Apportionment: What's Your Fair Share?

The Constitution of the United States mandates that a census be conducted every ten years. The importance of this count cannot be overstated: the number of persons in a given state on, e.g., April 1, 2020, will determine for the next ten years how many Representatives that state sends to Congress. Moreover, since a state's number of electoral votes is based on its number of Representatives (plus Senators, two for each state), each state's importance in Presidential politics is likewise fixed by the census outcome for the next decade.[1]

The basic problem of apportionment is this: You have a certain number of objects (e.g., congresspersons), and want to allocate them among several recipients (e.g., states). If the objects cannot be subdivided, what's the fairest way to distribute them? We might break the problem into two parts:

- What is the (mathematically exact) "fair share"; and
- How do we round that "fair share" into an apportionment?

In this activity, we will look at how these two questions have been answered, historically and today, for the specific problem of fairly apportioning Congressional representatives among all states.

1. Initial Thoughts on Sharing Fairly

Suppose we want to allocate a number of Representatives based on the fairness principle that states with larger populations are entitled to more Representatives. The first important question is: How many representatives should there be in all?

The Constitution mandates that there be no *more* than one representative for 30,000 persons. Based on the results of the first census (discussed further below), the total apportionment population for 1790 was found to be 3,615,922. At 1 representative per 30,000 persons, we would have 3,615,922 ÷ 30,000 ≈ 120.531 representatives. Since we can't have a fraction of a Congressperson, we need to round this figure. Ordinarily, we'd round up, to 121. However, this would cause us to exceed the "one representative per 30,000 persons," so we have to round down: There can be at most 120 Representatives. The next question to answer is then: How should these 120 Representatives be allocated among the different states?

[1] The Constitution actually identifies two purposes for the apportionment population count. First, the apportionment population determines the number of representatives a state sends to Congress, which in turn partly determines the number of electoral votes it has for selecting the President. Second, the apportionment population determines how much tax revenue a state is expected to contribute to the federal budget. This secondary purpose is no longer relevant, as it is currently fulfilled by individual contributions through federal income tax, instead of by state contributions.

Brainstorming Warm-up One

The following table shows the apportionment population for each of the 15 states that were part of the union in 1790 (listed roughly by their geographic location from north to south). Given that there can be at most "one representative per 30,000 persons," what ideas do you have about how to allocate 120 representatives among these states? For each method that you come up with, what difficulties arise when you try implementing it?

State	Apportionment Population
VT	85,533
NH	141,822
MA	475,327
RI	68,446
CT	236,840
NY	331,590
NJ	179,570
PA	432,878
DE	55,539
MD	278,514
VA	630,559
KY	68,705
NC	353,522
SC	206,235
GA	70,842
TOTAL	**3,199,355**

2. Hamilton's Approach

Alexander Hamilton (1755–1804) suggested the following approach. As we concluded above, a House with 120 Representatives would meet the "no more than one per 30,000 persons" requirement of the Constitution. Obviously, there could be fewer Representatives, but it seems reasonable to accept that the needs of a democratic and free society are best met when each Congressman represents as *few* constituents as possible.[2] (We use the gendered term because, in 1790, they were all men.) So, to begin with, divide each state's apportionment population by 30,000 to get the number of representatives to which it is "officially" entitled; this is referred to as the *quota*. Since no state's population is an exact multiple of 30,000, the numbers all have a decimal portion, as we see in the following table.

[2] By way of comparison, in 2020, a U.S. Congressperson has far more constituents (approximately 760,000 persons) than their colleagues in almost every other democracy. For instance, in Great Britain, a member of Parliament represents about 101,000 persons; in Germany, a member of the Bundestag represents about 138,000 persons; in France, a member of the National Assembly represents about 116,000; and in Japan, a member of the Diet represents about 272,000.

State	Apportionment Population	Quota
VT	**85,533**	**2.8511**
NH	**141,822**	**4.7274**
MA	**475,327**	**15.84423333**
RI	68,446	2.281533333
CT	**236,840**	**7.894666667**
NY	331,590	11.053
NJ	**179,570**	**5.985666667**
PA	432,878	14.42926667
DE	**55,539**	**1.8513**
MD	278,514	9.2838
VA	630,559	21.01863333
KY	68,705	2.290166667
NC	**353,522**	**11.78406667**
SC	**206,235**	**6.8745**
GA	70,842	2.3614

Now, if each of these quotas is rounded down, only 112 Representatives would be apportioned. In order to apportion a total of 120 Representatives, it thus will be necessary to give some states extra representatives. But which ones? The obvious answer, which Hamilton suggested, is to choose the states with the highest fractional parts.[3] These are the eight states in **bold**.

Brainstorming Warm-up Two
What problems or objections might Hamilton's contemporaries have raised about allocating the additional 8 Representatives to the states indicated in bold in the table on page 3? Try to come up with at least three problems or objections.

3. A Divisor Method

While Hamilton's views prevailed in Congress, he found himself at odds with two of the other Founding Fathers. Thomas Jefferson (1743–1826) objected to this approach, based as it was on "a doctrine so difficult and inobvious, as to be rejected at first sight by the very persons who afterwards became its most zealous advocates?"[4] Given that the "difficult and inobvious" doctrine is deciding whether a fractional part such as 0.8511 is greater than a fractional part such as 0.7274, and the fact that Jefferson's personal library included works on calculus and probability, we must regard Jefferson's objection as disingenuous.

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[3] Coincidentally, if we had rounded normally, these would have been the states that had the numbers rounded up; however, this will not always happen. We note that Hamilton supported this method in spite of the fact that his own home state (New York) did *not* gain any Representatives. We might regard this as a reflection of his commitment to the idea of a *nation*, and not a union of independent states.
[4] "Opinion on Apportionment Bill," April 4, 1792, https://founders.archives.gov/documents/Jefferson/01-23-02-0324.

George Washington (1732–1799) had a more understandable objection: increasing *any* state's representation would cause it to have more than one representative per 30,000 persons. Consequently, when the apportionment bill came to his desk, he used the first Presidential veto to send it back to Congress. Congress proved unable to override the Presidential veto, and Hamilton's method was not used for the first apportionment.

The easiest way to resolve Washington's objection would be to simply round all fractions down. As we found above, this would have made a Congress of 112 Representatives, which was slightly smaller than the maximum size of 120 persons. Instead, Jefferson suggested a *modified divisor* method: rather than compute the representation based on a ratio of 30,000 persons per Congressman, choose a different (larger) number of persons per Congressman.[5] For the 1790 census results, Jefferson suggested using a divisor of 33,000 and rounding all quotients down, which would give each state the following allocations:

State	Apportionment Population	Representatives
VT	85,533	2
NH	141,822	4
MA	475,327	14
RI	68,446	2
CT	236,840	7
NY	331,590	10
NJ	179,570	5
PA	432,878	13
DE	55,539	1
MD	278,514	8
VA	630,559	19
KY	68,705	2
NC	353,522	10
SC	206,235	6
GA	70,842	2

Congress, and Washington, approved this plan.

Brainstorming Warm-up Three

How many representatives were actually allocated in the 1790 apportionment?

In what ways might this result be seen as unfair, and by whom?

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[5] It's not clear why Jefferson argued for a modified divisor of 33,000. However, it may be worth noting that under Hamilton's plan, Virginia (Washington and Jefferson's home state) would have had 21 of the 120 congressional delegates: 17.5% of the total. But under Jefferson's plan, Virginia had 19 of 105 delegates: 18.1% of the total. We shouldn't read too much into this, because if Jefferson had used Hamilton's own 30,000 and rounded all fractions down, Virginia's representation would be 21 out of 112 delegates, giving it an even larger share (18.75%) of Congress.

4. The 3/5 Compromise

The mathematical problem of congressional apportionment is further complicated by political considerations, which in turn can raise serious moral issues. For instance, the number of representatives is based on the number of persons in a state. But what constitutes a "person"? The southern states, with large slave populations, argued that slaves (who couldn't vote and were accorded no civil rights whatsoever) should be counted as persons. As this would give southern states a larger number of representatives, the northern states argued that slaves shouldn't be counted. As a compromise, slaves were counted as 3/5 of a person for purposes of congressional apportionment. The following table gives the details of the official 1790 census that were used to compute the apportionment populations based on this rule.

State	Free Population	Slave Population	Total Population	Apportionment Population
VT	85,523	16	85,539	85,533
NH	141,727	158	141,885	141,822
MA	475,327	0	475,327	475,327
RI	67,877	948	68,825	68,446
CT	235,182	2,764	237,946	236,840
NY	318,796	21,324	340,120	331,590
NJ	172,716	11,423	184,139	179,570
PA	430,636	3,737	434,373	432,878
DE	50,207	8,887	59,094	55,539
MD	216,692	103,036	319,728	278,514
VA	454,983	292,627	747,610	630,559
KY	61,247	12,430	73,677	68,705
NC	293,179	100,572	393,751	353,522
SC	141,979	107,094	249,073	206,235
GA	53,284	29,264	82,548	70,842
TOTAL	**3,199,355**	**694,280**	**3,893,635**	**3,615,922**

But no matter how you view the apportionment problem, counting slaves as 3/5 of a person is indefensible. If we take the viewpoint that the apportionment population determines the state's congressional representation, then a state can't expect its democratic strength to be based in part on a population to whom it won't extend civil rights. And if we take the viewpoint that the apportionment population determines how much a state should contribute, then slaves, whose forced labor provided the backbone of the economies of the slaveholding states, should count as full persons so that the value of their unpaid labor should be properly accounted for. One of the greatest shames of American democracy is thus this rule that slaves count as 3/5 of a person. Although slavery would be ended with the ratification of the 13th Amendment in 1865, the wording was modified only by the 1868 ratification of the 14th Amendment.[6]

Activity One

a. Determine, based on Jefferson's method and N = 33,000, the state representations if slaves were not included in the population totals.
b. Determine, based on Jefferson's method and N = 33,000, the state representations if slaves counted as full persons.
c. Remember that the number of Electors each state receives is equal to the number of Representatives, determined by its apportionment population; and the number of Senators, always equal to 2. Choose an election between 1792 and 1860.[7] How would the results have changed if slaves did *not* count for the apportionment population, or if slaves counted as full persons for purposes of apportionment?

5. The Polestar of Equal Representation

In general, apportionments between 1790 and 1930 followed the same pattern suggested by Jefferson:

- A divisor *N* would be chosen. This can be viewed as the "ideal district size," the number of persons each congressional delegate represents.
- A rounding rule would then be applied.

The problem is that no matter how you apportion congressional delegates, some state will argue that it has been slighted and received too few representatives.[8] What's needed is some way to measure fairness. In the words of the Supreme Court, what is the "polestar of equal representation?"[9]

Consider the actual 1790 apportionment (shown in the table on page 5). Under this plan, Massachusetts received 15 congressional delegates, while Rhode Island, with its smaller population, received 2. In what sense is this a "fair" apportionment? Since the apportionment is supposed to be based on the apportionment population, we might consider how many persons each delegate represents. Massachusetts's apportionment population of 475,327 persons was represented by 15 Congressmen, so each Congressman represented 475,327 ÷ 15 ≈ 31,688 persons; we may view this as the size of the ideal congressional district in the state of Massachusetts. Meanwhile, each of Rhode Island's 2 Congressmen represented 68,446 ÷ 2 = 34,223 persons. Since each of Rhode Island's

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[6] The original copies of the founding documents of the United States, of course, are preserved by the National Archives; the Constitution may be examined online at https://www.archives.gov/founding-docs/constitution.
[7] For historical Electoral College results by state, see the nonpartisan website owned by Allan Keiter: https://www.270towin.com/historical-presidential-elections/. A table of free, slave, and total populations by state for the 1790 through 1850 censuses was published in *The Seventh Census of the United States* (Washington, DC, 1853), p. ix: https://www2.census.gov/library/publications/decennial/1850/1850a/1850a-02.pdf.
[8] Divisor methods are subject to a number of other paradoxes and peculiarities, which are dealt with extensively elsewhere; see the Bibliography for more details.
[9] *Department of Commerce v. Montana,* 503 U.S. 442 (1992), https://tile.loc.gov/storage-services/service/ll/usrep/usrep503/usrep503442/usrep503442.pdf.

Congressmen represented more persons, then in some sense, each person in Rhode Island has a smaller representation in Congress. Looked at another way, the *difference* in the district size is a measure of the *unfairness* of the apportionment.

There are two ways to measure this difference. First, we might take a look at the *absolute difference*: the greater minus the lesser, or 34,223 – 31,688 = 2,535. The virtue of absolute difference is that everyone agrees on this value.

However, we don't typically compare absolute differences. For instance, the absolute difference between a 1-pound weight and a 2-pound weight is 1 pound, while the absolute difference between a 100-pound weight and a 101-pound weight is also 1 pound. But we typically regard the difference between the first pair of weights to be more significant than the difference between the second pair. Because of this, we are more likely to consider the *relative* difference.

The problem with relative difference is that we need to know what the difference is relative *to*. That is, while everyone would agree that the absolute difference between Massachusetts's and Rhode Island's apportionment populations is 2,535 persons, computing the relative difference requires making a choice of our comparison quantity. The usual choice is the smaller of the quantities being compared, in this case the apportionment population of Massachusetts. This gives the relative difference in district size as 2,535 ÷ 31,688 ≈ 8%. Consequently, we might say that the districts in Rhode Island represent about 8% more people than the districts in Massachusetts.

It should now be clear that there is unfairness in the congressional delegations, and we ask the question responsible for all human progress: Can we do better? In particular, given that we've now quantified the unfairness, can we find a way to reduce that number?

To think about how to do that, let's return to Hamilton's idea of a Congress with 120 members. As shown in the table on page 5, the 1790 apportionment based on Jefferson's method produced a Congress with 105 members. If we wanted to add 15 members to this body, what's the fairest way to do so? To simplify the problem, suppose we had a choice of giving Massachusetts or Rhode Island one of the extra delegates. Which of these two states is more deserving?

Since the district sizes in Massachusetts are already smaller than those in Rhode Island, giving Massachusetts the extra Congressman would also increase the disparity. Thus common sense dictates that Rhode Island should get the extra seat. However, common sense is sometimes a poor substitute for mathematics.

If Rhode Island had 3 Congressmen instead of 2, then on average, each Congressman in Rhode Island would represent 68,446 ÷ 3 ≈ 22,815 persons. But the absolute difference between the congressional districts in Massachusetts and in Rhode Island would then be 31,688 – 22,815 = 8,873, an *increase* in disparity! And if we look at the relative disparity, that would also increase to about 40%. In other words, giving Rhode Island an extra congressman would have actually exacerbated the unfairness of the apportionment.

Activity Two

Consider the 1790 (Jefferson) apportionment.

a. Suppose you could give an extra seat to either New Hampshire or Massachusetts. Would giving the extra seat to New Hampshire increase or decrease the absolute difference? What happens with the relative difference?

b. Suppose you could give the extra seat to either New Jersey or Massachusetts. Would giving the extra seat to New Jersey increase or decrease the absolute difference? What happens with the relative difference?

c. Suppose you had a choice of giving an extra seat to one of Rhode Island, New Hampshire, Massachusetts, or New Jersey. Which state should you give the extra seat to, and why?

d. Suppose you had a choice of giving an extra seat to either Maryland or Delaware. Which state should you give the extra seat to, and why?

6. The Huntington-Hill Method

The preceding section suggests a way we can apportion Congressional seats fairly. Since the Constitution requires each state to have at least one representative, we can begin by assigning one representative to each state. Then assign additional Congresspersons, one at a time, in a way that reduces the disparity.

There are two problems with this approach. The first is a minor one: *Which* disparity do we want to reduce? Absolute disparity has the advantage of being well-defined, but it's not how we typically measure inequalities. Relative disparity has the advantage of being consistent with how we measure inequalities, but we have to decide on a value against which to compare the disparity.

In 1910, Joseph A. Hill (1860–1938), the Chief Statistician of the Division of Research and Results for the Bureau of the Census, suggested that the relative difference be the measure of disparity. To bolster support for relative difference, Hill consulted Edward V. Huntington (1874–1952), a mathematician at Harvard University. Huntington wrote a series of articles supporting the use of relative disparity. Since 1940, apportionment has been based on minimizing the relative disparity using what is now called the Huntington-Hill method.[10]

A bigger problem that the Huntington-Hill method solves for us is this: our goal of giving additional Congresspersons in a way that reduces the relative disparity would seem to require making pairwise comparisons between every possible pairing of states, for every additional Congressperson we wish to assign. Even doing this with just 15 states would require 105 pairwise comparisons; with

[10] There have been some challenges to the use of relative disparity. The most noteworthy occurred after the 1990 census, where Montana lost a congressional seat. The Supreme Court, in *Department of Commerce v. Montana* (1992), rejected the argument that apportionment should be based on absolute disparity on legislative, not mathematical, grounds: Congress was permitted to choose how it would apportion congressional seats, and it happened to choose the Huntington-Hill method. In addition to the decision, which is hyperlinked in note 9, readers who want to learn more about the legal history might start with the Library of Congress's annotation on Article I, Section 2, Clause 3: https://constitution.congress.gov/browse/essay/artI_S2_C3_1/.

today's 50 states, the number of pairwise comparisons that would be needed goes up to 1,275. Fortunately, there's another approach.

Imagine we've apportioned all the Congresspersons, but haven't bothered to try to make things fair. Consequently, some states will have more than their fair share, while others will have less. Clearly the solution is to transfer delegates from one state to another.

Consider two states, with populations A and B, and current delegations of *n* and *m*, respectively. For simplicity, let's also call the states themselves A and B, and suppose we're considering transferring a seat *from* B *to A*. To begin with, this must mean $\frac{A}{n} > \frac{B}{m}$ (since if A had *smaller* districts than B, we wouldn't consider giving A more congresspersons). If $\frac{A}{n+1} > \frac{B}{m-1}$, then A's district size after the transfer would *still* be larger, and there's no question but that we should do it. Thus, the only case that requires analysis is where $\frac{B}{m-1} > \frac{A}{n+1}$.

Our goal is to reduce the relative disparity in district sizes. Since $\frac{A}{n} > \frac{B}{m}$, the relative inequality before the transfer of a seat was $\frac{\frac{A}{n} - \frac{B}{m}}{\frac{B}{m}}$. After the transfer of a seat, we have $\frac{B}{m-1} > \frac{A}{n+1}$ and the relative inequality will be $\frac{\frac{B}{m-1} - \frac{A}{n+1}}{\frac{A}{n+1}}$. If the transfer is warranted, then the relative inequality after transfer will be less than it was when we started, so we must have

$$\frac{\frac{A}{n} - \frac{B}{m}}{\frac{B}{m}} > \frac{\frac{B}{m-1} - \frac{A}{n+1}}{\frac{A}{n+1}}.$$

Rearranging and simplifying gives us

$$\frac{A^2}{n(n+1)} > \frac{B^2}{m(m-1)}$$

or, taking the square roots of both sides (which is possible since *B* is the population and *m, n* are greater than 1, all quantities have to be positive)

$$\frac{A}{\sqrt{n(n+1)}} > \frac{B}{\sqrt{m(m-1)}}.$$

Note that all of our operations are reversible, so it follows that if this inequality holds, then a transfer of a seat from B to A is warranted.

It seems we still have to perform a comparison between every possible pair of states. But let's consider a change in viewpoint. The right-hand side of our final inequality may be regarded as the effect of a state losing a seat. It follows that state B should *lose* a seat to *any* state that satisfies the inequality.

With this change in viewpoint, we can focus on the left-hand side only. If the inequality holds at all, it will hold for the largest value of $\frac{A_i}{\sqrt{n_i(n_i+1)}}$, where A_i is state *i*'s population and n_i is the size of its (current) congressional delegation. This value is known as the *priority value*. The state with the highest priority value should receive a delegate from some other state. In practice, we don't have to decide *which* state that delegate comes from, and we can proceed as follows:

1. Assign every state $n_i = 1$ representative.
2. Compute the priority values $\frac{A_i}{\sqrt{n_i(n_i+1)}}$ for each state.
3. Assign an additional representative to the state with the highest priority value, and recompute that state's priority value. Note the other priority values are unchanged.
4. Repeat Step 3 until all desired representatives are assigned.

This method has been used since the 1940 apportionment.

Activity Three

a. Apply the Huntington-Hill method to the 1790 apportionment populations.

b. Consider the election you chose to examine in Activity One. Would the results have changed if the Electoral College representation had been based on the Huntington-Hill method?

c. Show that, if the goal is to decrease the *absolute* disparity, then the priority values should be computed as $\frac{A_i(2n_i+1)}{n_i(n_i+1)}$.

   Using the priority values for the absolute disparity to allocate additional representatives to states is known as the *Dean method,* after James Dean (1748–1823), Professor of Astronomy and Mathematics at Dartmouth College, who suggested it in 1832 to former student and current Massachusetts Senator Daniel Webster (1782–1852).[11]

d. Apply the Dean method to the 1790 apportionment populations.

e. Return one last time to the election you chose to examine in Activity One. Would the results have changed if the Electoral College representation had been based on the Dean method?

Bibliography

Huntington, Edward V. "A New Method of Apportionment of Representatives." *Quarterly Publications of the American Statistical Association*, Vol. 17, No. 135 (Sept. 1921), pp. 859–870.

Huntington, Edward V. "The Mathematical Theory of the Apportionment of Representatives." *Proceedings of the National Academy of Science*, Vol. 7 (1921).

Robinson, E. Arthur, and Ullman, Danile. *A Mathematical Look at Politics*. CRC Press, 2011.

Suzuki, Jeff. *Constitutional Calculus*. Johns Hopkins University Press, 2015.

U.S. Census Bureau. Results of the 1790 Census.

[11] Webster ignored his former teacher's advice and presented his own method, which relied on the ordinary rules of rounding.