, $g(z)$ takes on every value between z_{n+1} and z_n as z goes from z_n to z_{n-1}. Let w_n be a number in $[z_{n+1}, z_n]$; by induction on n, there is an associated sequence (not necessarily unique) of numbers $w_k \in [z_{k+1}, z_k]$ with $w_{k+1} = g(w_k)$ for $0 \leq k < n$. Since $|f(g(z)) - 1| = |f(z) - 1|/2$, it follows also by induction that $|f(w_n) - 1| \leq 2^{-n}/3$ for all $w_n \in [z_{n+1}, z_n]$. Here and for the rest of the proof, we regard z_n as fixed for all $n \geq 0$, but we regard w_n as an arbitrary number in $[z_{n+1}, z_n]$, with a corresponding sequence of predecessors w_k as described above.

Next, let $h(z) = (f(z) - 1)/z$ for $z > 0$. Then

$$h(g(z)) = \frac{f(g(z)) - 1}{g(z)} = \frac{(f(z) - 1)/2}{zf(z/2)/2} = \frac{h(z)}{f(z/2)}.$$

So for $n \geq m \geq 0$,

$$h(w_m) = h(w_n) \prod_{k=m}^{n-1} f(w_k/2).$$

Since $w_k/2 \in [z_{j+1}, z_j]$ for some $j \geq k$, we have $|f(w_k/2) - 1| \leq 2^{-k}/3$, so

$$\prod_{k=m}^{\infty} (1 - 2^{-k}/3) \leq \prod_{k=m}^{n-1} f(w_k/2) \leq \prod_{k=m}^{\infty} (1 + 2^{-k}/3).$$

Then since the infinite sum of $2^{-k}/3$ converges, the infinite products above converge, and both products approach 1 as $m \to \infty$. Thus, we can write $|h(w_m)/h(w_n) - 1| \leq \delta_m$ where $\delta_m \to 0$ as $m \to \infty$. Choose m such that $\delta_m < 1$. Since h is continuous, $h(w_m)$ is bounded for $w_m \in [z_{m+1}, z_m]$. It follows that $h(w_n)$ can be bounded independently of both n and w_m. Thus, $h(z)$ is bounded for $z \in (0, z_m]$, since every such z is equal to a w_n that satisfies the inequalities above. Also,

$$|h(w_m) - h(w_n)| \leq \delta_m |h(w_n)|.$$

Let a and $a + b$ be respectively the lim inf and the lim sup of $h(z)$ as $z \to 0^+$, both of which are finite according to the preceding paragraph. To complete the proof of the Claim, we must show that $b = 0$; if not, then $b > 0$. In that case, there exist arbitrarily small values of $z > 0$ such that $h(z) \geq a + 19b/20$. For such a z that is less than z_1, choose $m \geq 0$ such that $z \in [z_{m+2}, z_{m+1})$. Choose $w \in (0, z_m]$ such that $a - b/20 \leq h(w) \leq a + b/20$, choose $n \geq m$ such that $w \in [z_{n+1}, z_n]$, and write $w = w_n$, where w_k has the properties described above for $0 \leq k < n$. In particular, $w_m \geq z_{m+1} > z$, and

$$h(w_m) \leq h(w) + \delta_m |h(w)| \leq a + b/20 + \delta_m(|a| + |b/20|).$$

Returning to the identity (*) at the beginning of this solution, we subtract 2 from both sides and rewrite, for example, $f(z) - 1 = zh(z)$ to get

$$xf\left(\frac{z}{f(x)}\right) h\left(xf\left(\frac{z}{f(x)}\right)\right) + zh(z) = \left(x + \frac{z}{f(x)}\right) h\left(x + \frac{z}{f(x)}\right).$$

Notice that $x + z/f(x)$ goes from z to ∞ as x goes from 0 to ∞; since $w_m > z$, by continuity we can choose $x > 0$ so that $x + z/f(x) = w_m$. Then $x < w_m \leq z_m$, so $|f(x) - 1| \leq 2^{-m}/3$. Thus,

$$\left|\frac{z}{f(x)} - z\right| = \frac{|z(1 - f(x))|}{f(x)} \leq \frac{z2^{-m}/3}{1 - 2^{-m}/3} \leq z2^{-m}/2.$$

In particular, $z/f(x) \le z + z2^{-m}/2 \le 3z/2$. Since $z \le z_{m+1} = g(z_m) \le 2z_m/3$, we have $z/f(x) \le z_m$, so $|f(z/f(x)) - 1| \le 2^{-m}/3$. Next, we calculate

$$\begin{aligned} h\left(xf\left(\frac{z}{f(x)}\right)\right) &= \frac{(x + z/f(x))h(x + z/f(x)) - zh(z)}{xf(z/f(x))} \\ &\le \frac{(x + z + z2^{-m}/2)(a + b/20 + \delta_m(|a| + |b/20|)) - z(a + 19b/20)}{x(1 - 2^{-m}/3)}. \end{aligned}$$

As $m \to \infty$, the right side approaches

$$\frac{(x+z)(a + b/20) - z(a + 19b/20)}{x} = a + b/20 - \frac{z}{x}(9b/10).$$

Since $z \ge z_{m+2} = g(z_{m+1}) \ge z_{m+1}/3 = g(z_m)/3 \ge z_m/9 \ge x/9$, we have $a + b/20 - (z/x)(9b/10) \le a + b/20 - b/10 = a - b/20$. Also, as z becomes arbitrarily small, so does $xf(z/f(x))$, because $x \le 9z$, and m becomes arbitrarily large. We have shown that $h(xf(z/f(x)))$ can be bounded above by a quantity that approaches $a - b/20 < a$ as $m \to \infty$. This contradicts the definition of a if $b > 0$, so we conclude that $b = 0$ and that a is the limit of $h(z)$ as $z \to 0^+$. $\square$

We can now write $f'(0) = a$, where $f'(0)$ represents the derivative from the right. Then $x + z/f(x)$ is right-differentiable at $x = 0$, with derivative $1 - zf'(0)/f(0)^2 = 1 - az$. Assume that $1 - az > 0$; then $x + z/f(x)$ is continuous and strictly increasing for $x \ge 0$ sufficiently small, with value z at $x = 0$. We next show that f is (two-sided) differentiable at z for all $z > 0$ with $1 - az > 0$. First, we subtract $f(z) + 1$ from each side of (*) and divide by $x + z/f(x) - z$ to get, for $x > 0$ sufficiently small,

$$\begin{aligned} \frac{f(x + z/f(x)) - f(z)}{x + z/f(x) - z} &= \frac{f(xf(z/f(x))) - 1}{x + z/f(x) - z} = \frac{f(xf(z/f(x))) - 1}{xf(z/f(x))} \frac{xf(z/f(x))}{x + z/f(x) - z} \\ &= \frac{f(xf(z/f(x))) - 1}{xf(z/f(x))} \frac{f(x)f(z/f(x))}{f(x) + z(1 - f(x))/x} \end{aligned}$$

The right side has limit $af(z)/(1 - az)$ as $x \to 0^+$, so the left side has the same limit, which is then the derivative of f from the right at z.

Next, we substitute $x = z - y$ into the functional equation from the problem statement to get

$$f((z-y)f(y)) + f(yf(z-y)) = 1 + f(z).$$

Now $(z - y)f(y)$ is right-differentiable at $y = 0$, with derivative $zf'(0) - f(0) = az - 1$. Thus, $(z - y)f(y)$ is strictly decreasing for $y \ge 0$ sufficiently small, with value z at $y = 0$. We rearrange terms in the identity above and divide by $z - (z - y)f(y)$ to get, for $y > 0$ sufficiently small,

$$\begin{aligned} \frac{f(z) - f((z-y)f(y))}{z - (z-y)f(y)} &= \frac{f(yf(z-y)) - 1}{z - (z-y)f(y)} = \frac{f(yf(z-y)) - 1}{yf(z-y)} \frac{yf(z-y)}{z - (z-y)f(y)} \\ &= \frac{f(yf(z-y)) - 1}{yf(z-y)} \frac{f(z-y)}{z(1 - f(y))/y + f(y)} \end{aligned}$$

The right side has limit $af(z)/(1 - az)$ as $y \to 0^+$, so the left side has the same limit, which is then the derivative of f from the left at z.

We conclude that

$$f'(z) = \frac{af(z)}{1 - az}$$

for all $z > 0$ such that $1 - az > 0$. This linear differential equation, together with the initial condition $f(0) = 1$, has a unique solution on its domain of definition; this solution can be verified to be $f(z) = 1/(1 - az)$. Then if $a > 0$, it is impossible for f to be continuous at $1/a$. Thus, we must have $a \leq 0$, in which case $1 - az > 0$ for all $z > 0$. Therefore, every solution must have the form $f(z) = 1/(1 - az)$ for all $z > 0$, for some constant $a \leq 0$. One can check that this function is in fact a solution of the functional equation in the problem statement, for each $a \leq 0$.