# Online supplement to the manuscript: "Get infinitely rich! (while definitely going broke)"

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In what follows, we define $\mathbb{N}$ to be $\{1, 2, 3, \ldots\}$, the set of positive integers.

## 1 Why composition of polynomials works

The following result appears in multiple places (see for instance [5], and also Section XII.1 of [2]):

**Theorem 1.1.** *To get the next polynomial $p_n(x)$ from $p_{n-1}(x)$ and $q_n(x)$, replace every $x$ in $p_{n-1}(x)$ with $q_n(x)$. In other words:*

$$p_n = p_{n-1} \circ q_n.$$

*Proof.* Consider a monomial $r_k x^k$, where $r_k \in [0, 1]$; suppose that this is one term of the polynomial $p_{n-1}(x)$. This monomial then records the probability (after the first $(n-1)$ machines) of having exactly $k$ coins; that probability would be $r_k$. Now, suppose that we do indeed have exactly $k$ coins, and place them all into machine $M_n$. Each coin now represents a random variable, as it either wins or loses when placed into $M_n$. By assumption, these random variables are all independent. We are interested in the *sum* of these random variables, meaning the total number of coins after all plays on $M_n$ are complete.

If $A$ and $B$ are independent random variables, then their sum $(A + B)$ is distributed according to the *convolution* of the probability distributions of $A$ and $B$. It so happens that the convolution rule is exactly the same as the rule for multiplication of polynomials. If $A$ takes values $a_1, a_2, \ldots$ with probabilities $s_1, s_2, \ldots$ respectively, and $B$ takes values $b_1, b_2, \ldots$ with probabilities $t_1, t_2, \ldots$ respectively, and $A, B$ are independent, then for each $u$,

$$P(A + B = u) = \sum_{i,j} s_i t_j \begin{cases} 1, & \text{if } u = a_i + b_j \\ 0, & \text{else.} \end{cases}$$

Meanwhile if $f(x) = \sum s_i x^{a_i}$ and $g(x) = \sum t_j x^{b_j}$, then for each $u$, the coefficient on $x^u$, in the product $f(x)g(x)$, is also

$$\sum_{i,j} s_i t_j \begin{cases} 1, & \text{if } u = a_i + b_j \\ 0, & \text{else.} \end{cases}$$

By induction on $k$ it follows that, given $k$ independent random variables $X_1, \ldots, X_k$, all identically distributed according to the probability generating function $q_n(x)$, the sum $X_1 + \cdots + X_k$ follows the probability generating function $q_n(x)^k$. Finally we weight this result by $r_k$ and add it to the weighted results from all the other monomials, according to the law of total probability. The result is precisely $p_{n-1}(q_n(x))$. □

Essentially the same proof still works even if we have multiple variables $x_1, x_2, \ldots, x_m$, standing for various "types" of coins. Also, we may replace "polynomial" everywhere with "formal power series"—however, the above proof does require all exponents to be nonnegative integers.

## 2 A proof that we go broke with probability 1

**Conjecture 2.1.** *Let $p_0(x) = x$, and for each $n \in \mathbb{N}$, let*

$$q_n(x) = \frac{x^{n+1} + (n-1)}{n},$$

*and for each $n \in \mathbb{N}$, let*

$$p_n(x) = p_{n-1}(q_n(x)).$$

*Then*

$$\lim_{n\to\infty} p_n(0) = 1.$$

This section is devoted to showing that Conjecture 2.1 is true. Notice that, by induction, all $p_n$ and $q_n$ are polynomials with nonnegative coefficients and at least one non-constant coefficient strictly positive; hence all $p_n$ and $q_n$ are strictly increasing on $[0,1]$. Below, we use this result repeatedly.

**Proposition 2.2.** *For all $n \in \mathbb{N}$, and for all $x \in [0,1]$,*

$$q_{n+1}(x) \geq q_n(x).$$

*Further, if $x \in [0,1)$ then $q_{n+1}(x) > q_n(x)$.*

*Proof.* Certainly $q_n(1) = 1$ for all $n$, so assume $0 \leq x < 1$. The following statements are equivalent:

$$\begin{aligned}
q_{n+1}(x) &> q_n(x) \\
\frac{x^{n+2}+n}{n+1} &> \frac{x^{n+1}+n-1}{n} \\
nx^{n+2}+n^2 &> (n+1)x^{n+1}+n^2-1 \\
nx^{n+2}-nx^{n+1} &> x^{n+1}-1 \\
nx^{n+1}(x-1) &> x^{n+1}-1 \\
nx^{n+1} &< \frac{1-x^{n+1}}{1-x} \\
nx^{n+1} &< 1+x+x^2+\cdots+x^n.
\end{aligned}$$

But $0 \le x < 1$, so

$$x \ge x^2 \ge \ldots \ge x^n \ge x^{n+1},$$

and therefore

$$nx^{n+1} \le x+x^2+\cdots+x^n < 1+x+x^2+\cdots+x^n. \qquad \square$$

**Proposition 2.3.** *For each $n \in \mathbb{N}$, there exists a unique $Q_n \in [0,1)$ such that $q_n(Q_n) = Q_n$.*

*Proof.* Let $h_n(x) = x^{n+1} - nx + n - 1$. Then $h_n(x) = 0$ if and only if $q_n(x) = x$. We know that $h_n(1) = 0$, and synthetic division yields

$$h_n(x) = (x-1)((x^n + x^{n-1} + \cdots + x) + 1 - n).$$

Let $g_n(x) = x^n + x^{n-1} + \cdots + x + 1 - n$, so we want to show that $g_n$ has a unique zero in $[0,1)$. Uniqueness is clear because $g_n'$ is positive on $(0,1)$, so it suffices to show existence. But $g_n(0) = 1 - n \le 0$ while $g_n(1) = 1 > 0$, so $g_n$ has a zero in $[0,1)$ by the Intermediate Value Theorem. We call this number $Q_n$. $\square$

**Proposition 2.4.** *Let $n \in \mathbb{N}$. For all $x \in [0,1)$,*

$$x < q_n(x) \iff x < Q_n,$$

*and*

$$x = q_n(x) \iff x = Q_n.$$

*Proof.* Let $x \in [0,1)$. First suppose $x < q_n(x)$. We calculate that $q_n'(1) > 1$, so by continuity there exists $\varepsilon > 0$ such that $q_n' > 1$ on $(1-\varepsilon, 1]$. Recall that $q_n(1) = 1$ so by the Mean Value Theorem, $q_n(t) < t$ for all $t \in (1-\varepsilon, 1)$. In particular $x \le 1 - \varepsilon$. But now by the Intermediate Value Theorem $q_n$ has a fixed-point in $(x, 1-\varepsilon/2)$. By uniqueness, this fixed-point must be $Q_n$, hence $x < Q_n$. For the other direction, suppose $x \ge q_n(x)$. Then since $0 \le q_n(0)$, by the Intermediate Value Theorem there exists a fixed-point of $q_n$ in $[0,x]$; hence $x \ge Q_n$.

The second equivalence is simply existence and uniqueness of $Q_n$. (We include the statement here to show that the strict inequalities in the first equivalence can be replaced with non-strict inequalities, and/or reversed, as desired.) $\square$

**Proposition 2.5.** *For all $n \in \mathbb{N}$,*

$$Q_n \geq 1 - \frac{2}{n^2}.$$

*Proof.* Using Proposition 2.4, the following inequalities are equivalent:

$$\begin{aligned}
1 - \frac{2}{n^2} &\leq Q_n \\
1 - \frac{2}{n^2} &\leq q_n\left(1 - \frac{2}{n^2}\right) \\
1 - \frac{2}{n^2} &\leq \frac{\left(1 - \frac{2}{n^2}\right)^{n+1} + n - 1}{n} \\
n - \frac{2}{n} &\leq \left(1 - \frac{2}{n^2}\right)^{n+1} + n - 1 \\
1 - \frac{2}{n} &\leq \left(1 - \frac{2}{n^2}\right)^{n+1} \\
1 - \frac{2}{n} &\leq 1 - \binom{n+1}{1}\left(\frac{2}{n^2}\right) + \binom{n+1}{2}\left(\frac{4}{n^4}\right) - \ldots \pm \binom{n+1}{n+1}\left(\frac{2^{n+1}}{n^{2(n+1)}}\right) \\
\frac{2}{n^2} &\leq \binom{n+1}{2}\left(\frac{4}{n^4}\right) - \binom{n+1}{3}\left(\frac{8}{n^6}\right) + \ldots \pm \binom{n+1}{n+1}\left(\frac{2^{n+1}}{n^{2(n+1)}}\right).
\end{aligned}$$

But the right-hand side is an example of a (finite) Alternating Series. Certainly its terms alternate in sign, and we claim that they are monotone decreasing in absolute value. Proof of claim: Let $a_k$ be the signed term $\pm\binom{n+1}{k}\left(\frac{2^k}{n^{2k}}\right)$ from the right-hand side, beginning with $k = 2$, and for $k > n+1$ let $a_k = 0$. For all $k > n+1$ we have $a_k = 0 = a_{k+1}$; hence $|a_{k+1}| \leq |a_k|$ for those $k$. For $2 \leq k \leq n+1$, we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\binom{n+1}{k+1}(2)}{\binom{n+1}{k}(n^2)} = \frac{2(n+1-k)}{(k+1)n^2} \leq \frac{2n}{3n^2} < 1.$$

Therefore by the explicit bounds in the Alternating Series Test, the right-hand side is bounded below by its second partial sum, so we are done if we can show that

$$\frac{2}{n^2} \leq \binom{n+1}{2}\left(\frac{4}{n^4}\right) - \binom{n+1}{3}\left(\frac{8}{n^6}\right).$$

Multiplying both sides by $6n^6$ to clear denominators, it is equivalent to show that

$$\begin{aligned} 12n^4 &\leq 12n^2(n+1)(n) - 8(n+1)(n)(n-1) \\ 0 &\leq 12n^3 - 8(n^3 - n) \\ 0 &\leq 4n^3 + 8n. \end{aligned}$$

Clearly this last inequality holds for all $n \in \mathbb{N}$. □

*Remark.* The above proof can also be modified to show that $Q_n \leq 1 - \frac{1}{n^2}$, but we will not really need an upper bound on $Q_n$ (other than $Q_n < 1$).

**Corollary 2.6.**

$$\lim_{n\to\infty} Q_n = 1.$$

*Proof.* This follows from the Squeeze Theorem, since $1 - \frac{2}{n^2} \leq Q_n \leq 1$. □

**Proposition 2.7.** *For each $n \in \mathbb{N}$,*

$$Q_n < Q_{n+1}.$$

*Proof.* We have

$$Q_{n+1} = q_{n+1}(Q_{n+1}) > q_n(Q_{n+1}),$$

by Proposition 2.2, since $Q_{n+1}$ is strictly less than 1 by its original definition in Proposition 2.3. But $Q_{n+1} > q_n(Q_{n+1}) \iff Q_{n+1} > Q_n$, by Proposition 2.4. □

**Proposition 2.8.** *There exists $L \leq 1$ such that*

$$\lim_{n\to\infty} p_n(0) = L.$$

*Proof.* We said that $p_n(0)$ is the probability of being broke after machine $M_n$. But if we are broke after $M_n$, then we are still broke after $M_{n+1}$. Therefore

$$p_n(0) \leq p_{n+1}(0),$$

for all $n$. But also, $p_n(0) \leq 1$ for all $n$, since each $p_n(0)$ is a probability. Therefore by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} p_n(0)$$

exists, and is at most 1. □

**Definition 2.9.** We define $L$ to be the limit in Proposition 2.8.

**Proposition 2.10.** *Let $x \in [0,1)$, and $n \in \mathbb{N}$, and write $q_n^k$ to mean the composition of $k$ copies of $q_n$. Then:*
*(i) If $x < Q_n$, then $x < q_n(x) < Q_n$.*
*(ii) If $x > Q_n$ then $x > q_n(x) > Q_n$.*
*(iii) If $x = Q_n$ then $x = q_n(x) = Q_n$.*
*(iv)*

$$\lim_{k\to\infty} q_n^k(x) = Q_n.$$

*Proof.* Statement (iii) is clear. For (i), suppose that $x < Q_n$. Then

$$x < q_n(x)$$

by Proposition 2.4, and

$$q_n(x) < q_n(Q_n)$$

because $q_n$ is strictly increasing on $[0, 1]$. The proof of statement (ii) is similar.

It remains to prove statement (iv). By (i), (ii), and (iii), the sequence

$$(x, q_n(x), q_n(q_n(x)), \ldots, q_n^k(x), \ldots)$$

is monotone, and it is bounded by 0 and 1. Let $G$ be the limit of this sequence. Then

$$G = \lim_{k\to\infty} q_n^k(x) = q_n\left(\lim_{k\to\infty} q_n^{k-1}(x)\right) = q_n(G)$$

by continuity, so $G$ is a fixed-point of $q_n$. If the sequence is decreasing then $G \leq x < 1$; if it is increasing then $G \leq Q_n < 1$. Either way, $G$ is a fixed-point belonging to $[0, 1)$, so $G = Q_n$. □

Thus, applying $q_n$ results in movement toward $Q_n$, and repeatedly applying $q_n$ moves a point arbitrarily close to $Q_n$.

**Proposition 2.11.** *The sequence*

$$(p_1(Q_1), p_2(Q_2), \ldots, p_n(Q_n), \ldots)$$

*is monotone increasing, and convergent.*

*Proof.* Let $n \geq 2$. Then

$$p_{n-1}(Q_{n-1}) = p_{n-1}(q_{n-1}(Q_{n-1})) \leq p_{n-1}(q_n(Q_{n-1})) = p_n(Q_{n-1}) \leq p_n(Q_n),$$

by Propositions 2.2 and 2.7, and the sequence is bounded above by 1. □

**Proposition 2.12.**

$$\lim_{n\to\infty} p_n(Q_n) = L.$$

*Proof.* For each $n \geq 2$, we know that $\lim_{k\to\infty} q_n^k(0) = Q_n > Q_{n-1}$, by Propositions 2.7 and 2.10 (iv). Thus for each $n \geq 2$ we may choose $K_n \in \mathbb{N}$ such that $q_n^m(0) > Q_{n-1}$ for all $m \geq K_n$; then let $k_1 = 1$, and for $n \geq 2$ let $k_n = \max(K_n, k_{n-1})$. Now by Proposition 2.2, for all $n \geq 2$,

$$\begin{aligned} p_{n-1}(Q_{n-1}) &\leq p_{n-1}(q_n^{k_n+1}(0)) = ((q_1 \circ q_2 \circ \ldots \circ q_{n-1}) \circ (q_n \circ q_n \circ \ldots \circ q_n))(0) \\ &\leq ((q_1 \circ q_2 \circ \ldots \circ q_{n-1}) \circ (q_n \circ q_{n+1} \circ \ldots \circ q_{n+k_n}))(0) \\ &= p_{n+k_n}(0) \\ &\leq p_{n+k_n}(Q_{n+k_n}). \end{aligned}$$

Thus for each $n \geq 2$,

$$p_{n-1}(Q_{n-1}) \leq p_{n+k_n}(0) \leq p_{n+k_n}(Q_{n+k_n}). \tag{1}$$

But $(p_{n+k_n}(Q_{n+k_n}))$ is a subsequence of $(p_n(Q_n))$, because we required $k_{n+1} \geq k_n \geq 1$, and $(p_n(Q_n))$ is convergent by Proposition 2.11. Therefore $(p_{n+k_n}(Q_{n+k_n}))$ and $(p_n(Q_n))$ must share the same limit, say $T$. Thus in Inequality 1, the two outside terms approach $T$ as $n \to \infty$, while the middle term approaches $L$. By the Squeeze Theorem, $L = T$. $\square$

**Theorem 2.13.**

$$L = 1.$$

*That is, Conjecture 2.1 is true.*

*Proof.* Given $n \in \mathbb{N}$, let $L_n(x)$ be the linear approximation to $p_n(x)$, taken at base point $a = 1$. That is,

$$\begin{aligned} L_n(x) &= p_n'(1)(x-1) + p_n(1) \\ &= (n+1)(x-1) + 1. \end{aligned}$$

Since $p_n''$ is nonnegative on $(0, 1)$, we claim that

$$p_n(x) \geq L_n(x)$$

for all $x \in [0, 1)$. Proof of claim: we show the contrapositive, that if $p_n(x) < L_n(x)$ for some $x \in [0, 1)$, then there exists $d \in (0, 1)$ such that $p_n''(d) < 0$. Suppose that $x \in [0, 1)$ and $p_n(x) < L_n(x)$. By the Mean Value Theorem, there exists $c \in (x, 1)$ such that

$$\begin{aligned} p_n'(c) - L_n'(c) &= \frac{p_n(x) - L_n(x) - (p_n(1) - L_n(1))}{x - 1} \\ &= \frac{p_n(x) - L_n(x)}{x - 1}. \end{aligned}$$

Therefore $p_n'(c) - L_n'(c) > 0$, since both the top and bottom of the fraction are negative. Now by the MVT again, there exists $d \in (c, 1)$ such that

$$p_n''(d) - L_n''(d) = \frac{p_n'(c) - L_n'(c) - (p_n'(1) - L_n'(1))}{c - 1}.$$

But $L_n$ is linear so its second derivative is 0 everywhere; meanwhile $L_n'(1) = p_n'(1)$ by definition of $L_n$. Thus

$$p_n''(d) = \frac{p_n'(c) - L_n'(c)}{c - 1}.$$

Above, we had $p_n'(c) - L_n'(c) > 0$, so $p_n''(d) < 0$. This proves the claim.

So we conclude that $p_n(x) \geq L_n(x)$ for all $x \in [0, 1)$. Therefore for all $n \in \mathbb{N}$,

$$p_n(Q_n) \geq L_n(Q_n) \geq L_n\left(1 - \frac{2}{n^2}\right),$$

since $L_n$ is increasing (and using Proposition 2.5). Thus for all $n$,

$$1 \geq p_n(Q_n) \geq (n+1)\left(-\frac{2}{n^2}\right) + 1 = 1 - \frac{2n+2}{n^2}.$$

By the Squeeze Theorem, $p_n(Q_n) \to 1$; hence $L = 1$ by Proposition 2.12. □

# 3 A fair(ish) game where you don't necessarily go broke

In this section we examine a sequence of machines which approach fairness, but where going broke has probability $< 1$. Below, we'll say that "round $n$" means the procedure of putting all your coins into machine $n$, and collecting your winnings.

Consider a sequence of slot machines where the $n$th machine returns either 2 coins, with probability $\alpha_n$, or 0 coins with probability $1 - \alpha_n$. The probability of never going broke is certainly at least as large as the probability of always having at least $n + 1$ coins after playing the $n$th machine. And this probability is at least as large as the infinite product

$$Q = \prod_{n=1}^{\infty} P\left(\text{win at least } n+1 \text{ coins in round } n \middle| \text{start round } n \text{ with } n \text{ coins}\right),$$

for if we ever win strictly more than $n + 1$ coins in a round $n$, we can either (1) discard the excess, or (2) put it into a separate "account" which we can play separately on the side. If anything, choice (2) will improve our chances of reaching a given number of coins in the future, relative to choice (1).

Let $X_n$ be the number of coins after playing rounds 1 through $n$. We want to find $\alpha_n$ such that $\alpha_n \to \frac{1}{2}$ but $Q > 0$. If

$$P\left(X_n \geq n+1 \middle| X_{n-1} = n\right) \geq 1 - \frac{1}{(n+1)^2} \tag{2}$$

for all $n \geq 1$, then $Q$ will be at least

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{(n+1)^2}\right) = \prod_{n=1}^{\infty}\left(\frac{n(n+2)}{(n+1)^2}\right) = \left(\frac{1 \cdot 3}{2 \cdot 2}\right)\left(\frac{2 \cdot 4}{3 \cdot 3}\right)\left(\frac{3 \cdot 5}{4 \cdot 4}\right) \cdots = \frac{1}{2} > 0.$$

Of course (provided that $\alpha_n > 0$ for all $n$) it actually suffices to show that Inequality 2 holds for all sufficiently large $n$; it need not be true immediately at $n = 1$.

Now $X_n$ is a random variable, given by $X_n = 2Y_n$, where $Y_n$ is the number of wins in round $n$. We assume that round $n$ begins with exactly $n$ coins; hence $Y_n \sim Binomial(n, \alpha_n)$. We have

$$P(X_n \geq n+1) = P\left(Y_n \geq \frac{n+1}{2}\right) \geq P\left(Y_n > \frac{n+1}{2}\right) = 1 - P\left(Y_n \leq \frac{n+1}{2}\right)$$

so we wish to show that $P\left(Y_n \leq \frac{n+1}{2}\right) \leq \frac{1}{(n+1)^2}$ (for all sufficiently large $n$). We set $\alpha_n = \frac{n+1}{2n} + d_n$, and try to find $d_n$ satisfying all desired properties, including $d_n \to 0$.

We will choose $d_n \geq 0$, so that $\frac{n+1}{2} \leq n\alpha_n$. Let $Z_n = n - Y_n$, so $Z_n$ is the number of failures out of $n$ trials, and $Z_n \sim Binomial(n, 1-\alpha_n)$. Then

$$E(Z_n) = n(1-\alpha_n) = n - \frac{n+1}{2} - nd_n = \frac{n-1}{2} - nd_n,$$

so

$$P\left(Y_n \leq \frac{n+1}{2}\right) = P\left(Z_n \geq \frac{n-1}{2}\right) = P\left(Z_n - E(Z_n) \geq nd_n\right).$$

Recall that $d_n \geq 0$. If $d_n > 0$ then by Hoeffding's Inequality (Theorem 2 of [4]),

$$P\left(\frac{Z_n - E(Z_n)}{n} \geq d_n\right) \leq \exp\left(\frac{-2n^2 (d_n)^2}{n}\right) = \exp\left(-2n\left(d_n^2\right)\right).$$

(On the other hand if $d_n = 0$ then $\exp\left(-2n\left(d_n^2\right)\right) = \exp(0) = 1$, so the same bound holds trivially in this case.)

Now take $d_n = \min\left(1 - \frac{n+1}{2n}, \frac{1}{n^{1/4}}\right)$, so $d_n \geq 0$ for all $n$, and $d_n \to 0$, and $2n(d_n^2) = 2\sqrt{n}$ for large $n$ (specifically, $n \geq 20$). Finally

$$\frac{1}{\exp(2\sqrt{n})} \leq \frac{1}{(n+1)^2} \tag{3}$$

for all $n \geq 0$, so $Q > 0$. (To verify Inequality 3: we show that $2\sqrt{x} \geq 2\ln(x+1)$ for all $x \in [0, \infty)$. The two sides are equal at $x = 0$, and we claim that $\frac{d}{dx}\sqrt{x} \geq \frac{d}{dx}\ln(x+1)$, for all $x > 0$. It is equivalent to show that $x + 1 \geq 2\sqrt{x}$ for all $x > 0$; this is equivalent to $(\sqrt{x} - 1)^2 \geq 0$.) □

*Remark.* Indeed, for all $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $(1-Q)^N < \varepsilon$. Thus by beginning with one coin, then playing a machine that returns $N$ coins with probability 1, and then following it up with the $(\alpha_n)$ sequence of machines, we can guarantee that the probability of eventually going broke is less than $\varepsilon$, even though $\alpha_n \to \frac{1}{2}$ so the machines approach fairness.

*Remark.* Above, we only needed Hoeffding's Inequality for the case of a binomial random variable. This narrower result appears as Lemma 1 in [7], and it is a direct consequence of Theorem 1 in [1]; both of these sources predate [4]. In addition, Theorem 1 in [4] would be sufficient for our purposes. However, Theorem 2 in [4] is stated in a form which is convenient for us, and [4] gives a complete, self-contained proof of the more general result.

*Remark.* Above, we used Hoeffding's Inequality instead of, say, the normal approximation to the binomial distribution (Central Limit Theorem). The problem with the Central Limit Theorem approximation

$$\begin{aligned} P\left(Y_n \le \frac{n+1}{2}\right) &= P\left(\overline{Y}_n - \alpha_n \le \frac{n+1}{2n} - \alpha_n\right) \\ &= P\left(\frac{\overline{Y}_n - \alpha_n}{\sqrt{n\alpha_n(1-\alpha_n)}} \le \frac{-d_n}{\sqrt{n\alpha_n(1-\alpha_n)}}\right) \\ &\approx \Phi\left(\frac{-d_n}{\sqrt{n\alpha_n(1-\alpha_n)}}\right) \\ &\le \Phi\left(\frac{-d_n}{\sqrt{n/4}}\right), \end{aligned}$$

where $\overline{Y}_n = Y_n/n$ and $\Phi$ is the cumulative distribution function for the standard normal distribution, is that the "$\approx$" relation is not very tight: the error is bounded by a term on the order of $\frac{1}{\sqrt{n}}$, which will dominate the desired bound of $\frac{1}{(n+1)^2}$, regardless of what bounds we may find for the $\Phi$ term. (This is the Berry-Esséen Theorem; see for instance [3], section XVI.5.)

*Remark.* Let

$$\alpha_n = \frac{n+1}{2n} + \min\left(1 - \frac{n+1}{2n},\ \frac{1}{n^{1/4}}\right) = \min\left(1,\ \frac{n+1}{2n} + \frac{1}{n^{1/4}}\right),$$

and consider a sequence of slot machines, where the $n$th machine has PGF

$$q_n(x) = \alpha_n x^2 + (1-\alpha_n)x^0.$$

Let us solve for the fixed-points of $q_n$. By the Quadratic Formula, the fixed-points are

$$x = 1, \qquad x = \frac{1-\alpha_n}{\alpha_n}.$$

Let $H_n = (1-\alpha_n)/\alpha_n = 1/\alpha_n - 1$. Since $1/2 < \alpha_n \le 1$, we find that $0 \le H_n < 1$. Therefore each $q_n$ has a unique fixed-point $H_n \in [0,1)$, and since $\lim_{n\to\infty} \alpha_n = 1/2$, we get

$$\lim_{n\to\infty} H_n = \frac{1-1/2}{1/2} = 1.$$

Thus the sequence of minimal fixed-points approaches 1, but the probability of going broke does not approach 1.

# 4 Code for simulating the "infinite slot machines" game

The following code is written in the programming language R [6]. Instead of simulating every individual coin as a Bernoulli random variable, it is (much) more efficient to simulate an entire round of coins, as a binomial random variable.

```
##############

# Note: Change these values to desired quantities.

number_of_trials_simulated <- 100000
monitor <- 5000

# Setting number_of_trials_simulated to 100000 means that it will
# simulate 100000 separate, independent trials of playing the
# sequence of machines.

# Setting monitor to 5000 means that, in each trial, it will show you
# the current status after every 5000th round.  This also means that
# if you go broke before the 5000th round, then it won't show any
# output for that trial of the game.

# Set monitor to 0 if you don't want to see any output printed while
# the code is running.

##############


result <- function(n, coins){
    return((n+1) * rbinom(1, coins, 1/n))
}

one_trial <- function(monitor=0, game_number=1){
    #
    # The argument monitor is used if you want to view the progress
    # while the function is still running.  With the default value
    # monitor = 0,
    # it doesn't show any output while running. But this can make it
    # look like the computer is frozen, in the case of games
    # that take a very long time.
    #
    # If you set monitor to, say, 1000, then it will show you the
    # current status of the game after every 1000th round. This also
    # means that it won't print anything unless the game reaches
    # at least the 1000th round.
    #
    coins <- 1
    round <- 0 # number of rounds completed so far
    best_coins <- 1
    while (coins > 0){
```

```
        round <- round + 1
        coins <- result(round, coins)
        best_coins <- max(best_coins, coins)
        if (monitor > 0){
            if (round %% monitor == 0){
                cat("Current game is:", game_number, "\n")
                cat("Current round is:", round, "\n")
                cat("Current number of coins is:", coins, "\n")
                cat("\n")
                flush.console() # otherwise it waits,
                # and does all printing at the end
            }
        }
    }
    return(c(round, best_coins))
}

repeated_games <- function(num_trials, monitor=0){
    d = c()
    for (i in 1:num_trials){
        d = c(d, one_trial(monitor, i))
    }
    return(matrix(d, nrow=2, ncol=num_trials))
}

y <- repeated_games(number_of_trials_simulated, monitor)
rounds <- y[1,]
coins <- y[2,]

cat("Total number of games played: ", number_of_trials_simulated, "\n")
cat("Highest number of rounds before going broke: ", max(rounds), "\n")
cat("Highest number of coins achieved: ", max(coins), "\n")
```

## References


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