Online supplement to the manuscript:
“Get infinitely rich! (while definitely going broke)"

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In what follows, we define \( \mathbb{N} \) to be \{1, 2, 3, \ldots \}, the set of positive integers.

1 Why composition of polynomials works

The following result appears in multiple places (see for instance [5], and also Section XII.1 of [2]):

**Theorem 1.1.** To get the next polynomial \( p_n(x) \) from \( p_{n-1}(x) \) and \( q_n(x) \), replace every \( x \) in \( p_{n-1}(x) \) with \( q_n(x) \). In other words:

\[ p_n = p_{n-1} \circ q_n. \]

**Proof.** Consider a monomial \( r_k x^k \), where \( r_k \in [0, 1] \): suppose that this is one term of the polynomial \( p_{n-1}(x) \). This monomial then records the probability (after the first \((n-1)\) machines) of having exactly \( k \) coins; that probability would be \( r_k \). Now, suppose that we do indeed have exactly \( k \) coins, and place them all into machine \( M_n \). Each coin now represents a random variable, as it either wins or loses when placed into \( M_n \). By assumption, these random variables are all independent. We are interested in the sum of these random variables, meaning the total number of coins after all plays on \( M_n \) are complete.

If \( A \) and \( B \) are independent random variables, then their sum \((A + B)\) is distributed according to the convolution of the probability distributions of \( A \) and \( B \). It so happens that the convolution rule is exactly the same as the rule for multiplication of polynomials. If \( A \) takes values \( a_1, a_2, \ldots \) with probabilities \( s_1, s_2, \ldots \) respectively, and \( B \) takes values \( b_1, b_2, \ldots \) with probabilities \( t_1, t_2, \ldots \) respectively, and \( A, B \) are independent, then for each \( u \),

\[
P(A + B = u) = \sum_{i,j} s_i t_j \begin{cases} 1, & \text{if } u = a_i + b_j \\ 0, & \text{else.} \end{cases}
\]

Meanwhile if \( f(x) = \sum s_i x^{a_i} \) and \( g(x) = \sum t_j x^{b_j} \), then for each \( u \), the
coefficient on $x^n$, in the product $f(x)g(x)$, is also

$$\sum_{i,j} s_it_j \begin{cases} 1, & \text{if } u = a_i + b_j \\ 0, & \text{else.} \end{cases}$$

By induction on $k$ it follows that, given $k$ independent random variables $X_1, \ldots, X_k$, all identically distributed according to the probability generating function $q_n(x)$, the sum $X_1 + \cdots + X_k$ follows the probability generating function $q_n(x)^k$. Finally we weight this result by $r_k$ and add it to the weighted results from all the other monomials, according to the law of total probability. The result is precisely $p_{n-1}(q_n(x))$. \hfill \Box

Essentially the same proof still works even if we have multiple variables $x_1, x_2, \ldots, x_m$, standing for various “types” of coins. Also, we may replace “polynomial” everywhere with “formal power series”—however, the above proof does require all exponents to be nonnegative integers.

### 2 A proof that we go broke with probability 1

**Conjecture 2.1.** Let $p_0(x) = x$, and for each $n \in \mathbb{N}$, let

$$q_n(x) = \frac{x^{n+1} + (n-1)}{n},$$

and for each $n \in \mathbb{N}$, let

$$p_n(x) = p_{n-1}(q_n(x)).$$

Then

$$\lim_{n \to \infty} p_n(0) = 1.$$

This section is devoted to showing that Conjecture 2.1 is true. Notice that, by induction, all $p_n$ and $q_n$ are polynomials with nonnegative coefficients and at least one non-constant coefficient strictly positive; hence all $p_n$ and $q_n$ are strictly increasing on $[0, 1]$. Below, we use this result repeatedly.

**Proposition 2.2.** For all $n \in \mathbb{N}$, and for all $x \in [0, 1]$,

$$q_{n+1}(x) \geq q_n(x).$$

Further, if $x \in [0, 1)$ then $q_{n+1}(x) > q_n(x)$.

**Proof.** Certainly $q_n(1) = 1$ for all $n$, so assume $0 \leq x < 1$. The following
statements are equivalent:

\[
q_{n+1}(x) > q_n(x) \\
x^{n+2} + n > x^{n+1} + n - 1 \\
x^{n+2} + n^2 > (n + 1)x^{n+1} + n^2 - 1 \\
x^{n+2} - nx^{n+1} > x^{n+1} - 1 \\
x^{n+1}(x - 1) > x^{n+1} - 1 \\
x^{n+1} < \frac{1 - x^{n+1}}{1 - x} \\
x^{n+1} < 1 + x + x^2 + \cdots + x^n.
\]

But \(0 \leq x < 1\), so

\[
x \geq x^2 \geq \ldots \geq x^n \geq x^{n+1},
\]

and therefore

\[
x^{n+1} \leq x + x^2 + \cdots + x^n < 1 + x + x^2 + \cdots + x^n.
\]

**Proposition 2.3.** For each \(n \in \mathbb{N}\), there exists a unique \(Q_n \in [0, 1)\) such that \(q_n(Q_n) = Q_n\).

**Proof.** Let \(h_n(x) = x^{n+1} - nx + n - 1\). Then \(h_n(x) = 0\) if and only if \(q_n(x) = x\). We know that \(h_n(1) = 0\), and synthetic division yields

\[
h_n(x) = (x - 1)((x^n + x^{n-1} + \cdots + x) + 1 - n).
\]

Let \(g_n(x) = x^n + x^{n-1} + \cdots + x + 1 - n\), so we want to show that \(g_n\) has a unique zero in \([0, 1)\). Uniqueness is clear because \(g_n\) is positive on \((0, 1)\), so it suffices to show existence. But \(g_n(0) = 1 - n \leq 0\) while \(g_n(1) = 1 > 0\), so \(g_n\) has a zero in \([0, 1)\) by the Intermediate Value Theorem. We call this number \(Q_n\). \(\square\)

**Proposition 2.4.** Let \(n \in \mathbb{N}\). For all \(x \in [0, 1)\),

\[
x < q_n(x) \iff x < Q_n,
\]

and

\[
x = q_n(x) \iff x = Q_n.
\]

**Proof.** Let \(x \in [0, 1)\). First suppose \(x < q_n(x)\). We calculate that \(q_n'(1) > 1\), so by continuity there exists \(\varepsilon > 0\) such that \(q_n' > 1\) on \((1 - \varepsilon, 1)\). Recall that \(q_n(1) = 1\) so by the Mean Value Theorem, \(q_n(t) < t\) for all \(t \in (1 - \varepsilon, 1)\). In particular \(x \leq 1 - \varepsilon\). But now by the Intermediate Value Theorem \(q_n\) has a fixed-point in \((x, 1 - \varepsilon/2)\). By uniqueness, this fixed-point must be \(Q_n\), hence \(x < Q_n\). For the other direction, suppose \(x \geq q_n(x)\). Then since \(0 \leq q_n(0)\), by the Intermediate Value Theorem there exists a fixed-point of \(q_n\) in \([0, x]\); hence \(x \geq Q_n\).
The second equivalence is simply existence and uniqueness of $Q_n$. (We include the statement here to show that the strict inequalities in the first equivalence can be replaced with non-strict inequalities, and/or reversed, as desired.)

**Proposition 2.5.** For all $n \in \mathbb{N}$,

$$Q_n \geq 1 - \frac{2}{n^2}.$$

**Proof.** Using Proposition 2.4, the following inequalities are equivalent:

$$1 - \frac{2}{n^2} \leq Q_n$$

$$1 - \frac{2}{n^2} \leq q_n \left(1 - \frac{2}{n^2}\right)$$

$$1 - \frac{2}{n^2} \leq \frac{(1 - \frac{2}{n^2})^{n+1} + n - 1}{n}$$

$$n - \frac{2}{n} \leq \left(1 - \frac{2}{n^2}\right)^{n+1} + n - 1$$

$$1 - \frac{2}{n} \leq \left(1 - \frac{2}{n^2}\right)^{n+1}$$

$$1 - \frac{2}{n} \leq 1 - \frac{(n + 1)}{1} \left(\frac{2}{n^2}\right) + \frac{(n + 1)}{2} \left(\frac{4}{n^4}\right) - \cdots \pm \frac{(n + 1)}{n + 1} \left(\frac{2^{n+1}}{n^{2(n+1)}}\right)$$

$$\frac{2}{n^2} \leq \frac{(n + 1)}{2} \left(\frac{4}{n^4}\right) - \frac{(n + 1)}{3} \left(\frac{8}{n^6}\right) + \cdots \pm \frac{(n + 1)}{n + 1} \left(\frac{2^{n+1}}{n^{2(n+1)}}\right).$$

But the right-hand side is an example of a (finite) Alternating Series. Certainly its terms alternate in sign, and we claim that they are monotone decreasing in absolute value. Proof of claim: Let $a_k$ be the signed term $\pm \left(\frac{n+1}{k}\right) \left(\frac{2^k}{n^{2k}}\right)$ from the right-hand side, beginning with $k = 2$, and for $k > n + 1$ let $a_k = 0$. For all $k > n + 1$ we have $a_k = 0 = a_{k+1}$; hence $|a_{k+1}| \leq |a_k|$ for those $k$. For $2 \leq k \leq n + 1$, we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(n+1)(2)}{(n+1)(n^2)} = \frac{2(n+1-k)}{(k+1)n^2} \leq \frac{2n}{3n^2} < 1.$$ 

Therefore by the explicit bounds in the Alternating Series Test, the right-hand side is bounded below by its second partial sum, so we are done if we can show that

$$\frac{2}{n^2} \leq \frac{(n + 1)}{2} \left(\frac{4}{n^4}\right) - \frac{(n + 1)}{3} \left(\frac{8}{n^6}\right).$$

Multiplying both sides by $6n^6$ to clear denominators, it is equivalent to show
that

\[
12n^4 \leq 12n^2(n+1)(n) - 8(n+1)(n)(n-1) \\
0 \leq 12n^3 - 8(n^3 - n) \\
0 \leq 4n^3 + 8n.
\]

Clearly this last inequality holds for all \( n \in \mathbb{N} \).

\[ \text{Remark.} \] The above proof can also be modified to show that \( Q_n \leq 1 - \frac{1}{n^2} \), but we will not really need an upper bound on \( Q_n \) (other than \( Q_n < 1 \)).

**Corollary 2.6.**

\[
\lim_{n \to \infty} Q_n = 1.
\]

**Proof.** This follows from the Squeeze Theorem, since \( 1 - \frac{2}{n^2} \leq Q_n \leq 1 \).

**Proposition 2.7.** For each \( n \in \mathbb{N} \),

\[ Q_n < Q_{n+1}. \]

**Proof.** We have

\[ Q_{n+1} = q_{n+1}(Q_{n+1}) > q_n(Q_{n+1}), \]

by Proposition 2.2, since \( Q_{n+1} \) is strictly less than 1 by its original definition in Proposition 2.3. But \( Q_{n+1} > q_n(Q_{n+1}) \iff Q_{n+1} > Q_n \), by Proposition 2.4.

**Proposition 2.8.** There exists \( L \leq 1 \) such that

\[
\lim_{n \to \infty} p_n(0) = L.
\]

**Proof.** We said that \( p_n(0) \) is the probability of being broke after machine \( M_n \). But if we are broke after \( M_n \), then we are still broke after \( M_{n+1} \). Therefore

\[ p_n(0) \leq p_{n+1}(0), \]

for all \( n \). But also, \( p_n(0) \leq 1 \) for all \( n \), since each \( p_n(0) \) is a probability. Therefore by the Monotone Convergence Theorem,

\[
\lim_{n \to \infty} p_n(0)
\]

exists, and is at most 1.

**Definition 2.9.** We define \( L \) to be the limit in Proposition 2.8.

**Proposition 2.10.** Let \( x \in [0,1) \), and \( n \in \mathbb{N} \), and write \( q^k_n \) to mean the composition of \( k \) copies of \( q_n \). Then:

(i) If \( x < Q_n \), then \( x < q_n(x) < Q_n \).

(ii) If \( x > Q_n \), then \( x > q_n(x) > Q_n \).

(iii) If \( x = Q_n \), then \( x = q_n(x) = Q_n \).

(iv) \[
\lim_{k \to \infty} q^k_n(x) = Q_n.
\]
Proof. Statement (iii) is clear. For (i), suppose that \( x < Q_n \). Then
\[
x < q_n(x)
\]
by Proposition 2.4, and
\[
q_n(x) < q_n(Q_n)
\]
because \( q_n \) is strictly increasing on \([0, 1]\). The proof of statement (ii) is similar.

It remains to prove statement (iv). By (i), (ii), and (iii), the sequence
\[
(x, q_n(x), q_n(q_n(x)), \ldots, q_n^k(x), \ldots)
\]
is monotone, and it is bounded by 0 and 1. Let \( G \) be the limit of this sequence. Then
\[
G = \lim_{k \to \infty} q_n^k(x) = q_n \left( \lim_{k \to \infty} q_n^{k-1}(x) \right) = q_n(G)
\]
by continuity, so \( G \) is a fixed-point of \( q_n \). If the sequence is decreasing then \( G \leq x < 1 \); if it is increasing then \( G \leq Q_n < 1 \). Either way, \( G \) is a fixed-point belonging to \([0, 1)\), so \( G = Q_n \).

Thus, applying \( q_n \) results in movement toward \( Q_n \), and repeatedly applying \( q_n \) moves a point arbitrarily close to \( Q_n \).

**Proposition 2.11.** The sequence
\[
(p_1(Q_1), p_2(Q_2), \ldots, p_n(Q_n), \ldots)
\]
is monotone increasing, and convergent.

**Proof.** Let \( n \geq 2 \). Then
\[
p_{n-1}(Q_{n-1}) = p_{n-1}(q_{n-1}(Q_{n-1})) \leq p_{n-1}(q_n(Q_{n-1})) = p_n(Q_{n-1}) \leq p_n(Q_n),
\]
by Propositions 2.2 and 2.7, and the sequence is bounded above by 1.

**Proposition 2.12.**
\[
\lim_{n \to \infty} p_n(Q_n) = L.
\]

**Proof.** For each \( n \geq 2 \), we know that \( \lim_{k \to \infty} q_n^k(0) = Q_n > Q_{n-1} \), by Propositions 2.7 and 2.10 (iv). Thus for each \( n \geq 2 \) we may choose \( K_n \in \mathbb{N} \) such that \( q_n^m(0) > Q_{n-1} \) for all \( m \geq K_n \); then let \( k_1 = 1 \), and for \( n \geq 2 \) let \( k_n = \max(K_n, k_{n-1}) \). Now by Proposition 2.2, for all \( n \geq 2 \),
\[
p_{n-1}(Q_{n-1}) \leq p_{n-1}(q_n^{k_n+1}(0)) = ((q_1 \circ q_2 \circ \ldots \circ q_{n-1}) \circ (q_n \circ q_n \circ \ldots \circ q_n))(0)
\]
\[
= p_{n+k_n}(0)
\]
\[
\leq p_{n+k_n}(Q_{n+k_n}).
\]
Thus for each $n \geq 2$,
\[ p_{n-1}(Q_{n-1}) \leq p_{n+k_n}(0) \leq p_{n+k_n}(Q_{n+k_n}). \tag{1} \]

But $(p_{n+k_n}(Q_{n+k_n}))$ is a subsequence of $(p_n(Q_n))$, because we required $k_{n+1} \geq k_n \geq 1$, and $(p_n(Q_n))$ is convergent by Proposition 2.11. Therefore $(p_{n+k_n}(Q_{n+k_n}))$ and $(p_n(Q_n))$ must share the same limit, say $T$. Thus in Inequality 1, the two outside terms approach $T$ as $n \to \infty$, while the middle term approaches $L$. By the Squeeze Theorem, $L = T$.

**Theorem 2.13.**

$L = 1$.

That is, Conjecture 2.1 is true.

**Proof.** Given $n \in \mathbb{N}$, let $L_n(x)$ be the linear approximation to $p_n(x)$, taken at base point $a = 1$. That is,
\[
L_n(x) = p_n'(1)(x - 1) + p_n(1) = (n + 1)(x - 1) + 1.
\]

Since $p''_n$ is nonnegative on $(0, 1)$, we claim that
\[ p_n(x) \geq L_n(x) \]
for all $x \in [0, 1)$. Proof of claim: we show the contrapositive, that if $p_n(x) < L_n(x)$ for some $x \in [0, 1)$, then there exists $d \in (0, 1)$ such that $p''_n(d) < 0$. Suppose that $x \in [0, 1)$ and $p_n(x) < L_n(x)$. By the Mean Value Theorem, there exists $c \in (x, 1)$ such that
\[
p_n'(c) - L_n'(c) = \frac{p_n(x) - L_n(x) - (p_n(1) - L_n(1))}{x - 1} = \frac{p_n(x) - L_n(x)}{x - 1}.
\]

Therefore $p_n'(c) - L_n'(c) > 0$, since both the top and bottom of the fraction are negative. Now by the MVT again, there exists $d \in (c, 1)$ such that
\[
p_n''(d) - L_n''(d) = \frac{p_n'(c) - L_n'(c) - (p_n'(1) - L_n'(1))}{c - 1}.
\]

But $L_n$ is linear so its second derivative is 0 everywhere; meanwhile $L_n'(1) = p_n'(1)$ by definition of $L_n$. Thus
\[
p_n''(d) = \frac{p_n'(c) - L_n'(c)}{c - 1}.
\]

Above, we had $p_n'(c) - L_n'(c) > 0$, so $p_n''(d) < 0$. This proves the claim.
So we conclude that \( p_n(x) \geq L_n(x) \) for all \( x \in [0,1) \). Therefore for all \( n \in \mathbb{N} \),

\[
p_n(Q_n) \geq L_n(Q_n) \geq L_n \left( 1 - \frac{2}{n^2} \right),
\]
since \( L_n \) is increasing (and using Proposition 2.5). Thus for all \( n \),

\[
1 \geq p_n(Q_n) \geq (n+1) \left( -\frac{2}{n^2} \right) + 1 = 1 - \frac{2n+2}{n^2}.
\]
By the Squeeze Theorem, \( p_n(Q_n) \to 1 \); hence \( L = 1 \) by Proposition 2.12.

\[\square\]

3 A fair(ish) game where you don’t necessarily go broke

In this section we examine a sequence of machines which approach fairness, but where going broke has probability \(< 1\). Below, we’ll say that “round \( n \)” means the procedure of putting all your coins into machine \( n \), and collecting your winnings.

Consider a sequence of slot machines where the \( n \)th machine returns either 2 coins, with probability \( \alpha_n \), or 0 coins with probability \( 1 - \alpha_n \). The probability of never going broke is certainly at least as large as the probability of always having at least \( n+1 \) coins after playing the \( n \)th machine. And this probability is at least as large as the infinite product

\[
Q = \prod_{n=1}^{\infty} P \left( \text{win at least } n+1 \text{ coins in round } n \big| \text{start round } n \text{ with } n \text{ coins} \right),
\]
for if we ever win strictly more than \( n+1 \) coins in a round \( n \), we can either (1) discard the excess, or (2) put it into a separate “account” which we can play separately on the side. If anything, choice (2) will improve our chances of reaching a given number of coins in the future, relative to choice (1).

Let \( X_n \) be the number of coins after playing rounds 1 through \( n \). We want to find \( \alpha_n \) such that \( \alpha_n \to \frac{1}{2} \) but \( Q > 0 \). If

\[
P \left( X_n \geq n+1 \big| X_{n-1} = n \right) \geq 1 - \frac{1}{(n+1)^2} \quad (2)
\]
for all \( n \geq 1 \), then \( Q \) will be at least

\[
\prod_{n=1}^{\infty} \left( 1 - \frac{1}{(n+1)^2} \right) = \prod_{n=1}^{\infty} \left( \frac{n(n+2)}{(n+1)^2} \right) = \left( \frac{1 \cdot 3}{2 \cdot 2} \right) \left( \frac{2 \cdot 4}{3 \cdot 3} \right) \left( \frac{3 \cdot 5}{4 \cdot 4} \right) \cdots = \frac{1}{2} > 0.
\]
Of course (provided that \( \alpha_n > 0 \) for all \( n \)) it actually suffices to show that Inequality (2) holds for all sufficiently large \( n \); it need not be true immediately at \( n = 1 \).
Now $X_n$ is a random variable, given by $X_n = 2Y_n$, where $Y_n$ is the number of wins in round $n$. We assume that round $n$ begins with exactly $n$ coins; hence $Y_n \sim \text{Binomial}(n, \alpha_n)$. We have

$$P(X_n \geq n + 1) = P\left(Y_n \geq \frac{n + 1}{2}\right) \geq P\left(Y_n > \frac{n + 1}{2}\right) = 1 - P\left(Y_n \leq \frac{n + 1}{2}\right)$$

so we wish to show that $P\left(Y_n \leq \frac{n + 1}{2}\right) \leq \frac{1}{(n + 1)^2}$ (for all sufficiently large $n$). We set $\alpha_n = \frac{n + 1}{2} + d_n$, and try to find $d_n$ satisfying all desired properties, including $d_n \to 0$.

We will choose $d_n \geq 0$, so that $\frac{n + 1}{2} \leq n\alpha_n$. Let $Z_n = n - Y_n$, so $Z_n$ is the number of failures out of $n$ trials, and $Z_n \sim \text{Binomial}(n, 1 - \alpha_n)$. Then

$$E(Z_n) = n(1 - \alpha_n) = n - \frac{n + 1}{2} - nd_n = \frac{n - 1}{2} - nd_n,$$

so

$$P\left(Y_n \leq \frac{n + 1}{2}\right) = P\left(Z_n \geq \frac{n - 1}{2}\right) = P\left(Z_n - E(Z_n) \geq nd_n\right).$$

Recall that $d_n \geq 0$. If $d_n > 0$ then by Hoeffding’s Inequality (Theorem 2 of [4]),

$$P\left(\frac{Z_n - E(Z_n)}{n} \geq d_n\right) \leq \exp\left(-\frac{2n^2(d_n)^2}{n}\right) = \exp\left(-2n(d_n^2)\right).$$

(On the other hand if $d_n = 0$ then $\exp(-2n(d_n^2)) = \exp(0) = 1$, so the same bound holds trivially in this case.)

Now take $d_n = \min\left(1 - \frac{n + 1}{2n}, \frac{1}{\sqrt{n}}\right)$, so $d_n \geq 0$ for all $n$, and $d_n \to 0$, and $2n(d_n^2) = 2\sqrt{n}$ for large $n$ (specifically, $n \geq 20$). Finally

$$\frac{1}{\exp(2\sqrt{n})} \leq \frac{1}{(n + 1)^2} \tag{3}$$

for all $n \geq 0$, so $Q > 0$. (To verify Inequality 3: we show that $2\sqrt{x} \geq 2\ln(x + 1)$ for all $x \in [0, \infty)$. The two sides are equal at $x = 0$, and we claim that $\frac{d}{dx}2\sqrt{x} \geq \frac{d}{dx}\ln(x + 1)$, for all $x > 0$. It is equivalent to show that $x + 1 \geq 2\sqrt{x}$ for all $x > 0$; this is equivalent to $(\sqrt{x} - 1)^2 \geq 0$.)

Remark. Indeed, for all $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $(1 - Q)^N < \varepsilon$. Thus by beginning with one coin, then playing a machine that returns $N$ coins with probability 1, and then following it up with the $(\alpha_n)$ sequence of machines, we can guarantee that the probability of eventually going broke is less than $\varepsilon$, even though $\alpha_n \to \frac{1}{2}$ so the machines approach fairness.

Remark. Above, we only needed Hoeffding’s Inequality for the case of a binomial random variable. This narrower result appears as Lemma 1 in [7], and it is a direct consequence of Theorem 1 in [1]; both of these sources predate [4]. In addition, Theorem 1 in [4] would be sufficient for our purposes. However, Theorem 2 in [4] is stated in a form which is convenient for us, and [4] gives a complete, self-contained proof of the more general result.
Remark. Above, we used Hoeffding’s Inequality instead of, say, the normal approximation to the binomial distribution (Central Limit Theorem). The problem with the Central Limit Theorem approximation
\[
P(Y_n \leq \frac{n+1}{2}) = P\left(\frac{Y_n - \alpha_n}{\sqrt{n\alpha_n(1-\alpha_n)}} \leq \frac{-d_n}{\sqrt{n\alpha_n(1-\alpha_n)}}\right)
\approx \Phi\left(\frac{-d_n}{\sqrt{n/4}}\right)
\leq \Phi\left(\frac{-d_n}{\sqrt{n/4}}\right),
\]
where $Y_n = Y_n/n$ and $\Phi$ is the cumulative distribution function for the standard normal distribution, is that the “$\approx$” relation is not very tight: the error is bounded by a term on the order of $\frac{1}{\sqrt{n}}$, which will dominate the desired bound of $\frac{1}{(n+1)^2}$, regardless of what bounds we may find for the $\Phi$ term. (This is the Berry-Esséen Theorem; see for instance [3], section XVI.5.)

Remark. Let $\alpha_n = \frac{n+1}{2n} + \min\left(1 - \frac{n+1}{2n}, \frac{1}{n^{1/4}}\right) = \min\left(1, \frac{n+1}{2n} + \frac{1}{n^{1/4}}\right)$, and consider a sequence of slot machines, where the $n$th machine has PGF $q_n(x) = \alpha_n x^2 + (1 - \alpha_n)x^0$.

Let us solve for the fixed-points of $q_n$. By the Quadratic Formula, the fixed-points are $x = 1$, $x = \frac{1 - \alpha_n}{\alpha_n}$. Let $H_n = (1 - \alpha_n)/\alpha_n = 1/\alpha_n - 1$. Since $1/2 < \alpha_n \leq 1$, we find that $0 \leq H_n < 1$. Therefore each $q_n$ has a unique fixed-point $H_n \in [0,1)$, and since $\lim_{n \to \infty} \alpha_n = 1/2$, we get
\[
\lim_{n \to \infty} H_n = \frac{1 - 1/2}{1/2} = 1.
\]
Thus the sequence of minimal fixed-points approaches 1, but the probability of going broke does not approach 1.

4 Code for simulating the “infinite slot machines” game

The following code is written in the programming language R [6]. Instead of simulating every individual coin as a Bernoulli random variable, it is (much) more
efficient to simulate an entire round of coins, as a binomial random variable.

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# Note: Change these values to desired quantities.

number_of_trials_simulated <- 100000
monitor <- 5000

# Setting number_of_trials_simulated to 100000 means that it will
# simulate 100000 separate, independent trials of playing the
# sequence of machines.

# Setting monitor to 5000 means that, in each trial, it will show you
# the current status after every 5000th round. This also means that
# if you go broke before the 5000th round, then it won't show any
# output for that trial of the game.

# Set monitor to 0 if you don't want to see any output printed while
# the code is running.

########################

result <- function(n, coins){
  return((n+1) * rbinom(1, coins, 1/n))
}

one_trial <- function(monitor=0, game_number=1){
  #
  # The argument monitor is used if you want to view the progress
  # while the function is still running. With the default value
  # monitor = 0,
  # it doesn't show any output while running. But this can make it
  # look like the computer is frozen, in the case of games
  # that take a very long time.
  #
  # If you set monitor to, say, 1000, then it will show you the
  # current status of the game after every 1000th round. This also
  # means that it won't print anything unless the game reaches
  # at least the 1000th round.
  #
  coins <- 1
  round <- 0 # number of rounds completed so far
  best_coins <- 1
  while (coins > 0){
round <- round + 1
coins <- result(round, coins)
best_coins <- max(best_coins, coins)
if (monitor > 0){
  if (round %% monitor == 0){
    cat("Current game is:", game_number, "\n")
    cat("Current round is:", round, "\n")
    cat("Current number of coins is:", coins, "\n")
    cat("\n")
    flush.console() # otherwise it waits,
    # and does all printing at the end
  }
}
return(c(round, best_coins))

repeated_games <- function(num_trials, monitor=0){
  d = c()
  for (i in 1:num_trials){
    d = c(d, one_trial(monitor, i))
  }
  return(matrix(d, nrow=2, ncol=num_trials))
}

y <- repeated_games(number_of_trials_simulated, monitor)
rounds <- y[1,]
coins <- y[2,]

cat("Total number of games played: ", number_of_trials_simulated, "\n")
cat("Highest number of rounds before going broke: ", max(rounds), "\n")
cat("Highest number of coins achieved: ", max(coins), "\n")

References


