MAA AMC
American Mathematics Competitions

American Mathematics Competitions

36th Annual

AIME I

American Invitational Mathematics Examination I
Tuesday, March 6, 2018

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor penalties for wrong answers.
3. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
4. A combination of your AIME score and your American Mathematics Contest 12 score are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of your AIME score and your American Mathematics Contest 10 score are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO and USAJMO will be given on WEDNESDAY and THURSDAY, April 18 and 19, 2018.
5. Record all your answers, and identification information, on the AIME answer sheet. Only the answer sheet will be collected from you.

The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

1. Let S be the number of ordered pairs of integers (a, b), with $1 \leq a \leq 100$ and $b \geq 0$, such that the polynomial $x^2 + ax + b$ can be factored into the product of two (not necessarily distinct) linear factors with integer coefficients. Find the remainder when S is divided by 1000.

2. The number n can be written in base 14 as $\underline{a}\,\underline{b}\,\underline{c}$, can be written in base 15 as $\underline{a}\,\underline{c}\,\underline{b}$, and can be written in base 6 as $\underline{a}\,\underline{c}\,\underline{a}\,\underline{c}$, where $a > 0$. Find the base-10 representation of n.

3. Kathy has 5 red cards and 5 green cards. She shuffles the 10 cards and lays out 5 of the cards in a row in a random order. She will be happy if and only if all the red cards laid out are adjacent and all the green cards laid out are adjacent. For example, card orders RRGGG, GGGGR, or RRRRR will make Kathy happy, but RRRGR will not. The probability that Kathy will be happy is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. In $\triangle ABC$, $AB = AC = 10$ and $BC = 12$. Point D lies strictly between A and B on $\overline{AB}$ and point E lies strictly between A and C on $\overline{AC}$ so that $AD = DE = EC$. Then AD can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

5. For each ordered pair of real numbers (x, y) satisfying

$$\log_2(2x + y) = \log_4(x^2 + xy + 7y^2),$$

there is a real number K such that

$$\log_3(3x + y) = \log_9(3x^2 + 4xy + Ky^2).$$

Find the product of all possible values of K.

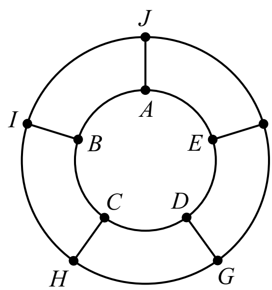
6. Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

7. A right hexagonal prism has height 2. The bases are regular hexagons with side length 1. Any 3 of the 12 vertices determine a triangle. Find the number of these triangles that are isosceles (including equilateral triangles).

8. Let $ABCDEF$ be an equiangular hexagon such that $AB = 6$, $BC = 8$, $CD = 10$, and $DE = 12$. Denote by d the diameter of the largest circle that fits inside the hexagon. Find d^2.

9. Find the number of four-element subsets of $\{1, 2, 3, 4, \ldots, 20\}$ with the property that two distinct elements of the subset have a sum of 16, and two distinct elements of the subset have a sum of 24. For example, $\{3, 5, 13, 19\}$ and $\{6, 10, 20, 18\}$ are two such subsets.

10. The wheel shown below consists of two circles and five spokes, with a label at each point where a spoke meets a circle. A bug walks along the wheel, starting at point A. At every step of the process, the bug walks from one labeled point to an adjacent labeled point. Along the inner circle the bug only walks in a counterclockwise direction, and along the outer circle the bug only walks in a clockwise direction. For example, the bug could travel along the path $AJABCHCHIJA$, which has 10 steps. Let n be the number of paths with 15 steps that begin and end at point A. Find the remainder when n is divided by 1000.



11. Find the least positive integer n such that when 3^n is written in base 143, its two right-most digits in base 143 are 01.

12. For each subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

13. Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

14. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

15. David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, and C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin \varphi_A = \frac{2}{3}$, $\sin \varphi_B = \frac{3}{5}$, and $\sin \varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Your competition manager will receive a copy of the
2018 AIME Solution Pamphlet with the scores.

Questions and comments about problems
and solutions for this exam should be sent to:

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Send questions and comments about administrative arrangements to:

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