Core Mathematics

This chapter contains the report of the Subpanel on Core Mathematics of the CUPM Panel on a General Mathematical Sciences Program, reprinted with minor changes from Chapter III of the 1981 CUPM report entitled RECOMMENDATIONS FOR A GENERAL MATHEMATICAL SCIENCES PROGRAM.

New Roles for Core Mathematics

In the 1960's CUPM extensively examined curriculum in core mathematics—upper division subjects that comprise the trunk from which the other specialized branches and applications of mathematics emerge. It reviewed and revised its recommendations in 1972. See the *Compendium of CUPM Recommendations* published by the Mathematical Association of America, especially the 1972 revision of the *General Curriculum for Mathematics in Colleges.* With the current restructuring of the mathematics major into a mathematical sciences major, new questions have been raised about core mathematics curriculum. These questions concern the role of core mathematics in a mathematical sciences major as much as syllabi of individual courses. This chapter focuses on four questions that were addressed to the Core Mathematics Subpanel by the parent Panel on a General Mathematical Sciences Program.

The members of this subpanel represent a variety of institutions, public and private, liberal-arts colleges, and research-oriented universities. All the members have seen at their institutions a divergence of the mathematics major from its form during their own undergraduate training, as career opportunities for mathematics majors have changed. In part, the members lament the passing of the mathematics program that nurtured their love of mathematics. At the same time they acknowledge the challenge of the diversity of the present and future. They realize that it is not now realistic for CUPM to recommend a core set of pure mathematics courses to be taken by all mathematical sciences majors in every institution.

While the mathematics major has generally broadened towards a mathematical sciences major, it is still possible for an institution, large or small, to elect to retain a traditional pure mathematics major, alone or in conjunction with an applied mathematics major. But it is clearly more appropriate to work within current realities to fashion a unified mathematical sciences major with diminished pure content, a major incorporating both breadth and selective depth. (If size warrants, the unified major can have several tracks, one for preparation for graduate study in mathematics.) This subpanel is concerned with the role in a mathematical sciences major of upper-level core mathematics courses, and more generally with appreciation of the depth and power of mathematics.

A prime attribute of a person educated in mathematical sciences is his or her ability to respond when confronted with a mathematical problem, whether in pure mathematics, applied mathematics, or one which uses mathematics that the person has not seen before. Our students should be prepared to function as professionals in areas needing mathematics not by having learned stock routines for stock classes of problems but by having developed their ability in problem solving, modeling and creativity. This general pedagogical theme, that was stressed throughout the first chapter "Mathematical Sciences," guided the thinking of the Core Mathematics Subpanel.

The report of the Core Mathematics Subpanel is meant to be supportive rather than directive. What an individual department does should reflect its constituency of students, their needs, their numbers, and the goals, character and size of the institution.

Four Questions

QUESTION 1: ***Is there a minimal set of upper-level core mathematics (algebra, analysis, topology, geometry) that every mathematical sciences major should study?***

ANSWER: No. There is no longer a common body of pure mathematical information that every student should know. Rather, a department's program must be tailored according to its perception of its role and the needs of its students. Whether pure mathematics is required of all in some substantial way; whether it is used as an introduction to advanced work of applied nature or as a completion to an applied program; or whether pure mathematics is simply one track in a collection of programs in a large department will be an institutional option. Departments must recognize this fact, establish their programs with a clear understanding of objectives that are being met, and be prepared to share and explain these perceptions with their students. The limited resources of smaller departments must be exploited with great efficiency and wisdom. Such departments may face a difficult decision of whether to abandon certain traditional branches of mathematics entirely in order to offer courses and tracks best suited to their students.

The underlying problem is that students enter college with much less mathematics than they used to, but they expect to leave with more. There is a wide span of preparation among entering college students, they want an education that is specific to chosen career goals, and the levels of mathematical and computational skills and sophistication that accompany these goals have risen. Core courses such as abstract algebra and analysis are valuable for continuing study in many fields, but they are not essential for all careers.

The Core Mathematics Subpanel and the parent Mathematical Sciences Panel jointly recommend that all mathematical sciences students take a sequence of two courses leading to the study of some subject in depth (see the first chapter, "Mathematical Sciences").

QUESTION 2: *Should there be major changes in the content or mode of instruction of upper-level core mathematics courses?*

ANSWER: While there will continue to be some students who plan to move toward a doctorate in pure, or applied, mathematics and an academic career, the mathematical sciences major is seen by most students as preparation for immediate employment or for Masters-level graduate training in areas outside of mathematics (but where mathematical tools are needed). Thus mathematics departments can no longer view their upper-division courses as a collection of courses that faculty wish they had had prior to admission to graduate school. Rather, departments must offer pure mathematics courses that are compatible with the overall goals of a mathematical sciences major, courses that are intellectually and pedagogically complete in themselves, courses that are both the beginning and the end of most students' study of the subject. The main objective in such courses now is developing a deeper sense of mathematical analysis and associated abstract problem-solving abilities. In these courses students learn how to learn mathematics.

There is always a continuing need to re-examine the nature and content of any course. Some courses carry baggage that may be there largely for historical reasons. A frequent example of this is the traditional course in differential equations which is populated by isolated discoveries of the Bernoulli clan (and lacking in discussion of numerical methods). Instructors are slow to discard topics that have a strong aesthetic appeal (for the instructors) but are no longer important building blocks in the field. Syllabi and approaches in pure mathematics courses must be adapted to changing constituencies with a careful balance of learning new concepts and modes of reasoning and of using these constructs, a balance of "listening" and "doing." Students should emerge from a course feeling that they have become junior experts in some topics: they should know facts and relationships, know some of the "whys" behind this mathematics.

It would be desirable for courses to be structured with review stages that require reflection by students of what analysis to use to solve a problem. The courses need to contain assignments that ask for short proofs of results and for application of concepts and techniques from one problem to another (apparently unrelated) problem. Proper judgment in the selection of a method of analysis is the key both to constructing mathematical proofs and to problem solving in applied mathematics; nurturing this ability is the critical challenge to instructors. Students should be required to present material both orally and in writing on a regular basis. Since students do not have to know a standard body of theorems for graduate study, the course content in algebra, analysis, topology and geometry can vary according to faculty interests and possible ties with stronger quantitative areas of an institution (e.g., physics or biology).

The density of proofs in an upper-level course is always a controversial issue. It is traditional to feel that one objective of such a course is to teach students how to construct proofs. However, this skill comes slowly and seldom arouses the same pleasure in students as it does in instructors. Some proofs are needed in any upper-level mathematics course to knit together the entire structure that is being presented, but one should probably aim at piecewise rigor rather than a Landauesque totality. Students' mathematical maturity will develop as much, and it will be far less painful.

The preceding pedagogical goals in core mathematics must accommodate the reality that courses such as abstract algebra may only be offered in alternate years and that two-semester sequences or courses with core mathematics prerequisites will be difficult to schedule. With a broad mixture of students in infrequently-offered courses, instructors must be sensitive to the discouragement some students may feel in the presence of more sophisticated seniors.

QUESTION 3: *How can the full scope of mathematics be conveyed to students? Should this be done by one-semester survey courses that cover a range of fields?*

ANSWER: Students pursuing specific career goals in mathematical sciences and those taking upper-level mathematical "service" courses need to be made aware of the depth and breadth of mathematics and the greater mathematical maturity that their subsequent careers may demand. Mathematical survey courses do not appear to be the answer. They will not be able to move beyond vocabulary and notation to give any sense of global structure in any of the fields covered.

Physicists seem to have been remarkably successful in communicating some understanding about the "big picture" to their students and laymen through expository articles that treat highly technical subjects by presenting only a projection or shadow of the true structure, but doing so in a way that does not seem to offend their consciences. Similar approaches should be possible in mathematics using expository *American Mathematical Monthly, Mathematics Magazine,* or *Scientific American* articles. Following the reading of such an article, a (once-a-week) class would discuss concepts, technicalities and applications in the article plus additional examples. Natural topic areas are complex analysis and two-dimensional hydrodynamics; number theory and public key cryptography; calculus of variations and soap films; queueing theory and, say, toll road design.

More traditional ways of projecting the wide-ranging nature of mathematics are by rotation of courses and by providing seminars, extracurricular mathematical activities, summer work opportunities, and by references and linkages to mathematics in courses in other departments. This breadth should also give a sense of the rapidly changing nature of uses of mathematics and of the need of learning how to learn mathematics.

QUESTION 4: *Should pure mathematics courses be postponed for most students until the senior year to follow and abstract from more applied courses earlier in the curriculum?*

ANSWER: Many mathematical sciences students who prefer problem solving to theory appear to have considerable difficulty in their sophomore or junior years with abstract core mathematics. For these students, core mathematics may better wait until a senior year "capstone" course(s) that builds on maturity developed in earlier problem solving courses. This course (preferably year-long if only one such course is required) in a subject such as analysis or abstract algebra would build a student's capacity (and appetite) for abstraction and proof and for solving complex problems involving a combination of analytical techniques. The course would seek depth rather than breadth. The course should link abstract concepts with their concrete uses in previous courses, such as integration concepts used in limiting probability distributions. It should illustrate in several ways the power and usefulness of mathematical abstraction and generalization.

There are two important provisos about senior-year courses. First, when core courses cannot be offered every year, they obviously must be accessible to most juniors. Second, the mathematically gifted student (whether a mathematics major or not) must be able to take such senior core courses in the sophomore year without needing applied prerequisites that other students naturally take before the core course. Such gifted students today are often directed towards popular careers such as engineering or medicine and by their senior year would be too immersed in professional training to take the pure mathematics course that would reveal their mathematical research potential.

It is worthwhile recalling that before 1950 few colleges offered regular courses in abstract algebra, topology, or up-to-date advanced calculus. The 1950's and 1960's were memorable in mathematics education, but today's students must be viewed as in the historical mainstream rather than as slow in learning to handle abstractions.

Individual institutions will differ greatly in the design of such senior courses. As noted in the discussion of Question 2, these courses should require oral and written student presentations. The spirit of this recommendation could be achieved with a year-long course in a subject such as differential equations or combinatorics that begins with applications and leads to abstraction or a course that begins with abstraction and leads to applications.

Sample Course Outlines

In this section we discuss two approaches to the fundamental upper-level core subjects of abstract algebra and analysis. We suggest an ideal treatment and then a more modest version that is appropriate for most current mathematical sciences students. The descriptions are stated in terms of student objectives.

The philosophy behind each of the course descriptions is that the student needs a working understanding of the subject far more than a detailed intensive and critical knowledge. The instructor's central goal is to teach the student how to learn mathematics, expecting that students will correctly retain only a tiny portion of what was taught, but that when they need to refresh their knowledge, they will be far better able to do so than if they had never taken the course. Proofs are not of major importance, but in both approaches students should be able to understand what the hypotheses of a theorem mean and how to check them. They should also be able to detect when seemingly plausible statements are false (and should be shown counterexamples to such statements; e.g., integrals that should converge but do not).

Abstract Algebra I (Ideal)

A. Give the student a guided tour through the algebraic "zoo," so that he or she knows what it means to be a group, a ring, a field, an associative algebra, etc. Include associated concepts such as category, morphism, isomorphism, coset, ideal, etc.
B. Show the student useful ways for generating one algebraic structure out of another, such as automorphism groups, quotient groups, algebras of transformations, etc.
C. Give the student an understanding of the basic structure theorems for each of the algebraic systems discussed, as well as an understanding of their proofs.
D. Give the student experience in using the preceding ideas and constructions and seeing how these ideas arise in other branches of mathematics (analysis, number theory, geometry, etc.).
E. Show the student how algebra is used in fields outside of mathematics, such as physics, genetics, information theory, etc.

Abstract Algebra II (Modest)

A. Combine parts of A and B of Course I by showing students at least two different types of algebraic structures and several instances in which such an algebraic structure evolved or is constructed out of another mathematical structure. The goal is for a student to be able to recognize when a situation has aspects that lend themselves to an algebraic formulation; e.g., rings out of polynomials.
B. Describe part of the theory for one of the structures introduced in A and illustrate several of the deductive steps in the theory. Students should see the nature of tight logical reasoning and the usefulness of algebraic concepts, as well as come to appreciate the cleverness of the theory's discoverers.
C. Discuss at least one application of algebra outside of mathematics.
D. Assign students a variety of problems which require recognition of algebraic structures in unfamiliar forms, proof of small deductive steps, and use of theory in B.

Analysis I (Ideal)

A. Give the student a working knowledge of point set topology in R^n and analogous concepts for a metric space.
B. Study the class of continuous maps from a region in R^n into R^m, and the special properties of maps in class C' and C''.
C. Study integration of continuous and piecewise continuous functions over appropriately chosen sets, bounded and unbounded, and then extend this to integration with respect to set functions.
D. Extend to the theory of differential forms and develop a relationship between differentiation of forms and the boundary operator, via Stokes' theorem.

Analysis II (Modest)

A. Give the student a glossary of terms in point set topology, appropriate also to a metric space and applied to R^n, and practice in their meanings. (Do not prove inter-relations, but state them clearly.)
B. Introduce the class of C'' maps from R^n into R^m, and discuss a few problems involving such functions, each motivated by a concrete "real" situation. Solve each of the problems by stating and illustrating the appropriate general theorems, and in a few cases, sketching part of the proofs.
C. Discuss integration in terms of measurement and averaging, extend this to R^n, and explain briefly techniques of numerical integration. At all stages give attention to improper integrals.
D. Extend the notion of function to differential forms, illustrated with physical and geometric examples. Motivate Stokes' theorem as the analogue of the fundamental theorem of calculus, and arrive at a correct formulation of it without proof. Illustrate the theorem with examples, including some involving the geometric topology of surfaces; if students' background is appropriate, examples in physics (hydrodynamics or electromagnetism) should be given.

An analysis course can also be given an "advanced calculus" emphasis including topics such as Fourier series and transforms, special functions, and fixed-point theorems, with applications of these topics to differential equations. For further discussion of this approach, see versions one and three of Mathematics 5 in the CUPM recommendations for a *General Curriculum in Mathematics for Colleges* (revised 1972).

Subpanel Members

PAUL CAMPBELL, CHAIR, Beloit College.
LIDA BARRETT, Northern Illinois University.
R. CREIGHTON BUCK, University of Wisconsin.
MARGARET HUTCHINSON, University of St. Thomas.