USING THE MATHEMATICS OF ASIA IN YOUR CLASSROOM

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THE “PYTHAGOREAN THEOREM” IN CHINA

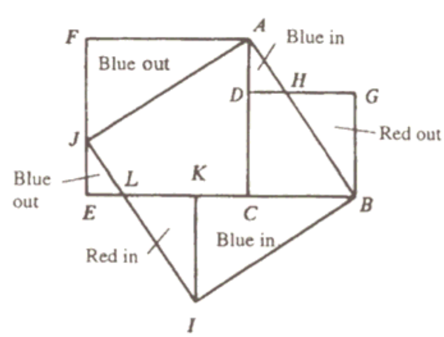
The “Pythagorean Theorem” is used in the Chinese work, *Nine Chapters on the Mathematical Art*, written around the beginning of our era, but edited and commented on by Liu Hui in the third century. In Chinese, the shorter side of a right triangle is called the *gou*, while the longer side is called the *gu*. So at the beginning of chapter 9 of the *Nine Chapters* appears the

***Gou-gu* Rule**: Add the squares of the *gou* and the *gu*, take the square root [of the sum] giving the hypotenuse. Further, the square of the *gu* is subtracted from the square on the hypotenuse. The square root of the remainder is the *gou*. Further, the square of the *gou* is subtracted from the square on the hypotenuse. The square root of the remainder is the *gu*.

Liu comments as follows:

The shorter side [of the perpendicular sides] is called the *gou*, and the longer side the *gu*. The side opposite to the right angle is called the hypotenuse [*xian*]. The *gou* is shorter than the *gu*. The *gou* is shorter than the hypotenuse. They apply in various problems in terms of rates of proportion. Hence [I] mention them here so as to show the reader their origin. Let the square on the *gou* be red in color, the square on the *gu* be blue. Let the deficit and excess parts be mutually substituted into corresponding positions, the other parts remain unchanged. They are combined to form the square on the hypotenuse. Extract the square root to obtain the hypotenuse.

Although the diagram that Liu refers to is not in the manuscripts we have, there have been many suggested reconstructions. Here is one of them, which illustrates what is called the “out-in” complementarity principle in Chinese mathematics:



Is this a “proof”?

CHINA – SOLVING SYSTEMS OF LINEAR EQUATIONS

This example is from chapter 8 of the *Nine Chapters*, in which a method is displayed for solving systems of linear equations. The method of the *Nine Chapters* is equivalent to our modern method of Gaussian elimination, which turns the original matrix of coefficients into a triangular matrix, from which the original system can be solved. But Liu Hui, in his commentary, addresses the question as to why the method works.

Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grade yield? Answer: Top grade paddy yields 9 1/4 *dou* [per bundle]; medium grade paddy 4 1/4 *dou*; [and] low grade paddy 2 3/4 *dou*.

This problem can be represented in modern terms by a system of three equations in three unknowns x, y, and z, where x is the yield per bundle of top grade paddy, y is the yield of middle grade paddy, and z is the yield of low grade paddy. The equations are:

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

The text explains that we should set up a "matrix" of the coefficients as on the left, and then transform this matrix step by step into the one on the right with the zeroes. Basically, the procedure is to multiply the middle column by the entry in the top right and then subtract the right hand column until one gets a 0 in the top middle. In this case we multiply the middle column by 3 and then subtract the right hand column twice. We then repeat this procedure with the left hand column and the right hand column, and then again with the middle column and the left hand column.

1	2	3	1	0	3	0	0	3	0	0	3
2	3	2	2	5	2	4	5	2	0	5	2
3	1	1	3	1	1	8	1	1	36	1	1
26	34	39	26	24	39	39	24	39	99	24	39

The equations formed by the right hand matrix can then be easily solved. That is, since $36z = 99$. we calculate that $z = 99/36 = 33/12 = 2\ 3/4$, where z is the yield of low grade paddy. We can then substitute this value in the second equation to solve for y and substitute both values in the first equation to solve for x, Liu elaborates on this, but then gives the following explanation:

The meaning of this rule is: subtract the column with smallest [top entry] repeatedly from the columns with larger [top entries], then the top entry must vanish. With the top entry gone, the column has one item absent. However, if the rates in one column are subtracted [from another column], this does not affect the proportions of the remainders.

The key idea – that adding and subtracting what amount to equations “does not affect the proportions of the remainders”. This is essentially the same as the axiom, if equals are added to (or subtracted from) equals, the results are equal. So Liu here has given what amounts to a proof of the procedure.

QUADRATIC EQUATIONS IN INDIA

Indian mathematicians were aware of the quadratic formula for solving quadratic equations before such ideas were discussed in Islam. Here is a verse from the *Brāhmasphutasiddhānta* of Brahmagupta in the seventh century, giving a version of the quadratic formula. The context is the solution of a quadratic equation where *a* squares plus *b* unknowns equal *c rupas*:

Diminish by the middle number the square root of the *rupas* multiplied by four times the square and increased by the square of the middle number; divide the remainder by twice the square. The result is the middle number.

Brahmagupta applied this to several interesting problems. But here is one due to Mahāvīra a few centuries later:

One-third of a herd of elephants and three times the square root of the remaining part of the herd were seen on a mountain slope; and in a lake was seen a male elephant along with three female elephants constituting the ultimate remainder. How many were the elephants here?

SUMS OF SQUARES IN INDIA

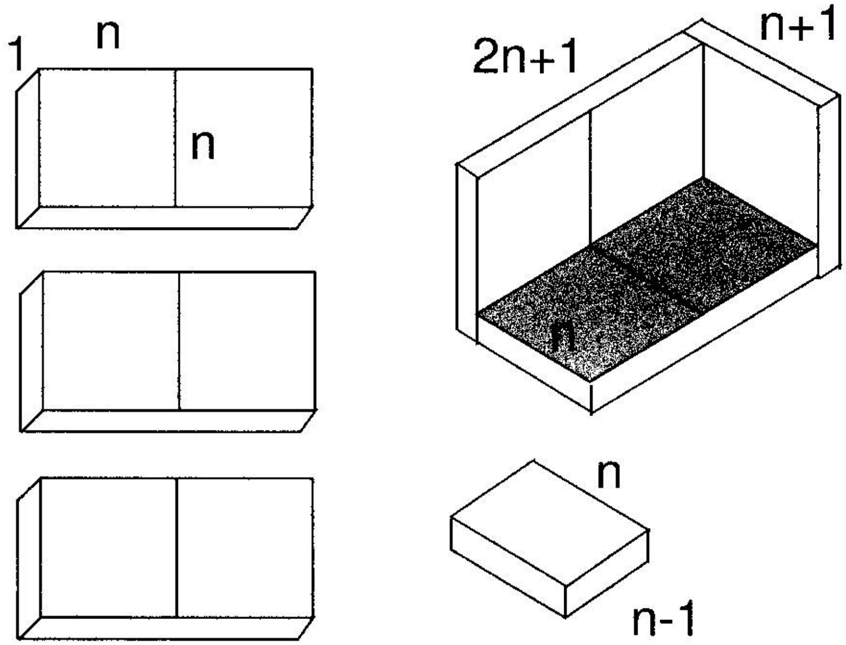
One of the earliest extant compilations of Indian mathematical knowledge was the *Āryabhatīya* of Āryabhata (5^{th} c.). This work, composed of numerous brief verses, contained very brief pieces of mathematics covering numerous areas, from algebra to trigonometry. Here we look at just one verse and then consider the commentary by the fifteenth century Keralese mathematician Nīlakantha – who was part of the school of Madhava that developed power series representations for various trigonometric functions, and also presented derivations and proofs, often absent from earlier Indian works.

So here is the verse of Āryabhata:

A sixth part of the triple product of the [term-count n] plus one, [that sum] plus the term-count, and the term-count, in order, is the total of the series of squares.

That is, Āryabhata was asserting the result that the sum of the first n integral squares is $(1/6)(n+1)(2n+1)n$, a standard result that was known to Archimedes. Nīlakantha aims to demonstrate this. As he says, "being that this is demonstrated if there is equality of the total of the series of squares multiplied by six and the product of the three quantities, their equality is to be shown. A figure with height equal to the term-count, width equal to the term-count plus one, and length equal to the term-count plus one plus the term-count is equal to the product of the three quantities. But that figure can be made to construct the total of the series of squares multiplied by six."

Nīlakantha then describes the construction of this figure. At each stage k he uses three "dominoes" of thickness 1, width k, and length $2k$. Thus, the total volume of the dominoes is $6k^2$. From the largest set, he constructs four walls of the desired figure. One of the dominoes forms one wall, a second forms the floor. The third is broken into two pieces, one of length $n+1$ and one of length $n–1$, and these form the two ends. The total length of the "box" is now $2n+1$, its width is $n+1$ and its height is n. The inside space of the box has length $2n–1 = 2(n–1) + 1$, width $n = (n–1)+1$, and height $n–1$. Thus we can create the walls of this new space with the three dominoes of thickness 1, width $n–1$, and length $2(n–1)$. We then continue until the entire box is filled. The result is now obvious. Is this a proof of the formula? What is the author assuming?

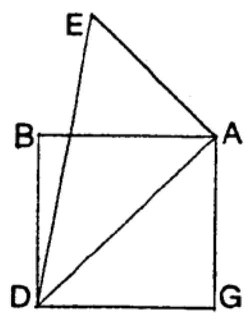


1
n
n
2n+1
n+1
n
n
n-1

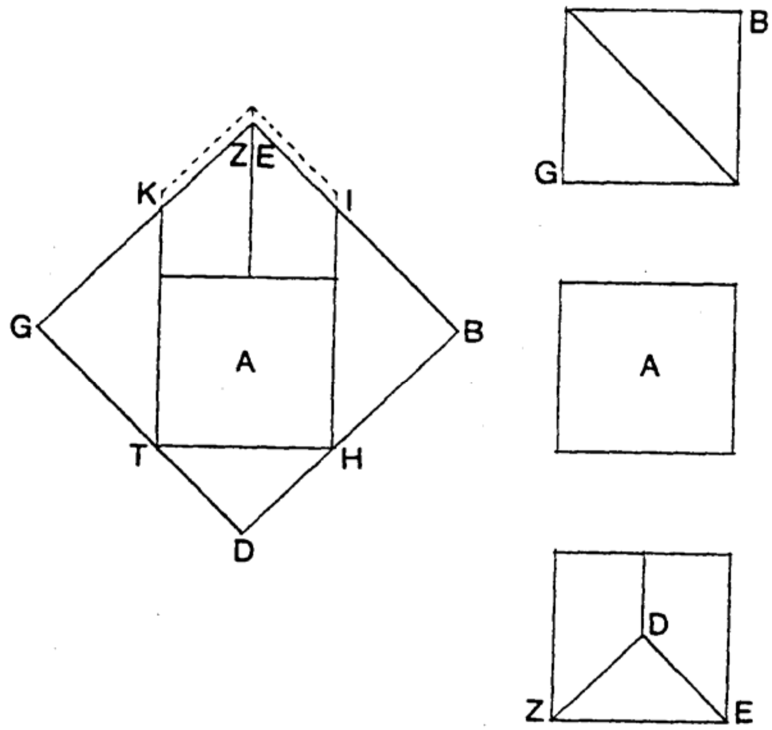
GEOMETRIC PROOF IN ISLAM

This example is taken from the *Book on the Geometrical Constructions Necessary to the Artisan* by Abu al-Wafā'al-Būzjānī (940-997). Abu al-Wafā served in the court of the Būyid dynasty in Baghdad in the second half of the tenth century. Here he was considering the problem of constructing a large square out of three identical squares. Abu al-Wafā wrote, "A number of geometers and artisans have erred in the matter of these squares and their assembling. The geometers [have erred] because they have little practice in constructing, and the artisans [have erred] because they lack knowledge of proofs."

Further, he notes, "I was present at some meetings in which a group of geometers and artisans participated. They were asked about the construction of a square from three squares. A geometer easily constructed a line such that the square of it is equal to the three squares, but none of the artisans was satisfied with what he had done. (The construction simply uses the Pythagorean Theorem twice to construct the square root of 3. But this construction does not help the artisan at all. He wants physically to divide those squares into pieces from which one square can be assembled.)

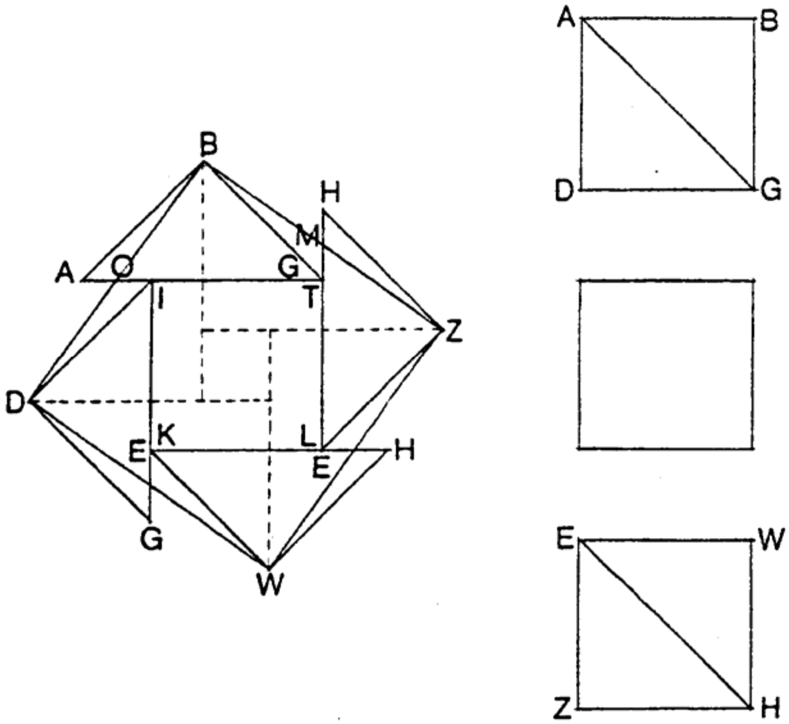


Abu al-Wafā then presented one of the methods of the artisans, in order that "the correct ones may he distinguished from the false ones and someone who looks into this subject will not make a mistake by accepting a false method, God willing." Here is the diagram:



Abu'l-Wafã said: “But this figure which he constructed is fanciful, and someone who has no experience in the art or in geometry may consider it correct, but if he is informed about it he knows that it is false.” He goes on to say that the angles are correct, and it looks like a good construction. But, in fact, the side of the proposed large square is equal to the side of the smaller square plus half the diagonal. And a quick calculation shows him that, in effect, the square of $1 + \sqrt{2}/2$ is not equal to 3.

Abu al-Wafã then gives a geometrically correct construction, with the proof: We bisect two of the squares along their diagonals. Each of those is applied to one side of the third square; we place one of the angles of the triangle which is half a right angle at one angle of the square, and the hypotenuse of it [the triangle] at the side [of the square]. Thus, part of the triangle sticks out at the other angle [of the square]. Then we join the right angles of the triangles by means of straight lines. That becomes the side of the desired square. From each big triangle, a small triangle is cut off for us [by a straight line], and we transfer it to the [empty] position of the triangle that is produced at the other angle.



To prove that this is correct, Abu al-Wafā needs to prove that, for example, triangle *BGM* is equal to triangle *MZH*. This is so because angle *G* is half a right angle, angle *H* is half a right angle, the two opposite angles of the triangles at *M* are equal, and side *BG* is equal to side *ZH*. So an appropriate Euclidean congruence theorem applies (AAS).

ALGEBRAIC EQUATIONS IN ISLAM

The Egyptian mathematician, Abū Kāmil, who lived from 850-930, wrote an algebra book based on Al-Khwārizmī's earlier text. But he gave a lot more complicated problems, especially those involving several variables.

For example, consider the following: "And if it is said to you that you will divide ten *dirham*s into three parts so that you will multiply the smallest by its like and the middle by its like and it [the result of putting the two together] is like the largest [multiplied] by its like. And you multiply the smallest by the largest and it will be the like of the middle multiplied by its like."

In other words, we have three equations in three unknowns, $x < y < z$:

$$\begin{gathered} x + y + z = 10 \\ x^2 + y^2 = z^2 \\ xz = y^2 \end{gathered}$$

How would you (or your students} solve this system? Abū Kāmil evidently noted that the three equations are homogeneous, so that he could use what has become known as the method of false position. Namely, he began by ignoring the first equation and "guessing" the value 1 for x. Substituting this into the second and third equations and then substituting the third equation into the second gives the equation

$$y^4 = y^2 + 1$$

Although this is a fourth degree equation, it is of quadratic type, so he could solve for y^2 by the quadratic formula:

$$z = y^2 = \frac{1+\sqrt{5}}{2} \text{ and } y = \sqrt{\frac{1+\sqrt{5}}{2}}$$

Now, returning to the first equation, we add the values we have for x, y, and z:

$$1 + \sqrt{\frac{1+\sqrt{5}}{2}} + \frac{1+\sqrt{5}}{2}$$

This value is not 10, as required by the first equation, but is – as you can calculate – and irrational number approximately equal to 3.9. But because the equations are homogeneous, he could divide 10 by this value, and then multiply each of the values already calculated for the unknowns by this quotient. Of course, dividing 10 by this expression is not a trivial exercise. Nevertheless, he works this through very carefully – using words, not our symbols. Of course, the quotient is itself the value of x. He gets

$$x = \frac{10}{1+\frac{1+\sqrt{5}}{2}+\sqrt{\frac{1+\sqrt{5}}{2}}} = 5-\sqrt{\sqrt{3125}-50} \approx 2.57$$

To finish the problem, Abū Kāmil, instead of trying to multiply this value by the "wrong" values of y and z, begins again, setting $z = 1$. Once he gets the correct value for z, he simply finds y by subtracting the correct values for x and z from 10. I leave the details as an exercise.

The online magazine *Convergence: Where Mathematics, History and Teaching Interact* is available at http://convergence.mathdl.org. (This url actually gets you to the general website for the Mathematics Digital Library, but you can then click on Convergence to enter the Convergence site.) This magazine provides articles, reviews, classroom suggestions, and many other items that will help any teacher of grades 9-14 use the history of mathematics in the teaching of mathematics. When you access the magazine, you will note that the home page mentions just a few articles. But if you do a search, you will find very many more in the archive, which are all available to you. Also, we welcome your articles. Please send manuscripts electronically to vkatz@udc.edu.

In addition, the Mathematical Association of America has published a CD entitled *Historical Modules for the Teaching and Learning of Mathematics*. This CD has numerous lesson plans in all areas of secondary mathematics which demonstrate the use of the history of mathematics in teaching mathematics. There are eleven modules in all, covering topics from Negative Numbers through Functions. Each module contains many lesson plans, with each lesson plan containing both teacher notes and pages for the students. Some samples from the CD are included here. To order, go to www.maa.org and then go to Bookstore and search for the title.

Combinations
Teacher Notes

Placement in Course: This section is appropriate as an introduction to combinations in middle school. It would be a replacement for any textbook introduction. Of course, a textbook would provide many more problems for practice with this idea.

Time Frame: This lesson should take approximately three class periods.

Materials: Each pair of students will need 10 manipulatives in different colors to represent strings. Since colored strings themselves are difficult to work with, you can use beads or cubes or any other small objects which are available in 10 colors. You may even have success by having students pretend actually to be the strings and to move around to form various combinations. The colors used in this discussion are red, orange, yellow, green, blue, violet, pink, white, teal, and sepia. Each color is designated by its first letter: R, O, Y, G, B, V, P, W, T, and S. (If you have a different set of colors, you may need to change the letters on the worksheets.)

Suggested lesson plan:

Give each pair of students 10 manipulatives in different colors to represent strings. Allow time for them to form the combinations and record the results on the three tables on the student activity sheet. The questions from that sheet follow, along with suggested answers and further discussion.

1. How many different choices can one make of 1 thread out of 10?

   The answer should be 10, if students understood the problem. If you began with only 9 colors then the answer should be 9. In general if you start with n colors the answer would be n choices.

2. List the different sets of 2 threads one can pick from the ten different threads. List the sets which can be made if you start with only 9 threads?

   Students are to pick out all possible pairs and record them. Students should give some thought to how they pick out the threads in order to form all possible pairs. Do not tell students how many combinations they should have constructed.

   From 10 threads, there are 45 ways of choosing two, listed below. From 9 threads there are 36 ways of choosing two.

OR	*YR*	*YO*	*GR*	*GO*	*GY*	*BR*	*BO*	*BY*	*BG*	*VR*	*VO*	*VY*	*VG*	*VB*
PR	*PO*	*PY*	*PG*	*PB*	*PV*	*WR*	*WO*	*WY*	*WG*	*WB*	*WV*	*WP*	*TR*	*TO*
TY	*TG*	*TB*	*TV*	*TP*	*TW*	*SR*	*SO*	*SY*	*SG*	*SB*	*SV*	*SP*	*SW*	*ST*

3. Study Ibn Mun'im's words. Fill in the "2" column, following his reasoning. The numbers in the column down the left tell you how many threads you have to choose from. The numbers along the top tell you how many threads are in each bundle. The first few numbers are done for you as an example. Work out a formula to determine the number in column 2. You may know this formula from previous work.

   Show the overhead transparency (#7) with Ibn Mun'im's words. Discuss the number of combinations that should have been listed for 2 out of 5 threads and 2 out of 6. Students should determine that the number of possible pairs out of ten threads is given by the sum $1 + 2 + 3 + 4 = 10$. Similarly, if we begin with 5 threads there are $1 + 2 + 3 + 4 + 5 = 15$ ways to pick pairs; if we begin with 7 threads there are 21 ways, etc. The students may already know the formula for the sum of the first n integers and therefore will realize that the number of ways to choose 2 colors out of n is given by $n(n - 1)/ 2$. If not, this is an opportunity to discuss the derivation of this formula.

	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

The column labeled 1 records the number of ways one can choose 1 color out of n colors. The column labeled 2 records the number of ways of choosing 2 colors out of n as determined in the previous example. Note that the method to determine the number of row n of column 2 is to add the numbers of column 1 together from the top down to row n–1. It is convenient to put in the column labeled 0 partly because one can then note that the number in column 1 of row n is determined by adding together the numbers in column 0 from the top down to row n–1. One can also think of column 0 as recording the number of ways one can choose zero colors out of a set of n threads. It is convenient to name the entry in row n column k as ${}_nC_k$ and think of this number as representing the number of ways of choosing k elements out of a set of n. For example, ${}_{10}C_2 = 45$ and represents the number of ways of choosing 2 elements out of a set of 10.

4. In how many ways can one pick three colors out of ten? Before attempting to answer this question first determine how many ways one can pick three colors out of four? Then list the possibilities for picking three colors out of five and continue filling in this chart.

Number of Threads	Ways of Choosing Three	#
4 (ROGY)	ROY, ROG, OYG, RGY	4
5 (ROGYB)	ROY, ROG, ROB, RYG, RYB, RGB, OYG, OYB, OGB, YGB	10
6		
7		
8		
9		
10		120

5. Generalize the results for ${}_4C_3$ and ${}_5C_3$ to a formula for ${}_nC_3$. Determine these values for $n = 6$, 7, 8, 9, 10. Go back to the table in problem #3 and fill in the "3" column.

   Students who have discovered a systematic method of listing the possibilities here should write it out and/or demonstrate it to others. The class can then consider Ibn Mun'im's method of determining the number of combinations of three colors as shown on transparency #8. Students should compare their method with that of Ibn Mun'im. At each stage one is combining a given color with all the pairs of colors preceding the given one. Therefore ${}_5C_3 = {}_2C_2 + {}_3C_2 + {}_4C_2$ and, in general,

$$ {}_nC_3 = {}_2C_2 + {}_3C_2 + {}_4C_2 + \ldots + {}_{n-1}C_2 $$

6. Develop a formula for ${}_nC_4$ in terms of the values ${}_kC_3$ and determine the number of ways of choosing 4 colors out of 6 by using the formula. Fill in the column for 4. Then give a rule in words for determining any number in the table. Complete the remaining columns.

   The formula for ${}_nC_4$ is ${}_nC_4 = {}_3C_3 + {}_4C_3 + {}_5C_3 + \ldots + {}_{n-1}C_3$. In general, any number in the table is the sum of the numbers in the previous column up to the row before the row of the given number. Once students complete the table, they may recognize this table as the table of binomial coefficients usually known as Pascal's triangle. This would then be a good time to point out that although Pascal did extensive work with his triangle, many of the patterns in it were known hundreds of years before Pascal, not only in North Africa, but also in the Middle East, India, and China.

7. We now want to develop a formula for ${}_nC_k$ which does not require knowing the values from the previous row or column. We already have a formula for ${}_nC_2$, so begin by calculating ${}_4C_3$ using that formula. The idea is to list the pairs of letters and associate each pair with the

remaining letters. We know there are ${}_4C_2 = 6$ pairs of 4 letters. Using R, O, Y, G the pairs are {RO}, {RY}, {RG}, {OY}, {OG}, and {YG}. List all the triplets which can be made by adding one color to each of these pairs. How many times does each triplet now appear? Hint: The triplet consisting of the colors red, orange and green occurs as {RO,G}, {RG,O} and {OG,R}. Show that the formula for ${}_4C_3$ is ${}_4C_3 = (4–2)({}_4C_2/3) = 2({}_4C_2/3) = 4$

The method presented is basically due to a 14th century Arabic mathematician Abu l'Abbas Ahmad al-Marrakushi ibn al-Banna, from Marrakech in what is now Morocco. The triplets which can be made are [{RO},Y], [{RO},G], [{RY},O], [{RY},G], [{RG},O], [{RG},Y], [{OY},R], [{OY},G], [{OG},R], [{OG},Y], [{YG}, R] and [{YG},O]. Each triplet occurs 3 times in this list of 12. Since we found 12 triplets by multiplying the ${}_4C_2 = 6$ pairs of colors by 4 − 2 = 2 remaining letters, and since each triplet is repeated 3 times, we can derive the formula indicated for ${}_4C_3$.

8. Repeat the process from 7 to calculate formulas for ${}_5C_3$ and ${}_6C_3$. Then generalize this to give a formula for ${}_nC_3$. First, write the formula for ${}_nC_3$ in terms of ${}_nC_2$. Then use the formula for ${}_nC_2$ developed in step 3 to write a general formula for ${}_nC_3$ only involving n.

   The answers are that ${}_5C_3 = (5–2)({}_5C_2/3)$ and ${}_6C_3 = (6–2)\ {}_6C_2/3$. The general formula is ${}_nC_3 = (n–2)({}_nC_2/3)$ or ${}_nC_3 = [(n–2)(n–1)n]/[3\times2\times1]$.

9. Repeat this process to calculate ${}_nC_4$. Begin by calculating ${}_5C_4$, ${}_6C_4$, and ${}_7C_4$. Then figure out a general formula for ${}_nC_4$.

   To calculate ${}_5C_4$, we begin with ${}_5C_3$ triplets, then find quartets of numbers by adding one of the unused colors to each triplet. Since for each triple, there are 5−3 =2 unused colors, we get $2({}_5C_3)$ quartets. But we then find that each quartet is repeated 4 times. So ${}_5C_4 = (5–3)({}_5C_3/4) = 5$. Similarly, we find that ${}_6C_4 = (6–3)({}_6C_3/4) = 15$ and ${}_7C_4 = (7–3)({}_7C_3/4) = 35$. We can now generalize these formulas to get ${}_nC_4 = (n–3)({}_nC_3/4)$ or, since we already know a formula for ${}_nC_3$, we have ${}_nC_4 = [n(n–1)(n–2)(n–3)]/[1\times2\times3\times4]$.

10. Finally, determine the general formula for ${}_nC_k$, by looking at the formulas already determined. Check some of your values against the known values from the table.

    The formulas already developed generalize to

$${}_nC_k = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{1\times2\times3\times\cdots\times k}$$

   This is the standard multiplicative formula for ${}_nC_k$. Students should be encouraged to calculate various values for ${}_nC_k$ and compare with the numbers in their table. You also may want to show the students the equivalent formula

$${}_nC_k = \frac{n!}{k!(n-k)!}.$$

If one thinks about the problem, one realizes that the sets of two colors may be obtained by combining the second color with the first [{O,R}], the third color with the first and with the second [{Y,R}, {Y,O}], the fourth color with the first, with the second and with the third [{G,R}, {G,O}, {G,Y}], the fifth color with the first, with the second, with the third and with the fourth [{B,R}, {B,O}, {B,Y}, {B,G}], and so on.

Ibn Mun'im (early 13th century, Morocco)

[Transparency #7]

Funded by the N.S.F.

To determine the number of combinations of three colors, one first combines the third color with the first and the second, then combines the fourth color with each pair of colors among the three colors which precede it – the first, the second and the third – then combines the fifth color with each pair of colors among the four preceding colors, … up to the combination of the tenth color with each pair of colors among the nine preceding colors. But each pair of colors is a combination from the second line. For this reason, … we write in the case of the fourth color, that the number of combinations obtained by the combination of the fourth color with each pair among the preceding colors is equal to the number of bunches of two colors composed of colors preceding the fourth color, and this is also equal to the sum of the two first numbers of the second line, namely 3. We therefore write 3 in the second column in the third line.

Ibn Mu'nim

[Transparency #8]

Combinations
Student Pages

In the previous sections we have looked at permutations, that is, arrangements. For example, we can arrange the letters {A, B,C} into six possible "words" namely, ABC, ACB, BAC, BCA, CAB, and CBA. We now want to consider just picking sets of letters, where we are not concerned with arrangements. In this case, {A, B,C} will form only one set, instead of six. In particular, in this activity we would like to answer the question:

> If you have 10 threads of different colors, how many ways can you chose smaller groups of colors, if order is not critical?

We are assuming you have threads of ten different colors: red, orange, yellow, green, blue, violet, pink, white, teal, and sepia, each abbreviated by its first letter.

1. How many different choices can you make of 1 thread out of 10? List them.

   We now determine the number of sets of two colors. We consider the words of Ibn Mun'im written in the early part of the 13th century in Morocco:

   > If one thinks about the problem, one realizes that the sets of two colors may be obtained by combining the second color with the first [{O,R}], the third color with the first and with the second [{Y,R}, {Y,O}], the fourth color with the first, with the second and with the third [{G,R}, {G,O}, {G,Y}], the fifth color with the first, with the second, with the third and with the fourth [{B,R}, {B,O}, {B,Y}, {B,G}], and so on.

2. List the different sets of 2 threads one can pick from the four different threads. Write your answer in the first row. List the sets which can be made if you start with only 5 threads. Write your answer in the second row. etc.

Number of Threads	Ways of Choosing Two	#
4		
5		
6		
7		
8		
9		
10		

3. Study Ibn Mun'im's words. Fill in the "2" column, following his reasoning. The numbers in the column down the left tell you how many threads you have to choose from. The numbers along the top tell you how many threads are in each bundle. The first few numbers are done for you. Work out a formula to determine the number in column 2. You may know this formula from previous work.

	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3								
4	1	4	6								
5	1	5									
6	1	6									
7	1	7									
8	1	8									
9	1	9									
10	1	10									

4. In how many ways can one pick three colors out of ten? First determine how many ways one can pick three colors out of four? (There is only one way to pick three colors out of three.) Then list the possibilities for picking three colors out of five and continue filling in this chart.

Number of Threads	Ways of Choosing Three	#
4		
5		
6		
7		
8		
9		
10		

We need a notation for this concept of choosing sets of so many colors out of a larger set. We will use the notation ${}_nC_k$ to represent the number of ways we can choose sets of k objects out a set of n objects. Thus, the numbers in column 2 of the first table are the numbers ${}_nC_2$, where n ranges from 2 to 10. Similarly, the numbers in the last column of the second table are the numbers ${}_nC_3$, where n ranges 4 to 10. We want to find formulas to calculate ${}_nC_k$ in various cases.

5. Generalize the results for ${}_4C_3$ and ${}_5C_3$ to a formula for ${}_nC_3$.
   Determine these values for $n = 6, 7, 8, 9, 10$.
   Go back to the table in problem #3 and fill in the "3" column.

6. Develop a formula for ${}_nC_4$ in terms of the values ${}_kC_3$ and determine the number of ways of choosing 4 colors out of 6 by using the formula. Fill in the column for 4. Then give a rule in words for determining any number in the table. Complete the remaining columns.

7. We now want to develop a formula for ${}_nC_k$ which does not require knowing the values from the previous row or column. We already have a formula for ${}_nC_2$, so begin by calculating ${}_4C_3$ using that formula. The idea is to list the pairs of letters and associate each pair with the remaining letters. We know there are ${}_4C_2 = 6$ pairs of 4 letters. Using R, O, Y, G the pairs are {RO}, {RY}, {RG}, {OY}, {OG}, and {YG}. List all the triplets which can be made by adding one color to each of these pairs.
   How many times does each triplet now appear? Hint: The triplet consisting of the colors red, orange and green occurs as {RO,G}, {RG,O} and {OG,R}.
   Show that the formula for ${}_4C_3$ is ${}_4C_3 = (4–2)({}_4C_2/3) = 2({}_4C_2/3) = 4$

8. Repeat the process from 7 to calculate formulas for ${}_5C_3$ and ${}_6C_3$. Then generalize this to give a formula for ${}_nC_3$. First, write the formula for ${}_nC_3$ in terms of ${}_nC_2$. Then use the formula for ${}_nC_2$ developed in step 3 to write a general formula for ${}_nC_3$ only involving n.

9. Repeat this process to calculate ${}_nC_4$. Begin by calculating ${}_5C_4$, ${}_6C_4$, and ${}_7C_4$. Then figure out a general formula for ${}_nC_4$.

10. Finally, determine the general formula for ${}_nC_k$, by looking at the formulas already determined. Check some of your values against the known values from the table.

Exercises

1. How many possible four letter words with four distinct letters can one form with the 26 letters of the alphabet? Note that once one picks a set of four letters, one then has to calculate the number of permutations of this set. In general, the number of permutations of k things taken out of a set of n is called ${}_nP_k$. In doing this problem, you should convince yourself that ${}_nP_k = k!\,{}_nC_k$.

2. How many possible five letter words with five distinct letters can one form with the 26 letters of the alphabet?

3. How many words of four letters with one letter repeated once can one form out of the letters of the alphabet?

4. How many sets of four ingredients to make perfume can we find in a set of sixteen total ingredients?

5. In how many ways can one pick six numbers out of forty-four?

6. The school band is going to London for spring break. They can each take one small suitcase holding no more than 12 items of clothing (not including underwear) and one pair of shoes. What items of clothing would one take to have an optimal number of outfits? Are the combinatorial formulas developed here useful to answer this question?

Answers

1. ${}_{26}P_4 = 358{,}800$

2. ${}_{26}P_5 = 7{,}893{,}600$

3. ${}_{26}P_4/2! = 179{,}400$

4. ${}_{16}C_4 = 1820$

5. ${}_{44}C_6 = 7{,}059.052$

6. The optimal solution is 6 tops and 6 bottoms, producing 36 different outfits, if we assume all tops and bottoms can be worn together. You use the fact that ${}_6C_1 \bullet {}_6C_1 = 36$ to get this solution.

Ancient Chinese Problems Activity
Teacher Notes

Level: This activity is designed for middle school or high school students in Pre-algebra, Algebra I, Geometry, or Algebra II.

Objective: Students will experience the kinds of problems that were being solved and the methods used to solve them thousands of years ago in China.

Materials: Make copies of the Student Pages, including the Solutions from the *Nine Chapters on the Mathematical Art,* for the students. You also may wish to reproduce the Chinese Area Formulas and Chinese Volume Formulas from the **Lengths, Areas, and Volumes in China** section of this module for your students.

When to Use: Use this activity after formulas for areas and volumes have been studied.

How to Use: Let students work in groups of 2, 3, or 4 to solve the problems. Have them write a paragraph about each problem, comparing the method used to solve the problem today with the method used long ago in China.

Background Information: See the **Lengths, Areas, and Volumes in China** section of this module.

Solutions: See the Solutions from the *Nine Chapters on the Mathematical Art* in the Student Pages. Additional comments about Problems 5 (32) and 8 (18) follow.

5. **Problem 1-32:** Students should compute the area A using the formula $A = \pi r^2$ with radius $r = D/2 = 30\frac{1}{6}$, to obtain $A = \pi\left(910\frac{1}{36}\right) \approx 2858.9366$ square *bu.* The four methods from the *Nine Chapters* for finding the area of a circle with a given circumference C, diameter D, and/or radius r are described, respectively, by the formulas $A = \frac{C}{2}\cdot r$, $A = \frac{CD}{4}$, $A = \frac{3D^2}{4}$, and $A = \frac{C^2}{12}$. The first two formulas are correct, while the third and fourth formulas are correct if we replace 3 by π.
8. **Problem 5-18:** Note that the author of the *Nine Chapters* uses the formula $V = \frac{1}{6}(a+b+c)hl$ for the volume V of a wedge with a trapezoidal base, or, as it is described in the list of Chinese Volume Formulas in the **Lengths, Areas, and Volumes in China** section of this module, a tomb entrance tunnel sloping down into the ground, or a prism with three trapezoids and two triangles as faces. Substituting $a = 2$, $b = 4 = c$, $h = 1$, and $l = 3$ *zhang* into this formula yields the correct answer in cubic *zhang,* while substituting $a = 20$, $b = 40 = c$, $h = 10$, and $l = 30$ *chi* into the formula yields the correct answer in cubic *chi.*

Ancient Chinese Problems Activity
Student Pages

Solve each problem from the ancient Chinese text, *Nine Chapters on the Mathematical Art.* If you use a formula, write it out. Then compare your methods and formulas with those the Chinese used in 200 BCE. For each problem, write a paragraph comparing your method with the ancient Chinese method.

1. **Problem 1-1** (Shen, 61):
   Now given a field 15 *bu* (paces) broad and 16 *bu* long.
   Find [the area of] the field.

2. **Problem 1-21** (Shen, 82):
   Given another field, 4/5 *bu* in breadth and 5/9 *bu* in length.
   Find [the area of] the field.

3. **Problem 1-25** (Shen, 84):
   Now given a triangular field with base 12 *bu* and altitude 21 *bu*.
   Find [the area of] the field.

4. **Problem 1-29** (Shen, 86):
   Now given a trapezoidal field (*ji tian*) with bases 20 *bu* and 5 *bu* respectively and altitude 30 *bu*. Find [the area of] the field.

5. **Problem 1-32** (Shen, 87):
   Now given a circular field, the circumference is 181 *bu* and the diameter 60 1/3 *bu.* Find [the area of] the field. (Comment on the relationship between circumference and diameter given here.)

6. **Problem 5-4** (Shen, 257):
   Now there is a dike [in the shape of a prism with a cross-section that is a trapezoid] with a lower breadth of 2 *zhang*, an upper breadth of 8 *chi*, an altitude of 4 *chi*, and a length of 12 *zhang,* 7 *chi*. Find the volume [in cubic *chi*]. (Hint: 1 *zhang* = 10 *chi.)*
   Each laborer's standard winter quota for earthworks is 444 [cubic] *chi.* How many laborers are required?

7. **Problem 5-15** (Shen, 269):
   Now given a *yang ma* (pyramid with rectangular base), with a breadth of 5 *chi*, a length of 7 *chi* and an altitude of 8 *chi*. Find its volume.

8. **Problem 5-18** (Shen, 287):
   Now given a *chu meng* (wedge with rectangular base), with a lower breadth of 3 *zhang*, a length of 4 *zhang*; and upper length of 2 *zhang*, no breadth; and an altitude of 1 *zhang*. Find its volume [in cubic *chi*]. (Hint: 1 *zhang* = 10 *chi.*)

Solutions from the *Nine Chapters on the Mathematical Art*

1. **Problem 1-1** (Shen, 61): Answer says 1 *mu* (= 240 square *bu).*

   After several similar problems are given, a general method for solving such problems is stated as follows: Method of *fang tian* says: Multiply the number of *bu* in breadth by that in length to obtain the product. Dividing by the conversion factor of 240 [square] *bu* in 1 *mu* gives the number of *mu*.

2. **Problem 1-21** (Shen, 82): Answer says 4/9 [square] *bu*.

   Method of multiplying fractions (*chen fen*) says: Multiply the denominators to obtain the *fa*. Multiply the numerators to obtain the *shi*. Divide the *shi* by the *fa*.

3. **Problem 1-25** (Shen, 84): Answer says 126 [square] *bu*.

   Method says: Multiply half the base by the altitude.

4. **Problem 1-29** (Shen, 86): Answer says 1 *mu,* 135 [square] *bu* (= 375 square *bu).*

   Method says: Multiply half the sum of the bases by the altitude, or half the altitude by the sum [of the bases]; divide by the number of [square] *bu* per *mu*.

5. **Problem 1-32** (Shen, 87): Answer says 11 *mu,* $90\frac{1}{12}$ [square] *bu* $\left(= 2730\frac{1}{12}\ \textit{square bu}\right)$.

   Method says: Multiplying half the circumference by the radius yields the area of the circle in [square] *bu*.
   Next method says: One-fourth the product of the circumference and the diameter.
   Next method says: One-fourth the product of three times the diameter squares.
   Next method says: The circumference squared and divided by 12.

6. **Problem 5-4** (Shen, 257): Answer says 7112 [cubic] *chi*; answer says $16\frac{2}{111}$ men.

   Method says: Add the upper and lower breadths, then halve [the sum]; multiply by the altitude or depth; then multiply by the length, giving the volume.

   Method says: Take the number [cubic] *chi* [of the volume] as dividend; the number of [cubic] *chi* of the standard quota as divisor. Divide, giving the number of laborers required.

7. **Problem 5-15** (Shen, 269): Answer says $93\frac{1}{3}$ [cubic] *chi*.

   Method says: Multiply the breadth by the length, then multiply by the altitude. Divide by 3.

8. **Problem 5-18** (Shen, 287): Answer says 5000 [cubic] *chi* (= 5 cubic *zhang*).

   Method says: Double the lower length, add it to the upper length, multiply the sum by the breadth, then multiply by the altitude. Divide by 6.

Altars of India Activity
Teacher Notes

Level: This activity is designed for students in middle school through high school.

Objective: Students will solve geometric problems relating to altar construction from ancient India.
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Materials: Copy the Student Pages to distribute to students.

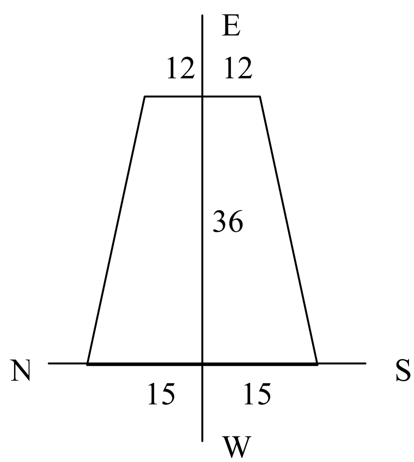
How to Use: Have students work individually or in small groups.

When to Use: Use this activity after students have studied areas of polygons. Problems 2, 3, and 5 require use of the Pythagorean Theorem.

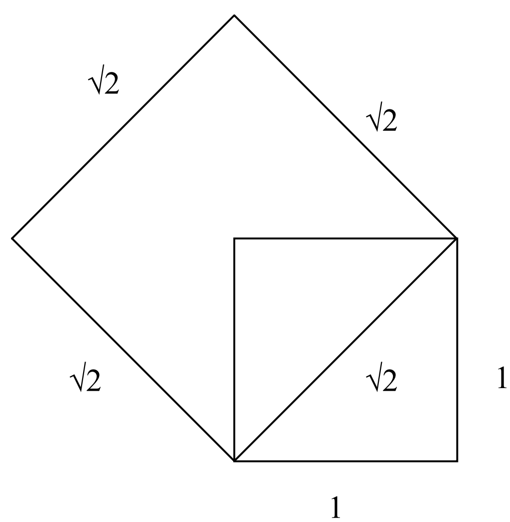
Background Information: In addition to the information provided in the Student Pages of this activity, see the **Lengths, Areas, and Volumes in India** section of this module.

Solutions:

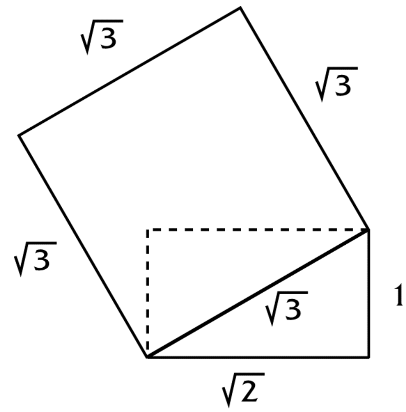
1.



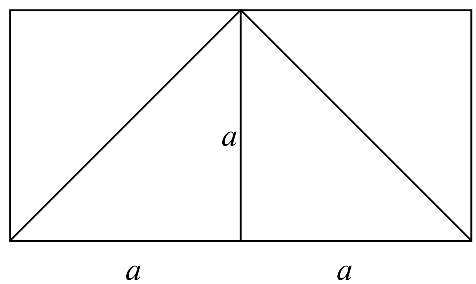
2. The diagram shows how to construct a square with twice the area of the square of side length 1. The square of side length $\sqrt{2}$ has area 2.



3. Assuming the original square has side length 1, use the line segments of lengths 1 and $\sqrt{2}$ from the solution to Problem 2 to construct a rectangle with sides of lengths 1 and $\sqrt{2}$ and diagonal of length $\sqrt{3}$. (Equivalently, use these line segments to construct a right triangle with legs of lengths 1 and $\sqrt{2}$ and hypotenuse of length $\sqrt{3}$.) The square of side length $\sqrt{3}$ has area three times that of the square of side length 1.



4. The isosceles triangle with base $2a$ and altitude a has the same area as the square of side length a, namely a^2.

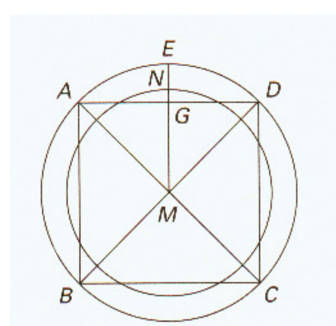


5. Note that the square with side AX has area AX^2, square $ABCD$ has area AD^2, and square $PQRS$ has area PQ^2. Since triangle ADX is a right triangle, then, by the Pythagorean Theorem, $AD^2 + DX^2 = AX^2$. Since $DX = PQ$, then this equation says that the sum of the areas of squares $ABCD$ and $PQRS$ is equal to the area of the square with side AX.

Extension Activity: Circling the Square

The *Sulbasutra* written by Apastamba sometime between 700 and 400 BCE contains instructions for constructing a circular altar having the same area as a given square altar (Katz, 41):

a) In square $ABCD$, let M be the intersection of the diagonals of the square.
b) Draw the circle with M as center and MA as radius.
c) Let ME be the radius of the circle perpendicular to AD and intersecting AD in G.
d) Let $GN = 1/3\ GE$. Then MN is the radius of a circle having area equal to the area of square $ABCD$.



Have students compute the area of the circle with radius *MN* in terms of the side length *a* of square *ABCD*. Does this circle have the same area as square *ABCD*? Since Apastamba asserts that the circle and square have the same area, what value must he have used for π?

If square *ABCD* has side length *a,* then *AG* and *MG* have length $\frac{a}{2}$ and *MA* has length $\frac{a\sqrt{2}}{2}$. Therefore, *ME* also has length $\frac{a\sqrt{2}}{2}$, *GE* has length $\frac{a\sqrt{2}}{2}-\frac{a}{2}=\frac{\left(\sqrt{2}-1\right)a}{2}$, *GN* has length $\frac{\left(\sqrt{2}-1\right)a}{6}$, and *MN* has length $\frac{a}{2}+\frac{\left(\sqrt{2}-1\right)a}{6}=\frac{\left(2+\sqrt{2}\right)a}{6}$. The circle of radius *MN* then has area $\pi\left(\frac{\left(2+\sqrt{2}\right)a}{6}\right)^2 \approx 1.0173a^2$, while square *ABCD* has area a^2. If we replace π by $\left(\frac{6}{2+\sqrt{2}}\right)^2 \approx 3.0883$, then the circle would have area a^2. Hence, Apastamba used $\pi \approx 3.0883$ in his construction of the circular altar.

Altars of India Activity
Student Pages

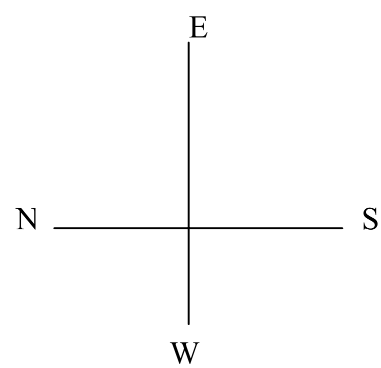
The earliest accounts of the use of area formulas in India appear in a series of religious texts called the *Sulbasutras* ("rules of the rope"), the first of which probably were written by priests before the sixth century BCE. The *Sulbasutras* gave specific instructions for constructing altars, which were used for religious ceremonies and rituals, including animal sacrifices. Altars were constructed in different shapes, depending on the ritual to be performed, and with surfaces of different areas, depending both on the type of sacrifice and on the rank of the person making the sacrifice. Ropes were used to lay out the dimensions of the altar, always employing a vertical east-west line of symmetry. Most of the altars were made in five layers of two hundred bricks each, again with the dimensions and surface areas specified in the *Sulbasutras* (Seidenberg, 319). Shapes of bricks that could be used to make the various types of altars included squares, rectangles, rhombuses, trapezoids, and a variety of triangles (Seidenberg 1962, 490-492, 494, 496; Tiwari, 91).

The priests Baudhayana, Apastamba, and Katyayana wrote the three *Sulbasutras* containing the most mathematics. The *Baudhayanasulbasutra* probably was written sometime between 800 and 600 BCE, and the *Sulbasutras* of Apastamba and Katyayana sometime between 700 and 400 BCE (Joseph, 226, 228).

Complete the following problems from the *Sulbasutras.*

1. The base of the Mahavedi altar has the shape of an isosceles trapezoid. With East at the top of your diagram, draw the base of the Mahavedi altar by following these instructions from the *Apastambasulbasutra:*

> From the post [intersection of the North-South and East-West axes], which is largest on the Eastern side of the hall, he strides . . . fifteen steps to the right and fixes a peg; this is the right hip. From the middle peg, he strides fifteen steps northwards and drives another peg; this is the left hip. From the middle peg, stride 36 steps eastward and drive a peg; this is the fore part. From the middle peg in front, he strides 12 steps to the right and [fixes] a peg there; this is the right shoulder. From the middle peg [in front], he strides 12 steps to the left; this is the left shoulder. This is the measure of the altar.



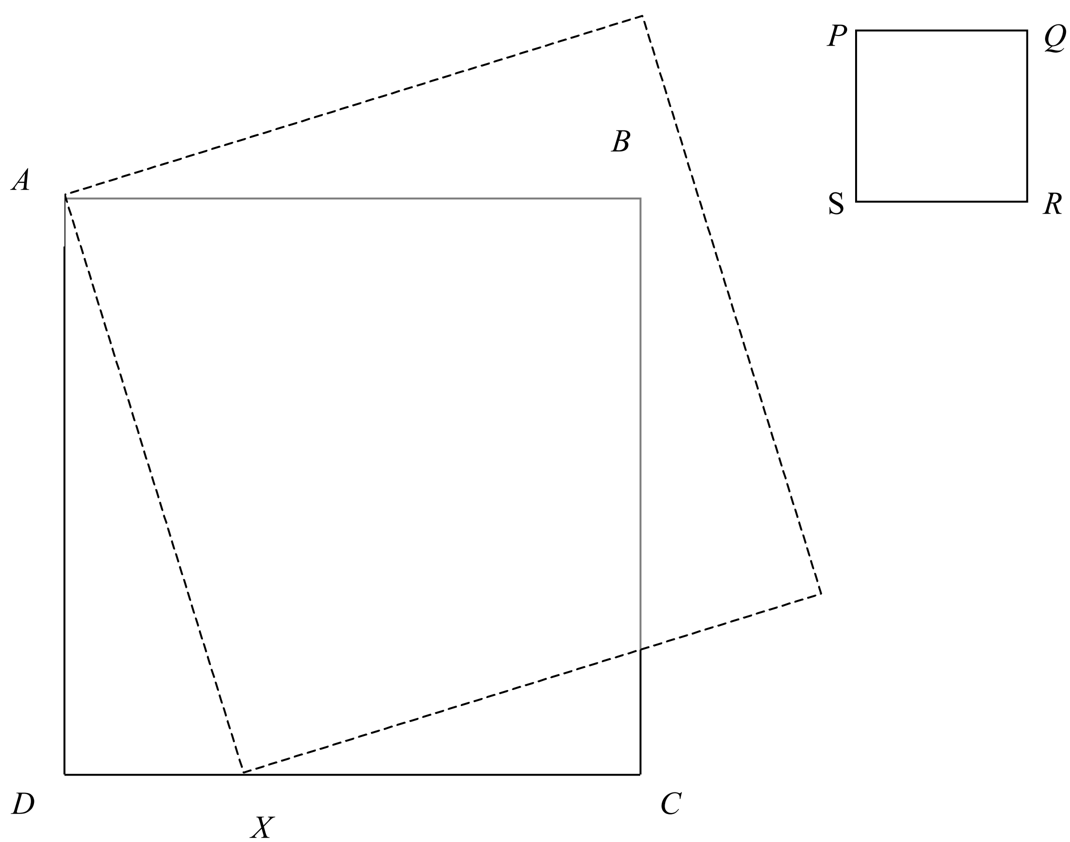
2. From a square with side length 1, find a square that has twice the area of the original square. A hint comes from the *Baudhayanasulbasutra:* "The diagonal of a square produces double the area of the square." Draw a diagram to show how this works.

3. Draw a diagram illustrating the following construction. To make a square having triple the area of a given square, form a rectangle using the diagonal of the given square as the length and the side of the given square as the width of the rectangle. Use the diagonal of this rectangle for the side of the triple-sized square (Grover, 22).

4. Draw a diagram to show what is meant in this passage from the *Sulbasutra* of Apastamba (Grover, 25):

> If it is desired to transform a square into an isosceles triangle [having the same area], double the area of the square [to form a rectangle, and] fix a pole at the middle point of the eastern [top] side. The two chords fastened to the pole and stretched to [one diagonal corner and then to the other diagonal corner] give an isosceles triangle equal in area to the [original] square.

5. The *Sulbasutras* written by Baudhayana, Apastamba, and Katyayana all contain instructions for constructing a square having area equal to the sum of the areas of two given squares (Joseph, 230-231). In the diagram (after Joseph, 231), squares *ABCD* and *PQRS* are given, and the point *X* is marked so that $DX = PQ$. Use the diagram to explain how you know that the square with side *AX* has area equal to the sum of the areas of squares *ABCD* and *PQRS*.



Part 3: Negative Numbers in India

The people of ancient India were interested in mathematics primarily as a means to understand astronomy and astrology and secondarily to aid in commercial dealings. In commerce, negative numbers represented debts and positive numbers represented assets. In the Bakhshali manuscript, a birch bark papyrus found in the village of Bakhshali in 1881 and believed to date from as early as 500 CE, an unknown author wrote mathematics using a fully developed base-10 place-value (positional) number system, very similar to the Indo-Arabic (also known as Hindu-Arabic) number system we use today. The author of the Bakhshali manuscript used a dot for zero and a cross (+) to represent a minus sign (Datta and Singh, 14).

Brahmagupta *c.* 625

Brahmagupta was an astronomer and mathematician at Bhillamala, India, who confessed in one of his books that some of his problems were written "simply for pleasure" (Struik, 66). Brahmagupta's most noted work is the *Brahmasphutasiddhanta*, which he wrote when he was 30 years old (Katz, 218) and in which he gave rules for working with fractions that are very similar to our rules today. About algebra, he said, "Since questions can scarcely be known [solved] without algebra, therefore, I shall speak of algebra with examples. By knowing the pulveriser, zero, negative and positive quantities, unknowns, elimination of the middle term, equations with one unknown, factum and the square-nature, one becomes the learned professor amongst the learned" (Datta and Singh, 130). The "pulveriser" was a method for solving equations.

Brahmagupta was one of the first Indian mathematicians to give rules for working with negative numbers. He referred to negative and affirmative (positive) quantities and gave the rules for working with such quantities.

> The sum of two positive numbers is positive,
> Of two negatives numbers is negative;
> Of a positive and a negative number is their difference.
> In subtraction, the less is to be taken from the greater.
> [The final result is] positive, if positive from positive,
> And negative, if negative from negative.
> If, however, the greater is subtracted from the less,
> That difference is reversed [in sign],
> Negative becomes positive and positive becomes negative.
> When positive is to be subtracted from negative or negative from positive,
> Then they must be added together. (Datta and Singh, 20)

Brahmagupta's rules for multiplication and division of signed numbers are as follows:

> The product of a positive and a negative [number] is negative;
> Of two negatives is affirmative;
> Positive multiplied by positive is affirmative.
> Positive divided by positive or negative by negative is affirmative.
> Positive divided by negative, is negative.

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> Negative divided by positive remains negative. (Boyer, 242)

He also stated rules for working with zero, claiming that zero divided by zero is zero, and

> A positive or negative, divided by zero, is a fraction with that for denominator. … [T]his fraction, of which the denominator is zero, is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth. (Katz, 226)

Today, we would consider zero divided by zero to be indeterminate and any other number divided by zero to be undefined. Brahmagupta also gave a rule for working with squares of signed numbers, declaring, "The square of a positive or a negative number is positive" (Datta and Singh, 23-24).

Mahavira, a ninth century Indian mathematician, who studied Brahmagupta's work and improved on it, asserted the utility and importance of mathematics in the introduction to his book, *Ganitasarasangraha*:

> In all those transactions which relate to worldly … or … religious affairs, calculation is of use. In the science of love, in the science of wealth, in music and in the drama, in the art of cooking, and similarly in medicine and in things like the knowledge of architecture; in prosody, in poetics and poetry, in logic and grammar and such other things,…the science of computation is held in high esteem. In relation to movements of the sun and other heavenly bodies, in connection with eclipses and the conjunction of planets…it is utilized. The number, the diameter and the perimeter of islands, oceans and mountains, the extensive dimensions of the rows of habitations and halls belonging to the inhabitants of the world, ... all of these are made out by means of computation. (Katz, 229)

Bhaskara II 1114-1185

A later Indian mathematician, Bhaskara, also called Bhaskara II because there was an earlier, seventh century mathematician named Bhaskara, was an astronomer in the Sahyadri area in southwestern India (Plofker, 2001 lecture). The Bhaskara in this part of the Story of Negative Numbers lived in the twelfth century. A common practice before language was written was to put knowledge in the form of verse to make it easier to memorize. The verse form continued in many cultures even after written languages became common. Bhaskara was one who liked to combine his mathematics with verse, as typified by the following problem:

> The square root of half the number of bees in a swarm
> Has flown out upon a jasmine bush;
> Eight ninths of the swarm has remained behind;
> A female bee flies about a male who is buzzing inside a lotus flower;
> In the night, allured by the flower's sweet odor, he went inside it

And now he is trapped!
Tell me, most enchanting lady, the number of bees. (Anglin, 115)

In modern algebraic form, Bhaskara is asking for the solution to the equation $\sqrt{\frac{x}{2}} + \frac{8}{9}x + 2 = x$, where x is the numbers of bees. You can check that $x = 72$ is a solution to this equation.

When Bhaskara wrote his mathematics, addition was indicated by juxtaposition, subtraction by a dot over the number to be subtracted (subtrahend), division by placing the divisor under the dividend, and multiplication and taking of roots by abbreviation of appropriate words (Boyer, 244). Bhaskara gave rules for working with positive and negative numbers. He stated:

> In addition of two negatives or two positive numbers the result is their sum.
> The sum of a positive and a negative is their difference.
> A positive [number] while being subtracted becomes negative,
> And a negative becomes positive; then addition as explained before.
> The product of two positive or two negative [numbers] is positive;
> The product of positive and negative is negative.
> In the case of division also, such are the rules [the same as for multiplication].
> The square of a positive and a negative number is positive;
> The square root of a positive number is positive as well as negative.
> There is no square root of a negative number, because it is non-square.
> (Datta and Singh, 20-24)

Bhaskara also found solutions to quadratic equations, including negative solutions. Although he was comfortable working with negative number solutions, he called them inadequate. Upon arriving at solutions of $x = 50$ and $x = -5$ to the quadratic equation $x^2 - 45x = 250$, he concluded, "The second value is in this case not to be taken, for it is inadequate; people do not approve of negative solutions" (Kline, 185). You can see from his statement that negative solutions were unacceptable during Bhaskara's time, even though he did acknowledge their existence. This is because Bhaskara and his contemporaries almost always dealt with problems describing physical objects, as in the problem about bees, so that a negative solution often did not make sense. (See the **Problems from India Activity**.)

Problems from India Activity
Teacher Notes

Level: This activity is designed for middle school through high school students who are familiar with linear and quadratic equations.

Materials: Make copies of the Student Pages to distribute to students.

Objective: Students will be exposed to mathematical problems and formulas from ancient and medieval India and will see how Indian mathematicians solved these problems. In particular, they will see how the Indian mathematician Bhaskara II dealt with negative numbers in his work.

When to Use: Use this activity when teaching problem solving with linear and quadratic equations.

How to Use: Read the background information below and Part 3: Negative Numbers in India from the Story of Negative Numbers. We recommend that you discuss this information with students and/or have them read it. In Part 3, the example of a problem about bees posed in verse form by Bhaskara II may be especially helpful. Have the students work in small groups of 2 to 4 students each to solve the problems. You could have students work the problems using modern methods, or using the Indian methods explained below, or both. In Problems 1, 3, and 4, students using modern methods would set up and solve linear equations. In Problems 5 and 6, students are asked to set up and check solutions to quadratic equations.

Background: Brahmagupta (*c.* 625) and Bhaskara II (1114-1185) were Indian mathematicians. They used negative numbers in the process of solving problems and found negative solutions to problems. They usually rejected these negative solutions (and even some positive solutions, as we shall see) because negative numbers of physical objects, such as monkeys or bees, did not make sense in the problems they were solving.

Other Ideas: Have students figure out how and why the Indian methods work. Compare these problems with the problems from the **Chinese Problems Activity.**

Historical and modern solutions to the problems in the activity:

1. The solution given in the Bakhshali manuscript is:

   [Put] any desired quantity in the vacant place, then construct the series $\left|\begin{array}{c|c|cc|cc} 1 & 2 & 2 & 3 & 6 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right|$;

   multiplied [becomes] $\left|\,1\,\middle|\,2\,\middle|\,6\,\middle|\,24\,\right|$; added [becomes] 33. Divide the visible quantity

   [given total] $\left|\begin{array}{c} 132 \\ 33 \end{array}\right|$; on reduction becomes $\left|\begin{array}{c} 4 \\ 1 \end{array}\right|$, the amount given [to the first].

   In modern notation, let x be the amount given to the first, then form the equation $x + 2x + 3(2x) + 4(3(2x)) = 132$. Simplify to $33x = 132$. Therefore $x = 4$ and the first had 4.

2. When $d = 0$, the quadrilateral is a triangle and Brahmagupta's formula reduces to Heron's formula.

3. According to Bhaskara II,

> Here the statement for equal worth is $6x + 300 = 10x - 100$. Now, by the rule, subtract the unknown on one side from that on the other. [The] unknown on the first side being subtracted from the unknown on the other side, the remainder is $4x$. The absolute term on the second side being subtracted from the absolute term on the first side, the remainder is 400. The residual known number 400 being divided by the coefficient of the residual unknown $4x$, the quotient is recognized to be the value of x, [or] 100.

   Notice that Bhaskara avoids any subtraction that would result in a negative number or coefficient, but that he does subtract –100 from 300, resulting in 400.

4. The equation $\frac{1}{5}x + \frac{1}{3}x + 3\left(\frac{1}{3}x - \frac{1}{5}x\right) + 1 = x$ or $\frac{1}{5}x + \frac{1}{3}x + \frac{2}{5}x + 1 = x$ describing the number x of bees has the solution $x = 15$ bees. Bhaskara identifies the coefficients $\frac{1}{5}, \frac{1}{3}, \frac{2}{5}$, and the "given" 1. He then subtracts $\frac{1}{5}, \frac{1}{3}$, and $\frac{2}{5}$ from 1, obtaining $\frac{1}{15}$, and divides 1 by $\frac{1}{15}$ to obtain 15 as the "resulting number of the swarm of bees." What Bhaskara is doing is assuming there is only one bee ($x = 1$), then noting that, under the conditions of the problem, this implies that there would be only $\frac{1}{15}$ bee, rather than the given 1 bee, left to hover between the jasmine and *pandanus* flowers. In order to obtain the correct number of bees, he must divide his assumed number of bees (1) by $\frac{1}{15}$, to obtain $x = 15$ bees. Equivalently, he must solve the equation $\frac{1}{15}x = 1$. Bhaskara's method, called the "method of false position," had been used to solve linear equations since ancient times in many cultures.

5. (a) The equation is $\left(\frac{1}{8}x\right)^2 + 12 = x$.

   (b) Students may check the given solutions using simple substitution, or they may solve the equation, which can be rewritten as $x^2 - 64x + 768 = 0$, by factoring, completing the square, or using the quadratic formula. Bhaskara solved the equation by completing the square. After adding $(32)^2$ to each side to obtain $x^2 - 64x + 1024 = 256$, or $(x - 32)^2 = 256$, he took the positive square root of each side to get $x - 32 = 16$, or $x = 48$. Returning to the equation $(x - 32)^2 = 256$, he checked to make sure that using the negative square root of 256 would not lead to a negative solution, noting that, since 16 is less than 32 in the equation $x - 32 = -16$, then $x = -16 + 32 = 16$ would be positive.

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6. (a) The equation is $\left(\frac{1}{5}x-3\right)^2+1=x$.

(b) Students may check the given solutions using simple substitution, or they may solve the equation, which can be rewritten as $x^2 - 55x + 250 = 0$, by factoring, completing the square, or using the quadratic formula. Bhaskara solved the equation by completing the square.

(c) Bhaskara rejected the solution $x = 5$ because one fifth of 5 monkeys, less 3 monkeys, yields –2 monkeys, an impossibility. Never mind that the number of monkeys in the cave would be $(-2)^2 = 4$ monkeys: Bhaskara says that the solution $x = 5$ is "incongruous. People do not approve a negative absolute number."

Sources: Anglin, 115; Burton, 204, 234-239; Katz, 218-230; Plofker 2000, 10

Problems from India Activity
Student Pages

1. Solve this problem from the Bakhshali manuscript (*c.* 500): The amount given to the first is not known. The second is given twice as much as the first; the third is given thrice as much as the second; and the fourth four times as much as the third. The total amount distributed is 132. What is the amount of the first?

2. Show that Heron's formula for the area of a triangle with sides of lengths *a, b,* and *c,* is a special case of Brahmagupta's formula for the area of a quadrilateral with sides of lengths *a, b, c,* and *d* inscribed in a circle. In each case, the quantity *s* is called the *semiperimeter* of the figure. Heron was a Greek mathematician living in Alexandria, Egypt, in the first century CE. Brahmagupta was an Indian mathematician from the seventh century CE.

   Heron's formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$

   Brahmagupta's formula: $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \dfrac{a+b+c+d}{2}$

The remaining problems were posed by the Indian mathematician Bhaskara II (1114-1185).

3. One person has three hundred coins and six horses. Another has ten horses of similar value and he has further a debt of one hundred coins. But they are of equal worth. What is the price of a horse?

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4. Out of a swarm of bees, one-fifth went to a *kadamba*-flower, one-third to a plantain-flower, and three times the difference of those … to a *kutaja*-flower. One remaining bee, tempted at the same time by the scent of a jasmine and a *pandanus,* hovered and wandered in the air. Tell me the number of bees.

5. The eighth part of a troop of monkeys, squared, was skipping in a grove and delighted with their sport. Twelve remaining monkeys were seen on the hill, amused with chattering to each other. How many were there in all?

   (a) Write an equation describing the number x of monkeys in the troop.

   (b) Show that $x = 16$ and $x = 48$ are solutions to your equation from part (a).

6. The fifth part of a troop of monkeys less three, squared, had gone to a cave; and one monkey was in sight having climbed on a branch. How many were there in all?

   (a) Write an equation describing the number x of monkeys in the troop.

   (b) Show that $x = 5$ and $x = 50$ are solutions to your equation from part (a).

   (c) Bhaskara II rejected the solution $x = 5$. Explain what, in the original problem, may have caused him to do so.

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