

Taking $t = 0$ we get $B = f(0) = \sin \beta$, while $t = \frac{\pi}{2}$ yields $A = f(\frac{\pi}{2}) = \sin(\frac{\pi}{2} + \beta) = \cos \beta$. Thus we get for all $\alpha, \beta \in \mathbb{R}$:

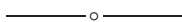
$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \sin \beta \cdot \cos \alpha.$$

We leave it to the reader to derive the identity for $\cos(\alpha + \beta)$.

Since this article was accepted, it has been observed that essentially the same approach appears in the book [4] (in Chapter 15). There the author defines π in terms of arclength instead of area, and starts with the inverse cosine instead of arcsin, but otherwise covers the same ground.

References

1. A. Budó, *Theoretische Mechanik*. Deutscher Verlag der Wissenschaften, Berlin, 1987.
2. A. von Braunnühl, *Vorlesungen über Geschichte der Trigonometrie*. 2 Volumes. Teubner, Leipzig, 1900–1903 (Reprinted Steiner-Verlag, Wiesbaden, 1971).
3. L. Euler, *Introductio in Analysin Infinitorum*, Bd. 1., Lausanne 1748 (Reprinted, W. Walter, ed., Springer, 1983).
4. M. Spivak, *Calculus*, 3rd ed., Publish or Perish Press, 1994.
5. M. C. Zeller, *The Development of Trigonometry from Regiomontanus to Pitiscus*, Edward, Ann Arbor, 1946.



An Upper Bound on the n th Prime

John H. Jaroma (jjaroma@austincollege.edu), Austin College, Sherman, TX 75090

In 1845, J. Bertrand conjectured that for any integer $n > 3$, there exists at least one prime p between n and $2n - 2$ [1]. In 1852, P. Tchebychev offered the first demonstration of this now-famous theorem. Today, *Bertrand's Postulate* is often stated as, "for any positive integer $n \geq 1$, there exists a prime p such that $n < p \leq 2n$."

Furthermore, if we let p_n denote the n th prime, then it is not difficult to show by induction that $p_n < 2^n$ for $n \geq 2$. Given this inequality, it also follows that $p_{n+1} < 2p_n$ for $n \geq 3$. Contemporary textbooks in number theory which allude to either or both of these two corollaries of Bertrand's Postulate include [2], [5], and [6].

Our purpose is to demonstrate that the textbook bound of 2^n on the n th prime can be improved considerably by using a similar technique involving the following 1952 result of J. Nagura [3]. The motivation for this note originated from a lecture the author recently prepared for his number theory class on the distribution of prime numbers.

Theorem 1 (Nagura). *There exists at least one prime number between n and $\frac{6}{5}n$ for $n \geq 25$.*

In particular, observe that the 26th prime is 101 and $(1.2)^{26} \approx 114.48$. Then, by induction on n , we now have the following result.

Theorem 2. $p_n < (1.2)^n$ for $n > 25$.

Proof. By the preceding observation, the theorem is true for $n = 26$. Now assume that for $n = k$, the result also holds. Hence, for $n = k + 1$, the induction hypothesis

and Theorem 1 imply that there exists a prime number p such that $p_n < (1.2)^k < p < (1.2)^{k+1}$. Therefore, $p_{n+1} < (1.2)^{k+1}$. ■

So, for example, if $n = 26$, we may compare the upper bound of $(1.2)^{26} \approx 114$ obtained by Theorem 2 to the present textbook bound of $2^{26} = 67108864$. It would appear that a significant improvement in the estimate is to be had for the same effort.

Finally, we remark that for $n \geq 7022$, an even better estimate is obtained by using the more recent result of G. Robin [4]. It states that

$$p_n \leq n \log n + n(\log \log n - 0.9385).$$

However, a succinct demonstration of Robin's result is likely to be beyond the scope of most elementary courses in number theory.

Acknowledgment. The author would like to thank an anonymous referee whose suggestions helped to improve the overall presentation of this note.

References

1. J. Bertrand, Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu'elle renferme, *J. Ecole Polyt.* **30** (1845) 123–140.
2. D. M. Burton, *Elementary Number Theory*, 5th ed., McGraw Hill, 2002.
3. J. Nagura, On the interval containing at least one prime number, *Proc. Japan Acad.* **28** (1952) 177–181.
4. G. Robin, Estimation de la fonction de Tschébychef θ sur le k -ième nombre premier et grandes valeurs de la fonction $\omega(n)$, nombre de diviseurs de n , *Acta Arith.*, **42** (1983) 367–389.
5. K. H. Rosen, *Elementary Number Theory and its Applications*, 4th ed., Addison Wesley, 2000.
6. P. D. Schumer, *Introduction to Number Theory*, PWS Publishing, 1996.

Solution to A Perplexing Polynomial Puzzle (See page 100)

Surprisingly, you can determine the whole polynomial from just two values. First ask for the value of $p(1)$. This is the sum of the coefficients, and so is an upper bound for all of them. Take b to be any larger integer, and ask for the value of $p(b)$. This number, written in base b , gives the sequence of coefficients. For example, if $p(n) = 3n^2 + 4n + 2$ then $p(1) = 9$. Take b to be 10. Then $p(10) = 342$. Of course, we usually write numbers in base 10, but this method works for any $b > p(1)$.

Adapted from *ContinuUM*, Newsletter of the Department of Mathematics at the University of Michigan.

Editor's Note: Submissions of similar items of general interest are most welcome; please send them to cmj@ipfw.edu.