Taking $t = 0$ we get $B = f(0) = \sin\beta$, while $t = \frac{\pi}{2}$ yields $A = f(\frac{\pi}{2}) = \sin(\frac{\pi}{2} + \beta) = \cos\beta$. Thus we get for all $\alpha, \beta \in \mathbb{R}$:

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \sin\beta \cdot \cos\alpha.$$

We leave it to the reader to derive the identity for $\cos(\alpha + \beta)$.

Since this article was accepted, it has been observed that essentially the same approach appears in the book [**4**] (in Chapter 15). There the author defines π in terms of arclength instead of area, and starts with the inverse cosine instead of arcsin, but otherwise covers the same ground.

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An Upper Bound on the *n*th Prime

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In 1845, J. Bertrand conjectured that for any integer $n > 3$, there exists at least one prime p between n and $2n - 2$ [**1**]. In 1852, P. Tchebychev offered the first demonstration of this now-famous theorem. Today, *Bertrand's Postulate* is often stated as, "for any positive integer $n \geq 1$, there exists a prime p such that $n < p \leq 2n$."

Furthermore, if we let p_n denote the nth prime, then it is not difficult to show by induction that $p_n < 2^n$ for $n \geq 2$. Given this inequality, it also follows that $p_{n+1} < 2p_n$ for $n \geq 3$. Contemporary textbooks in number theory which allude to either or both of these two corollaries of Bertrand's Postulate include [**2**], [**5**], and [**6**].

Our purpose is to demonstrate that the textbook bound of 2^n on the nth prime can be improved considerably by using a similar technique involving the following 1952 result of J. Nagura [**3**]. The motivation for this note originated from a lecture the author recently prepared for his number theory class on the distribution of prime numbers.

Theorem 1 (Nagura). *There exists at least one prime number between* n *and* $\frac{6}{5}n$ *for* $n \geq 25$.

In particular, observe that the 26th prime is 101 and $(1.2)^{26} \approx 114.48$. Then, by induction on n, we now have the following result.

Theorem 2. $p_n < (1.2)^n$ *for* $n > 25$.

Proof. By the preceding observation, the theorem is true for $n = 26$. Now assume that for $n = k$, the result also holds. Hence, for $n = k + 1$, the induction hypothesis and Theorem 1 imply that there exists a prime number p such that $p_n < (1.2)^k < p < (1.2)^{k+1}$. Therefore, $p_{n+1} < (1.2)^{k+1}$. ■

So, for example, if $n = 26$, we may compare the upper bound of $(1.2)^{26} \approx 114$ obtained by Theorem 2 to the present textbook bound of $2^{26} = 67108864$. It would appear that a significant improvement in the estimate is to be had for the same effort.

Finally, we remark that for $n \geq 7022$, an even better estimate is obtained by using the more recent result of G. Robin [**4**]. It states that

$$p_n \leq n \log n + n(\log \log n - 0.9385).$$

However, a succinct demonstration of Robin's result is likely to be beyond the scope of most elementary courses in number theory.

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Solution to A Perplexing Polynomial Puzzle (See page 100)

Surprisingly, you can determine the whole polynomial from just two values. First ask for the value of $p(1)$. This is the sum of the coefficients, and so is an upper bound for all of them. Take b to be any larger integer, and ask for the value of $p(b)$. This number, written in base b, gives the sequence of coefficients. For example, if $p(n) = 3n^2 + 4n + 2$ then $p(1) = 9$. Take b to be 10. Then $p(10) = 342$. Of course, we usually write numbers in base 10, but this method works for any $b > p(1)$.

Adapted from *ContinuUM*, Newsletter of the Department of Mathematics at the University of Michigan.

Editor's Note: Submissions of similar items of general interest are most welcome; please send them to cmj@ipfw.edu.