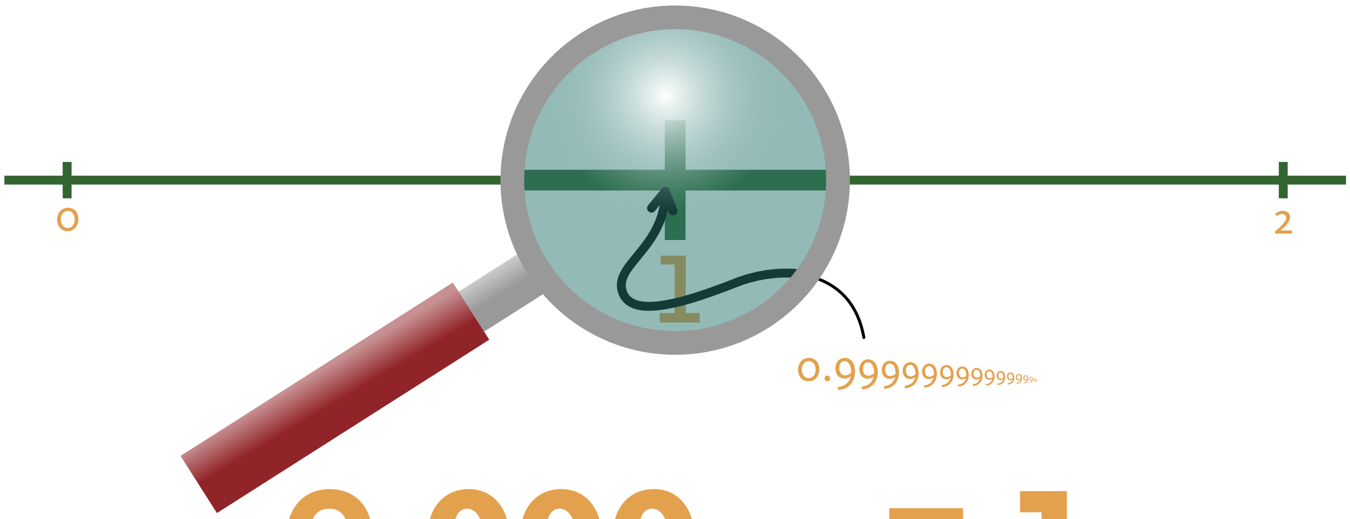


0.999… = 1
An Infinitesimal Explanation

Bryan Dawson

"I know the proofs, but I still don't believe it." Those words were uttered to me by a very good undergraduate mathematics major regarding $0.\overline{9} = 1$. This fact is possibly the most-argued-about result of arithmetic, one that can evoke great passion. But why?

According to Robert Ely [2] (see also Tall and Vinner [4]), the answer for some students lies in their intuition about the infinitely small: While they may understand that the difference between $0.\overline{9}$ and 1 is less than any positive real number, they still perceive a nonzero but infinitely small difference—an *infinitesimal* difference—between the two. And it's not just students; most professional mathematicians have not formally studied infinitesimals and their larger setting, the *hyperreal numbers*, and as a result sometimes wonder . . . if we look in the context of the hyperreal numbers, is it still true that $0.\overline{9} = 1$?

Larry Martinek, the educator behind the Mathnasium franchise, recently sent me an unpublished letter he and his then-12-year-old son Nick wrote in 1993, wherein they conjecture that perhaps $1 - 0.\overline{9} = \omega$, where "$\omega = 0.000\ldots1 = 0.\overline{0}1$, that is, an infinite number of 0s with a 1 'at the end.'" Given the latter description, ω is indeed an infinitesimal.

For reasons that will be apparent later, I will use the notation $\hat{0}$ instead of $\overline{0}$ to describe infinitely many repetitions of the given digit(s). Using this notation we would represent ω as $0.\hat{0}1$.

What exactly does that mean? Just as real numbers have decimal expansions, with one digit for each integer power of 10, so do hyperreal numbers. But the hyperreals contain "infinite integers," so there are digits representing not just 10^{-237} (the 237th digit past the decimal point) and $10^{-12{,}598}$ (the 12,598th digit), but also 10^{-Y} (the Yth digit past the decimal point), where $-Y$ is a negative infinite hyperreal integer. We have $10^{-5} = 0.00001$, four 0s followed by a 1 in the fifth decimal place, and also $10^{-Y} = 0.\hat{0}1$ where $\hat{0}$ represents $Y-1$ zeros, followed by a 1 in the Yth decimal place. (Since we'll see later that not all infinite hyperreal integers are equal, a more precise, but also uglier, notation would be

$$10^{-Y} = .\hat{0}^{(Y-1)}1 \text{ or } 10^{-Y} = .\overset{Y-1}{0}1).$$

Confused? Perhaps a little background information on the hyperreal numbers will help.

Diving Deeper

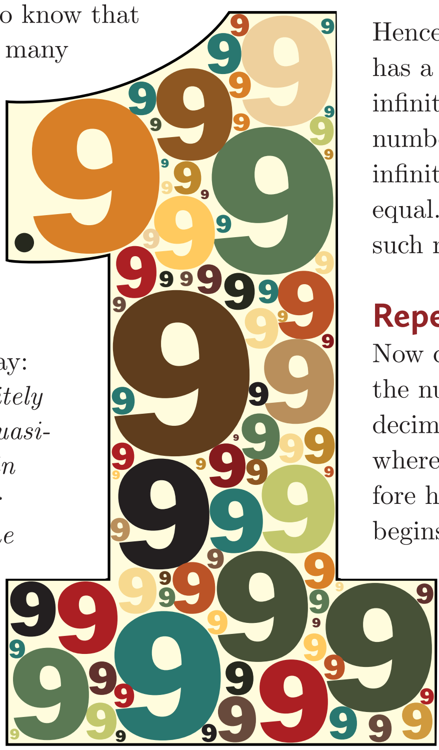
One method of exploring the hyperreal number system begins by postulating the existence of a single infinitesimal ω (not necessarily the specific ω given above), a number that is positive but is less than any positive real number. Then we stipulate that arithmetic must work as usual, and a vast wonderland of infinite and infinitely small numbers of varying sizes unfolds. That's the approach I use when teaching infinitesimal calculus.

Although this is the same approach for expanding the real numbers to the complex numbers, the details play out differently. Asserting the existence of a square root of -1 quickly leads to a complete characterization of complex numbers as having the form $a+bi$, where a and b are real numbers. But no such characterization is readily apparent when arguing that an infinitesimal exists.

The controversy over the legitimacy of infinitesimals, which began long before the development of calculus (see [1] for a fascinating historical account), raged long after complex numbers were generally accepted. Abraham Robinson finally resolved the issue of the existence of such a number system containing infinitesimals half a century ago, using techniques from mathematical logic.

So, how can we define a hyperreal number? To that end, we associate hyperreal numbers with (traditionally defined) sequences of real numbers. For instance, $(3,3,3,3,\ldots)=3$ and $(12.4,12.4,12.4,12.4,\ldots)=12.4$. But what about $(1,2,3,3,3,3,3,\ldots)$? Since that sequence agrees with $(3,3,3,3,3,3,\ldots)$ on all but two coordinates, we'll consider them to be equal: $(1,2,3,3,3,3,3,3,\ldots)=3$. In other words, we use not just sequences but *equivalence classes* of sequences to represent hyperreal numbers.

Although one needs to use what Henle and Kleinberg [3] call "quasi-big sets" to fully develop the equivalence classes, for our purposes it's enough to know that if two sequences differ in only finitely many coordinates, then they are in the same equivalence class and are considered equal (see [3] for details).



The equivalence relationship and the related *transfer principle* are keys to Robinson's work. The portion of the transfer principle that we need here can be described this way:

> *If all the coordinates (or all but finitely many, or, even more generally, a quasi-big set of coordinates) have a certain property, then the hyperreal number associated with that sequence has the property.*

For instance, $(1,2,3,4,5,6,\ldots)$ is an integer since every coordinate in the sequence is an integer. Also, $(1,2,3,4,5,6,\ldots)>1{,}000$ since all but finitely many coordinates are greater than 1,000. But that is true of any real number, not just 1,000: $(1,2,3,4,5,6,\ldots)>r$ for every real number r. Thus, $Z=(1,2,3,4,5,6,\ldots)$ is an infinite number, in fact, an infinite hyperreal integer.

Now consider 10^{-Z}. Since mathematical operations on these sequences take place coordinatewise,

$$\begin{aligned}10^{-Z} &= (10^{-1},10^{-2},10^{-3},10^{-4},\ldots)\\ &= (.1,.01,.001,.0001,.00001,\ldots).\end{aligned}$$

Then 10^{-Z} is a power of 10 since all its coordinates are. Also, $10^{-Z}<0.000024$ since all but four coordinates are less than 0.000024. In fact, 10^{-Z} is less than any positive real number. But $10^{-Z}>0$, since every coordinate is positive. Hence, 10^{-Z} matches our earlier description of a positive infinitesimal. And since Z is an integer, we can use our "hat" notation: $10^{-Z}=0.\hat{0}1$ with the 1 in the Zth decimal place.

Now try the above calculations for $2Z$. We have $2Z=(2,4,6,8,10,\ldots)$, which is also an infinite hyperreal integer, and thus $10^{-2Z}=0.\hat{0}1$, with the 1 in the $2Z$th decimal place. Notice, though, that $Z=(1,2,3,4,\ldots)\neq(2,4,6,8,\ldots)=2Z$, even though both are infinite, since they are unequal at each coordinate; not all infinite numbers are equal. Similarly,

$$\begin{aligned}10^{-Z} &= (.1,.01,.001,.0001,\ldots)\\ &\neq (.01,.0001,.000001,\ldots)=10^{-2Z}.\end{aligned}$$

Hence, $10^{-Z}\neq10^{-2Z}$, even though each has a decimal expansion that starts with infinitely many 0s. In other words, not all numbers described as $0.0000\cdots01$ with an infinite number of 0s followed by a 1 are equal. There are infinitely many different such numbers!

Repeating 9s

Now consider $1-10^{-Z}$. Because $10^{-Z}>0$, the number $1-10^{-Z}$ is less than 1. As a decimal, $1-10^{-Z}=1-0.\hat{0}1=0.\hat{9}9<1$, where ^ represents $Z-1$ digits. We therefore have a number whose decimal expansion begins with infinitely many 9s but is not equal to 1. In fact, we have infinitely many such numbers since $1-10^{-2Z}$, $1-10^{-3Z}$, and so on, also have the same property. But is that the same as saying $0.\overline{9}\neq1$?

Using sequences of real numbers,

again where operations are coordinatewise, we have

$$1-10^{-Z} = (1,1,1,1,\ldots) - (.1,.01,.001,.0001,\ldots) = (.9,.99,.999,.9999,\ldots).$$

But we also have $0.\overline{9} = (.\overline{9},.\overline{9},.\overline{9},.\overline{9},\ldots)$. Notice that none of the coordinates of $0.\overline{9}$ and $1-10^{-Z}$ match! The two numbers are therefore not equal.

Why are they not equal? One of the properties of a repeating decimal is that it does not terminate. Since every coordinate of the sequence $1-10^{-Z} = (.9,.99,.999,.9999,\ldots)$ is a terminating decimal, the hyperreal number $1-10^{-Z} = 0.\hat{9}9$ is a terminating decimal. The fact that it terminates after an infinite number of decimal places does not matter; it still eventually terminates!

Since $0.\overline{9} = (.\overline{9},.\overline{9},.\overline{9},.\overline{9},\ldots)$ is nonterminating in every coordinate, the hyperreal version of $0.\overline{9}$ also does not terminate. In the end, it comes down to the difference between "infinitely many" and "all"; $0.\overline{9}$ means that *all* digits past the decimal point are 9s, not merely the first infinitely many (that is, the first Y digits where Y is an infinite hyperreal integer). Hence, none of the numbers of the form $0.\hat{9}9$ are the same as $0.\overline{9}$.

Another way of looking at it is to compare the decimal expansions of the two numbers. Recall how we might compare $1-10^{-1{,}000{,}000}$ to $0.\overline{9}$:

$$0.\overline{9} = 0.99999999\ldots9999\ldots$$
$$1-10^{-1{,}000{,}000} = 0.\underbrace{99999999\ldots9}_{1{,}000{,}000 \text{ digits}}.$$

The difference begins in the 1,000,001st digit. Likewise, the difference between $1-10^{-Z} = 0.\hat{9}9$ and $0.\overline{9}$ is in the $(Z+1)$st digit:

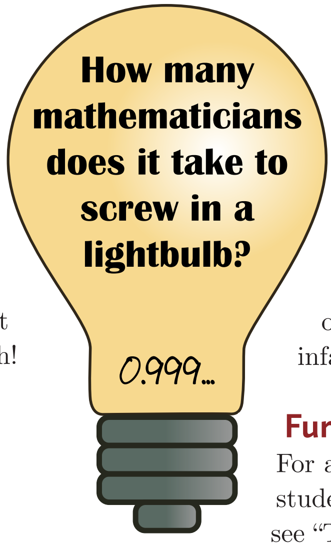
$$0.\overline{9} = 0.99999\ldots9999\ldots$$
$$1-10^{-Z} = 0.\underbrace{99999\ldots9}_{Z \text{ digits}}.$$

Since all the digits in $0.\overline{9}$ are 9, there is no room for even an infinitesimal space between it and 1. We are therefore back to $0.\overline{9} = 1$, even in the hyperreal number system. After all, if $0.\overline{9} = 1$ as a real number, then as hyperreal numbers

$$0.\overline{9} = (.\overline{9},.\overline{9},.\overline{9},.\overline{9},\ldots) = (1,1,1,1,\ldots) = 1$$

since they are equal on every coordinate.

Perhaps some of the confusion comes from thinking of $0.\overline{9}$ as $0.999\ldots$ with infinitely many 9s. As we have seen, if we only require an infinite number of 9s in the decimal expansion of our number, then there are infinitely many such hyperreals, only one of which is equal to 1. Unfortunately for those hoping otherwise, that one number is the infamous $0.\overline{9}$.



Further Reading

For another conjecture as to why some students do not believe that $0.\overline{9} = 1$, see "Teaching Tip: Accepting That $.999\ldots = 1$," by D. W. Cohen and J. M. Henle (*College Mathematics Journal* **40** no. 4 [2009] 258).

For an investigation of what properties an alternative number system would need in order to have $0.\overline{9} \neq 1$, read F. Richman's article "Is $0.999\ldots = 1$?" (*Mathematics Magazine* **72** no. 5 [1999] 396–400). ■

References

[1] A. Alexander, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World,* Scientific American/Farrar, Straus and Giroux, New York, 2014.

[2] R. Ely, Nonstandard student conceptions about infinitesimals, *Journal for Research in Mathematics Education* **41** no. 2 (2010) 117–146.

[3] J. M. Henle, E. M. Kleinberg, *Infinitesimal Calculus,* MIT Press, Cambridge, 1979.

[4] D. Tall, S. Vinner, Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics* **12** no. 2 (1981) 151–169.

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http://dx.doi.org/10.4169/mathhorizons.24.2.5