Supplement to "Letters to Joel," by David and Joel Stucki, *Math Horizons*, February 2015

Subject: mathematics as an art
Date: Mon, 24 Mar 2003 16:47:13 -0600
From: Joel Stucki
To: Dave Stucki

Sigh . . . this is the third time I've had to write this today. stupid computers.

OK. I saw *A Beautiful Mind* last night, and was especially struck by two things. First, Ron Howard said that in speaking to the real John Nash and other mathematicians, he found out that they don't think in numbers so much as in shapes and relationships. Can you explain that?

The second question is the more philosophical one, and I just about drove myself insane this morning trying to wrap my mind around it. Nash's character describes one student's solution to a problem as "elegant," and also calls math an art. I have always thought of mathematics as an objective, exact science, and so I fail to see how it can be described in subjective terms. Math is used extensively IN art, music, architecture, etc., but this of course is a mere application. How can I conceive of it as an art in itself? I realize of course this means I need to define art. This is the part where my brain explodes. I began to say that art was the indefinable expressive quality of something or other, but of course the word "indefinable" renders the rest of the definition moot. Then I said it was the original individualistic expression of an intellectual and/or technical pursuit. I liked that at first, then I realized it begs the question: what is expression? How can a human being, who judges everything by his own experiences, ever conceive of something totally original? I'm ultimately unable to define art, and I end up saying that anything subjective or with the possibility of variance can be artistic. My question, then, is this: "What is the nature of subjectivity in mathematics?"

I've probably just asked you to outline the entire body of mathematical study for the last few millennia. Sorry. Even a cursory understanding would be better than nothing. See, this is what I love about learning: interpretation; subjectivity. Put me in an algebra class and I'm asleep in five minutes. You learn some formula, and then you plug in any numbers and you get such and such an answer. No real thinking involved. The formula figures it out for you. But an equation, formula, pattern, etc., that can be INTERPRETED in different ways, that allows for creativity—that's interesting. I have no background in math, and so I don't know if I'm making sense. I hope I am.

Take your time,
Joel

Subject: Re: mathematics as an art
Date: Mon, 24 Mar 2003 21:10:36 -0500
From: David J. Stucki
To: Joel Stucki

Joel,

I'll give you a longer answer, but first a question: Do musicians think in notes, or is it more typical that they think in terms of melodic structure, rhythm, harmony, etc.? I suspect I know the answer, but getting you to confirm it will help me to know how best to answer the rest. Besides, it gives you an idea of where I'm going to go.

Wow, what a great email!

Thanks,
dave . . .

Subject: Re: mathematics as an art
Date: Tue, 25 Mar 2003 11:03:44 -0600
From: Joel Stucki
To: David Stucki

An excellent point. I guess I would say that we think in notes and rhythms while first learning a piece, and the phrases and dynamics come later, usually as a natural occurrence. And, of course, musicians at different levels go through this process at different rates. Beginners generally think only in notes and rhythms, and their teachers have to help them with more "musical" elements. At my stage of the game, I try to add some elements of phrasing and dynamics while sight-reading, and this usually changes several times while I learn the piece until I arrive at an interpretation that I feel does justice to the piece stylistically. Ultimately, the "shape of the line" becomes the most important concept musically. So I wouldn't say that I don't think in notes and rhythms, but that they are single elements in a much broader scope. I could go on forever about this. It's kind of the essence of music making.

I had a thought last night. Mathematical formulas (at least the basic ones I learned in high school) may be rigid, but someone had to think of them, refine them; someone had to pose the question to begin with, etc. That may be at least a small creative element in math. When I was a freshman in high school in Algebra 1, the teacher would present a formula and I would often try to think of a better one. I hoped to think of one that would be more efficient, or easier to remember. I never really nailed one down, but sometimes I would find one that worked maybe half the time (or less). I never did figure out why it would only work half the time—maybe I just executed the problem improperly. Anyhoo, just another thought.

Joel

Subject: Re: mathematics as an art
Date: Tue, 25 Mar 2003 23:25:41 -0500
From: David J. Stucki
To: Joel Stucki

Joel,

Mathematics starts with numbers, but mathematics isn't really *about* numbers. Rather, it is about abstract structures and patterns; whether these are structures and patterns that can be derived from the properties of numbers, or not. For example, logic is a branch of mathematics that doesn't have anything necessarily to do with numbers. The basic elements of logic are statements that can either be true or false. The mathematics of logic then describes the rules for how such statements can be combined, and how new statements can be deduced from given ones. Algebra is an abstraction of the rules of arithmetic to a context where particular numbers are not specified. It is the patterns of these rules in the abstract setting, not the specific formulas, that is interesting. What is so amazing about mathematics is that this process of abstraction can be continued to deeper and deeper levels. The algebra that is taught in high school is a very concrete, specific system of calculation. There is a more general subject of study called abstract algebra, where not only the numbers are abstracted away (as variables), but the operators (addition, multiplication, etc.) are abstracted out as well. So we can talk about "algebraic structures" that have nothing to do with numbers, addition, or multiplication, but still follow rules that are common to all algebras (logic is an example of a nonnumerical algebraic structure). The fact that we can recognize (and formalize) common structures in very different branches of mathematics is one of the senses in which mathematics is said to be aesthetic or beautiful. The myriad of connections between widely scattered areas of mathematics creates a sense of mystery and awe not unlike the reaction you might have to a well-crafted story (that has an intricately connected plot). The simplicity that is often found in mathematics is also a source of beauty.

Here's an example. Of course you are familiar with the numbers zero and one. You also know about addition and equality. You should also remember the number "pi" (the Greek letter p) that is the ratio of the circumference of a circle to its diameter. Two other

very important numbers in mathematics are "*e,*", which plays a crucial role in compounded interest and logarithms, and "*i*," which is the symbol given to the square root of negative one (i is a complex or imaginary number). What is regarded as the most elegant equation in all of mathematics, and also one of the most surprising is the following: $e^{i*\mathrm{pi}} + 1 = 0$. The equation says that when you raise e to the power i times pi that the resulting value is negative one (subtracting one from both sides). The fact that these five numbers, which originate from very different areas of mathematics, come together in such a simple relationship is beautiful.

So much for the elegance and beauty of mathematics. What about "mathematics is an art"? I suspect that Nash was using the term "art" more in the sense of a craft—the mathematician is an artisan. I take this to mean that the mathematical process, what happens when you are "doing" math, is an art form. For all practical purposes, the doing of mathematics is mostly problem solving (not in a math homework sense, but more generally). Problem solving requires creative thinking, intuition, informed guessing, innovation, etc. The connection is subtle, but is illustrated in many engineering disciplines. Bridges must have functional, objective properties (stability, weight/load tolerances, etc.), but an engineer who is a craftsman will also make it beautiful. Dad not only hung speaker clusters, but he considered himself a craftsman in this same way. A problem can be solved, but that doesn't mean that the solution isn't clumsy and garish. A mathematician artisan will conceive of an elegant solution. Unfortunately, very little of primary and secondary school mathematics exposes students to these sorts of ideas. High school math is mostly utilitarian.

Your response to my question describes exactly what I was after. The process you described is paralleled in mathematics. One way to see what I was trying to get at above is to think about the evolution of mathematics over the centuries. Mostly what has changed is the way in which we conceptualize and represent numbers. Think about how clumsy multiplication must have been for the Romans. What is XLVI times XIX? Since Roman numerals don't have the place value representations that Arabic numerals do (ones place, tens place, etc.), the normal method (algorithm) for multiplying that you learned in grade school doesn't work. Our modern way of just representing the numbers we're working with makes the mathematical computations more aesthetic.

Anyway, I hope this comes somewhere close to addressing what you asked. It's great to see movies being produced that make people think.

dave . . .