AMC 10 Student Practice Questions

You will find these and additional problems for the AMC 10 and AMC 12 on AMC's web site: http://www.unl.edu/amc, available from the current and previous AMC 10/12 Teacher Manuals, (http://www.unl.edu/amc/e-exams/e6-amc12/archive12.shtml) or from our Problems page archives (http://www.unl.edu/amc/a-activities/a7-problems/problem81012archive.shtml).

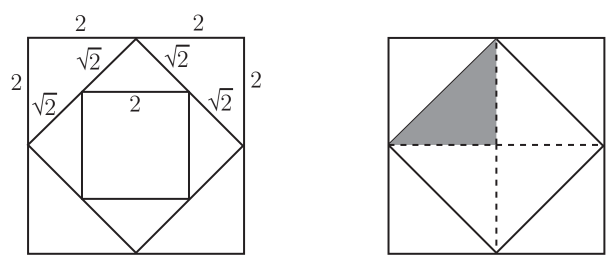
Each of the sides of a square S_1 with area 16 is bisected, and a smaller square S_2 is constructed using the bisection points as vertices. The same process is carried out on S_2 to construct an even smaller square S_3. What is the area of S_3?

(A) $\frac{1}{2}$ **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2008 AMC 10 A, Problem #10— "The sides of S_2 have length $\sqrt{2^2+2^2} = 2\sqrt{2}$."

Solution

Answer (E): The sides of S_1 have length 4, so by the Pythagorean Theorem the sides of S_2 have length $\sqrt{2^2+2^2} = 2\sqrt{2}$. By similar reasoning the sides of S_3 have length $\sqrt{(\sqrt{2})^2+(\sqrt{2})^2} = 2$. Thus the area of S_3 is $2^2 = 4$.



OR

Connect the midpoints of the opposite sides of S_1. This cuts S_1 into 4 congruent squares as shown. Each side of S_2 cuts one of these squares into two congruent triangles, one inside S_2 and one outside.
Thus the area of S_2 is half that of S_1. By similar reasoning, the area of S_3 is half that of S_2, and one fourth that of S_1.

Difficulty: Medium
NCTM Standard: Geometry Standard: apply transformations and use symmetry to analyze mathematical situations.
Mathworld.com Classification: Geometry > Plane Geometry > Squares

Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

(A) 120 **(B)** 130 **(C)** 140 **(D)** 150 **(E)** 160

2008 AMC 10 A, Problem #15—
"Set up equation to represent the relations with Ian's total time, h hours, and average speed, r miles."

Solution

Answer (D): Suppose that Ian drove for t hours at an average speed of r miles per hour. Then he covered a distance of rt miles. The number of miles Han covered by driving 5 miles per hour faster for 1 additional hour is

$$(r+5)(t+1) = rt + 5t + r + 5.$$

Since Han drove 70 miles more than Ian,

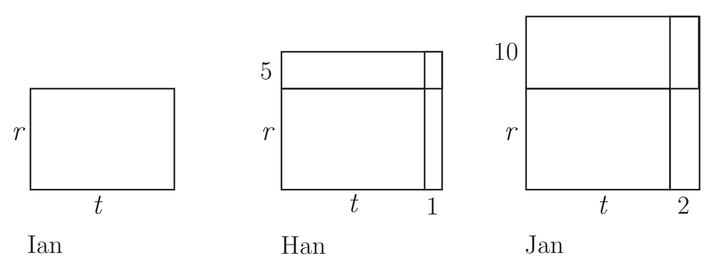
$$70 = (r+5)(t+1) - rt = 5t + r + 5, \quad \text{so} \quad 5t + r = 65.$$

The number of miles Jan drove more than Ian is consequently

$$(r+10)(t+2) - rt = 10t + 2r + 20 = 2(5t+r) + 20 = 2 \cdot 65 + 20 = 150.$$

OR

Represent the time traveled, average speed, and distance for Ian as length, width, and area, respectively, of a rectangle as shown. A similarly formed rectangle for Han would include an additional 1 unit of length and 5 units of width as compared to Ian's rectangle. Jan's rectangle would have an additional 2 units of length and 10 units of width as compared to Ian's rectangle.



Given that Han's distance exceeds that of Ian by 70 miles, and Jan's $10 \times t$ and $2 \times r$ rectangles are twice the size of Ian's $5 \times t$ and $1 \times r$ rectangles, respectively, it follows that Jan's distance exceeds that of Ian by

$$2(70-5) + 20 = 150 \text{ miles.}$$

Difficulty: Medium-hard

NCTM Standard: Algebra Standard: use symbolic algebra to represent and explain mathematical relationships.

Mathworld.com Classification: Algebra > Algebraic Equations > Linear Equation

Assume that x is a positive real number. Which is equivalent to $\sqrt[3]{x\sqrt{x}}$?

(A) $x^{1/6}$ **(B)** $x^{1/4}$ **(C)** $x^{3/8}$ **(D)** $x^{1/2}$ **(E)** x

2008 AMC 10 B, Problem #3— "Use the fact that $\sqrt[a]{x} = x^{\frac{1}{a}}$."

Solution

Answer (D): The properties of exponents imply that

$$\sqrt[3]{x\sqrt{x}} = \left(x \cdot x^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} = x^{\frac{1}{2}}.$$

Difficulty: Medium-hard

NCTM Standard: Number and Operations Standard: judge the effects of such operations as multiplication, division, and computing powers and roots on the magnitudes of quantities.

Mathworld.com Classification: Calculus and Analysis > Roots > Root

Suppose that (u_n) is a sequence of real numbers satisfying $u_{n+2} = 2u_{n+1} + u_n$, and that $u_3 = 9$ and $u_6 = 128$. What is u_5?

(A) 40 **(B)** 53 **(C)** 68 **(D)** 88 **(E)** 104

2008 AMC 10 B, Problem #11—
"Rewrite u_6 in term of u_4 and solve for u_4."

Solution

Answer (B): Note that $u_5 = 2u_4+9$ and $128 = u_6 = 2u_5+u_4 = 5u_4+18$. Thus $u_4 = 22$, and it follows that $u_5 = 2 \cdot 22 + 9 = 53$.

Difficulty: Medium-hard

NCTM Standard: Algebra Standard: understand relations and functions and select, convert flexibly among, and use various representations for them.

Mathworld.com Classification: Number Theory > Sequences > Sequence

Three red beads, two white beads, and one blue bead are placed in a line in random order. What is the probability that no two neighboring beads are the same color?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

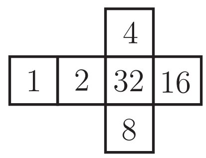
2008 AMC 10 B, Problem #22—
"Note that there are $6!/(3!2!1!) = 60$ distinguishable orders of the beads on the line."

Solution

Answer (C): There are $6!/(3!2!1!) = 60$ distinguishable orders of the beads on the line. To meet the required condition, the red beads must be placed in one of four configurations: positions 1, 3, and 5, positions 2, 4, and 6, positions 1, 3, and 6, or positions 1, 4, and 6. In the first two cases, the blue bead can be placed in any of the three remaining positions. In the last two cases, the blue bead can be placed in either of the two adjacent remaining positions. In each case, the placement of the white beads is then determined. Hence there are $2 \cdot 3 + 2 \cdot 2 = 10$ orders that meet the required condition, and the requested probability is $\frac{10}{60} = \frac{1}{6}$.

Difficulty: Medium-hard
NCTM Standard: Data Analysis and Probability Standard: understand and apply basic concepts of probability.
Mathworld.com Classification: Probability and Statistics > Probability > Probability

Three cubes are each formed from the pattern shown. They are then stacked on a table one on top of another so that the 13 visible numbers have the greatest possible sum. What is that sum?



(A) 154 **(B)** 159 **(C)** 164 **(D)** 167 **(E)** 189

2008 AMC 12 A, Problem #11— "The three pairs of opposite faces have numbers with sums $1+32=33$, $2+16=18$, and $4+8=12$."

Solution

Answer (C): The sum of the six numbers on each cube is $1+2+4+8+16+32=63$. The three pairs of opposite faces have numbers with sums $1+32=33$, $2+16=18$, and $4+8=12$. On the two lower cubes, the numbers on the four visible faces have the greatest sum when the 4 and the 8 are hidden. On the top cube, the numbers on the five visible faces have the greatest sum when the 1 is hidden. Thus the greatest possible sum is $3 \cdot 63 - 2 \cdot (4+8) - 1 = 164$.

Difficulty: Medium-easy

NCTM Standard: Geometry Standard: analyze properties and determine attributes of two- and three-dimensional objects.

Mathworld.com Classification: Geometry > Solid Geometry > Polyhedra > Cubes > Cube

What is the area of the region defined by the inequality $|3x - 18| + |2y + 7| \leq 3$?

(A) 3 **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 5

2008 AMC 12 A, Problem #14—

"The boundaries of the region are the two pairs of parallel lines $(3x - 18) + (2y + 7) = \pm 3$ and $(3x - 18) - (2y + 7) = \pm 3$."

Solution

Answer (A): The boundaries of the region are the two pairs of parallel lines

$$(3x - 18) + (2y + 7) = \pm 3 \quad \text{and} \quad (3x - 18) - (2y + 7) = \pm 3.$$

These lines intersect at $(6, -2)$, $(6, -5)$, $(5, -\frac{7}{2})$, and $(7, -\frac{7}{2})$. Thus the region is a rhombus whose diagonals have lengths 2 and 3. The area of the rhombus is half the product of the diagonal lengths, which is 3.

Difficulty: Hard

NCTM Standard: Number and Operations Standard: understand meanings of operations and how they relate to one another.

Mathworld.com Classification: Calculus and Analysis > Functions > Absolute Value
Calculus and Analysis > Inequalities > Inequality

Vertex E of equilateral $\triangle ABE$ is in the interior of unit square $ABCD$. Let R be the region consisting of all points inside $ABCD$ and outside $\triangle ABE$ whose distance from $\overline{AD}$ is between $\frac{1}{3}$ and $\frac{2}{3}$. What is the area of R?

(A) $\dfrac{12-5\sqrt{3}}{72}$ **(B)** $\dfrac{12-5\sqrt{3}}{36}$ **(C)** $\dfrac{\sqrt{3}}{18}$ **(D)** $\dfrac{3-\sqrt{3}}{9}$ **(E)** $\dfrac{\sqrt{3}}{12}$

2008 AMC 12 B, Problem #13—
"Sketch the figure, and identify region R on the figure."

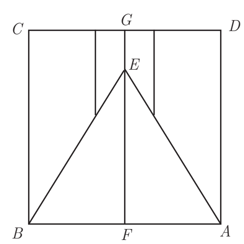
Solution

Answer (B): Draw a line parallel to $\overline{AD}$ through point E, intersecting $\overline{AB}$ at F and intersecting $\overline{CD}$ at G. Triangle AEF is a $30-60-90°$ triangle with hypotenuse $AE = 1$, so $EF = \frac{\sqrt{3}}{2}$. Region R consists of two congruent trapezoids of height $\frac{1}{6}$, shorter base $EG = 1 - \frac{\sqrt{3}}{2}$, and longer base the weighted average

$$\frac{2}{3}EG + \frac{1}{3}AD = \frac{2}{3}\left(1 - \frac{\sqrt{3}}{2}\right) + \frac{1}{3} \cdot 1 = 1 - \frac{\sqrt{3}}{3}.$$

Therefore the area of R is

$$2 \cdot \frac{1}{6} \cdot \frac{1}{2}\left(\left(1 - \frac{\sqrt{3}}{2}\right) + \left(1 - \frac{\sqrt{3}}{3}\right)\right) = \frac{1}{6}\left(2 - \frac{5\sqrt{3}}{6}\right) = \frac{12-5\sqrt{3}}{36}$$



OR

Place $ABCD$ in a coordinate plane with $B = (0,0)$, $A = (1,0)$, and $C = (0,1)$. Then the equation of the line BE is $y = \sqrt{3}x$, so $E = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and the point of R closest to B is $(\frac{1}{3}, \frac{\sqrt{3}}{3})$. Thus the region R consists of two congruent trapezoids with height $\frac{1}{6}$ and bases $1 - \frac{\sqrt{3}}{2}$ and $1 - \frac{\sqrt{3}}{3}$. Then proceed as in the first solution.

Difficulty: Hard
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Quadrilaterals > Trapezoid

A pyramid has a square base $ABCD$ and vertex E. The area of square $ABCD$ is 196, and the areas of $\triangle ABE$ and $\triangle CDE$ are 105 and 91, respectively. What is the volume of the pyramid?

(A) 392 **(B)** $196\sqrt{6}$ **(C)** $392\sqrt{2}$ **(D)** $392\sqrt{3}$ **(E)** 784

2008 AMC 12 B, Problem #18—
"Construct a triangle whose altitude is the altitude of the pyramid. Apply Heron's Formula to find the altitude."

Solution

Answer (E): Square $ABCD$ has side length 14. Let F and G be the feet of the altitudes from E in $\triangle ABE$ and $\triangle CDE$, respectively. Then $FG = 14$, $EF = 2 \cdot \frac{105}{14} = 15$ and $EG = 2 \cdot \frac{91}{14} = 13$. Because $\triangle EFG$ is perpendicular to the plane of $ABCD$, the altitude to $\overline{FG}$ is the altitude of the pyramid. By Heron's Formula, the area of $\triangle EFG$ is $\sqrt{(21)(6)(7)(8)} = 84$, so the altitude to $\overline{FG}$ is $2 \cdot \frac{84}{14} = 12$. Therefore the volume of the pyramid is $(\frac{1}{3})(196)(12) = 784$.

Difficulty: Hard

NCTM Standard: Geometry Standard: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Triangle Properties > Heron's Formula