How Do You Slice The Bread?

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When Gail and John make lunches for school, their six-year old twins, Jamie and Michael, frequently ask if they can share a peanut butter sandwich. They always want it cut in half, and always in "triangles." This article is the result of trying to find a method of locating a point on the top curved-crust of a slice of bread that halves the volume of the sandwich, or equivalently the area of the bread-slice face. In addition, we treat the problem of halving the crust of the sandwich with "triangles."

A slice of bread can be modeled in the xy-plane as a rectangle surmounted by a semi-ellipse with semi-major axis parallel to the bottom crust. The origin is located as shown in Figure 1, so that the equation for the semi-ellipse is in simplest form.

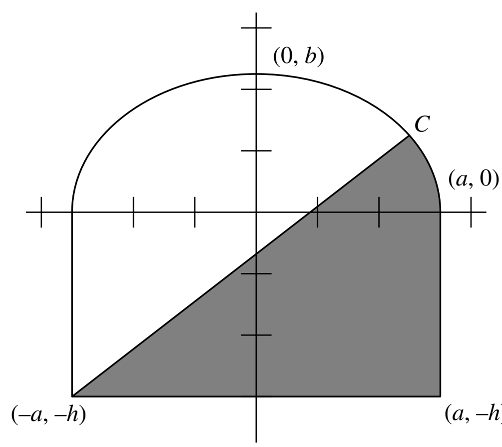

Figure 1. Two-dimensional model of a "triangulated" bread slice.

The line representing the path made by the knife blade passes from $(-a, -h)$ to the point C. C lies on the semi-ellipse with equation $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and has coordinates $(c, \frac{b}{a}\sqrt{a^2 - c^2})$. The equation for the line describing the path of the knife is

$$y = \frac{b/a\sqrt{a^2 - c^2} + h}{c + a}(x + a) - h. \tag{1}$$

We then seek to find values for c such that either the perimeter or the volume of the sandwich is halved. In either case, the x-coordinate of the point C, c, is parameterized in terms of a, b, and h.

The formulation derived above results in ellipse forms most recognizable to students. Dividing by the scalar a results in a non-standard ellipse equation with a cleaner integral form. Free of the dimension a, the problem produces the same relative locations as those determined below. A similar nondimensionalization can be made in the area problem to eliminate both a and b. We leave it up to the instructor to decide which way to present the two problems.

The arc length problem. We first address the problem of where to cut the slice so that the two resulting crust lengths are equal. Using the standard parameterization $x = a\cos\theta,\ y = b\sin\theta$ for the ellipse and the familiar arc length formula $\int \sqrt{dx^2 + dy^2}$ [**1**], one half of the perimeter of the bread slice is given by

$$\frac{1}{2}\left(2h + 2a + \int_0^{\pi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta\right)$$
$$= h + a + \int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta. \tag{2}$$

The "crust" perimeter of the shaded region in Figure 1 is then given by

$$2a + h + \int_0^{\arccos c/a} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta. \tag{3}$$

Equating expressions (2) and (3) and rearranging terms yields

$$a = \int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$
$$- \int_0^{\arccos c/a} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$
$$= \int_{\arccos c/a}^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta. \tag{4}$$

This trigonometric integral cannot be evaluated in closed form, so we instead consider the following special cases:

Case 1. When $b = a$ in Equation (4), that is, the bread slice is a rectangle surmounted by a semicircle, the following equation results:

$$a = \int_{\arccos c/a}^{\pi/2} a\, d\theta.$$

Carrying out the integration and solving for c yields

$$c = a\sin 1.$$

Case 2. Setting b equal to 0, Equation 4 becomes

$$a = \int_{\arccos c/a}^{\pi/2} \sqrt{a^2\sin^2\theta}\, d\theta = \int_{\arccos c/a}^{\pi/2} a\sin\theta\, d\theta.$$

Integrating and solving for c, we have the expected value $c = a$. We expect this result because when $b = 0$, the bread slice is shaped like a rectangle and therefore the cut should extend to the corner.

The area problem. We now address the problem of determining where to cut the slice so that the areas of the two "triangles" are equal. One-half of the area of the entire bread slice is given by

$$\frac{1}{2}\left(\frac{\pi}{2}ab + 2ah\right) = \frac{\pi ab}{4} + ah. \tag{5}$$

The area of the unshaded region in Figure 1 is given by integrating the difference of the y-coordinates of the ellipse and the cut line (1) from $x = -a$ to $x = c$:

$$\int_{-a}^{c} \left[\frac{b}{a}\sqrt{a^2 - x^2} - \left(\frac{b/a\sqrt{a^2 - c^2} + h}{c + a}(x + a) - h\right)\right] dx \tag{6}$$

Equating expressions (5) and (6) results in the integral equation

$$\frac{\pi ab}{4} + ah = \int_{-a}^{c} \left(\frac{b}{a}\sqrt{a^2 - x^2} - \frac{b/a\sqrt{a^2 - c^2} + h}{c + a}(x + a) + h\right) dx$$

Carrying out the integration and simplifying yields the equation

$$-b\sqrt{a^2 - c^2} - h(a - c) + ab \arcsin\frac{c}{a} = 0. \tag{7}$$

In this form, Equation (7) has no closed form solution for c. Assuming that the cut-point c is sufficiently close to a, we can apply the small-angle approximation $\sin\theta \approx \theta$ to obtain $\arcsin(c/a) \approx c/a$. We substitute this into Equation (7) to yield

$$c(b + h) + b\sqrt{a^2 - c^2} - ah = 0$$

with solutions

$$c = \frac{a}{(b + h)^2 + b^2}\left(h(b + h) \pm b\sqrt{2b(b + h)}\right).$$

Rearrangement yields

$$c = \frac{a\sqrt{1 + b/h}\left(\sqrt{1 + b/h} \pm b/h\sqrt{2b/h}\right)}{(1 + b/h)^2 + (b/h)^2}. \tag{8}$$

We now consider two special cases:

Case 1. Taking $\frac{b}{h} \to 0$ in Equation (8) yields $c = a$. As in the second arc length case, this bread slice is rectangular and so we expect a corner-to-corner cut to divide the slice in half.

Case 2. Taking $\frac{b}{h} \to 1$ in Equation (8), we have $c = \frac{4}{5}a$. Using *Mathematica* [**2**], we solve Equation (7) numerically to obtain the solution $c = 0.767132\,a$ in this case, which is within 5 percent of our approximate solution. Alternately, Newton's method can be used to determine this root of Equation (7).

Conclusion. These two bisection problems present material involving modeling, arc length, and area calculations suited for a first-year calculus course. Instructors looking to avoid parameterizing the ellipse could discuss only the area problem and the semicircular case of the arc length problem. It is also interesting to note that employing the usual approximation for the perimeter of a semi-ellipse, $\pi\sqrt{(a^2+b^2)/2}$, does not result in a simpler problem. Students and instructors looking to extend this problem could model the bread slices with curves other than ellipses.

References

1. G. B. Thomas, Jr. and R. L. Finney, *Calculus and Analytic Geometry*, 9th ed., Addison-Wesley, 1996.
2. S. Wolfram, *The Mathematica Book*, 3rd ed., Wolfram Media, 1996.

Limits of Functions of Two Variables

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A common way to show that a function of two variables is not continuous at a point is to show that the 1-dimensional limit of the function evaluated over a curve varies according to the curve that is used. For example one can show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is discontinuous at (0, 0) by showing that

$$\lim_{(x,mx)\to(0,0)} f(x,y) = \frac{m}{1+m^2},$$

which varies with m. The caveat is that the natural converse to this technique cannot be used to demonstrate that a function is continuous. One reminds students that

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

exists only when the limit of f exists as (x, y) approaches (a, b) over *all* curves that run through (a, b).

There is often some vagueness as to what is meant by *all* curves (e.g., all continuous curves, all differentiable curves) and we will see that such vagueness can lead to trouble.

A classic example (e.g., [**1**, exercise 8, p. 165]) is to demonstrate that for the function

$$f(x,y) = \begin{cases} 0 & \text{if } y \le 0 \quad \text{or} \quad y \ge x^n \\ 1 & \text{if } 0 < y < x^n \end{cases}$$

