Curriculum Burst 17: Distributive Rules

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Let @ denote the "averaged with" operator: $a @ b = \frac{a+b}{2}$. Which of the following distributive laws hold for all numbers x, y and z?

I: $x @ (y+z) = (x @ y) + (x @ z)$

II: $x + (y @ z) = (x+y) @ (x+z)$

III: $x @ (y @ z) = (x @ y) @ (x @ z)$

SOURCE: This is question # 15 from the 2011 MAA AMC 10b Competition.

QUICK STATS:

MAA AMC GRADE LEVEL
This question is appropriate for the 10th grade level.

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MATHEMATICAL TOPICS
Structure in algebraic equations

COMMON CORE STATE STANDARDS

A-SSE.1b: Interpret complicated expressions by viewing one or more of their parts as a single entity.

A-SSE.2: Use the structure of an expression to identify ways to rewrite it.

MATHEMATICAL PRACTICE STANDARDS
MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY
ESSAY 2: **DO SOMETHING**

THE PROBLEM-SOLVING PROCESS:

The right place to begin, as always, is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question is visually overwhelming! The @ symbol is throwing me as it is unfamiliar. But that is okay. I'll just take a deep breath and …

DO SOMETHING

I have three equations to contend with. I might as well examine them one at a time.

I: $x@(y+z)=(x@y)+(x@z)$

This still looks overwhelming! But let's take it in pieces.

The left side is $x@(y+z)$. If I keep my cool and remember that parentheses mean "group together," I can see that this is just the @ operator applied to the object on the left, x, and to the object on the right, $(y+z)$. And what does the @ operator do? It takes the average of two things. The left side is thus:

$$\frac{x+(y+z)}{2}.$$

So far so good!

Again following my order of operations I see that the right side, $(x@y)+(x@z)$, is the sum of two things: $x@y$ and $x@z$. Okay, the right side is:

$$\frac{x+y}{2}+\frac{x+z}{2}.$$

What was the question? We want to know which of the laws given hold for all numbers. Okay. So **I** says that

$$\frac{x+(y+z)}{2}=\frac{x+y}{2}+\frac{x+z}{2}$$

holds always. I doubt it! On the left we have $\frac{x+y+z}{2}$ and on the right we have $\frac{x+y+x+z}{2}$. Setting $x=14$ and $y=0$ and $z=0$, for example, shows a mismatch for sure. Equation **I** is out!

Okay … Keeping our cool and being clear with the role of parentheses in our order of operations let's now look at each side of equation **II**.

Left side: $x+(y@z)=x+\frac{y+z}{2}$

Right side: $(x+y)@(x+z)=\frac{(x+y)+(x+z)}{2}$

Do these match? Actually this right side is $\frac{2x+y+z}{2}$, which equals $x+\frac{y+z}{2}$. Yep! It's equivalent to the left! Equation **II** is a valid equation.

Looking at equation **III**:

Left side: $x@(y@z)=\frac{x+(y@z)}{2}=\frac{x+\frac{y+z}{2}}{2}$.

This is complicated, let's multiply the numerator and denominator each through by 2. (This won't change the fraction).

Left side: $\frac{\left(x+\frac{y+z}{2}\right)\times 2}{(2)\times 2}=\frac{2x+y+z}{4}$

Now for the other side:

Right side: $(x@y)@(x@z)=\frac{(x@y)+(x@z)}{2}$

$$=\frac{\frac{x+y}{2}+\frac{x+z}{2}}{2}$$

$$=\frac{\left(\frac{x+y}{2}+\frac{x+z}{2}\right)\times 2}{(2)\times 2}=\frac{x+y+x+z}{4}$$

This matches the left side!

We have the equations **II** and **III** are valid.

Extension: A nice way to think about a distributive rule is to think about one operation being "sprinkled over" another. For example, multiplication sprinkles over addition: $a\times(b+c)=a\times b+a\times c$. But addition does not "sprinkle" over multiplication: $a+(b\times c)$ does not equal $(a+b)\times(a+c)$, in general. Find some more valid distributive rules among the operators $+$, $\times$ and @.

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