Rubini and Determinants Classroom Activity

Instructions: Read the following excerpt from mathematician Raffaele Rubini's (1857) article "Application of the Theory of Determinants: Note," then complete "In class" problems 1 and 2.

About Rubini's subscript notation: Rubini explained in a footnote that the subscript $n-1$ at the lower right corner of the first determinant below (and two others in this excerpt) indicates that the matrix contains exactly $n-1$ identical entries involving x along its main diagonal except possibly in its upper left position. This matrix must therefore be an $n \times n$ matrix. For the second determinant shown below (left side of equation (2)), the subscript n indicates that the matrix contains exactly n identical entries involving x along its main diagonal except possibly in its upper left position. Therefore, this matrix also must be an $n \times n$ matrix. Looking ahead to equation (6), can you deduce which one of the three matrices is an $(n+1) \times (n+1)$ matrix? (The other two are $n \times n$ matrices.)

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3. If we change $h_{r,r}$ to x, the same formulas will yield:

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1+x & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1+x \end{vmatrix}_{n-1} = x^{n-1}; \tag{1}$$

$$\begin{vmatrix} 1+x & 1 & \dots & 1 \\ 1 & 1+x & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1+x \end{vmatrix}_{n} = nx^{n-1} + x^n; \tag{2}$$

[183] and changing in these formulas $1+x$ to x, and therefore x to $x-1$, we will have:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & x \end{vmatrix}_{n-1} = (x-1)^{n-1} = \begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^{n-1}; \tag{3}$$

$$\begin{vmatrix} x & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & x \end{vmatrix}_{n} = +n(x-1)^{n-1} + (x-1)^n. \tag{4}$$

Salvatore J. Petrilli, Jr., and Nicole Smolenski, "Analysis and Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'," *MAA Convergence* (July 2017): http://www.maa.org/press/periodicals/convergence/analysis-and-translation-of-raffaele-rubinis-1857-application-of-the-theory-of-determinants-note

... According to formula (3) [formula (10) in Rubini's article] we have:

$$\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^m \begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^n = \begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^{m+n}; \tag{5}$$

and from the comparison of formulas (4) and (3) [formulas (11) and (10) in Rubini's article] results:

$$\begin{vmatrix} x & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & x \end{vmatrix}_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & x \end{vmatrix}_n + n \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & x \end{vmatrix}_{n-1}. \tag{6}$$

[Formula (6) is formula (15) in Rubini's article.]

Example:

$$\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}_2 + 2\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}_1,$$

which could also be written as:

$$\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}_2 - 2\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}_2.$$

In class:

Mathematician(s):______________________________ Date:____________

Complete the first problem using Laplace expansion and the second problem using the method presented in the excerpt.

1. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix}$

Salvatore J. Petrilli, Jr., and Nicole Smolenski, "Analysis and Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'," *MAA Convergence* (July 2017): http://www.maa.org/press/periodicals/convergence/analysis-and-translation-of-raffaele-rubinis-1857-application-of-the-theory-of-determinants-note