On the divisibility and valuations of the Franel numbers

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Abstract

The Franel numbers are the sums of the cubes of binomial coefficients. Primes may be regarded with respect to their p-adic valuations of the Franel numbers. For some, the valuations are always 0. For others, the valuations seem to be equal to the number of occurrences of a particular digit in the base-p representation of the index. Furthermore, the 2-adic valuations of the Franel numbers have interesting properties that are explored.

Introduction

The sums of the first and second powers of the binomial coefficients are

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

respectively. The Franel numbers are the sums of the third powers.

Definition 1. Let $n \in \mathbb{N}$ (throughout, 0 is considered to be an element of $\mathbb{N}$). The $\boldsymbol{n}^{\text{th}}$ **Franel number**, denoted Fra_n, is

$$\sum_{k=0}^{n} \binom{n}{k}^3.$$

Franel, after whom the numbers are named, derives a second order recurrence formula for the Franel numbers in [Fr].

Theorem 1. *Let $n \geq 2$ be a natural number. Then*

$$n^2 \,\mathrm{Fra}_n = \left(7n^2 - 7n + 2\right) \mathrm{Fra}_{n-1} + 8(n-1)^2 \,\mathrm{Fra}_{n-2}$$

with $\mathrm{Fra}_0 = 1$ *and* $\mathrm{Fra}_1 = 2$.

The following is preliminary information to help understand the exploration of the Franel numbers.

Definition 2. Let p be a prime. Let $n \in \mathbb{N}$. The $\boldsymbol{p}$**-adic valuation of** $\boldsymbol{n}$, denoted $\nu_p(n)$, is the highest power of p that divides n.

Example. The 3-adic valuation of 24, $\nu_3(24) = 1$, since $3^1 \mid 24$ but $3^2 \nmid 24$.

For some primes p, the p-adic valuation of the n^{th} Franel number is always 0.

Definition 3. A prime is **type I** if it does not divide any Franel number.

Through much computer experimentation, it was also discovered that for other primes p, the p-adic valuation of the n^{th} Franel number seems to be determined by the base-p representation of the index n.

Definition 4. A prime is **type II** if $\nu_p(\mathrm{Fra}_n) = C_p\left(n, \frac{p-1}{2}\right)$ for all natural numbers n, where $C_p(n,k)$ is the number of k's in the base-p representation of n.

Additionally it was observed that the 2-adic valuations of the Franel numbers seem to have a different structure than valuations with other primes.

The primary goals of the research were to identify Type I and Type II primes and to explore the patterns of the 2-adic valuations of the Franel numbers.

Results I

The following theorem provides the basis for determining which primes are type I.

Theorem 2. *Let p be a prime. Let $n = n_d \cdots n_0$ be a natural number with its base-p representation. Then*

$$\mathrm{Fra}_n \equiv \prod_{j=0}^{d} \mathrm{Fra}_{n_j} \pmod{p}.$$

As a result, Corollary immediately follows.

Corollary. *A prime is type I if and only if the prime does not divide any of* $\mathrm{Fra}_0, \mathrm{Fra}_1, \ldots, \mathrm{Fra}_{p-1}$.

From this, a list of type I primes is found.

Corollary. *The only primes less than 100 that are type I are* 3, 11, 17, 19, 43, 83, 89, *and* 97.

Results II

While it has not been proven, 5 seems to be the first type II prime, as $\nu_5(\mathrm{Fra}_n) = C_5(n,2)$ for $n \leq 10^7$. It is also noted that all type II primes have particular modular residues.

Theorem 3. *If a prime is type II, then it is congruent to 5 or 7* $\pmod{8}$.

Another theorem gives additional credence to the belief that 5 is type II.

Theorem 4. *For all natural numbers n, if $C_5(n,2) \leq 2$, then $\nu_5(\mathrm{Fra}_n) = C_5(n,2)$.*

However, it is still only conjectured that 5 (and a number of other primes) are type II.

Conjecture. The only primes less than 100 that are type II are 5, 7, 13, 23, 31, 37, 47, 53, and 71.

While no proof has been found that any prime is type II, Theorem 5 is thought to be key.

Theorem 5. *Let p be an odd prime, $r \in \mathbb{Z}^+$, and $n \in \mathbb{N}$. Then*

$$\mathrm{Fra}_{np^r} \equiv \mathrm{Fra}_{np^{r-1}} \pmod{p^r}.$$

Results III

The 2-adic valuations of the Franel numbers provide certain recursions with a few exceptions, which appear to be categorizable. The following table is a list of indices that have exceptions to the recurrence found for the index $n \equiv 0 \pmod{8}$, the corresponding base-2 representation, and its index in the table, denoted by i.

n	base-2 of n	i
349528	$000\vert 01010101010101011000_2$	0
1398104	$001\vert 01010101010101011000_2$	1
2446680	$010\vert 01010101010101011000_2$	2
3495256	$011\vert 01010101010101011000_2$	3
4543832	$100\vert 01010101010101011000_2$	4
5592408	$101\vert 01010101010101011000_2$	5
6640984	$110\vert 01010101010101011000_2$	6
7689560	$111\vert 01010101010101011000_2$	7

It is conjectured that the terms of the form $n \equiv 0 \pmod{8}$ can be expressed as: $349528 + i \times 2^{20}$, where i is the corresponding index in the table. This has been verified for indices up to 7.7×10^6. A similar pattern is observed when $n \equiv 1, 2, 3 \pmod{8}$.

Conclusion

The following theorems and conjectures were found regarding the divisibility and valuations of the Franel numbers:

Theorem. *A prime p is type I if and only if p does not divide any of* $\mathrm{Fra}_0, \mathrm{Fra}_1, \ldots, \mathrm{Fra}_{p-1}$.

Theorem. *Let p be a prime. Let $n = n_d \cdots n_0 \in \mathbb{N}$ with its base-p representation. Then*

$$\mathrm{Fra}_n \equiv \prod_{j=0}^{d} \mathrm{Fra}_{n_j} \pmod{p}.$$

Theorem. *Let p be an odd prime, $r \in \mathbb{Z}^+$, and $n \in \mathbb{N}$. Then*

$$\mathrm{Fra}_{np^r} \equiv \mathrm{Fra}_{np^{r-1}} \pmod{p^r}.$$

Conjecture. For indices of the following modular equivalences, exceptions to the recurrences of 2-adic valuations of the Franel numbers can be expressed as shown, where i is the index of the exception:

- $n \equiv 0 \pmod{8}$ can be expressed as $349528 + i \times 2^{20}$,
- $n \equiv 1 \pmod{8}$ can be expressed as $87385 + i \times 2^{18}$,
- $n \equiv 2 \pmod{8}$ can be expressed as $349530 + i \times 2^{19}$, and
- $n \equiv 3 \pmod{8}$ can be expressed as $349531 + i \times 2^{19}$.

This conjecture has been verified for indices up to 7.7×10^6. New methods will have to be devised to prove this in general.

References

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