Challenges for College Mathematics:

An Agenda for the Next Decade

Report of a Joint Task Force

of the

Mathematical Association of America

and the

Association of American Colleges

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This report was completed in cooperation with a national review of arts and sciences majors initiated by the Association of American Colleges as part of its continuing commitment to advance and strengthen undergraduate liberal learning. The Mathematical Association of America was one of twelve learned societies contributing to this review. Each participating learned society convened a Task Force charged to address a common set of questions about purposes and practices in liberal arts majors; individual task forces further explored issues important in their particular fields. Reprints of this report on mathematics are available from the Mathematical Association of America, 1529 Eighteenth Street, NW, Washington, DC 20036.

In 1991, the Association of American Colleges published a single volume edition of all twelve learned society reports with a companion volume containing a separate report on "Liberal Learning and Arts and Science Majors." Inquiries about these two publications may be sent to Reports on the Arts and Sciences Major, Box R, Association of American Colleges, 1818 R Street, NW, Washington, DC 20009.

Generous funding for the project and dissemination of the reports was provided by the Fund for the Improvement of Postsecondary Education (FIPSE) and the Ford Foundation.

Contents

Preface

In response to a request from the Association of American Colleges (AAC), the Mathematical Association of America (MAA) convened a special Task Force to address a range of issues concerning the undergraduate major as a sequel to the 1985 AAC report *Integrity in the College Curriculum.* Whereas the 1985 *Integrity* study examined the effectiveness of general education, the new AAC study addresses the contribution that "study in depth" makes to liberal education. Each of the several AAC disciplinary task forces examined how study in depth relates to goals for the major, assurance of intellectual development, and relations with other fields.

The MAA-AAC Task Force operated in the context of two other simultaneous MAA reviews of the mathematics major. One was conducted by a subcommittee of CUPM, the Committee on the Undergraduate Program in Mathematics; the other by COMET, the Committee on the Mathematical Education of Teachers. Since these other committees are charged by MAA to provide specific advice to the mathematical community about requirements for the mathematics major, the MAA-AAC Task Force dealt only with broader questions that are central to the AAC study. Hence this report is not intended as a detailed statement about curricular content, but as a statement of issues and priorities that determine the context of undergraduate mathematics majors.

In preparing the present draft, the Task Force held several open hearings on issues concerning the undergraduate major at national meetings of the Mathematical Association of America and the American Mathematical Society. A draft of this report was mailed for review to several hundred persons including every Governor, Chair, and Secretary of the 29 sections of the Mathematical Association of America; department heads, deans, and provosts at a variety of institutions; and many leaders of mathematical professional societies. This draft was also reviewed at meetings of AAC and MAA in January 1990. At the latter meeting, the document was discussed extensively at an invitational three-hour roundtable meeting that involved about forty experienced college and university mathematicians, including many who have been working actively to improve opportunities for women and minorities in the mathematical sciences.

Sectional Governors of the MAA were asked to nominate exemplary departments whose programs illustrate issues discussed in the report. Descriptions of some of these programs have been adapted as illustrations of promising practices in the final draft. During the spring of 1989 graduating seniors on several campuses were surveyed as part of a multi-disciplinary AAC effort to gather student opinion. Subsequently, leaders of several student chapters of the Mathematical Association of America were asked to review a draft of the report. Their careful and thoughtful responses underscore our belief in the value to students of the recommendations contained in this report.

The report has benefited enormously from these many external reviews. We believe that it now represents a consensus of the informed mathematical community concerning urgent issues of importance to the undergraduate mathematics major. In August, 1990 the report was unanimously approved by the MAA Board of Governors as an official MAA statement concerning the undergraduate major. We hope that widespread discussion of this report will help focus the efforts at reform that are already underway on many campuses.

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Mathematics in Liberal Education

The 1985 AAC report *Integrity in the College Curriculum* [6] sets forth a vision of undergraduate education steeped in the tradition of liberal education. It describes study in depth in terms of the capacity to master complexity, to undertake independent work, and to achieve critical sophistication. To achieve the kind of depth envisioned by the authors of this report, students must grapple with connections, progress through sequential learning experiences, and enhance their capacity to discern patterns, coherence, and significance in their learning. Study in depth should enhance students' abilities to apply the approaches of their majors to a broad spectrum of problems and issues, and at the same time develop a critical perspective on inherent limitations of these approaches.

We are especially concerned in this study with how the experience in the major contributes to the education of the great majority of students who do not pursue advanced study in the field of their undergraduate major. Hence we focus more on the *quality* of students' engagement with their collegiate major than on curricular content which may be required for subsequent study or careers. This emphasis on general benefits of the major rather than on specific things learned gives the AAC undertaking a distinctive perspective that is not often emphasized in discussions of the mathematics major by mathematicians.

Historical Perspective

For over 35 years the Committee on the Undergraduate Program in Mathematics (CUPM) has helped provide coherence to the mathematics major by monitoring practice, advocating goals, and suggesting model curricula. Until the 1950s, most mathematics departments functioned primarily as service departments for science and engineering. CUPM was established in 1953 to "modernize and upgrade" the mathematics curriculum and "to halt the pessimistic retreat to remedial mathematics." At that time total enrollment in mathematics courses in the United States was approximately 800,000; each year about 4,000 students received a bachelor's degree in mathematics, and about 200 received Ph.D. degrees.

Following an unsuccessful initial effort to introduce a "universal" first-year course in college mathematics, CUPM concentrated in its second decade on proposals to strengthen undergraduate preparation for Ph.D. study in mathematics [15]. Spurred on by Sputnik and assisted by significant support from the National Science Foundation, interest in mathematics rose to levels never seen before (or since) in the United States. By 1970 total undergraduate mathematics enrollments had expanded to over three million students; U.S. mathematics departments produced 24,000 bachelors and 1,200 doctoral degrees a year.

But then the bubble burst. As student interest shifted from personal goals to financial security, and as computer science began to attract increasing numbers of students who in earlier years might have studied mathematics, the numbers of mathematics bachelor's degrees dropped by over 50% in ten years, as did the number of U.S. students who went on to a Ph.D. in mathematics. However, total undergraduate mathematics enrollments continued to climb as students shifted from studying mathematics as a major to enrolling in selected courses that provided tools necessary for other majors. In 1981, at the nadir of B.A. productivity, CUPM published its second major comprehensive report on the undergraduate curriculum [16] which advocated a broad innovative program in *mathematical sciences.*

During the 1980s the number of undergraduate majors rebounded, although both the number and the percentage of U.S. mathematics students who persist to the Ph.D. degree continued to decline [1, 21]. As a consequence, one of the most urgent problems now facing American mathematics is to restore an adequate flow in the pipeline from college study to Ph.D. attainment. Three recent reports from the National Academy of Sciences [39, 9, 44] document this crisis of renewal now confronting U.S. mathematics.

The precipitous decline in the 1970s and early

1980s in the number of undergraduate mathematics majors paralleled both a significant decrease in mathematical accomplishment of secondary school graduates and an erosion of financial support for mathematical research. Strangely, all this occurred at precisely the same time that the full scope of mathematical power was unfolding to an unprecedented degree in scientific research and business policy. These conflicting forces produced a spate of self-studies within the mathematical community, some devoted to school education [13, 19, 40, 41], some to college education [2, 50, 59], a few to graduate education [8, 12], and others to research issues [9, 24, 29]. The crisis of confidence among mathematicians reflected broad public concern about the quality and effectiveness of mathematics education [24, 45, 32]. Even research mathematicians are now taking seriously the crisis in school mathematics [27, 62]. A recent report [43] by the National Research Council recommends sweeping action based on an emerging consensus with broad support that extends well beyond the mathematical community.

Market-driven demand for undergraduate mathematics majors is strong. There is a shortage of mathematical scientists with doctor's degrees; there is consistent demand (with high salaries) for those who hold master's degrees in the mathematical sciences; there is steady demand for undergraduate mathematics majors in entry-level positions in business, finance, and technology-intensive industry; and there is increasing need for well-qualified high school teachers of mathematics, especially as a consequence of the national effort to raise standards for school mathematics. Moreover, the increasingly analytical nature of other professions (business, law, medicine) as well as the continued mathematization of science and engineering provide strong indicators of the long-range value of an undergraduate mathematics major for students who are entering other professions.

Today, mathematics is the second largest discipline in higher education. Indeed, over 10% of college and university faculty and student enrollments are in departments of mathematics. However, more than half of this enrollment is in high school-level courses, and most of the rest is devoted to elementary service courses. Less than 10% of the total post-secondary mathematics enrollment is in post-calculus courses that are part of the mathematics major. Even in these advanced courses many students are not mathematics majors—they are enrolled to learn mathematical techniques used in other fields. As a consequence, the major has suffered from neglect brought about in part by the overwhelming pressure of elementary service courses.

In spite of the 1965 and 1981 reports from CUPM and dozens of collateral studies published in the last decade, there is no national consensus about the undergraduate mathematics curriculum. However, examination of practices at many campuses reveals common threads that are highly compatible with goals of the AAC *Integrity* study. These include a multiple track system that addresses diverse student objectives, emphasis on breadth of study in the major, and requirements for depth that help students achieve critical sophistication. In this report we build on these common threads to suggest new goals for study in depth that will both enhance the mathematics major and better suit students who will live and work in the twenty-first century.

Common Goals

Mathematics shares with many disciplines a fundamental dichotomy of instructional purpose: mathematics as an object of study, and mathematics as a tool for application. These different perspectives yield two quite different paradigms for a mathematics major, both of which are reflected in today's college and university curricula. The first, reminiscent of the CUPM recommendations of the 1960s, focuses on a core curriculum of basic theory that prepares students for graduate study in mathematics. The second, reflecting the broader objectives of CUPM's 1981 mathematical sciences report, focuses on a variety of mathematical tools needed for a "life-long series of different jobs." Typically faculty interests favor the former, whereas student interests are usually inclined towards the latter. Most campuses support a mixed model representing a locally devised compromise between the two standards, although accurate descriptions of present practice are virtually non-existent. (Partly in response to the paucity of information about degree programs in mathematics, the National Research Council through its project Mathematical Sciences in the Year 2000 is seeking to stimulate a comprehensive examination of degree patterns in mathematics.)

The 1985 AAC *Integrity* study, echoing themes reflected in similar studies [10, 38, 42], provides a broad context in which to examine practices of individual disciplines. The AAC report highlights "study in depth" as an essential component of liberal education, but not for the reasons commonly advanced by students and their parents—preparation for vocation, for professional study, or for graduate school. *Integrity* views study in depth as a means to master complexity, to grasp coherence, and to explore subtlety. "Depth cannot be reached merely by cumulative exposure to more and more . . . subject matter." The AAC goals for study in depth are framed by twin concerns for intellectual coherence intrinsic to the discipline and for development of students' capacity to make connections, both within their major and with other fields.

Both previous models advanced by CUPM for the undergraduate major reflect these same potentially conflicting concerns. The earlier major, which emphasized a traditional core mathematics curriculum as preparation for graduate study, was motivated principally by the internal coherence of mathematics; the more recent proposal stressed an interplay of problem solving and theory across both the broad spectrum of the mathematical sciences and the boundaries that mathematics shares with other disciplines. Although many believe that the latter model, emphasizing broadly applicable mathematical methods, is better geared to graduates' future employment, others—including many liberal educators—believe that broader abilities such as the art of reasoning and the disposition for questioning are of greater long-term practical value. However, both types of majors—and the many mixtures that exist on today's campuses—help students discern patterns, formulate and solve problems, and cope with complexity. In this sense, present practice of mathematical science majors in U.S. colleges and universities matches well the overall objectives of *Integrity in the College Curriculum.*

Certain principles articulated by CUPM in 1981 (reprinted in a recent compendium [17] of CUPM curriculum recommendations from the past decade) make explicit areas where *Integrity's* objectives and those of the mathematical community align:

- The primary goal of a mathematical sciences major should be to develop a student's capacity to undertake intellectually demanding mathematical reasoning.
- A mathematical sciences curriculum should be designed for all students with an interest in mathematics, with both appropriate opportunities for average mathematics majors and appropriate challenges for more advanced students.
- Every student who majors in the mathematical sciences should complete a year-long course sequence at the upper-division level that builds on two years of lower-division mathematics.
- Instructional strategies should encourage students to develop new ideas and discover new mathematics for themselves, rather than merely master the results of concise, polished theories.
- Every topic in every course should be well motivated, most often through an interplay of applications, problem-solving, and theory. Applications and interconnections should motivate theory so that theory is seen by students as useful and enlightening.
- Students majoring in mathematics should undertake some real-world mathematical modelling project.
- Mathematics majors should complete a minor in a discipline that makes significant use of mathematics.

Emphasis on coherence, connections, and the intellectual development of all students are evident in these principles from the 1981 CUPM report. At the level of broad goals, the prevailing professional wisdom concerning undergraduate mathematics matches well the intent of AAC's *Integrity.*

Diverse Objectives

Once one moves beyond generalities and into specifics of program development, however, mainstream mathematical practice often diverges from many of the explicit AAC goals. Most students study mathematics in depth not to achieve broad goals of liberal education but for some professional purpose—for example, to support their study of science or to become a systems analyst, teacher, statistician, or computer scientist. Others study mathematics as a liberal art, an enjoyable and challenging major that can serve many ends. It is as true in mathematics as in any other field that the great majority of undergraduate mathematics majors do not pursue advanced study that builds on their major.

Because so few U.S. students pursue graduate study in the mathematical sciences, many mathematicians believe that the mathematics major should be strengthened in ways that will prepare students better for graduate study in mathematics. Whereas formerly this view may have been based on the myopic academic view of undergraduate education as the first step in the reproduction of university professors, today in mathematics this perspective is reinforced by vigorous argument based on an impending shortage of adequately trained mathematics faculty. Some argue for greater depth to ensure that all mathematics majors are capable of pursuing further study; others argue for greater breadth to attract more students to the mathematical sciences. National need has now reinforced self-interest in emphasizing preparation of the next generation of Ph.D. mathematicians as an important priority for many college and university mathematics departments [20, 39].

This said, there remains considerable room for debate about strategies for achieving the several different (but overlapping) objectives that are common among the majors offered by the 2500 mathematics departments in U.S. colleges and universities:

- Advanced Study. Preparation for graduate study in various mathematical sciences (including statistics and operations research) or in other mathematically based sciences (e.g., physics, computer science, economics).
- Professional Preparation. Skills required to pursue a career that requires considerable background in mathematics:
  - *Natural and Social Science.* Background for careers in science or engineering, in biology (including agriculture, medicine, and biotechnology), in many computing fields, and in such emerging interdisciplinary fields as cognitive science or artificial intelligence.
  - *Business and Industry.* Preparation for careers in management, finance, and other business areas that use quantitative, logical, or computer skills normally developed through undergraduate study of mathematics.
  - *School Teaching.* Preparation for teaching secondary school mathematics, or for careers as mathematics-science specialists for elementary school.
- Liberal Education. General background for non-mathematical professions such as law, medicine, theology, or public service, or for other employment which does not directly use mathematical skills.

There is no lack of sources for advice about objectives. Recent documents outline priorities for teacher training [14], applied mathematics [25], and service courses [33]. The diversity of U.S. education ensures that different departments will have different priorities that recognize differences in entering students, in goals of graduates, and in institutional missions.

For many students the link between their undergraduate major and their post-graduate plans is very elastic. Such students study mathematics for the same reason that hikers climb mountains—for challenge, for fun, and for a sense of accomplishment. Often mathematics is paired with another major to provide complementary strengths. At one institution this has helped strengthen the mathematics major while enhancing connections to other fields of study:

> An analysis of the senior class shows that nearly half of the graduates had double majors. In many cases the mathematics major was taken to provide theoretical and computational grounding necessary for a modern approach to primary majors in economics, physics, chemistry, or biology. In other cases availability of a double major gave students who enjoyed mathematics an opportunity to continue their studies in mathematics while also gaining a desired major in an apparently unrelated field—for example, English, philosophy, or music. Many of our majors chose mathematics as a sound liberal arts approach to a general education. The success of the major program in mathematics is due in part to the belief of both faculty and students that the study of mathematics is not just for those intending to pursue a career in the area.

Some argue that the goals of liberal education are best served by a mathematics major designed to prepare students for graduate school. Even though most mathematics majors never undertake further study of mathematics, advocates of pre-graduate preparation argue that the special combination of robust problem solving with rigorous logical thinking achieved by a solid pre-graduate major also serves well the more general objectives of sequential study, intellectual development, and connected learning. This view is substantiated in part by a strong history of mathematics majors being eagerly sought by employers and graduate programs in other areas (e.g., law, economics) for a wide variety of non-mathematical professions.

Others argue that since most of today's college students do not foresee graduate study in mathematics as a desirable goal, it is only through a more general major stressing links to other fields that enough students can be recruited to major in mathematics to serve as a proper basis for the nation's needs in teaching, in science, and in mathematics itself. In this view, a broad major stressing mathematics as part of liberal education is an effective strategy for strengthening the pool of potential mathematical doctoral students as well as students in other mathematically oriented professions. It is rare to find a department of mathematics that would naturally place top priority on a major that specifically serves liberal education, as is common, for example, in many departments of English or philosophy. The needs of society and the constraints of client disciplines in science and engineering do not permit mathematicians this luxury. But in the present climate, departments are rediscovering the strategic value of a broad major, even for those who do continue professionally in mathematics. Here's how one institution expresses its objectives for the mathematics major:

> The mathematics major is designed to include students with a wide variety of goals, tastes, and backgrounds. Mathematics is an excellent preparation for fields from technical to legal, from scientific to managerial, and from computational to philosophic. It is also a source of satisfaction for people in every line of endeavor. Recognizing this, we have constructed a program to welcome interested students of all sorts. Our comparatively unstructured program reflects not only the diversity of interests of our students, but their increasingly diverse backgrounds, and the increasingly diverse nature of what are now being called the "mathematical sciences."

Since department goals must match institutional missions, it would not be right for any committee to recommend uniform goals for individual departments. We can, however, urge increased attention to an important and distinctive feature of undergraduate mathematics: *National need requires greater encouragement for students to continue their study of mathematics beyond the bachelor's level—whether as preparation for school teaching, for university careers, or for employment in government and industry.* In some institutions this encouragement may arise from a thriving pregraduate program; in others it may evolve from an emphasis on liberal education. In all cases, departmental objectives must be realistically matched to student aspirations and to institutional goals. Wherever faculty and students share common objectives, mathematics can thrive.

Multiple Tracks

Most mathematics departments resolve the dilemma of diverse goals for the major with some sort of track system. In some institutions there are separate departments such as Applied Mathematics, Operations Research, Statistics, or Computer Science, whereas in others these options are accommodated by means of explicit or implicit choices within the offerings of a single Department of Mathematics or Department of Mathematical Sciences. Tracks within the major are a sensible strategy to respond to competing interests of students, of faculty, and of institutions.

Although tracks do accommodate student interests—and thereby help sustain enrollments—they can produce a fragmented curriculum. Whereas in the late 1960s most mathematics majors took six or seven standard courses before branching into electives, by 1981 CUPM found that there was no longer any national agreement on such an extensive core of the undergraduate mathematics major. At that time, only elementary calculus and linear algebra were universally recognized as required courses within the mathematics major. Branching occurred after the third or fourth course, rather than after the sixth or seventh.

Today however, despite institutional diversity, there is striking uniformity in the elementary and intermediate courses pursued by mathematics majors: all begin with calculus for two, three, or four semesters; most introduce linear algebra in the sophomore year and require one or two semesters of abstract algebra; virtually all require some upper division work in analysis—the "theory of calculus." Nowadays, most require some computer work (programming, computer use, or computer science) as well as some applied work (statistics, differential equations, etc.) among the electives. This restores a *de facto* six-course core to the major, typically half the total.

The rise, fall, and restoration of a core curriculum in the mathematics major paralleled similar patterns in other arts and sciences. Whereas the CUPM recommendations of the late 1960s may have had too narrow a focus, the subsequent curricular chaos of the mid-1970s may have been too unstructured. Mathematicians began worrying then—as AAC is now—about whether the typical student's experience with the mathematics major may lack appropriate coherence and depth. Too often, it seemed to many, the mathematics major had become just an accumulation of courses without sufficient structure to ensure a common core of learning.

In most colleges courses in the mathematical sciences are taught in many different departments. Even upper division mathematics courses are commonly taught in departments of application: a recent survey [28] shows that as many students study post-calculus mathematics outside departments of mathematics as do so within traditional mathematics course offerings. Examples include discrete mathematics taught in departments of computer science; methods of mathematical physics taught in departments of physics; logic and model theory taught in departments of philosophy; optimization and operations research taught in departments of economics; and mathematical foundations of linguistics taught in departments of linguistics. The practice of cross-listing such courses or counting them as electives in the major varies enormously (and arbitrarily) from campus to campus.

Whether for good or ill, the diffusion of mathematics courses both within departments of mathematical sciences and into other departments has moved the mathematics major away from a strict linear vertical pattern towards a more horizontal structure typical of the humanities or social science major. Today's major, however, retains a distinctive strength of mathematics: sequenced learning. By its very nature, mathematics builds on itself and reinforces links among related fields. All mathematics courses build on appropriate prerequisites. A student who progresses from calculus to probability to operations research sees just as many connections as does one who moves through the more traditional sequence of advanced calculus and real analysis. Although the focus of each student's work is different, the contributions made by each track to the general objectives of study in depth are comparable, and equally valuable.

Moreover, it is common for advanced courses to be offered in sequences (e.g., Abstract Algebra I, II; Real Analysis I, II; Probability and Statistics I, II) that begin with a three or four course chain of prerequisites. Many departments, following the 1981 CUPM recommendations, require mathematics majors to take some advanced sequence without specifying which particular sequence it should be. Thus most mathematics majors today take a substantial sequence of courses, but they no longer all take the *same* sequence of core courses. This is a wise policy for undergraduate mathematics in today's diverse climate: *Each student who majors in mathematics should experience the power of deep mathematics by taking some upper-division course sequence that builds on lower-division prerequisites. It is neither necessary nor wise, however, to require that all mathematics majors take precisely the same sequence.*

Flexibility with rigor can be administered in a variety of ways. In one institution with a flourishing mathematics major, the mechanism is a personal "contract" developed to suit each student's own objectives:

> Students arrange their major sequence according to a contract system. Potential majors meet with a member of the department and prepare a list of courses and activities that will constitute the major. This allows the student to arrange his or her program to suit special needs. The faculty member judges the appropriateness of the student proposal in terms of post-graduate plans, other studies, and general departmental guidelines. This contract system has two distinct advantages: it serves the personal needs of students, and the process itself enhances students' education. The process of developing the contract provides an opportunity for the student to work closely with a faculty member, to understand the variety of mathematical options in a personal framework, and to see how a program ensuring depth and breadth of study can be achieved.

Emphasizing Breadth

At the same time as we stress the value of sequential courses to study in depth, we must also emphasize the essential contribution of breadth to building mathematical insight and maturity. Whereas course sequences demonstrate depth by building in expected fashion on prior experience, the links that emerge among very different courses (tying geometry to calculus, group theory to computer science, number theory to analysis) reveal depth by indirection: such links point to deeper common principles that lie beyond the student's present understanding but are within grasp with further study. They show the mountain yet to be climbed—to shift metaphors from depth to height—and offer hints of the explanatory panorama to be revealed by some future and more profound principles.

There are other good reasons to recommend breadth as an important objective of an undergraduate mathematics major. Students who are introduced to a variety of areas will more readily discern the power of connected ideas in mathematics: unexpected links discovered in different areas provide more convincing examples of a deep logical unity than do the expected relationships in tightly sequenced courses.

For the many majors who will teach (either in high school or college), it is vitally important that their undergraduate experience provide a broad view of the discipline—since further study generally is more narrow and specialized. For those seeking their niche in the world of mathematics, a broad introduction to many different yet interconnected subjects, styles, and techniques helps pique interest and attract majors. And for the many students who may never make professional use of mathematics, depth through breadth offers a strong base for appreciating the true power and scope of the mathematical sciences. Graduates of programs that emphasize breadth will become effective ambassadors for mathematics.

Every student who majors in mathematics should study a broad variety of advanced courses in order to comprehend both the breadth of the mathematical sciences and the powerful explanatory value of deep principles. Such breadth can sometimes be achieved with courses offered by the department of mathematics, but more often than not it would be educationally advantageous for students to also select a few mathematically-based courses offered by other departments.

Effective Programs

Departments of mathematics in colleges and universities exhibit enormous variety in goals and effectiveness. In various universities, the percentage of bachelor's degrees awarded to students with majors in mathematics ranges from well under one-half of 1 percent to over 20 percent. In some departments the major is designed primarily to prepare students for graduate school. Other departments focus much of their major on preparing students to teach high school mathematics, or on preparing students for employment in business and industry. Most departments fail to attract or retain many Afro-American, Hispanic, or Native American students, whereas a few succeed in this very difficult arena.

Many measures can be used to monitor effectiveness of a mathematics major. Indicators of numbers of majors, of employability, of graduate school admissions, of eventual Ph.D.s, or of placement in teaching jobs are used by different departments according to their self-determined missions. Many mathematics departments work hard to improve their effectiveness in one or another of these different dimensions. Exploration, experimentation, and innovation—along with occasional failures—are the hallmarks of a department that is committed to effective education.

Mathematics programs that work can be found in all strata of higher education, from small private colleges to large state universities, from average to highly selective campuses. The variety of mathematics programs that work reveal what can be achieved when circumstance and commitment permit it. When faculty resolve is backed by strong administrative support, most mathematics departments can easily adopt strategies to build vigorous majors even while meeting other service obligations.

One department that has had great success in attracting students to major in mathematics bases its work on two "articles of faith:"

- We believe that faculty should relate to their students in such a way that each student in the department will know that someone is personally interested in him and his work.
- We believe that careful and sensitive teaching that helps students develop confidence and self-esteem is far more important than curriculum or teaching technique.

Another department builds strength on a foundation of excellent introductory instruction:

> We put our best teachers in the introductory courses. We put the most interesting material in the introductory courses. We try to make the statements of problems fun, not dry. We work very hard to motivate all topics, drawing on applications in other disciplines and in the working world. We are less interested in providing answers than in motivating students to ask the right questions.

Effective mathematics programs reflect sound principles of psychology as much as important topics in mathematics:

> We try to make students proud of their efforts in mathematical problem-solving, and especially proud of their partial solutions—what some might call mistakes. We look at how much is right in an answer and teach how to detect and correct the parts that are wrong.

Regular, formal recognition of student achievement at different stages of the major serves to build students' confidence and helps attract students to major in mathematics. Students know mathematics' reputation for being challenging, so recognition of honest accomplishment can provide a tremendous boost to a student's fragile self-esteem. *Effective programs teach students, not just mathematics.*

Challenges for the 1990s

Changes in the practice of mathematics and in the context of learning pose immense challenges for college mathematics. Many of those issues that pertain directly to course content, curricular requirements, and styles of instruction are under review by committees of the mathematical community. We focus here on challenges that transcend particular details of courses and curriculum:

- The *learning* problem: To help students learn to learn mathematics.
- The *teaching* problem: To adopt more effective styles of instruction.
- The *technology* problem: To enhance mathematics courses with modern computer methods.
- The *foundation* problem: To provide intellectually stimulating introductory courses.
- The *connections* problem: To help students connect areas of mathematics and areas of application.
- The *variety* problem: To offer students a sufficient variety of approaches to a mathematics major to match the enormous variety of student career goals.
- The *self-esteem* problem: To help build students' confidence in their mathematical abilities.
- The *access* problem: To encourage women and minorities to pursue advanced mathematical study.
- The *communication* problem: To help students learn to read, write, listen, and speak mathematically.
- The *transition* problem: To aid students in making smooth transitions between major stages in mathematics education.
- The *research* problem: To define and encourage appropriate opportunities for undergraduate research and independent projects.
- The *context* problem: To ensure student attention to historical and contemporary contexts in which mathematics is practiced.
- The *social support* problem: To enhance students' personal motivation and enthusiasm for studying mathematics.

These challenges have more to do with the success of a mathematics program than any curricular structure. In the diverse landscape of American higher education, successful programs differ enormously in curricular detail, but they all have in common effective responses to many of these broader challenges. The agenda for undergraduate mathematics in the 1990s must focus at least as much on these issues of context, attitude, and methodology as on traditional themes such as curricula, syllabi, and content.

Learning

One principal goal of the undergraduate mathematical experience is to prepare students for lifelong learning in a sequence of jobs that will require new mathematical skills. Departments of mathematics often interpret that goal as calling for breadth of study. But another interpretation is just as important: because mathematics changes so rapidly, undergraduates must become independent learners of mathematics, able to continue their own mathematical education once they graduate.

Most college students don't know how to learn mathematics, and most college faculty don't know how students do learn mathematics. It is a tribute to the efforts of individual students and teachers that any learning takes place at all. Effective programs pay as much attention to learning as they do to teaching.

First-year students need special attention to launch their college career on a suitable course. Typically, they carry with them a high school tradition of passive learning which emphasizes bite-sized problems to be solved by techniques provided by the textbook section in which the problem appears. Unfortunately, by maintaining this traditional teaching format which perpetuates the myth of passive mathematics learners, college calculus teachers typically contribute more to the problem than to its solution.

For example, calculus, the common entry point for potential mathematics and science majors, often fails to come alive intellectually as it should or as it is now at many institutions where calculus reform efforts are underway. One school has found that new goals for calculus can significantly enhance the entrée of students into the study of college mathematics:

> The larger goals of the major are reflected in the calculus sequence, which is founded on three principles: context, collaboration, and communication. "Context" means that we focus on the meaning and significance of calculus in the world. "Collaboration" means that students work in groups and support each other. "Communication" means the recognition that calculus is first of all a language, not only for scientists, but for economists and social scientists. Our goal is fluency.

Another institution uses calculus as a vehicle to broaden radically the view of mathematics that students bring with them from high school:

> Calculus should give students a solid base for advanced study. It is our opinion that our calculus courses were the weakest part of our program. We had, in effect, allowed the high schools to set the tone for our entire program. Our new course is so radically different from traditional calculus that our students are forced to confront the transition from school to college mathematics. It carries several important messages, e.g., mathematics is crucial for understanding science; mathematics has a strongly experimental side; mathematics is something we all are capable of understanding deeply; and mathematics is the most powerful of all the sciences.

Some institutions offer special freshman seminars as a way to encapsulate the ideal of liberal education in an intimate setting that permits students to identify with faculty mentors. However, in mathematics the massive tradition of calculus often stands in the way, so very few mathematics majors can trace the origin of their college major to a freshman seminar. Ideally calculus itself would be seen by colleges as the intellectual equivalent of a freshman seminar in which students learn to speak a new language. If that analogy were to be accepted, mathematics departments would teach calculus only in a context that placed a great deal of emphasis on one-to-one communication between student and teacher. Unfortunately, in too many institutions calculus is taught in large impersonal settings that make meaningful person-to-person dialogue unrealistic. Many efforts are now underway to reform the teaching of calculus [64]; most of these experiments emphasize student motivation and styles of learning as a primary factor in reshaping the course.

One way or another, students should learn early in their college years how to study and learn mathematics. They should learn psychological as well as mathematical strategies for solving problems. They should come to recognize that it is common even for mathematicians to hear lectures or read material that they cannot grasp, and they must learn how to pick up clues from such experiences that will fit into their personal mathematical puzzles only some time later. They should learn the value of persistence and the strategic value of going away and coming back. These "metacognitive" skills to control one's own learning are virtually never learned in high school mathematics, so they must be planned into the early stages of the college curriculum.

As students progress through their mathematical study, they need to learn the value of library and electronic resources as tools for learning. Mathematics students rarely use the library or other sources of information, concentrating instead on mastering material in course texts. They need specific assignments that focus on the big map of mathematics in order to gain perspective on their brief undergraduate tour. *Undergraduate students should not only learn the subject of mathematics, but also learn how to learn mathematics.* The major in mathematics should become more than the sum of its courses. By conscious effort to help students negotiate in unfamiliar terrain, instructors can provide them with the tools of inquiry necessary to approach the literature and learn whatever they need to know.

Teaching

The purpose of teaching, and its ultimate measure, is student learning. So in some sense one cannot discuss one without the other. However, as students must learn to learn, so teachers must learn to teach. In mathematics more than in most other subjects, the role of teaching assistants and part-time instructors is particularly important, especially in the first year [12]. Although there is no formula for successful teaching, there is considerable evidence that separates certain practices that have proven successful from those that are generally ineffective [55]. Teachers who study this evidence can learn much from the experiences of others.

Despite the general reputation of mathematics as one of the most desirable environments for developing rigorous habits of mind, criticism of undergraduate mathematics has been mounting in recent years for failure in this, mathematics' distinctive area of strength. Those who study cognitive development criticize standard teaching practices for failing to develop fully students' power to apply their mathematical knowledge in unfamiliar terrain. These critics conclude that present teaching practice in undergraduate mathematics does not do as much as it should to develop students' intellectual power.

The evidence of failure is persuasive, both locally and globally [57]. Data on the inability of the profession to attract and retain the best and brightest college graduates is confirmed by case studies of students who cannot make effective use of what they have learned. Although some very good students use a mathematics major as a platform for substantial accomplishment, the majority of those who major in mathematics never move much beyond technical skills with standard textbook problems. Passive teaching and passive learning results from an unconscious conspiracy of minimal expectations among students and faculty, both of whom find advantages in a system that avoids the challenges of active learning that fully engages both students and teacher. Both the curriculum and teaching practices must respond to this challenge of intellectual malnutrition that is all too common in today's major.

Much of the research that bears on how students learn college mathematics has been conducted in the setting either of high school mathematics or college physics. The results from these efforts are often surprising, yet not well known among university mathematicians. They show, among other things, that formal learning by itself rarely influences behavior outside the artificial classroom context in which the concept was learned [53, 54]. Students who know how to solve differential equations of motion often have no better insight into the behavior of physical phenomena described by these equations than do others who never studied the equations; students who have learned course-based tests of statistical significance frequently do not recognize statistical explanations for events in the world around them [47].

Additional evidence of how young adults learn mathematics—or more often, why they fail to learn—has accumulated in recent years as a result of many innovations in teaching tried on different campuses. For example, intervention programs designed to improve the mathematical performance of minority students show the importance of a supportive environment: constructive teamwork in a context of challenging problems in which instructors and students know each other personally builds mathematical self-esteem and, as a consequence, leads to greatly improved learning [5, 30, 63]. Very different but equally striking lessons emerge from experiences of students who study calculus in a technology-intensive environment: by forcing students (and instructors) to focus on the behavior of mathematical objects (functions, algorithms, operators) rather than on their formalism, and by integrating visual, numerical, and symbolic clues into the mathematical environment, computers reveal to students and faculty both avenues for insight and common sources of misconception [36].

A third example, but by no means the last that could be cited, can be found in evidence of improved student motivation and self-reliance that occurs in those contexts where research-like experiences are used to enrich traditional classroom and textbook experiences: students whose minds and eyes become engaged in the challenge of true discovery are frequently transformed by the experience [56].

The evidence from such diverse but non-traditional instructional environments shows clearly the effectiveness of instruction that builds self-confidence on the foundation of significant accomplishment in a context that is meaningful to the student. Here is an especially dramatic example:

> In 1986 we began a critical evaluation of our program, course offerings, and teaching methods. This examination led to profound changes in our understanding of the teacher-student relationship, and of our role in the educational process. We found, for example, that we had not engaged our students sufficiently to assume an active role in their learning of mathematics. So we deliberately modified our courses and attitude to experiment with active student participation in doing mathematics problems and theory both in class and outside class.
>
> Results were strikingly positive, and we largely discontinued the typical mathematics lecture format, since lecturing kept students in a passive role. With an active participation method, students studied the text and worked problems before class; faculty and students discussed difficult points in class. Students presented problems and results on the board in class with encouragement and guidance from the instructor. We found that students became actively and enthusiastically involved in their learning of mathematics, with the instructor acting as a coach.
>
> As a consequence of these changes, our faculty

> and students have become a community of learners and scholars. Students now do mathematics in groups outside class, and more graduating seniors are seeking advanced degrees in mathematics. The number of mathematics majors rose from 69 in 1986 to 104 in 1988; the Mathematics Department is now the largest unit in the School of Natural Sciences. Finally, and perhaps most important, faculty affirm the belief that many more students can realize their mathematical abilities.

Several barriers separate educational studies and experiments from the larger community of college and university mathematicians. First, there are very few individuals who conduct formal research dealing directly with college mathematics. Second, mathematicians tend to distrust educational research. Third, and perhaps more important, mathematicians follow habit more than evidence in their teaching styles: even well-documented reports of better methods are insufficient to influence mathematicians to change their teaching habits. (This is not really too surprising, since neither do convincing classroom explanations of effective mathematical methods suffice to eradicate deep-seated misconceptions among students.)

Too often mathematicians assume with little reflection that what was good for their education is good enough for their students, not realizing that most of their students, not being inclined to become mathematicians, have very different styles of learning. College faculty must begin to recognize the proven value of various styles of instruction that engage students more directly in their own learning. *Those who teach college mathematics must seek ways to incorporate into their own teaching styles the findings of research on teaching and learning.*

Studies of metacognition and problem solving have yielded some insights that could be useful in pedagogy, but they have also been frustrated by barriers that confront all teachers of mathematics (for example, the difficulty of assessing just what has been learned, and the great length of time required to develop effective problem-solving heuristics). Such studies may yield insights that will change for the better the way teachers teach and students learn. But so far, college-level evidence is sufficiently slim to make the case unconvincing to those who most need to be persuaded. We really don't know how to induce most students to rise to the challenge of mathematical thinking; we have much to learn about what works and what does not. *To improve our understanding of the intellectual development of college mathematics students, mathematicians should increase their efforts to conduct research on how college students learn mathematics.*

We need to experiment with new ways to evaluate teaching. One key factor in good teaching is how much students learn; other factors include such issues as how many students decide to major in mathematics, to go on to graduate school, or to work in mathematical careers. These are measures of the quality of teaching done both by an individual and by a department. They look not only to indicators such as demonstrable knowledge, but also to motivation, attitude, and enthusiasm for the discipline. *Evaluation of teaching must involve robust indicators that reflect the broad purposes of mathematics education.*

Technology

Computing has changed profoundly—and permanently—the practice of mathematics at every level of use. College mathematics departments, however, often lag behind other sciences in adapting their curricula to computing, although considerable momentum is now building within the community for greater use of computing. The delay in response may have been due in part to conservatism of mathematicians, but at least as important is the simple fact of computer power: only in the last few years have desk-top machines achieved sufficient power to provide a legitimate aid to undergraduate (and research) mathematics. As a consequence, scientific computation is becoming a third paradigm of scientific investigation—alongside experimental and theoretical science—and the role of experiment in the practice of mathematics itself is increasing [52].

Computing can enhance undergraduate study in many ways. It provides natural motivation for many students, and helps link the study of mathematics to study in other fields. It offers a tool with which mathematics influences the modern world and a means of putting mathematical ideas into action. It alters the priorities of courses, rendering certain favorite topics obsolete and making others, formerly inaccessible, now feasible and necessary [34, 68]. Computers facilitate earlier introduction of more sophisticated models, thus making instruction both more interesting and more realistic. The penetration of computing into undergraduate mathematics is probably the only force with sufficient power to overcome the rigidity of standardized textbooks [59, 66, 22].

The power of technology serves also an epistemological function by forcing mathematicians to ask anew what it means to know mathematics. Those who explore the impact of technology on education indict introductory mathematics courses for imparting to students mostly skills that machines can now do more accurately and more efficiently. It is certainly true that typical indicators of student performance document primarily that mathematics students can carry out prescribed algorithms—just what computers (or calculators) can do. College faculty can no longer avoid the deep challenge posed by computers for undergraduate mathematics: once calculations are automated, what is left that can be taught effectively to average students?

Responses to this challenge are taking shape in experimental programs in many departments of mathematics. It is, therefore, too early to describe the impact computing will have on the mathematics major. Certainly in those courses and tracks devoted to applied mathematics, computing must exert a major influence on the shape of the curriculum. In this age it would be unconscionable to offer a major in applied mathematics, statistics, or operations research without substantial and fully integrated use of computer methods. Change will come more slowly in core subjects such as topology, analysis, and algebra. In each of these subjects there are impressive computer-based applications (e.g., fractals, coding theory, dynamical systems), yet none of these applications has been central to the traditional methodology of the subject as taught in introductory courses. Despite differences in the pace of change, however, there is no turning back: computers have dramatically altered the practice of mathematics. *To ensure an effective curriculum for the twenty-first century, undergraduate mathematics should change—both in objectives and in pedagogy—to reflect the impact of computers on the practice of mathematics.*

Early experiments that make significant use of computing in undergraduate mathematics courses show that as the balance of student work shifts from computation—which machines do better than humans—to thought, the course becomes more difficult, more unsettling, and less closely attuned to student expectations [58]. As the ground rules of mathematics change from carrying out prescribed procedures to formulating problems and interpreting results, it will become more important than ever for faculty to communicate clearly to students the goals of the curriculum and how they might differ from what students have been led to believe by their prior study of school mathematics.

One institution reports that computers have changed the context of education in significant and unexpected dimensions:

> We constructed a strong computer-experimental component at all levels. Besides the obvious advantages for building experience, context, and intuition, there are less obvious payoffs. For example, laboratories are places where students spend lots of time and which become, in reality, their habitats outside of their dormitory rooms. Students form allegiances and friendships in laboratories.

Different types of surprises were revealed on another campus that has pioneered use of computers in advanced courses:

> The use of computer software made possible the introduction of topics previously reserved for graduate students. Examples include the use of MACSYMA, REDUCE, and MACAULEY in commutative algebra and algebraic geometry. For example, a 1989 honors thesis gave us strong evidence of the advances possible in learning mathematics with the help of computational aids.

Computers change not only how mathematics is practiced, but also how mathematicians think. Both changes are unsettling, yet ripe with opportunity for effective education. Indeed, in the realm of computing, students and faculty must grope together towards a new balance of power among the many components of undergraduate mathematics.

The transition of mathematics from a purely cerebral paper-and-pencil (or chalk-and-blackboard) discipline to a high technology laboratory science is not inexpensive. Space must be expanded for laboratories; classrooms and offices need to be equipped with computers and display devices; support staff must be hired to maintain both hardware and software; faculty must be given time to learn to use computers, to learn to teach with computers, and to redesign courses and entire curricula to reflect the impact of computing. Institutions must plan not only for an expensive transition, but also for continued operation at a higher plateau comparable to the traditional laboratory sciences. *Colleges must recognize in budgets, staffing, and space the fact that undergraduate mathematics is rapidly becoming a laboratory discipline.*

Foundations

Because of the highly sequenced nature of the mathematics curriculum, no student can complete an undergraduate mathematics major without having secured a proper foundation of calculus, linear algebra, and computing in the first two years of college. For many students, half of the credits required for the major are taken in the first two years. So the nature of mathematical learning in these years is of crucial importance both for individual success in completing a strong mathematics major, and for programmatic success in building a critical mass of upper class mathematics majors.

One-third of the first and second year college students in the United States are enrolled in two-year colleges, including over two-thirds of Afro-American, Hispanic, and Native American students. It is clear from these figures that any effort to strengthen the undergraduate mathematics major, especially to recruit more majors among minority students, must be carried out in a manner that includes two-year colleges as a full partner in preparing the foundation for study in depth.

The tradition of common texts and relatively standard syllabi for standard mathematics courses in the first two years has facilitated transfer of students and credits during these years even as it has mitigated against the open intellectual environment many believe to be essential for effective learning. Now, however, as momentum builds for reform of courses in the first two years, and as departments experiment in an effort to reshape the entire mathematics major, there is some risk that students from lower socio-economic backgrounds—the predominant clientele of the two-year colleges—will find themselves pursuing a course of study that is inconsistent with the efforts of four-year colleges to improve the undergraduate mathematics major.

Some four-year institutions that are engaged in curricular reform are extending the scope of their mathematics program to include informal consortia with other nearby institutions. One private Eastern liberal arts college is building just such arrangements into its mathematics program:

> Plans are underway to create a partnership with a local community college and a public school system to interest students, especially minority students, in mathematics.

Many institutions maintain regular ties with local high schools or community colleges, but it is rare to find such arrangements related specifically to mathematics departments. What is now rare should become common: *To ensure equal opportunity for access to undergraduate mathematics majors, mathematics departments should work with nearby two-year colleges to maintain close articulation of programs.*

Connections

Recent studies of the mathematical sciences [7, 8] point to two special features that have characterized twentieth-century research: the extensive growth in areas of application (no longer just limited to physics and engineering) and the impressive unity of mathematical theories (revealed by the frequent use of methods from one specialty to solve problems in another specialty). Connectedness, therefore, is inherent in mathematics. It is what gives mathematics its power, what establishes its truth, and what reveals its beauty.

Mathematics is widely recognized as the language of science. Its enabling role in the development of the physical sciences formed the paradigm of the scientific method. Today it is beginning to play a similar role in the biological sciences, where mathematical tools as diverse as knot theory, nonlinear dynamics, and mathematical logic are being applied to model the structure of DNA, the flow of blood, and the organization of the brain.

Similar connections have emerged in the human, social, and decision sciences. Statistical models undergird virtually every study of human behavior; axiomatic studies have helped establish a rigorous theory of social choice; and multi-dimensional mathematical analysis is employed widely to model the multitudinous attributes of economic, psychological, or social behavior. Today mathematics is truly the language of all science—physical, biological, social, behavioral, and economic.

Even as the connections multiply between abstract ideas of mathematics and concrete embodiments in the world, so too have the internal connections within the mathematical sciences proliferated. Key theorems and deep problems that link separate mathematical specialties have provided a force for vast growth of interdisciplinary research. Examples abound, including such areas as stochastic differential equations at the interface of probability theory and analysis; combinatorial geometry joining arithmetical methods of discrete mathematics to problems of space, shape, and position;

and control theory that employs tools from analysis, linear algebra, statistics, and computer science to formulate effective mechanisms of control for automated processes.

If the undergraduate major does not reveal connections, it has not revealed mathematics. Most mathematics courses and most mathematics majors do make substantial contributions to this objective. Indeed, it is not uncommon for sophomores to select mathematics as a major instead of chemistry or biology precisely because in their mathematics courses they can see more clearly the logical connections among different parts: in mathematics they can "figure things out" rather than just memorizing results. (Of course, many students make the opposite choice, but usually for other reasons.)

At its best, mathematics overflows with connections, both internal and external. But one must be honest: undergraduate courses do not always show mathematics at its best. At their worst, especially in lower-division courses through which both majors and non-majors must pass, they reveal mathematics as a bag of isolated tricks: problems in elementary courses are often solved more by recognition of which section of the text they come from than by any real understanding of fundamental principles. *Dealing with open-ended problem situations should be one of the highest priorities of undergraduate mathematics.* For example:

- Mathematics teachers could bring in outside ("real-world") examples to illustrate applications of material being studied in regular coursework.
- Student projects could emphasize connections, either to fields that use mathematics or from one part of mathematics to another.
- Greater emphasis on multi-step problems amenable to a variety of approaches would wean students away from the school tradition of bite-sized, self-contained problems.
- Problem-oriented seminars provide wonderful opportunities to explore links between various branches of mathematics.

Such problems would be pregnant with ambiguity, ripe with subtle connections, and overflowing with opportunities for multi-faceted analyses.

Variety

Mathematicians are fond of talking about an elusive concept called "mathematical maturity" that is the Holy Grail of undergraduate mathematics [60]. Maturity is one objective of study in depth, but its meaning must be derived from the context of a student's level and goals. Depth itself is a metaphor for many things. To a mathematician it signals knowledge, insight, complexity, abstraction, and proficiency; to some others it connotes such elusive concepts as ownership, empowerment, and control. Although most colleges equate study in depth with the major—a circumstance reflected also in this report—it is important to recognize that for some students the major may not achieve the objectives that many have for study in depth. For these students, curricular structures other than the traditional major may better approach their goals for study in depth.

Many college students study mathematics as an important adjunct to another field which is their primary interest (e.g., economics, education, biology). Some colleges offer joint majors that combine study of mathematics with study in a related field, usually tied together with some type of joint project. The ever-present danger in such options is that they merely combine two shallow minors without ever achieving the depth traditionally required in a major. Notwithstanding this risk, one must acknowledge that some objectives of study in depth are well within the range of an effective joint major, say, in mathematics and biology where senior students employ mathematical models based primarily on lower division mathematics to model a biological phenomenon and then test and modify the model based on laboratory data.

Teacher education poses a special case of particular significance, since mathematics is one of the few disciplines taught throughout all twelve grades of school. It is obviously important for our nation that school teachers be both competent and enthusiastic about mathematics. Special committees recommend standards for preparation of mathematics teachers [14, 18], and these recommendations provide one particular perspective on study in depth.

Prospective secondary school teachers of mathematics generally pursue an undergraduate degree that includes a major in mathematics, often constrained in special ways to ensure breadth appropriate to the responsibilities of high school mathematics teachers. However, the appropriate mathematical preparation of prospective elementary and middle school teachers—who commonly teach several subjects, and sometimes teach the whole curriculum—is subject to much debate these days. Many national studies have recommended that prospective elementary school teachers, like secondary school teachers, major in a liberal art or science rather than in the discipline of education. However, the traditional mathematics major is generally inappropriate for teachers at this level, and today there appears to be virtually no example of a viable alternative. Even more vexing is the question of achieving depth in mathematics appropriate to an elementary school teacher within a major in some other field. Some interesting ideas can be found in the "new liberal arts" initiative sponsored by the Alfred P. Sloan Foundation which has attempted to infuse quantitative methods in traditional liberal arts subjects [38].

Most of the issues, guidelines, and recommendations in this study focus on the traditional mathematics major, which is where most students who study mathematics in depth are to be found. However, study in depth can be done at any level and in many contexts. *Mathematics departments should take seriously the need to provide appropriate mathematical depth for students who wish to concentrate in mathematics without pursuing a traditional major.*

Self-Esteem

One of the greatest impediments to student achievement in mathematics is the widespread reputation of mathematics as a discipline for geniuses. Many facets of school and college practice conspire to portray mathematics in "macho" terms: only those who are bright, aggressive, and inclined towards arrogance are likely to succeed. Those who do not instantly understand—including many thoughtful, reflective, creative students—are made to feel "deeply dumb," like outsiders who don't get the point of an in-joke.

It is hard to overstate the power of intimidation to erode students' self-confidence. Many calculus teachers recognize the problem: bright freshman "show-offs"—usually white males—whose questions are designed not so much to elicit answers and build understanding as to demonstrate their superior intelligence to their classmates. The ritual is not unlike the bluffing maneuvers that male animals employ to claim dominant status in a herd. Many who are concerned about equality of opportunity believe that the widespread display of "geniusism" as a measure of worth in mathematics is in part a mask for sexism—an unconscious emphasis on behavior intended to preserve the *status quo* regarding access to leadership in teaching and research.

Fortunately there is a growing recognition in the mathematical community that old traditions must be replaced with new approaches better suited to the demographic realities of our age. We need to recognize that individuals bring very different but equally valuable strengths to the study of mathematics. A multiplicity of approaches that encourage student growth in many different dimensions is far more effective than a single-minded focus leading to a linear ranking in one narrow dimension of "brightness." Not every value in mathematical talent can be measured well by timed tests or intercollegiate competitions; the "Putnam powerhouse" is not the only standard by which undergraduate majors should be judged. (The William Lowell Putnam Examination, a national contest for undergraduates, is the Nobel competition of collegiate mathematics. It stumps even faculty with questions so hard that the median national score for undergraduates is frequently 0.)

Specific efforts to focus the mathematics curriculum on the interests and abilities of all students can bring dramatic results, as this campus report shows:

> At the time of the first registration for first-year students, fewer than 20 individuals in the entire freshman class indicate that mathematics is a possible major. A year later, the number is in the 50's, and by the junior year the number is over 100. One reason for this impressive increase in student interest in a mathematics major is the departmental position that mathematics is for everyone, not just the gifted. We attempt to demonstrate the power and applicability of mathematics by emphasizing breadth of study during the second and third years.

Self-confidence increases when students succeed, and decreases when they fail. "What students need to build self-confidence are genuine *small successes* of their own" [67]. Initial successes come from routine homework, but these are insufficient to the task. More effective are instructional strategies that engage the student in active learning: open-ended problems, team work that builds diverse problem-solving skills; undergraduate research experiences; independent study. *Building students' well-founded self-confidence should be a major pri-*

ority for all undergraduate mathematics instruction.

Access

Data from many sources [3, 48, 65] show that women and members of certain minority groups often discontinue their study of mathematics prematurely, before they are prepared appropriately for jobs or further school. Afro-American and Hispanic students drop out of mathematics at very high rates throughout high school and college; only a tiny fraction complete an undergraduate mathematics major [46]. In college, women major in mathematics almost as often as men do, but they persist in graduate studies at much lower rates. (Interestingly, mathematics comes closer to achieving an even balance of men and women among its undergraduate majors than virtually any other discipline; this record of equality disappears, however, in graduate school.)

Evidence from various intervention programs shows that the high drop-out rate among minority students can be reduced [4, 5, 30, 63]. Appropriate expectations that provide challenges without the stigma of "remediation" together with assignments and study environments that reinforce group learning have proved successful on many campuses. Mentoring programs of various types open doors of opportunity to women and minorities who have traditionally been under-represented in mathematically based fields. What becomes clear from these programs is that the tradition of isolated, competitive individual effort that dominates much mathematics instruction does not provide a supportive learning environment for all students.

Assignments that stress teamwork on problems chosen to relate to student interests can help many students succeed in mathematics. The experiences of students who work in teams to solve large computer science projects and of those who participate in science research groups show clearly the benefit of incentives for careful work that is created by the team atmosphere. Mathematicians must learn that the teaching strategies they recall as being successful in their own education—and in the education of a mostly white male professional class—do not necessarily work as well for those who grow up in vastly different cultures within the American mosaic.

Programs that work for minority students are built on the self-evident premise that students do not all learn mathematics in the same way. Classroom methods must fit both the goals of the major (e.g., to help students to learn to communicate mathematically) and the learning styles of individual students (e.g., need for peer support and positive feedback). These same principles apply to all students, not just to students of color. *To provide effective opportunities for all students to learn mathematics, colleges must offer a broader spectrum of instructional practice that is better attuned to the variety of students seeking higher education.*

Communication

College graduates with majors in mathematically-based disciplines are often perceived by society as being verbally inept: the stereotype of the computer hacker who cannot communicate except with a computer has permeated the business world, and tainted mathematics graduates with the same reputation. Recognizing the legitimate basis for this concern in the incomprehensible writing of their own upper-division students, many mathematics departments are beginning to emphasize writing in mathematics courses at all levels.

The forms of writing employed in mathematics courses include the standard genres used in other disciplines (expository essays, personal journals, laboratory reports, library papers, research reports) as well as some that are particularly relevant to mathematics (proofs, computer programs, solutions to problems). Many students and professors are uneasy about what writing means in a mathematics class, about how to grade it, and how to improve it. Few mathematicians know how to teach students to improve their writing or speaking, although there is increasing professional interest in this issue [31, 37, 61].

One department focuses on communication throughout the major, and stresses writing and speaking mathematics in a required senior colloquium:

> The conclusion of the major features the colloquium course "Mathematical Dialogues." The emphasis here, as in earlier courses, is on communication, as well as on the connections among the different branches of mathematics. Mathematical Dialogues consists of lectures from invited scholars, discussions, and independent work. Students are expected to read papers and write reviews, to listen to talks and to deliver them.

In industry, one of the most important tasks for a mathematician is to communicate to nonmathematicians in writing and orally the mathematical formulation and solution of problems. Each student's growth in mathematical maturity depends in essential ways on continual growth in the ability to communicate in the language of mathematics: to read and write, to listen and speak. Students must learn the idioms of the discipline, and the relation of mathematical symbols to English words. They need to learn how to interpret mathematical ideas arising in many different sources, and how to suit their own expression of mathematics to different audiences. *Mathematics majors should be offered extensive opportunities to read, write, listen, and speak mathematical ideas at each stage of their undergraduate study.* Indeed, writing and speaking is the preferred test of comprehension for most of the broad goals of study in depth.

Transitions

As students grow in mathematical maturity from early childhood experiences to adult employment, they face a series of difficult transitions where the nature of mathematics seems to change abruptly. These "fault lines" that cross the terrain of mathematics education appear at predictable stages:

- Between arithmetic and algebra, when letter symbols, variables, and relationships become important.
- Between algebra and geometry, when logical proof replaces calculation as the methodology of mathematics.
- Between high school and college, when the expectation for learning on one's own increases significantly.
- Between elementary and upper-division college mathematics, when the focus shifts from techniques to theory, from solving problems to writing proofs.
- Between college and graduate school, when the level of abstraction accelerates at a phenomenal rate.
- Between graduate school and college teaching, when the realities of how others learn must take precedence.
- Between graduate school and research, when the new Ph.D. must not just solve a serious problem, but learn to find good problems as well.

Students experience real trauma in crossing these transitions; many drop out of the mathematics pipeline as a consequence, often to the detriment of their future study in many disciplines. Mathematics education at all levels, from grade school through graduate school, should take as a goal to smooth out the roughness caused by these difficult transitions. College mathematics departments should, in particular, seek to streamline the transition of students to college, to upper-division mathematics, and to graduate school.

In college, students often experience a different type of transitional problem that applies in virtually all courses: to understand the relation between theory and applications. This is probably the most common complaint that students and faculty in collateral disciplines raise about undergraduate mathematics courses: they are often perceived as being too theoretical and insufficiently applied. Although in some cases this perception may be well justified, in many other instances the problem rests more with insufficient effort to demonstrate the value of theory to application than with an actual excess of theory. The problem is not that the transition from application to theory is inappropriate, but that it is often taken without sufficient effort to build appropriate motivation or connections. *Smooth curricular transitions improve student learning and help maintain momentum for the study of mathematics.*

Research

The role of so-called "capstone experiences" such as undergraduate research, theses, or senior projects is one of the more controversial ingredients in discussions of the mathematics major. Typically, such requirements are common in the humanities and the sciences, especially in more selective institutions. In the humanities they are viewed as opportunities for integration; in the sciences, as opportunities for research. In both science and humanities, capstone requirements offer apprenticeships in the investigative methods of the field.

In mathematics, however, there has been little consensus about objectives, feasibility, or benefits of this type of requirement. Very few institutions heeded the 1981 CUPM call for a required course in mathematical modelling for all majors. Many mathematicians believe in coverage as more crucial to understanding: standard theorems, paradigms of proof, and significant counterexamples in all major areas must be covered before a student is ready to advance to the next stage of mathematical maturity. In this view, learning what is already known is a prerequisite to discovering the unknown. Moreover, special capstone courses appear superfluous since each course provides its own capstone—the fundamental theorem of calculus, the central limit theorem in statistics, the fundamental theorem of algebra—which ties together a long chain of prior study. When forced to choose between a capstone experience or another advanced course, advocates of coverage will unhesitatingly choose the latter.

Because of mathematics' austere definition of "research"—a definition which, incidentally, rules out the professional work of more than half the nation's mathematics faculty—many mathematicians believe that except in very rare cases, undergraduates cannot do research in mathematics. Moreover, in most areas of mathematics, students cannot even assist in faculty research, as they do quite commonly in the laboratory sciences. The exceptions in mathematics are principally where computer investigation—the mathematician's laboratory—can aid the research effort. As a consequence, many mathematicians believe that further coursework would better serve the goals of integration (because the higher one progresses in mathematics, the more internal links one can see) and at the same time help advance the student towards better preparation for further study or application of mathematics.

Others feel that any encounter with a substantial problem that a student does not know how to solve can provide a legitimate and rewarding research experience. Indeed, many colleges have used summer experiences with undergraduate research as an effective strategy to recruit students to careers in the mathematical sciences [26, 56], and the National Science Foundation is actively supporting such programs. There are now many diverse programs offering research experiences for undergraduate mathematics majors.

In applied areas—especially in statistics, computing, and operations research—it is easier to develop projects that are sufficiently rich and varied so that students can make progress along various lines of investigation. Computers now are making inroads in theoretical areas of mathematics, permitting exploration of conjectures that heretofore were beyond the range of any undergraduate. Students preparing to teach mathematics in high school also have open an enormous range of appropriate projects to translate interesting newer mathematics into curriculum appropriate to the schools. In some cases students may want to undertake research into how people learn mathematics, to explore for themselves the effectiveness of various instructional strategies and the impact of computers on development of mathematical understanding.

Internships in industry, co-op programs that mix study with work, and summer research opportunities in industrial or government laboratories provide rich environments for breaking down the artificial barriers of courses and classrooms. They enable students to integrate mathematics learned in several different courses; to experience the role of mathematical models; to extend their mathematical repertoire beyond just what has been taught; and to establish mathematical concepts in a context of varied use, applications, and connections.

Experiences of departments with long-standing traditions of undergraduate research or senior projects confirm both the value of such work and the effort required for success:

> While these projects require a great deal of time and effort on the part of students and faculty, we generally feel that it is well worth it. Most of the students report that they had worried about the senior project for their first three years, but had ultimately found it to be a very worthwhile and stimulating part of their college experience. All recommended that this important aspect of the undergraduate experience be retained.

One department in an institution whose academic calendar permits extended blocks for full-time study in one subject requires all majors to complete a major project in the senior year:

> The final project in the major field should demonstrate application of the skills, methods, and knowledge of the discipline to the solution of a problem that would be representative of the type to be encountered in one's career. Project activities encompass research, development and application, involve analysis or synthesis, are experimental or theoretical, emphasize a particular subarea of the major or combine aspects of several subareas.

Another department uses summers to provide opportunities for research experiences: student participants range from freshmen to seniors, and engage in a wide variety of mathematical investigations:

> We are convinced that everyone working in mathematics can find problems appropriate for undergraduates. Many problems can be attacked

> without any knowledge of the complex machinery which generated them. Mathematicians know how exciting mathematical research can be. The best way to generate interest in mathematics is to provide undergraduates with the chance to experience that excitement.

Since hard work by itself is insufficient to ensure reasonable progress on a mathematical problem, there is ever-present danger that undergraduates confronted with difficult theoretical problems will flounder and become discouraged. Strong faculty intervention can prevent disaster, but excessive supervision undermines the independence that is supposed to result from the project. Effective undergraduate research experiences require careful planning and steady, unobtrusive leadership. One must carefully choose problems to be suggested to undergraduates for the research experiences: they must be tailored to the individual undergraduate.

Effective programs provide stepping stones to help students progress from routine homework to independent investigation. For example, one institution plans a progression leading to the senior project:

> Mathematics majors enroll in a Junior Seminar where they are asked to read critically two senior projects from earlier years to describe the strengths and weaknesses of these papers, and to suggest how they would improve on these papers had they written them. This Seminar also helps acquaint these students with appropriate standards of exposition in mathematics.

The range of opportunities for independent investigation is so broad and the evidence of benefit so persuasive as to make unmistakably clear that research-like experiences should be part of every mathematics student's program. *Undergraduate research and senior projects should be encouraged wherever there is sufficient faculty to provide appropriate supervision.* Effective programs must be tailored to the needs and interests of individual students; no single mode of independent investigation can lay claim to absolute priority over others. Flexibility of implementation is crucial to ensure that all experience the exhilaration of discovery which accompanies involvement with mathematical research.

Context

Mathematics courses—especially those taken by majors—have traditionally been taught as purely utilitarian courses in techniques, theory, and applications of mathematics. Most courses pay no more than superficial attention to the historical, cultural, or contemporary context in which mathematics is practiced. Today, however, as mathematical models are used increasingly for policy and operational purposes of immense consequence, it is vitally important that students of mathematics learn to think through these issues even as they learn the details of mathematics itself.

Examples abound of mathematical activity that leads directly to decisions of great human import. Software written for the Strategic Defense Initiative depends on mathematical theories of orbital dynamics for its performance, and on the ability of logicians and computer scientists to verify that complex untestable programs will perform correctly under any possible situation. Debates about the relation of carbon dioxide buildup to global warming and consequent implications for governmental and industrial policies center in large part on different interpretations of statistical and mathematical projections. Computer-controlled trading of stocks, epidemiological studies of AIDS, and implications of various voting rules offer other examples where mathematics really matters in important decisions affecting daily life.

Students of mathematics should be encouraged to see mathematics as a human subject whose theories often begin in ambiguity and controversy. It takes decades, sometimes centuries, for scholars to sculpt and polish the precise theories that are expounded in today's textbooks. Historical analogs provide useful yardsticks to students (and faculty) who seek to understand the limits of what mathematics can contribute to public policy. As society comes to rely increasingly on mathematical analyses—often well-disguised—of social, economic, or political issues, mathematics majors must confront the social and ethical implications of such activity. All such issues can be enlightened by appropriate historical case studies, and motivated by compelling debates of our age. *All mathematics students should engage in serious study of the historical context and contemporary impact of mathematics.*

One possible strategy to achieve both this objective and several others as well is to adapt a modelling project or course to problems of significant societal impact. In such a setting students could undertake original investigation, gain experience in reading, writing, listening, and speaking about mathematically rich material, explore historical antecedents and contemporary debates, and gain experience in team work to address complex, open-ended problems. For many students a capstone project on a public policy issue would be a fitting way to relate their mathematics major to liberal education.

Social Support

The abstract, austere nature of mathematics provides relatively few intrinsic rewards for the typical undergraduate who is trying to pursue a field of study and at the same time learning to establish and maintain personal friendships. In this context the social support provided by departmental activities can be decisive in tipping the balance either for or against a mathematics major. Peer group support helps build mathematical self-confidence and enhances the intrinsic rewards that come from mathematical achievement.

Virtually all successful mathematics departments instigate and support a variety of extracurricular activities. Examples include mathematics clubs, student chapters of the Mathematical Association of America, or chapters of the mathematics honorary society Pi Mu Epsilon. Another common feature of successful departments is informal faculty-led sessions to help students solve problems posed in collegiate periodicals or to prepare for national contests such as the Putnam Examination or the Mathematical Modelling Contest. Banquets, picnics, and barbecues lend a light touch that help students become acquainted with each other and with the faculty of the entire department.

Other activities can enrich students' experiences with their courses by providing links to the world beyond the campus. Undergraduate colloquia with visiting mathematicians from industry or universities is one common mechanism. Alumni involvement through career nights or other activities can help students imagine what they too could do with their major. Current students will be inspired when departments make visible the variety of accomplishments of their graduates—not only those who have become mathematicians but also the majority who have used their undergraduate mathematics for other ends. *Mathematics departments should exert active leadership in promoting extracurricular activities that enhance peer group support among mathematics majors.*

Mechanisms for Renewal

Constant vigilance is needed to maintain quality in any academic department. This is especially true in mathematics, where the subject is continually evolving, where external departments impose their own often-conflicting demands, where so much teaching effort is devoted to remedial, elementary, and lower division work, and where the very ability of the discipline to attract sufficient numbers of students to careers in the mathematical sciences is now in serious doubt. We focus here on five mechanisms of renewal:

- Dialogue: To talk with students and colleagues.
- Assessment: To measure what is happening.
- Faculty Development: To improve intellectual vitality.
- Departmental Review: To listen to colleagues and clients.
- Graduate Education: To provide leadership for improvement.

The key ingredient is listening—to one's students, to one's discipline, to one's colleagues, to one's friends, and to one's critics. Departments that listen—and learn—will thrive.

Dialogue

Departments often know very little about their students' views of the undergraduate mathematics major. That different students pursue mathematics for very different reasons is clear. Most departments must accommodate students with quite different purposes, although certain departments tend to focus their programs on one or another objective (for example, preparation for jobs, preparation for teaching, preparation for graduate school). Many departments, especially small departments, find it impossible to sustain several different programs of equal high quality.

Mathematicians also frequently know almost nothing about the expectations held by their colleagues in cognate disciplines for the mathematical preparation of students with other majors. It is not uncommon for the three interested parties—mathematics professors, science faculty advisors, and students—never to discuss goals or objectives, but only credit hour requirements. It should come as no surprise that in the absence of good communication, misunderstandings flourish.

Undergraduate mathematics shares many borders with other subjects and institutions: vertically with high schools below and graduate schools above; horizontally with science, business, and engineering. Each border is a potential impediment to the smooth flow of ideas and students. Mathematics departments must work hard to maintain effective articulation across these many boundaries:

- With high schools whose curriculum is also changing and whose students will arrive at college with new expectations.
- With departments in the physical sciences and engineering whose students use advanced mathematics.
- With graduate schools in the mathematical sciences, which attract and retain far too few U.S. students.
- With employers of bachelors degree graduates who expect employees who can function effectively in a work environment.

Regular discussion is essential to maintain effective policies that will satisfy these many boundary conditions.

To the extent that resources permit, departments should seek to determine and then accommodate different student career interests. This means that even small departments should provide mechanisms (e.g., independent study, special seminars) to allow students of diverse interests to receive a major suitable to their career objectives. Mathematics is too diverse and student purposes too different for any single set of eight to ten courses to meet all needs equally well.

Students too must recognize that the practice of mathematics is quite different from the textbook image they usually bring with them from high school. Often students expect of college mathematics merely advanced topics in the spirit of school mathematics: a succession of techniques, exercises, and test problems, each explained by the instructor with sufficient clarity that what remains for the student is only the requirement of practice and memorization. Such expectations do little to foster creativity, independence, criticism, and perspective—the more important goals of liberal education.

The different perspectives of mathematics student and mathematics professor often approach caricature. Eager students expect of college classes directed instruction in tools of the trade with which they can, upon graduation, get jobs that pay more than their professors earn. Professors, in contrast, expect students who are eager to take on challenging problems and who will learn on their own whatever they need to make progress. Students, in this exaggerated portrait, feel responsible only for what they have been taught, whereas faculty judge as truly significant only those things students can do which they have not been taught.

It is important for mathematics departments to help faculty and students recognize their own perspectives on mathematics and understand the perspectives of others. Doing this is not the same as covering a syllabus of mathematical topics; it involves instead various strategies to enable faculty and students to discuss mathematics in informal ways. Such discussions are an important part of the process by which students grow from the limited school perspective to the self-directed stance of a professional.

Announcing or publishing department goals is not sufficient to achieve this important objective. What is required is a process that engages all students in significant and repeated discussion of individual goals throughout their undergraduate study of mathematics. In particular, *careful and individualized advising is crucial to students' success.* Effective advising builds an atmosphere of mutual respect among faculty and students. Courses, career objectives, motivations, fears, celebrations are all part of advising, and of special importance in the long, slow process of building students' self-confidence.

Assessment

Many would argue that goals for study in depth can be effective only if supported by a plan for assessment that persuasively relates the work on which students are graded to the objectives of their education. Assessment in courses and of the major as a whole should be aligned with appropriate objectives, not just with the technical details of solving equations or doing proofs. Many specific objectives can flow from the broad goals of study in depth, including solving open-ended problems; communicating mathematics effectively; close reading of technically-based material; productive techniques for contributing to group efforts; recognizing and expressing mathematical ideas embedded in other contexts. Open-ended goals require open-ended assessment mechanisms;

although difficult to use and interpret, such devices yield valuable insight into how students think.

Relatively few mathematics departments now require a formal summative evaluation of each student's major. The few that do often use the Graduate Record Examination (or an undergraduate counterpart) as an objective test, together with a local requirement for a paper, project, or presentation on some special topic. Many institutions, frequently pressured by mandates from on high, are developing comprehensive plans for assessing student outcomes; a few are exploring innovative means of assessment based on portfolios, outside examiners, or undergraduate research projects. Here's one example that blends a capstone course with a senior evaluation:

> The Senior Evaluation has two major components to be completed during the fall and spring semesters of the senior year. During the fall semester the students are required to read twelve carefully selected articles and to write summaries of ten of them. (Faculty-written summaries of two articles are provided as examples.) This work comprises half the grade on the senior evaluation. During the fall semester each student chooses one article as a topic for presentation at a seminar. During the spring semester the department arranges a seminar whose initial talks are presented by members of the department as samples for the students. At subsequent meetings, the students present their talks. Participation in the seminar comprises the other half of the grade for the Senior Evaluation.

Because of the considerable variety of goals of an undergraduate mathematics major, it is widely acknowledged that ordinary paper-and-pencil tests cannot by themselves constitute a valid assessment of the major. Although some important skills and knowledge can be measured by such tests, other objectives (e.g., oral and written communication; contributions to team work) require other methods. Some departments are beginning to explore portfolio systems in which a student submits samples of a variety of work to represent just what he or she is capable of. A portfolio system allows students the chance to put forth their best work, rather than judging them primarily on areas of weakness.

The recommendations [41] from the National Council of Teachers of Mathematics for evaluation and assessment of school mathematics convey much wisdom that is applicable to college mathematics. Assessment must be aligned with goals of instruction. If one wants to promote higher order thinking and habits of mind suitable for effective problem solving, then these are the things that should be tested. Moreover, assessment should be an integral part of the process of instruction: it should arise in large measure out of learning environments in which the instructor can observe how students think as well as whether they can find right answers. *Assessment of undergraduate majors should be aligned with broad goals of the major: tests should stress what is most important, not just what is easiest to test.*

Faculty Development

The relation of research and scholarship to faculty vitality is one of the most difficult issues facing many departments of mathematics, especially in smaller institutions. Professional activity is crucial to inspired teaching and essential to avoid faculty burn-out. Mathematical research in its traditional sense plays only a small role in the mechanisms required to maintain intellectual vitality of a mathematics department: only about one in five full-time faculty in departments of mathematics publish regularly in research journals, and fewer than half of those have any financial support for their research. Clearly the community needs to encourage and support a broader standard as a basis for maintaining faculty leadership both in curriculum and in scholarship.

The first step is to expand the definition of professional activity from "research" to "scholarship," more in a manner akin to that currently recognized in some other academic disciplines. Applied consulting work, software development, problem solving, software and book reviews, expository writing, and curriculum development are examples of activities that serve many of the same purposes as research: they advance the field in particular directions, they engage faculty in active original work, they serve as models for students of how mathematics is actually practiced, and they provide opportunities for student projects.

Teaching in new areas is also a form of scholarship in mathematics. Unlike many other disciplines where faculty rarely teach outside their own areas of specialty, mathematicians are generally expected to teach a wide variety of courses. Learning and then teaching a course far outside one's zone of comfort is an effective way to build internal connections which then spill over in all courses one teaches. A teacher who is still an active learner sets a fine example for students concerning the true meaning of scholarship.

The second step is to insist on greater communication about professional activity in mathematics so that it becomes public. Only the bright light of public scrutiny by colleagues in various institutions—not only on one's own campus—can affirm the quality and value of professional work. "Public" need not mean merely publication; lectures, workshops, demonstrations, reports of various sorts can serve the same objective. What matters is that the result become part of the profession, and be evaluated by the profession. *To ensure continued vitality of undergraduate mathematics programs, all mathematics faculty should engage in public professional activity, broadly defined.*

Department Review

More than any other academic discipline, mathematics is constrained to serve many masters: the many sciences that depend on mathematical methods; the demand of quantitative literacy that undergirds general education; the need to educate teachers for our nation's schools; the need of business and industry for mathematically literate employees; the expectation of mathematical proficiency by faculty and students in natural science, business, engineering, and social science; the professional standards of employers for entry-level technical personnel; and the requirements of the mathematical sciences themselves for well-prepared graduate students. It is an enormous challenge for a department of mathematics, one that very few are able to fulfill with distinction in every dimension.

Because of these diverse demands, it is especially important that departments of mathematics undergo regular review, with both external and internal mechanisms to provide evaluation and advice. External requirements mandate periodic review of all departments in many colleges and universities, especially in public institutions. But in other institutions, department goals are defined implicitly without self-reflection or benefit of external perspectives. At worst, the goals of such departments are defined by coverage of standard textbooks. Often it takes a crisis—such as when the engineering or business school complains about certain courses—for departments to step back and examine their objectives. Reviews should take place regularly, not just when some crisis threatens the *status quo.*

Client disciplines expect from mathematics departments an amazing repertoire of support services for students who will major in other fields [44]. Some demand a magic bullet—a perfect infusion of just those mathematical methods (and no more) needed in the other field; others expect a rigorous filter that will pass on only those students who are sufficiently bright to function ably in upper-division work in other fields. Occasionally, but all too rarely, an external discipline will require mathematics primarily to enable students to benefit from the intrinsic values of mathematics: logic, rigor, analysis, symbol-sense, etc. Since the expectations of other departments are often not clearly conveyed by the list of mathematics courses that they recommend or require, regular reviews provide a good mechanism—but not the only one—to ensure that different departments at least understand their differing perspectives and objectives.

Virtually all departments receive informal feedback from graduates, employers, and graduate schools. Speaker programs that bring students and faculty into contact with users of mathematics serve both to inform students about the broad world of mathematics beyond their classroom walls and to provide informal feedback to help regulate the curriculum and keep it properly tuned to the needs of graduates. All such informal means of feedback are valuable and must be encouraged. However, they are no substitute for formal, regular, external review. *Both external reviews and informal feedback are needed to assure quality in departments of mathematics.*

There are many advantages to a regular program of external reviews that should form the basis of all reviews:

- A broad-based review provides a strategic opportunity to document the accomplishments of a department. Well-structured reviews can effectively counter external (political) demands for narrow or inappropriate instruments of assessment such as a multiple-choice examination of all graduates.
- Reviews provide a structured and neutral forum for mathematicians to discuss with those who use mathematics both the mathematical needs of client disciplines and the common issues that both mathematics and the client discipline face

in accommodating changes that are underway in the mathematics curriculum.

- By involving members of the faculty outside the department in the review—especially those from fields that are served by mathematics and those involved in faculty curriculum committees—a department of mathematics can help educate colleagues across the campus about the special opportunities and challenges of teaching mathematics. Ignorance can usually be turned to understanding through discussions prompted by the occasion of a regular review.
- By including non-academic reviewers such as industrial executives, scientists, professionals, and community leaders, the department can gain valuable insight into the qualities that will be expected of graduates who enter the work force.
- Regular reviews encourage faculty members to think about the department's program as a whole, rather than only about the courses they teach. Such discussions make it more likely that the curriculum will remain responsive to student needs, and to the changing demands of the mathematical sciences. Reviews provide an ideal mechanism for the department to assert control over its own program.

The Mathematical Association of America can provide advice to departments both about the structure of effective reviews and about appropriate consultants or reviewers.

Graduate Education

Even though relatively few mathematics majors go on to receive a graduate degree in the mathematical sciences, the health of college mathematics is inextricably linked with the status of graduate education. As the sole agent for advanced degrees, graduate schools bear alone the responsibility for preparing college mathematics teachers; as the primary locus of mathematical research, graduate schools shape the nature of the discipline, and hence of the curriculum. Much of the responsibility for renewing undergraduate mathematics rests with the graduate schools, since it is they who provide the primary professional education of those who are responsible for undergraduate mathematics: college faculty.

Indicators from many sources [12, 35, 44] suggest that the match between undergraduate and graduate education in mathematics is not now serving U.S. interests especially well:

- Too few U.S. mathematics majors choose to enter graduate school in a mathematical science.
- U.S. mathematics students do less well in graduate school—and drop out more often—than foreign nationals.
- Many students finish graduate school ill-equipped for the breadth of teaching duties typically expected of undergraduate mathematics teachers.
- Relatively few who finish doctoral degrees in mathematics actually go on to effective research careers in mathematics.

In the 1970s, as the number of U.S. students applying to graduate school in mathematics began to decline, the graduate schools responded by increasing the number of international students, most of whom had completed a more intense and specialized education in mathematics than is typical of American undergraduates. Hence the level of mathematics expected of beginning graduate students gradually shifted upward to an international standard that is well above current U.S. undergraduate curricula. Consequently, the failure or drop-out rate of U.S. students increased, creating pressure for more international students and even higher entrance expectations.

It is time to break this negative feedback loop by encouraging better articulation of programs and standards between U.S. undergraduate colleges and U.S. graduate schools. Such cooperation is needed both to enhance the success of U.S. students and to enable the graduate schools to better match their programs with the needs of the colleges and universities who employ a majority of those who receive advanced degrees. *Renewal of undergraduate mathematics will require commitment, leadership, and support of graduate schools.*

One good mechanism for such cooperation would be an exchange of visitors between undergraduate and graduate institutions so that each can learn about the needs of the other. Especially as change occurs in the content and nature of the undergraduate major, it is very important that graduate schools maintain programs of study and research that are appropriately linked to the undergraduate program in mathematics.

Summary

Without becoming entangled in specific curriculum and course recommendations—which are the proper province of other committees of mathematics professional organizations—we can nevertheless enumerate several broad principles implied by our study of the undergraduate mathematics major:

Goals and Objectives

- The primary goal of a mathematical sciences major should be to develop a student's capacity to undertake intellectually demanding mathematical reasoning.
- The undergraduate mathematics curriculum should be designed for all students with an interest in mathematics.
- Applications should motivate theory so that theory is seen by students as useful and enlightening.
- Mathematics majors should be offered extensive opportunities to read, write, listen, and speak mathematical ideas at each stage of their undergraduate study.

Breadth and Depth

- All students who major in mathematics should study some sequence of upper division courses that shows the power of study in depth.
- Every student who majors in mathematics should study a broad variety of advanced courses.
- Mathematics departments should take seriously the need to provide appropriate mathematical depth to students who wish to concentrate in mathematics without pursuing a traditional major.
- Mathematics majors should complete a minor in a discipline that makes significant use of mathematics.

Learning and Teaching

- Instruction should encourage students to explore mathematical ideas on their own.
- Undergraduate students should not only learn the subject of mathematics, but also learn how to learn mathematics.
- Those who teach college mathematics should seek ways to incorporate into their own teaching styles the findings of research on teaching and learning.
- Mathematicians should increase their efforts to understand better how college students learn mathematics.
- Evaluation of teaching must involve robust indicators that reflect the broad purposes of mathematics education.

Access and Encouragement

- Effective programs teach students, not just mathematics.
- National need requires greater encouragement for students to continue their study of mathematics beyond the bachelor's degree.
- To provide effective opportunities for all students to learn mathematics, colleges should offer a broader spectrum of instructional practice that is better attuned to the variety of students seeking higher education.
- To ensure for all students equal access to higher mathematics education, mathematics departments should work with nearby two-year colleges to maintain close articulation of programs.
- Smooth curricular transitions improve student learning and help maintain momentum for the study of mathematics.

Using Computers

- The mathematics curriculum should change to reflect in appropriate ways the impact of computers on the practice of mathematics.
- Colleges must recognize in budgets, staffing, and space the fact that undergraduate mathematics is rapidly becoming a laboratory discipline.

Doing Mathematics

- Dealing with open-ended problem situations should be one of the highest priorities of undergraduate mathematics.
- All undergraduate mathematics students should undertake open-ended projects whose scope extends well beyond typical textbook problems.
- Undergraduate research and senior projects should be encouraged wherever there is sufficient faculty to provide appropriate supervision.
- Students majoring in mathematics should undertake some real-world mathematical modelling project.

Students

- Building students' well-founded self-confidence should be a major priority for all undergraduate mathematics instruction.

- Careful and individualized advising is crucial to students' success.
- All mathematics students should engage in serious study of the historical context and contemporary impact of mathematics.
- Mathematics departments should actively encourage extracurricular programs that enhance peer group support among mathematics majors.

Renewal

- It is important for mathematics departments to help faculty and students recognize their own perspectives on mathematics and understand the perspectives of others.
- Assessment of undergraduate majors should be aligned with broad goals of the major; tests should stress what is most important, not just what is easiest to test.
- To ensure continued vitality of undergraduate mathematics programs, all mathematics faculty should engage in public professional activity, broadly defined.
- Regular external reviews and informal feedback are needed to assure quality in departments of mathematics.
- Renewal of undergraduate mathematics will require commitment, leadership, and support of graduate schools.

In most respects both prevailing professional wisdom and current practice for the mathematics major reflect well the major goals of AAC's *Integrity.* Discussion continues on many campuses about whether the major should focus inward towards advanced study in the mathematical sciences or outward towards preparation for diverse careers in science and management. These discussions are more about strategies than long-term goals, however, since either emphasis can advance the broad AAC goals of coherence, connections, and intellectual development.

Liberal education provides a versatile background for a life of ever-changing challenges. Among the many majors from which students can choose, mathematics can help ensure versatility for the future. Habits of mind nurtured in an undergraduate mathematics major are profoundly useful in an enormous variety of professions. The challenge for college mathematicians is to ensure that the major provides—and is seen by students as providing—not just technical facility, but broad empowerment in the language of our age.

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